Quantitative recurrence for beta expansion

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Background

A well known theory on recurrence is Poincaré's recurrence theorem.

Theorem (Poincaré's Recurrence Theorem)

Let $T : X \to X$ be a measurable transformation on a probability space $(X, \mathcal{B}, \mu)$. Let $B \in \mathcal{B}$ with $\mu(B) > 0$. Then for almost all points $x \in B$, the orbit $\{T^n(x)\}_{n \geq 0}$ returns to B infinitely often.

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However, this information is only of qualitative nature and it is only concerned with a fixed target B.

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Problem 3

What happens if the ball B shrinks with time and more generally if the ball also moves around with time?

Beta expansion

Let $\beta > 1$ be a real number. The beta transformation with respect to β is given as

$$T_\beta(x) = \beta x (\text{mod } 1).$$

Hence, any $x \in [0,1)$ can be expanded uniquely as an finite or infinite series with the form

$$x = \frac{\epsilon_1(x,\beta)}{\beta} + \frac{\epsilon_2(x,\beta)}{\beta^2} + \cdots,$$

where $\epsilon_n(x,\beta) = [\beta T_\beta^{n-1}(x)]$ are called the digits of x and the sequence $\Big(\epsilon_1(x,\beta), \epsilon_2(x,\beta), \cdots\Big)$ is called the sequence of the beta expansion of x with respect to the base β.

Admissible sequence

Definition

An n-block $(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ is called admissible in base β if there exists $x \in [0, 1]$ such that $\epsilon_k(x, \beta) = \epsilon_k$ for all $1 \leq k \leq n$. An infinite sequence $(\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots)$ is admissible in base β if $(\epsilon_1, \epsilon_2, \cdots, \epsilon_k)$ is admissible in base β for all $k \geq 1$.

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Denote by Σ_β all infinite admissible sequence in the beta expansion with respect to the base β.

Characterizations on admissible sequence

Theorem (W. Parry)

Let $\beta > 1$ be a real number and $\epsilon(1,\beta)$ be the β-expansion of 1. We denote by w an infinite sequence of positive integer.
(1) If $\epsilon(1,\beta)$ is infinite, $w \in \Sigma_\beta$ if and only if $\sigma^k(w) <_{lex} \epsilon(1,\beta)$, for all $k \geq 0$.
(2)If $\epsilon(1,\beta)$ is finite, i.e., $\epsilon(1,\beta) = (\epsilon_1(1,\beta), \cdots, \epsilon_n(1,\beta), 0^\infty)$ with $\epsilon_n(1,\beta) \neq 0$. Then $w \in \Sigma_\beta$ if and only if $\sigma^k(w) <_{lex} \epsilon^*(1,\beta)$, for all $k \geq 0$. where $\epsilon^*(1,\beta) = \Big(\epsilon_1(1,\beta), \epsilon_2(1,\beta), \cdots, \epsilon_{n-1}(1,\beta), (\epsilon_n(1,\beta) - 1)\Big)^\infty$ is a purely periodic sequence.

Known results

Quantitative recurrence properties to general settings.

Theorem (Boshernitzan)

Let (X, μ, T, d) be a measure dynamical system with a metric d. Assume that, for some $\alpha > 0$, the Hausdorff α-measure $\mathcal{H}^{\alpha}$ is σ-finite on the space X. Then for μ-almost all $x \in X$,

$$\liminf_{n\to\infty} n^{\frac{1}{\alpha}} d(T^n x, x) < \infty.$$

If, moreover, $\mathcal{H}^{\alpha}(X) = 0$, then for μ-almost all $x \in X$,

$$\liminf_{n\to\infty} n^{\frac{1}{\alpha}} d(T^n x, x) = 0.$$

When applied to beta transformation, one has, almost surely,

$$\liminf_{n\to\infty} nd(T^n x, x) < \infty.$$

Quantitative recurrence properties to beta transformations.

Theorem (W. Philipp)

Let ϕ be a positive function defined on natural numbers $\mathbb{N}$. The set

$$R_\beta(\phi) := \left\{ x \in [0,1] : T_\beta^n x \in B(x, \phi(n)^{-1}),\ \text{infinitely often}\ n \in \mathbb{N} \right\}$$

is a full or null set if and only if the series $\sum_{n\geq 1} \phi^{-1}(n)$ diverges or not.

Our formulations

Given a general function $\phi : \mathbb{N} \to \mathbb{R}^+$, find the Hausdorff dimension of the set

$$\Big\{x \in [0,1] : |T_\beta^n x - x| < \phi(n)^{-1}, \text{ infinitely often } n \in \mathbb{N}\Big\}.$$

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Theorem (Main result)

For any $\beta > 1$,

$$\dim_H R_\beta(\phi) = \frac{1}{1+b}, \text{ where } b = \liminf_{n\to\infty} \frac{\log_\beta \phi(n)}{n}.$$

Difficulties

In studying the metric theory related to beta transformation, one of the big obstacle lies in estimating the length of a cylinder

$$I_\beta(\epsilon_1, \cdots, \epsilon_n) = \{x \in [0,1] : \epsilon_k(x,\beta) = \epsilon_k, 1 \le k \le n\}$$

for general $\beta > 1$.

Our strategy

Principle

As far as Haudorff dimension or metric theory is concerned, we can discard some bad points without affecting the final results.

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As a result, we confine our attention to "good" points:

$$\{x = (\epsilon_1, \cdots, \epsilon_n, \cdots) : (\epsilon_1, \cdots, \epsilon_n, \cdots) \in \Sigma_{\beta_N}\},$$

where β_N is defined as the solution to the equation

$$1 = \frac{\epsilon_1(1,\beta)}{\beta_N} + \frac{\epsilon_2(1,\beta)}{\beta_N^2} + \cdots + \frac{\epsilon_N(1,\beta)}{\beta_N^N}.$$

By the criterion for admissible sequence, one can know easy that

$$\Sigma_{\beta_N} \subset \Sigma_\beta.$$

More importantly, for any admissible block $(\epsilon_1, \cdots, \epsilon_n) \in \Sigma_{\beta_N}$, $(\epsilon_1, \cdots, \epsilon_n, 0^N, 1) \in \Sigma_{\beta_N}$, so is in Σ_β.

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As a consequence,

Proposition

For any admissible block $(\epsilon_1, \cdots, \epsilon_n) \in \Sigma_{\beta_N}$, then it holds the following length estimation

$$\left| I_\beta(\epsilon_1, \cdots, \epsilon_n) \right| \asymp \frac{1}{\beta^n}.$$

Cantor subset

It is easy to see that if the β-expansion of $T^n x$ and that of x coincide in a long block from the very beginning, one can make sure that $T^n x$ and x are close enough. So, our Cantor subset is defined as follows.

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$$E_1 = \bigcup_{\epsilon_1, \cdots, \epsilon_{n_1} \in \Sigma_{\beta_N}^{n_1}} I_\beta((\epsilon_1, \cdots, \epsilon_{n_1} 0^N)^{w_1+1} 0^N)$$

where w_1 is some rational with $w_1(n_1 + N) \in \mathbb{N}$ such that

$$\beta^{-(n_1+N)w_1} < \phi(n_1)^{-1}, \text{ but } \beta^{-(n_1+N)w_1+1} \geq \phi(n_1)^{-1}.$$

Then it follows that, for any $x \in I_\beta((\epsilon_1, \cdots, \epsilon_{n_1} 0^N)^{w_1+1} 0^N)$,

$$\left| T_\beta^{n_1} x - x \right| < \frac{1}{\phi(n_1)}. \tag{2.1}$$

Level 2. Write $m_1 = (w_1+1)(n_1+N)$, $t_2 = n_2 - m_1$. For each $I_\beta = I_\beta(\epsilon_1, \cdots, \epsilon_{m_1})$, set

$$E_2(I_\beta) = \bigcup_{\epsilon_{m_1+1},\cdots,\epsilon_{n_2} \in \Sigma_{\beta_N}^{t_2}} I_\beta((\epsilon_1, \cdots, \epsilon_{n_2} 0^N)^{w_2+1} 0^N),$$

where w_2 is some rational with $w_2(n_2+N) \in \mathbb{N}$ such that

$$\beta^{-(n_2+N)w_2} < \phi(n_2^{-1}), \text{ but } \beta^{-(n_2+N)w_2+1} \geq \phi(n_2^{-1}).$$

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where w_2 is some rational with $w_2(n_2+N) \in \mathbb{N}$ such that

$$\beta^{-(n_2+N)w_2} < \phi(n_2^{-1}), \text{ but } \beta^{-(n_2+N)w_2+1} \geq \phi(n_2^{-1}).$$

Then finally, define

$$E_2 = \bigcup_{I_\beta(\epsilon_1,\cdots,\epsilon_{m_1}) \in E_1} E_2(I_\beta(\epsilon_1, \cdots, \epsilon_{m_1})).$$

Further study

Size of the level set defined by the Birkhoff ergodic related to beta expansion.

Theorem (Fan, Feng, Wu)

Let (Σ_A, T) *be a topologically mixing subshift of finite type on an alphabet consisting of* m *symbols and let* $\Phi : \Sigma_A \to \mathbb{R}^d$ *be a continuous function. For any possible limit of the ergodic limit* α *of* $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \Phi(T^j x)$, *one has*

$$h_{\text{top}}\left(x \in \Sigma_A : \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \Phi(T^j x) = \alpha\right) = \sup\left\{h_\mu : \int \Phi d\mu = \alpha\right\}$$

where h_μ *denotes the entropy of* μ *and* h_{top} *denotes the topological entropy.c* d.

If we relate above system with beta shift, it deals with the β with finite expansion of the unit 1. Then how about the case for general β.

Question 1

Given $\beta > 1$. Denote by Σ_β the collection of all infinite admissible sequence in the beta expansion in base β. Let $\Phi : \Sigma_\beta \to \mathbb{R}$ be a continuous function. How about the size of the set

$$\left\{ x \in [0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \Phi(T^j(\epsilon(x,\beta))) = \alpha \right\},$$

where $\epsilon(x,\beta)$ denotes the sequence of the β-expansion of x.

Shrinking target problems for beta transformation.

Question 2

Given $\beta > 1$ and $x_0 \in [0, 1)$. Let Φ be a positive function defined on $\mathbb{N}$. Find the Hausdorff dimension of the set

$$\Big\{x \in [0, 1] : \Big|T^n x - x_0\Big| < \Phi(n), \quad \text{for infinite many } n \in \mathbb{N}\Big\}.$$

Thanks

THANK YOU!