Introduction to Rauzy fractals

Guangzhou, 2010
Several questions

- **Geometry** Efficient way to produce a pixelized line? A pixelized plane?
- **Number theory** Compute good rational approximations of a vector?
- **Physics** Appropriate location for atoms to enhance conductivity properties and build frying pans?
- **Mathematics** Provide codings for toral translations?

Behind all these questions, a similar process: self-induced objects
Common reformulation

Do fractal shapes generate a tiling of a plane?

Purpose of the lecture: Origin of the main properties of this fractal shape
Substitution / periodic point

Alphabet $\mathcal{A} = \{1, \ldots, n\}$.

Substitution: replacement rule on $\mathcal{A}$

$$
\sigma(1) = 12, \quad \sigma(2) = 13, \quad \sigma(3) = 1
$$

Periodic point

- Iterate $\sigma$: eventually, the image of a letter starts by the same letter.
  $$
  \sigma^d(a) = a, \ldots
  $$

- Iterate $\sigma^d$: Increasing set of words with the same beginnings.

  $1$
  $12$
  $1213$
  $1213121$
  $1213121121312$
  $121312112131212131211213$

- The limit is a periodic point of the substitution
  $$
  u = \sigma^\infty(a) \quad \sigma^k(u) = u
  $$
Substitution / periodic point

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12 13
12 13 12 1
12 13 12 1 12 13 12
12 13 12 1 12 13 12 12 13 12 1 12 13

- The limit is a periodic point of the substitution

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Primitivity

Definition A substitution $\sigma$ is primitive if there exists $k$ such that all letters appears in the image of all letters through $\sigma^k$.

Checking?

- Incidence matrix $M$: abelianization of the substitution
- $\sigma$ primitive iff $M^k$ has only positive coefficients for one $k$.

Examples

$\sigma(1) = 12, \sigma(2) = 13, \sigma(3) = 1$

$$M = \begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}$$

Primitive ($M^3 > 0$)

$\sigma(1) = 12, \sigma(2) = 32, \sigma(3) = 23$

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$M^n$ always has zeros in the first line.
Not primitive

Main interest The matrix $M$ has a simple dominant eigenvalue
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Not primitive

Main interest The matrix $M$ has a simple dominant eigenvalue
Existence of periodic points

Theorem Every substitution on a $n$-letter alphabet has at least one (and at most $n$) periodic points.

Proof

- if $w$ is a periodic point, then $\sigma^k(u_0)$ starts with $u_0$.

- This means that there exists a loop starting from $u_0$ in the graph:
  
  $a \rightarrow b$ iff $\sigma(a)$ starts with $b$

- The graph is finite and every node have an exiting edge.
  It contains at least one loop!
Represent a word?

How to see the fixed point of a substitution on a $n$-letters alphabet?

**Abelianization map** Linearly map letters to $n$ independent vectors $e_1, \ldots, e_n$

$$P : w_1 \ldots w_k \in \mathcal{A}^n \mapsto e_{w_1} + \cdots + e_{w_k} \in \mathbb{R}^n.$$ 

1211212112

12 13 12 1 12 13 12 12 13 12 12 13

**Commutation formula** applying $\sigma$ to a word is equivalent to applying $M$ to its linearized vector.

$$P(\sigma(W)) = MP(W)$$
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**Commutation formula**

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$$ P(\sigma(W)) = MP(W) $$
Specific case? The stair remains close to a line?

Substitution of Pisot type The dominant eigenvalue of its incidence matrix is a unit Pisot number

- the determinant is \( \pm 1 \)
- all the roots of the characteristic polynomial have a modulus less than or equal to 1.

In other words The matrix \( M \) has one expanding eigenvalue \( \beta \) and all other eigenvalues \( \beta^{(i)} \) are contracting.
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Projecting a stair on an appropriate plane

Hypothesis \( \sigma \) substitution unit of Pisot type on a \( n \) letter-alphabet.

Projections

- \( \mathbb{H}_e \) expanding line of \( \mathbb{M} \).
- \( \mathbb{H}_c \) contracting hyperplane of \( \mathbb{M} \).
- \( \pi : \mathbb{R}^n \rightarrow \mathbb{H}_c \) projection along \( \mathbb{H}_e \).

Commutation relation: \( h = \pi M \) is a contraction on the hyperplane \( \mathbb{H}_c \).

Application The projection of \( P(\sigma(W)) \) is smaller than \( P(W) \).
Projecting a stair on an appropriate plane

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**Projections**
- \( \mathbb{H}_e \) expanding line of \( M \).
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**Application** The projection of \( P(\sigma(W)) \) is smaller than \( P(W) \).
Distance property

**Proposition** If $\sigma$ is unit of Pisot type, then the stair of any periodic point remains at a finite distance from the expanding line.

**Proof** Node in the stair: $P(u_0 \ldots u_K)$.

- Embrace $u_0 \ldots u_k$ between $\sigma^d(u_0 \ldots u_l)$ and $\sigma^d(u_0 \ldots u_{l+1})$.

- There exists a part $P_0$ of $\sigma(u_{l+1})$ such that
  
  $$u_0 \ldots u_k = \sigma^d(u_0 \ldots u_l)P_0$$

- Iterate this decomposition
  
  $$u_0 \ldots u_k = \sigma^d(P_N)\sigma^d(P_{N-1})\ldots \sigma^d(P_1)P_0$$

- Linearize
  
  $$\pi P(u_0 \ldots u_k) = h^dP(P_N) + h^d(P_{N-1})P(P_{N_1}) + \ldots + h\sigma^dP(P_1) + P(P_0)$$

- $h$ is a contraction and the $p_k$'s are in finite number.

The projections of the stair along the expanding line are bounded.
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- Iterate this decomposition $u_0 \ldots u_k = \sigma^{dN}(P_N)\sigma^{d(N-1)}(P_{N_1}) \ldots \sigma^d(P_1)P_0$

- Linearize
  \[ \pi\mathbf{P}(u_0 \ldots u_k) = h^{dN}\mathbf{P}(P_N) + h^{d(N-1)}\mathbf{P}(P_{N_1}) \ldots + h\sigma^d\mathbf{P}(P_1) + \mathbf{P}(P_0) \]

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The projections of the stair along the expanding line are bounded
Rauzy fractals

Zoom on the remaining part...

Rauzy fractal
Project the nodes of the stair and take the closure.
\[ \mathcal{T} := \{ \pi(P(u_0 \ldots u_{N-1})) \mid N \in \mathbb{N} \}. \]

Subtiles Look at the last edge to be read.
\[ \mathcal{T}(i) := \{ \pi(P(u_0 \ldots u_{k-1})) \mid N \in \mathbb{N}, \ u_N = i \}. \]
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Decomposition

\[ \sigma(1) = 12, \sigma(2) = 13, \sigma(3) = 1. \]

\[ h \text{ is a contracting similarity of angle } \simeq 2/3 \]

- The letter 1 appears in the images of 1, 2 and 3, with no prefix behind.
  \[ T(1) = hT(1) \cup hT(2) \cup hT(3) \]

- The letter 2 appear only in the image of 1, behind a prefix 1.
  \[ T(2) = hT(1) + \pi P(1) \]

- The letter 3 appear only in the image of 2, behind a prefix 1.
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General decomposition rule

\[ T(i) = \bigcup_{\sigma(j) = \text{pis}} hT(j) + \pi P(p). \]
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*h* is a contracting similarity of angle \( \simeq 2/3 \)

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**General decomposition rule**

\[ \mathcal{T}(i) = \bigcup_{\sigma(j) = \text{pis}} h\mathcal{T}(j) + \pi P(p). \]
Measure of tiles

**Theorem** The measures of the subtiles are proportional to the expanding eigenvector of the incidence matrix.

**Proof**

- **Decomposition rule**

\[ T(i) = \bigcup_{\sigma(j)=pis} hT(j) + \pi P(p). \]

- **h** is a contraction with ratio $1/\beta$.

\[ \mu(T(i)) \leq 1/\beta \sum_{\sigma(j)=pis} \mu(T(j)) \quad (\mu(T(i)))_i \leq 1/\beta M(\mu(T(i)))_i \]

- **Perron-Frobenius theorem.** Let $M$ a matrix with positive entries and $\beta$ be its dominant eigenvalue. Let $X$ be a vector with positive entries. Then we have $MX \leq \beta X$. The equality holds only when $X$ is a dominant eigenvector of $M$. 
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GIFS : telling which tile to put in the tile

$$\mathcal{I}(i) = \bigcup_{\sigma(j)=pis} h\mathcal{I}(j) + \pi P(p).$$

Graph directed iterated function system \((G, \{\tau_e\}_{e \in E})\)
- Finite directed graph \(G\) with no stranding vertices
- With each edge \(e\) of the graph, is associated a contractive mapping \(\tau_e : \mathbb{R}^n \rightarrow \mathbb{R}^n\).

**GIFS attractors** unique compact sets \(K_i\) such that \(K_i = \bigcup_{i \xrightarrow{\tau_e} j} \tau_e(K_j)\)

Prefix-suffix graph : defined naturally from the decomposition formula
\(i \xrightarrow{(p,i,s)} j\) iff \(\sigma(j) = pis\)
GIFS : telling which tile to put in the tile

\[ \mathcal{T}(i) = \bigcup_{\sigma(j) = pis} h\mathcal{T}(j) + \pi P(p). \]

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Prefix-suffix graph : defined naturally from the decomposition formula

\[ i \xrightarrow{(p,i,s)} j \text{ iff } \sigma(j) = pis \]
Theorem (Arnoux & Ito’01, Sirvent & Wang’02) Let $\sigma$ be a primitive unit Pisot substitution over the alphabet $A$ of $n$ letters. The subtiles of $\mathcal{T}$ are solutions of the GIFS:

$$\forall i \in A, \mathcal{T}(i) = \bigcup_{j \in A, (p, i, s) \rightarrow j} h\mathcal{T}(j) + \pi P(p).$$

The pieces are disjoint in the decomposition of each subtile.

Self-similar? $h$ must have eigenvalues with the same modulus.
First consequence: dependency to the fixed point?

**Theorem** A Rauzy fractal is compact and it is independent from the periodic point that was chosen.

**Proof** Same GIFS equation whenever the periodic point.
Second consequence: condition for disjointness

Are the subtiles disjoint?

Strong coincidence condition For every pair of letters \((j_1, j_2) \in A^2\), we shall check that \(\sigma^k(j_1) = p_1i s_1\) and \(\sigma^k(j_2) = p_2 i s_2\) with \(P(p_1) = P(p_2)\) or \(P(s_1) = P(s_2)\).

Theorem (Arnoux & Ito'01) If \(\sigma\) satisfies the strong coincidence condition, then the subtiles \(\mathcal{T}(i)\) of the central tile \(\mathcal{T}\) have disjoint interiors.

Proof: \(h\mathcal{T}(j_1) + P(p_1)\) and \(h\mathcal{T}(j_2) + P(p_2)\) both appear in the GIFS decomposition of \(\mathcal{T}(i)\). Therefore they are disjoint.
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**Theorem** (Arnoux & Ito’01) If \(\sigma\) satisfies the strong coincidence condition, then the subtiles \(T(i)\) of the central tile \(T\) have disjoint interiors.

**Proof**: \(hT(j_1) + P(p_1)\) and \(hT(j_2) + P(p_2)\) both appear in the GIFS decomposition of \(T(i)\). Therefore they are disjoint.
Inflating the fractal?

We have a Rauzy fractal and a decomposition rule.

Can we produce a tiling from these pieces?
Formal definition

**Multiple tilings** Let $K_i$, $i \in A$ be a finite collection of compact sets of a Euclidean space $\mathbb{H}$.

A *multiple tiling* of the space $\mathbb{H}$ by the compact sets $K_i$ is given by a *translation set* $\Gamma \subset \mathbb{H} \times A$ such that

(A) **Covering property**

\[
\mathbb{H} = \bigcup_{(\gamma, i) \in \Gamma} K_i + \gamma
\]

(B) **Uniform covering property** almost all points in $\mathbb{H}$ are covered exactly $p$ times for some positive integer $p$.

(C) **Local “sparsity”** Each compact subset of $\mathbb{H}$ intersects a finite number of tiles.

**Tiling property** If $p = 1$, then the multiple tiling is called a *tiling*. 
How to define location of tiles?

Face of type $i$ located in $x$ orthogonal to the $i$-th axis of a translate of the unit cube located at $x$.

Stepped hyperplane (Arnoux&Berthe&Ito’02) Union of faces of unit cubes which cross the contracting plane

$x$ is above $\mathbb{H}_c$ and $x - e_i$ is below $\mathbb{H}_c$. 
How to define location of tiles?

**Translation set** $\Gamma \subset \mathbb{H} \times A$?

**Definition**
The Self-replicating translation set is the projection of the stepped surface on the contracting plane.

$$\Gamma_{sr} = \{[\gamma, i^*] \in \pi(\mathbb{Z}^n) \times A \mid \gamma = \pi(x), x \in \mathbb{Z}^n, \ x \ \text{above} \ H_c, \ x - e_i \ \text{below} \ H_c \}.$$
How to define location of tiles?

*Translation set* $\Gamma \subset \mathbb{H} \times \mathcal{A}$

- location of a tile
- name of the tile to draw

Definition

The *Self-replicating translation set* is the projection of the stepped surface on the contracting plane.

$$\Gamma_{sr} = \{ [\gamma, i^*] \in \pi(\mathbb{Z}^n) \times \mathcal{A} \mid \gamma = \pi(x), x \in \mathbb{Z}^n, \begin{array}{c} x \text{ above } \mathbb{H}_c, x - e_i \text{ below } \mathbb{H}_c \end{array} \}.$$
Covering property The tiles $T(i) + \gamma$ for all $[\gamma, i]$ in the translation set cover the full plane $\mathbb{R}^{n-1}$.

Proof

- $\mathbb{Z}^n$ is covered by translated copies of the stair projection along the stepped surface.
- From Kronecker's theorem, every point of the plane is approximated by projection of points in $\mathbb{Z}^n$.

Application to Rauzy fractal The Rauzy fractal has a nonempty interior.
From covering to Rauzy fractals

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Application to Rauzy fractal  The Rauzy fractal has a nonempty interior.
Self-replicating?

Substitution on faces $E_1^*$ is defined on the subsets of $\pi(\mathbb{Z}^n) \times A$:

$$E_1^* \{[\gamma, i^*]\} = \bigcup_{j \in A, \sigma(j) = \text{pis}} \{[h^{-1}(\gamma + \pi P(p)), j^*]\},$$

Similar to the rules of decomposition of the Rauzy fractal!

$$h^{-1}T(1) = T(1) \cup T(2) \cup T(3)$$

$$h^{-1}T(2) = T(1) + \pi(e_3)$$

$$h^{-1}T(3) = T(2) + \pi(e_3)$$

The translation set is a fixed point (Arnoux & Ito'01) $E_1^*(\Gamma_{sr}) = \Gamma_{sr}$

The substitution $E_1^*$ maps two different tips on disjoint patches
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Similar to the rules of decomposition of the Rauzy fractal!

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The translation set is a fixed point (Arnoux & Ito '01) $E_1^*(\Gamma_{sr}) = \Gamma_{sr}$

The substitution $E_1^*$ maps two different tips on disjoint patches
Proposition The boundary of the Rauzy fractal has a zero measure and the Rauzy fractal is the closure of its interior.

- From the GIFS equation, either all subtiles have zero measure boundaries or none.
- Play with the Perron-Frobenius formula
- Deeply use that patches in iterations of $E_1^*$ are disjoint and corresponding Rauzy pieces are measure disjoint.
Consequence: multiple tiling

**Proposition** The self-replicating translation set \( \Gamma_{sr} \) is **repetitive** and satisfies

\[
\mathbb{H}_c = \bigcup_{[\gamma,i^*] \in \Gamma_{sr}} (\mathcal{T}(i) + \gamma)
\]

where almost all points of \( \mathbb{H}_c \) are covered \( p \) times (\( p \) is a positive integer).

In other words: we have defined a **Self-replicating multiple tiling**
Proof for multiple tiling

Locally finite deduced from compactness.

Repetitivity Kronecker’s theorem \( \pm \) relatively dense.

Multiple tiling Boundary with zero measure \( \pm \) repetitivity.
Conclusion: quite general proofs!

Framework
- GIFS equation with a primitive graph
- Information on location of pieces (=properties of $E_1^*$)

Conclusions
- Compact set with nonempty interior.
- Closure of its interior.
- Boundary with zero measure.
- Covering with a almost constant degree.

How valid are these proofs for all GIFS’s??
No general topological properties

Zero inner point, connectivity, disklikeness are not always satisfied.
Relations to other talks in the conference

- Arnoux: Geometric representation of a symbolic dynamical system.
- Adamczewski: Rauzy fractal represent integers in non-real expansions bases.
- Berthe: Example of local rules to generate combinatorial tilings.
- Akiyama: The tiling property is very close to the Pisot conjecture.
- Harriss, Bressaud: can we exhibit matching rules for the Rauzy tiling?
- Sellami, Jolivet: properties of families of Rauzy fractals....
- Other talks: good chance that Rauzy fractals are hidden somewhere

To be continued... **Rauzy fractal represent self-induced actions, therefore they lie in many mathematical objects**