Uniform Sets and Super-Stationary Sets

based on a joint work with T. Kamae, H. Rao & Y.-M. Xue

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Notations

Σ: a countably infinite set

$\{0,1\}^{\Sigma}$: the set of 0-1-words defined on Σ; equivalently, the set of functions $\Sigma \to \{0,1\}$

π_S: the projection (or restriction) $\{0,1\}^{\Sigma} \to \{0,1\}^{S}$, where $S \subset \Sigma$

Example

$$\Sigma = \{ \quad \cdots \quad a \quad b \quad c \quad d \quad e \quad f \quad g \quad \cdots \quad \}$$

$$\Omega = \left\{ \begin{array}{ccccccccc} \cdots & . & . & . & . & . & . & . & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & . & . & . & . & . & . & . & \cdots \end{array} \right\}$$

Then, $\pi_{\{d,e,f\}}\Omega = \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right\}$

Hence, $\#\pi_{\{d,e,f\}}\Omega = 4$.

Uniform set

Definition

A closed set $\Omega \subset \{0,1\}^{\Sigma}$ is called a uniform set if $\#\pi_S\Omega$ depends only on $\#S$ for any finite set $S \subset \Sigma$.

Whence the function $p_{\Omega}(k) := \#\pi_S\Omega$ with $\#S = k$ is called the uniform complexity function of Ω (by convention $p_{\Omega}(0) = 1$.)

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Example (trivial cases: full and singleton)

- If $\Omega = \{0,1\}^{\Sigma}$, then Ω is a uniform set with $p_\Omega(k) = 2^k$.
- If $\Omega = \{\omega\}$, then Ω is a uniform set with $p_\Omega(k) = 1$.

Examples

Example (doubleton)

$\Omega = \{\omega, \eta\}$ is uniform with $p_\Omega(k) = 2$ if and only if $\omega = \overline{\eta}$, i.e. $\omega(\sigma) = 1 - \eta(\sigma)$ $(\forall \sigma \in \Sigma)$.

$$\Omega = \left\{ \begin{array}{ccccccc} \cdots & 1 & 1 & 0 & 1 & 1 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \end{array} \right\}$$

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Example (Dirac)

$\Omega = \{\omega;\ \sum_{\sigma\in\Sigma} \omega(\sigma) \leq 1\}$ is uniform with $p_\Omega(k) = k + 1$.

$$\Sigma = \{ \quad \cdots \quad a \quad b \quad c \quad d \quad e \quad f \quad \cdots \quad \}$$

$$\pi_{\{a,c,f\}}\Omega = \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right\}$$

Hence, $p_\Omega(3) = 4$.

Examples

Example (rotation word)

Let θ be an irrational number and $\Sigma = \{n\theta;\ n \in \mathbb{N}\} \subset \mathbb{R}/\mathbb{Z}$. For $x \in \mathbb{R}/\mathbb{Z}$, let $\omega_x \in \{0,1\}^\Sigma$ be such that

$$\omega_x(\sigma) = 0 \rightleftharpoons x + \sigma \in [0, 1/2) \pmod 1.$$

Then, $\Omega := \{\omega_x;\ x \in \mathbb{R}/\mathbb{Z}\}$ is a uniform set with $p_\Omega(k) = 2k$, since for any $\{\sigma_1, \sigma_2, \cdots, \sigma_k\} \subset \Sigma$, the partition generated by the sets $[-\sigma_i, 1/2 - \sigma_i)$ $(i = 1, 2, \cdots, k)$ on $\mathbb{R}/\mathbb{Z}$ has $2k$ elements.

Notation

Let $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\}$ be an infinite subset of $\mathbb{N}$. For $\omega \in \{0,1\}^{\mathbb{N}}$ and $\Omega \subset \{0,1\}^{\mathbb{N}}$, define $\omega[\mathcal{N}] \in \{0,1\}^{\mathbb{N}}$ and $\Omega[\mathcal{N}] \subset \{0,1\}^{\mathbb{N}}$ by

$$\begin{aligned} \omega[\mathcal{N}](n) &:= \omega(N_n) \quad (n \in \mathbb{N}) \\ \Omega[\mathcal{N}] &:= \{\omega[\mathcal{N}];\ \omega \in \Omega\}. \end{aligned}$$

$$\begin{array}{rcccccccccccccc} \Sigma = & \{ & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots & \} \\ \mathcal{N} = & \{ & & N_0 & & N_1 & N_2 & & & N_3 & & N_4 & N_5 & \cdots & \} \\ & & & & & & & & & & & & & & \\ \omega = & & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & \cdots & \end{array}$$

Then, $\omega[\mathcal{N}] = 0\ 1\ 0\ 0\ 0\ 1\ \cdots$.

Super-Stationary set

Definition

A closed set $\Omega \subset \{0,1\}^{\mathbb{N}}$ is called a super-stationary set if $\Omega[\mathcal{N}] = \Omega$ holds for any infinite subset $\mathcal{N}$ of $\mathbb{N}$.

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- Uniform sets are introduced to study the structure of maximal pattern sturmian words;
- Super-stationary sets are uniform; also they are stationary (i.e. shift invariant);
- Super-stationary sets are considered as phenomena which are independent of the time scale, but sensitive only to the direction of time.

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- For example:rotation, take $\{\sigma_0, \sigma_1, \sigma_2, \cdots\} \subset \Sigma$ such that either

$$\sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_0 + 1/2$$

or

$$\sigma_0 > \sigma_1 > \sigma_2 > \cdots > \sigma_0 - 1/2.$$

Let $\psi : \mathbb{N} \to \Sigma$ be such that $\psi(n) = \sigma_n$. Then, $\Omega \circ \psi$ is a super-stationary set. In fact, we have

$$\Omega \circ \psi = \{\omega \in \{0,1\}^{\mathbb{N}};\ \omega \text{ is monotone}\}.$$

super-stationary set in uniform set

In general, we can prove that

Theorem

For a uniform set $\Omega \subset \{0,1\}^{\Sigma}$*, there exists an injection* $\psi : \mathbb{N} \to \Sigma$ *such that* $\Omega \circ \psi$ *is a super-stationary set. Hence, any uniform complexity function is realized by a super-stationary set.*

Notation

For $\xi = \xi_1\xi_2\cdots\xi_l \in \{0,1\}^+$ and $\omega \in \{0,1\}^{\mathbb{N}}$, ξ is called a super-subword of ω, denoted $\xi \ll \omega$, if

$$\omega(s_1)\omega(s_2)\cdots\omega(s_l) = \xi$$

holds for some $0 \leq s_1 < s_2 < \cdots < s_l$.

For $\xi \in \{0,1\}^*$ and $\Xi \subset \{0,1\}^+$, denote

$$\begin{aligned} \mathcal{P}(\xi) &= \{\omega \in \{0,1\}^{\mathbb{N}};\ \xi \ll \omega \text{ does not hold}\} \\ \mathcal{P}(\Xi) &= \cap_{\xi\in\Xi}\mathcal{P}(\xi) \\ \mathcal{Q}(\Xi) &= \cup_{\xi\in\Xi}\mathcal{P}(\xi) \end{aligned}$$

cover

We call $\eta \in \{0,1\}^* \cup \{0,1\}^{\mathbb{N}}$ a cover of Ξ if $\xi \ll \eta$ for any $\xi \in \Xi$.

A cover η of Ξ is called a minimal cover of Ξ if any $\zeta \ll \eta$ but $\zeta \neq \eta$ is not a cover of Ξ.

The least common multiple of Ξ is the set of all minimal covers of Ξ, which is denoted by $\mathrm{L}(\Xi)$.

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- Note that $\mathrm{L}(\Xi)$ is a finite subset of $\{0,1\}^*$ for any finite subset Ξ of $\{0,1\}^*$.

Structure of Super-Stationary sets

Theorem

- *Ω is a super-stationary set other than $\{0,1\}^{\mathbb{N}}$ if and only if $\Omega = \mathcal{Q}(\Xi)$ for some nonempty finite set $\Xi \subset \{0,1\}^+$.*
- *For nonempty finite set $\Xi \subset \{0,1\}^+$, $\mathcal{Q}(\Xi) = \mathcal{P}(L(\Xi))$.*
- *Any uniform complexity function in k other than 2^k is written as a polynomial function of k for large k.*

Bigger Alphabet

Recently, Kamae generalized these results to a general alphabet.

Hereafter, the alphabet $\mathbb{A}$ consists of at least 2 letters. Uniform sets and super-stationary sets in $\mathbb{A}^{\mathbb{N}}$ can be defined in the same way.

Still OK

Theorem

For a uniform set $\Omega \subset \mathbb{A}^{\mathbb{N}}$*, there exists an infinite subset* $\mathcal{N} \subset \mathbb{N}$ *such that* $\Omega[\mathcal{N}]$ *is a super-stationary set. Hence, any uniform complexity function is realized by a super-stationary set.*

Still OK

Theorem

For a uniform set $\Omega \subset \mathbb{A}^{\mathbb{N}}$, there exists an infinite subset $\mathcal{N} \subset \mathbb{N}$ such that $\Omega[\mathcal{N}]$ is a super-stationary set. Hence, any uniform complexity function is realized by a super-stationary set.

The uniform complexity functions over $\mathbb{A}$ can be efficiently calculated.

Not yet OK

Recall that for $\xi \in \mathbb{A}^*$ and $\Xi \subset \mathbb{A}^+$, denote

$$
\begin{aligned}
\mathcal{P}(\xi) &= \{\omega \in \mathbb{A}^{\mathbb{N}};\ \xi \ll \omega \text{ does not hold}\} \\
\mathcal{P}(\Xi) &= \cap_{\xi \in \Xi} \mathcal{P}(\xi) \\
\mathcal{Q}(\Xi) &= \cup_{\xi \in \Xi} \mathcal{P}(\xi)
\end{aligned}
$$

Not yet OK

- $\mathcal{Q}(\Xi)$ is always super-stationary.
- When $\#\mathbb{A} \geq 3$, there is a super-stationary set other than $\mathbb{A}^{\mathbb{N}}$ which can not be written as $\mathcal{Q}(\Xi)$ for any finite $\Xi \subset \mathbb{A}^*$.
- On the other hand, a set is super-stationary if and only if it can be written as $\mathcal{P}(\Xi)$ with $\Xi \subset \mathbb{A}^*$ satisfying some technical condition.