

Uniform Sets and Super-Stationary Sets

based on a joint work with T. Kamae, H. Rao & Y.-M. Xue

Bo TAN

Huazhong University of Science & Technology

Guangzhou
July 8, 2010



Notations

Σ : a countably infinite set

$\{0, 1\}^\Sigma$: the set of 0-1-words defined on Σ ;
equivalently, the set of functions $\Sigma \rightarrow \{0, 1\}$

π_S : the projection (or restriction) $\{0, 1\}^\Sigma \rightarrow \{0, 1\}^S$,
where $S \subset \Sigma$

Example

$$\Sigma = \{ \dots a b c d e f g \dots \}$$

$$\Omega = \left\{ \begin{array}{cccccccccc} \dots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & \dots \\ \dots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{array} \right\}$$

$$\text{Then, } \pi_{\{d,e,f\}}\Omega = \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right\}$$

Hence, $\#\pi_{\{d,e,f\}}\Omega = 4$.

Uniform set

Definition

A closed set $\Omega \subset \{0, 1\}^\Sigma$ is called a **uniform set** if $\#\pi_S\Omega$ depends only on $\#S$ for any finite set $S \subset \Sigma$.

Whence the function $p_\Omega(k) := \#\pi_S\Omega$ with $\#S = k$ is called the **uniform complexity function** of Ω (by convention $p_\Omega(0) = 1$.)

Uniform set

Definition

A closed set $\Omega \subset \{0, 1\}^\Sigma$ is called a **uniform set** if $\#\pi_S\Omega$ depends only on $\#S$ for any finite set $S \subset \Sigma$.

Whence the function $p_\Omega(k) := \#\pi_S\Omega$ with $\#S = k$ is called the **uniform complexity function** of Ω (by convention $p_\Omega(0) = 1$.)

Example (trivial cases: full and singleton)

- If $\Omega = \{0, 1\}^\Sigma$, then Ω is a uniform set with $p_\Omega(k) = 2^k$.
- If $\Omega = \{\omega\}$, then Ω is a uniform set with $p_\Omega(k) = 1$.

Examples

Example (doubleton)

$\Omega = \{\omega, \eta\}$ is uniform with $p_{\Omega}(k) = 2$ if and only if $\omega = \bar{\eta}$, i.e. $\omega(\sigma) = 1 - \eta(\sigma) (\forall \sigma \in \Sigma)$.

$$\Omega = \left\{ \begin{array}{cccccccc} \cdots & 1 & 1 & 0 & 1 & 1 & \cdots & \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & \end{array} \right\}$$

Examples

Example (doubleton)

$\Omega = \{\omega, \eta\}$ is uniform with $p_{\Omega}(k) = 2$ if and only if $\omega = \bar{\eta}$, i.e. $\omega(\sigma) = 1 - \eta(\sigma)$ ($\forall \sigma \in \Sigma$).

$$\Omega = \left\{ \begin{array}{cccccccc} \cdots & 1 & 1 & 0 & 1 & 1 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \end{array} \right\}$$

Example (Dirac)

$\Omega = \{\omega; \sum_{\sigma \in \Sigma} \omega(\sigma) \leq 1\}$ is uniform with $p_{\Omega}(k) = k + 1$.

$$\Sigma = \left\{ \begin{array}{cccccccc} \cdots & a & b & c & d & e & f & \cdots \\ & 0 & & 0 & & & 0 & \\ \pi_{\{a,c,f\}}\Omega = & \left\{ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right\} & & & & & & \end{array} \right\}$$

Hence, $p_{\Omega}(3) = 4$.

Examples

Example (rotation word)

Let θ be an irrational number and $\Sigma = \{n\theta; n \in \mathbb{N}\} \subset \mathbb{R}/\mathbb{Z}$. For $x \in \mathbb{R}/\mathbb{Z}$, let $\omega_x \in \{0, 1\}^\Sigma$ be such that

$$\omega_x(\sigma) = 0 \iff x + \sigma \in [0, 1/2) \pmod{1}.$$

Then, $\Omega := \{\omega_x; x \in \mathbb{R}/\mathbb{Z}\}$ is a uniform set with $p_\Omega(k) = 2k$, since for any $\{\sigma_1, \sigma_2, \dots, \sigma_k\} \subset \Sigma$, the partition generated by the sets $[-\sigma_i, 1/2 - \sigma_i)$ ($i = 1, 2, \dots, k$) on \mathbb{R}/\mathbb{Z} has $2k$ elements.

Notation

Let $\mathcal{N} = \{N_0 < N_1 < N_2 < \dots\}$ be an infinite subset of \mathbb{N} .
 For $\omega \in \{0, 1\}^{\mathbb{N}}$ and $\Omega \subset \{0, 1\}^{\mathbb{N}}$, define $\omega[\mathcal{N}] \in \{0, 1\}^{\mathbb{N}}$ and $\Omega[\mathcal{N}] \subset \{0, 1\}^{\mathbb{N}}$ by

$$\begin{aligned}\omega[\mathcal{N}](n) &:= \omega(N_n) \quad (n \in \mathbb{N}) \\ \Omega[\mathcal{N}] &:= \{\omega[\mathcal{N}]; \omega \in \Omega\}.\end{aligned}$$

$$\begin{array}{rcccccccccccc} \Sigma = & \{ & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \dots & \} \\ \mathcal{N} = & \{ & & N_0 & & N_1 & N_2 & & & N_3 & & N_4 & N_5 & \dots & \} \\ \omega = & & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & \dots & \end{array}$$

Then, $\omega[\mathcal{N}] = 0\ 1\ 0\ 0\ 0\ 1\ \dots$.

Super-Stationary set

Definition

A closed set $\Omega \subset \{0, 1\}^{\mathbb{N}}$ is called a **super-stationary** set if $\Omega[\mathcal{N}] = \Omega$ holds for any infinite subset \mathcal{N} of \mathbb{N} .

Super-Stationary set

Definition

A closed set $\Omega \subset \{0, 1\}^{\mathbb{N}}$ is called a **super-stationary** set if $\Omega[\mathcal{N}] = \Omega$ holds for any infinite subset \mathcal{N} of \mathbb{N} .

- Uniform sets are introduced to study the structure of maximal pattern sturmian words;

Super-Stationary set

Definition

A closed set $\Omega \subset \{0, 1\}^{\mathbb{N}}$ is called a **super-stationary** set if $\Omega[\mathcal{N}] = \Omega$ holds for any infinite subset \mathcal{N} of \mathbb{N} .

- Uniform sets are introduced to study the structure of maximal pattern Sturmian words;
- Super-stationary sets are uniform; also they are stationary (i.e. shift invariant);

Super-Stationary set

Definition

A closed set $\Omega \subset \{0, 1\}^{\mathbb{N}}$ is called a **super-stationary** set if $\Omega[\mathcal{N}] = \Omega$ holds for any infinite subset \mathcal{N} of \mathbb{N} .

- Uniform sets are introduced to study the structure of maximal pattern Sturmian words;
- Super-stationary sets are uniform; also they are stationary (i.e. shift invariant);
- Super-stationary sets are considered as phenomena which are independent of the time scale, but sensitive only to the direction of time.

Examples

- `example:full` or `example:Dirac` with $\Sigma = \mathbb{N}$ is a super-stationary set.

Examples

- `example:full` or `example:Dirac` with $\Sigma = \mathbb{N}$ is a super-stationary set.
- `example:singleton` or `example:doubleton` with $\Sigma = \mathbb{N}$ is a super-stationary set if and only if $\omega = 0^\infty$ or 1^∞ .

Examples

- `example:full` or `example:Dirac` with $\Sigma = \mathbb{N}$ is a super-stationary set.
- `example:singleton` or `example:doubleton` with $\Sigma = \mathbb{N}$ is a super-stationary set if and only if $\omega = 0^\infty$ or 1^∞ .
- For `example:rotation`, take $\{\sigma_0, \sigma_1, \sigma_2, \dots\} \subset \Sigma$ such that either

$$\sigma_0 < \sigma_1 < \sigma_2 < \dots < \sigma_0 + 1/2$$

or

$$\sigma_0 > \sigma_1 > \sigma_2 > \dots > \sigma_0 - 1/2.$$

Let $\psi : \mathbb{N} \rightarrow \Sigma$ be such that $\psi(n) = \sigma_n$. Then, $\Omega \circ \psi$ is a super-stationary set. In fact, we have

$$\Omega \circ \psi = \{\omega \in \{0, 1\}^{\mathbb{N}}; \omega \text{ is monotone}\}.$$

super-stationary set in uniform set

In general, we can prove that

Theorem

For a uniform set $\Omega \subset \{0, 1\}^\Sigma$, there exists an injection $\psi : \mathbb{N} \rightarrow \Sigma$ such that $\Omega \circ \psi$ is a super-stationary set. Hence, any uniform complexity function is realized by a super-stationary set.

Notation

For $\xi = \xi_1 \xi_2 \cdots \xi_l \in \{0, 1\}^+$ and $\omega \in \{0, 1\}^{\mathbb{N}}$, ξ is called a **super-subword** of ω , denoted $\xi \ll \omega$, if

$$\omega(s_1)\omega(s_2)\cdots\omega(s_l) = \xi$$

holds for some $0 \leq s_1 < s_2 < \cdots < s_l$.

For $\xi \in \{0, 1\}^*$ and $\Xi \subset \{0, 1\}^+$, denote

$$\mathcal{P}(\xi) = \{\omega \in \{0, 1\}^{\mathbb{N}}; \xi \ll \omega \text{ does not hold}\}$$

$$\mathcal{P}(\Xi) = \bigcap_{\xi \in \Xi} \mathcal{P}(\xi)$$

$$\mathcal{Q}(\Xi) = \bigcup_{\xi \in \Xi} \mathcal{P}(\xi)$$

cover

We call $\eta \in \{0, 1\}^* \cup \{0, 1\}^{\mathbb{N}}$ a **cover** of Ξ if $\xi \ll \eta$ for any $\xi \in \Xi$.

A cover η of Ξ is called a **minimal cover** of Ξ if any $\zeta \ll \eta$ but $\zeta \neq \eta$ is not a cover of Ξ .

The **least common multiple** of Ξ is the set of all minimal covers of Ξ , which is denoted by $L(\Xi)$.

cover

We call $\eta \in \{0, 1\}^* \cup \{0, 1\}^{\mathbb{N}}$ a **cover** of Ξ if $\xi \ll \eta$ for any $\xi \in \Xi$.

A cover η of Ξ is called a **minimal cover** of Ξ if any $\zeta \ll \eta$ but $\zeta \neq \eta$ is not a cover of Ξ .

The **least common multiple** of Ξ is the set of all minimal covers of Ξ , which is denoted by $L(\Xi)$.

- Note that $L(\Xi)$ is a finite subset of $\{0, 1\}^*$ for any finite subset Ξ of $\{0, 1\}^*$.

Structure of Super-Stationary sets

Theorem

- Ω is a super-stationary set other than $\{0, 1\}^{\mathbb{N}}$ if and only if $\Omega = \mathcal{Q}(\Xi)$ for some nonempty finite set $\Xi \subset \{0, 1\}^+$.
- For nonempty finite set $\Xi \subset \{0, 1\}^+$, $\mathcal{Q}(\Xi) = \mathcal{P}(L(\Xi))$.
- Any uniform complexity function in k other than 2^k is written as a polynomial function of k for large k .

Bigger Alphabet

Recently, Kamae generalized these results to a general alphabet.

Hereafter, the alphabet \mathbb{A} consists of at least 2 letters. Uniform sets and super-stationary sets in $\mathbb{A}^{\mathbb{N}}$ can be defined in the same way.



Still OK

Theorem

For a uniform set $\Omega \subset \mathbb{A}^{\mathbb{N}}$, there exists an infinite subset $\mathcal{N} \subset \mathbb{N}$ such that $\Omega[\mathcal{N}]$ is a super-stationary set. Hence, any uniform complexity function is realized by a super-stationary set.

Still OK

Theorem

For a uniform set $\Omega \subset \mathbb{A}^{\mathbb{N}}$, there exists an infinite subset $\mathcal{N} \subset \mathbb{N}$ such that $\Omega[\mathcal{N}]$ is a super-stationary set. Hence, any uniform complexity function is realized by a super-stationary set.

The uniform complexity functions over \mathbb{A} can be efficiently calculated.



Not yet OK

Recall that for $\xi \in \mathbb{A}^*$ and $\Xi \subset \mathbb{A}^+$, denote

$$\mathcal{P}(\xi) = \{\omega \in \mathbb{A}^{\mathbb{N}}; \xi \ll \omega \text{ does not hold}\}$$

$$\mathcal{P}(\Xi) = \bigcap_{\xi \in \Xi} \mathcal{P}(\xi)$$

$$\mathcal{Q}(\Xi) = \bigcup_{\xi \in \Xi} \mathcal{P}(\xi)$$

Not yet OK

- $Q(\Xi)$ is always super-stationary.
- When $\#\mathbb{A} \geq 3$, there is a super-stationary set other than $\mathbb{A}^{\mathbb{N}}$ which can not be written as $Q(\Xi)$ for any finite $\Xi \subset \mathbb{A}^*$.
- On the other hand, a set is super-stationary if and only if it can be written as $\mathcal{P}(\Xi)$ with $\Xi \subset \mathbb{A}^*$ satisfying some technical condition.

