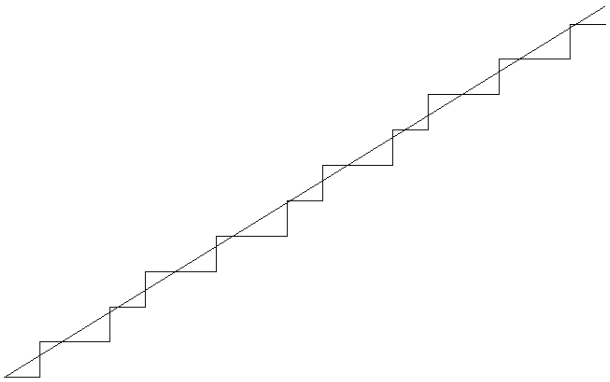
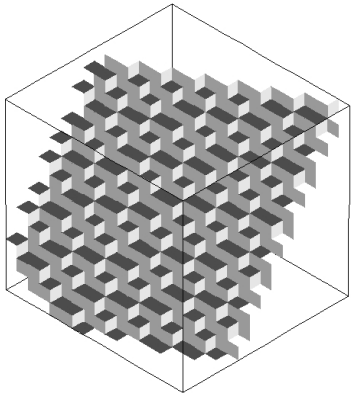
Introduction to Rauzy fractals

Guanghzhou, 2010

Several questions

- Geometry Efficient way to produce a pixelized line ? A pixelized plane ?
- Number theory Compute good rational approximations of a vector ?
- Physics Appropriate location for atoms to enhance conductivity properties and build frying pans ?
- Mathematics Provide codings for toral translations ?

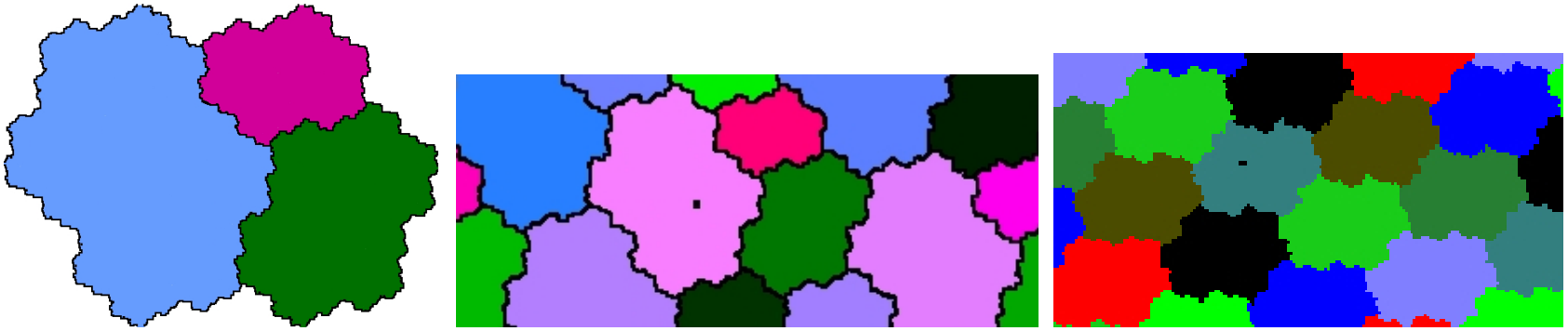




Behind all these questions, a similar process : self-induced objects

Common reformulation

Do fractal shapes generate a tiling of a plane ?



Purpose of the lecture : Origin of the main properties of this fractal shape

Substitution / periodic point

Alphabet $\mathcal{A} = \{1, \ldots, n\}$.

Substitution : replacement rule on $\mathcal{A}$

$$\sigma(1) = 12,\ \sigma(2) = 13,\ \sigma(3) = 1$$

Periodic point

- Iterate σ : eventually, the image of a letter starts by the same letter.

$$\sigma^d(a) = a\ldots.$$

- Iterate σ^d : Increasing set of words with the same beginnings.

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- The limit is a *periodic point of the substitution*

$$u = \sigma^\infty(a) \qquad \sigma^k(u) = u$$

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Primitivity

Definition A substitution σ is *primitive* if there exists k such that *all* letters appears in the image of all letters through σ^k.

Checking ?

- Incidence matrix $\mathbf{M}$: abelianization of the substitution
- σ primitive iff $\mathbf{M}^k$ has only positive coefficients for one k.

Examples

$\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 1$

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Primitive ($\mathbf{M}^3 > 0$)

$\sigma(1) = 12$, $\sigma(2) = 32$, $\sigma(3) = 23$

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$\mathbf{M}^n$ always has zeros in the first line.

Not primitive

Main interest The matrix $\mathbf{M}$ has a simple dominant eigenvalue

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Existence of periodic points

Theorem Every substitution on a n-letter alphabet has at least one (and at most n) periodic points.

Proof

- if w is a periodic point, then $\sigma^k(u_0)$ starts with u_0.
- This means that there exists a loop starting from u_0 in the graph :
  $a \longrightarrow b$ iff $\sigma(a)$ starts with b
- The graph is finite and every node have an exiting edge.
  It contains at least one loop !

Represent a word ?

How to see the fixed point of a substitution on a n-letters alphabet ?

Abelianization map Linearly map letters to n independent vectors $\mathbf{e}_1, \ldots \mathbf{e}_n$

$$\mathbf{P} : w_1 \ldots w_k \in \mathcal{A}^n \mapsto \mathbf{e}_{w_1} + \cdots + \mathbf{e}_{w_k} \in \mathbb{R}^n.$$

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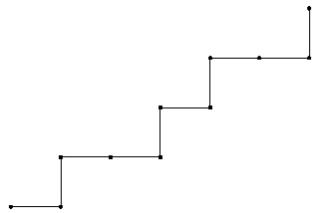

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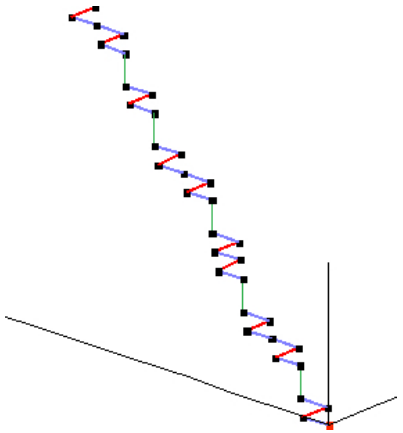

Commutation formula applying σ to a word is equivalent to applying $\mathbf{M}$ to its linearized vector.

$$\mathbf{P}(\sigma(W)) = \mathbf{M}\mathbf{P}(W)$$

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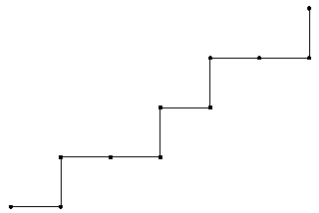

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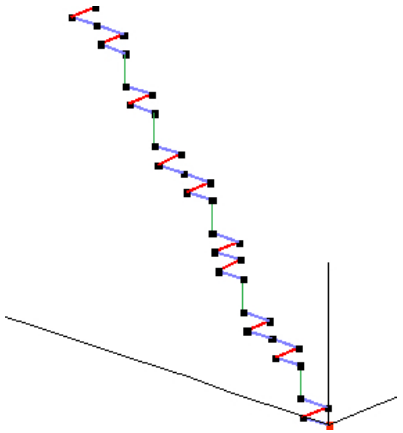

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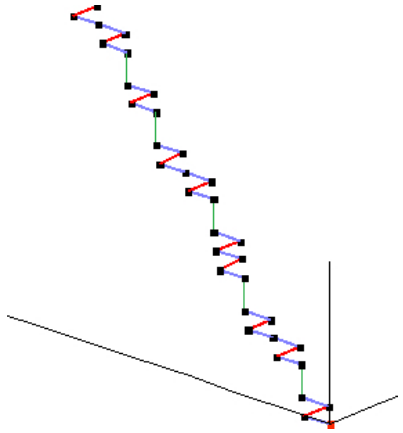
Pisot Case

Specific case ? The stair remains close to a line ?



Substitution of Pisot type The dominant eigenvalue of its incidence matrix is a unit Pisot number

- the determinant is ± 1
- all the roots of the characteristic polynomial have a modulus less than or equal to 1.

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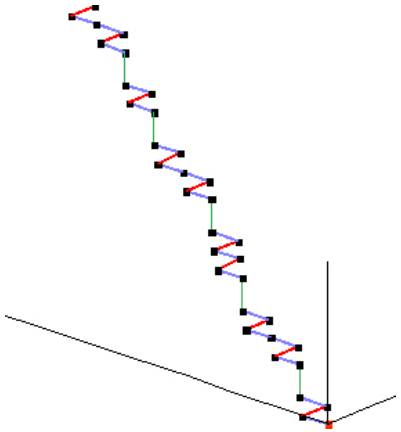

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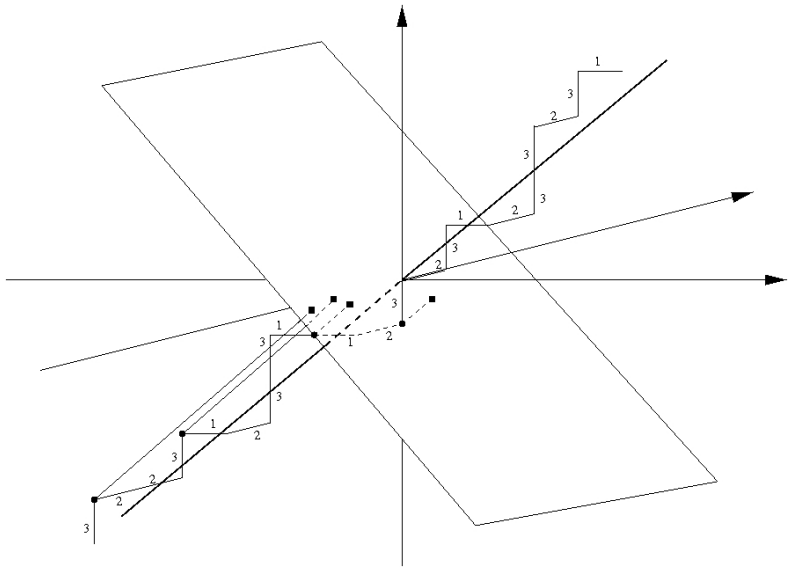
Projecting a stair on an appropriate plane

Hypothesis σ substitution unit of Pisot type on a n letter-alphabet.

Projections

- $\mathbb{H}_e$ expanding line of $\mathbf{M}$.
- $\mathbb{H}_c$ contracting hyperplane of $\mathbf{M}$.
- $\pi : \mathbb{R}^n \longrightarrow \mathbb{H}_c$ projection along $\mathbb{H}_e$.



Commutation relation : $\mathbf{h} = \pi\mathbf{M}$ is a contraction on the hyperplane $\mathbb{H}_c$.

Application **The projection of $\mathbf{P}(\sigma(W))$ is smaller than $\mathbf{P}(W)$.**

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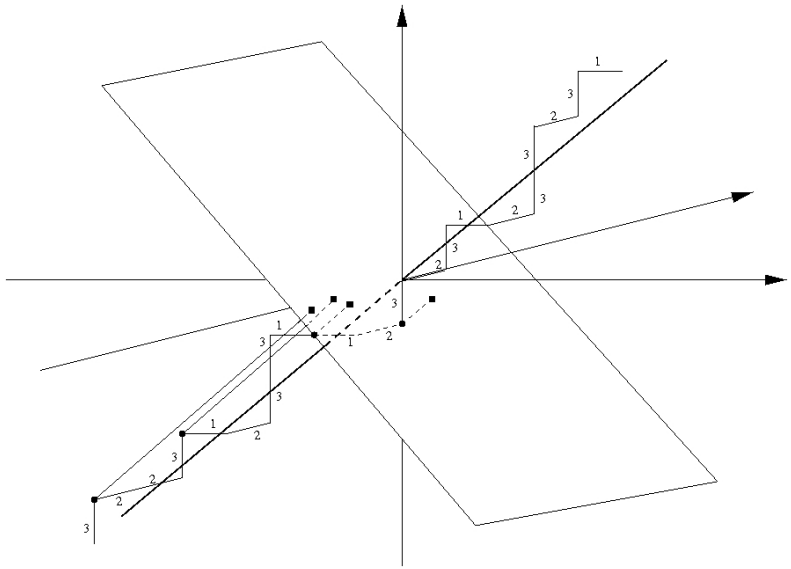

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Distance property

Proposition If σ is unit of Pisot type, then the stair of any periodic point remains at a finite distance from the expanding line.

Proof Node in the stair : $\mathbf{P}(u_0 \dots u_K)$.

- Embrace $u_0 \dots u_k$ between $\sigma^d(u_0 \dots u_l)$ and $\sigma^d(u_0 \dots u_{l+1})$
- There exists a part P_0 of $\sigma(u_{l+1})$ such that
$$u_0 \dots u_k = \sigma^d(u_0 \dots u_l)P_0$$
- Iterate this decomposition
$$u_0 \dots u_k = \sigma^{dN}(P_N)\sigma^{d(N-1)}(P_{N_1}) \dots \sigma^d(P_1)P_0$$
- Linearize
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The projections of the stair along the expanding line are bounded

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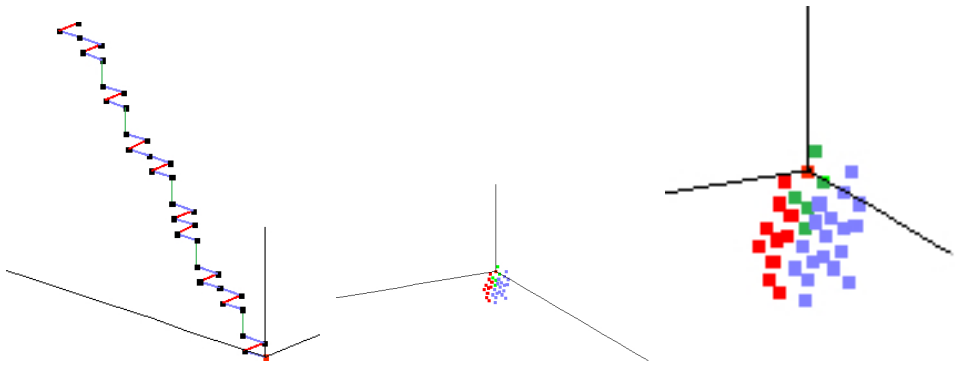
Rauzy fractals

Zoom on the remaining part...



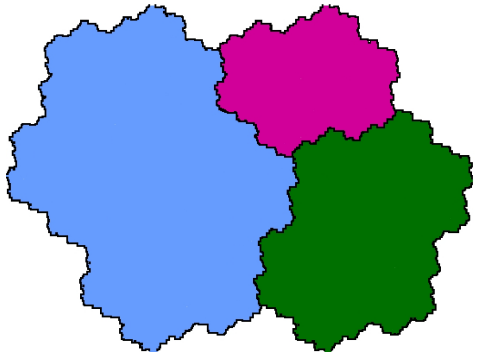
Rauzy fractal

Project the nodes of the stair and take the closure.

$$\mathcal{T} := \overline{\{\pi(\mathbf{P}(u_0 \dots u_{N-1})) \mid N \in \mathbb{N}\}}.$$

Subtiles Look at the last edge to be read.

$$\mathcal{T}(i) := \overline{\{\pi(\mathbf{P}(u_0 \dots u_{k-1})) \mid N \in \mathbb{N},\ u_N = i\}}.$$



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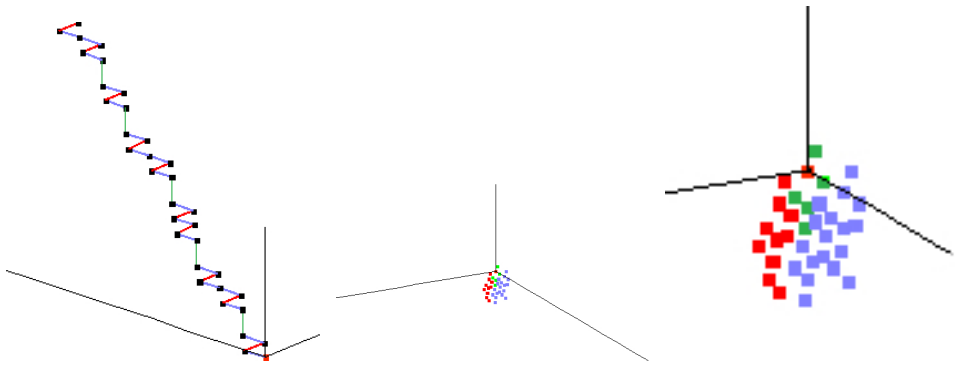

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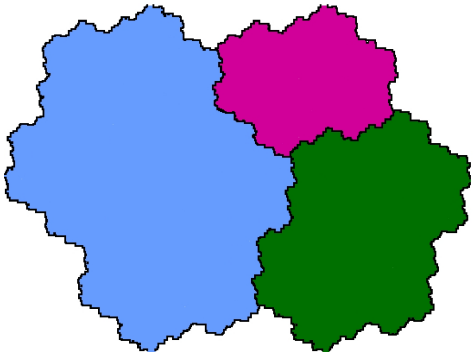

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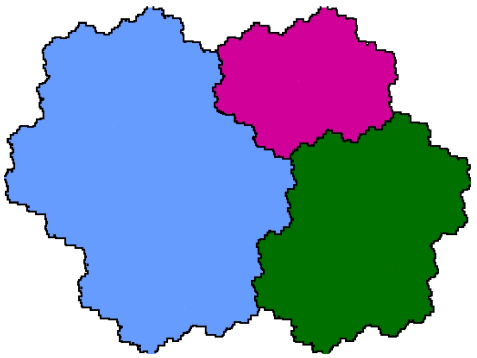
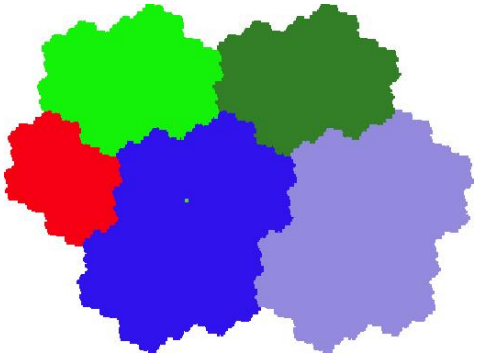
$\mathbf{h}$ is a contracting similarity of angle $\simeq 2/3$





- The letter 1 appears in the images of 1, 2 and 3, with no prefix behind.

$$\mathcal{T}(1) = \mathbf{h}\mathcal{T}(1) \cup \mathbf{h}\mathcal{T}(2) \cup \mathbf{h}\mathcal{T}(3)$$

- The letter 2 appear only in the image of 1, behind a prefix 1.

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- The letter 3 appear only in the image of 2, behind a prefix 1.

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General decomposition rule

$$\mathcal{T}(i) = \bigcup_{\sigma(j)=pis} \mathbf{h}\mathcal{T}(j) + \pi\mathbf{P}(p).$$

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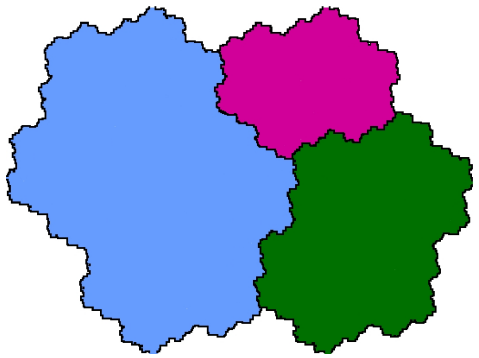

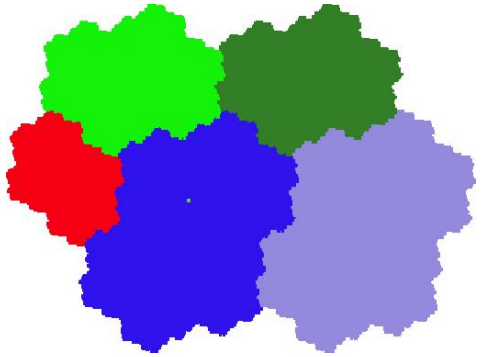

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Measure of tiles

Theorem The measures of the subtiles are proportional to the expanding eigenvector of the incidence matrix.

Proof

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- Perron-Frobenius theorem. Let $\mathbf{M}$ a matrix with positive entries and β be its dominant eigenvalue. Let $\mathbf{X}$ be a vector with positive entries. Then we have $\mathbf{MX} \leq \beta\mathbf{X}$. The equality holds only when $\mathbf{X}$ is a dominant eigenvector of $\mathbf{M}$.

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GIFS : telling which tile to put in the tile

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Graph directed iterated function system $(G, \{\tau_e\}_{e\in E})$

- Finite directed graph G with no stranding vertices
- With each edge e of the graph, is associated a contractive mapping $\tau_e : \mathbb{R}^n \to \mathbb{R}^n$.

GIFS attractors unique compact sets K_i such that

$$K_i = \bigcup_{i \xrightarrow{e} j} \tau_e(K_j)$$

Prefix-suffix graph : defined naturally from the decomposition formula

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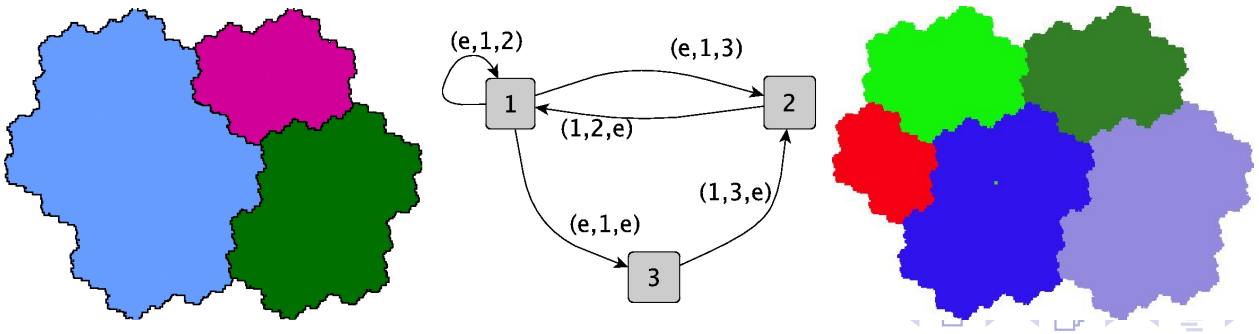
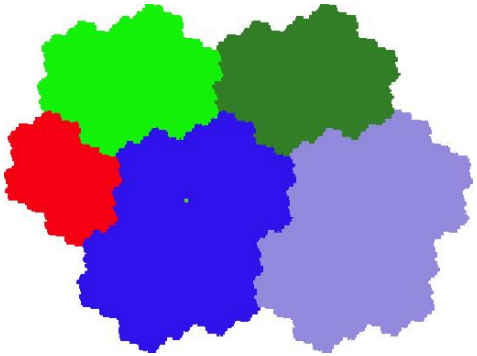


GIFS property

Theorem(Arnoux&Ito'01, Sirvent&Wang'02) Let σ be a primitive unit Pisot substitution over the alphabet A of n letters.
The subtiles of $\mathcal{T}$ are solutions of the GIFS :

$$\forall i \in A,\ \mathcal{T}(i) = \bigcup_{\substack{j \in A, \\ i \xrightarrow{(p,i,s)} j}} \mathbf{h}\mathcal{T}(j) + \pi \mathbf{P}(p).$$

The pieces are disjoint in the decomposition of each subtile



Self-similar ? **h** must have eigenvalues with the same modulus.

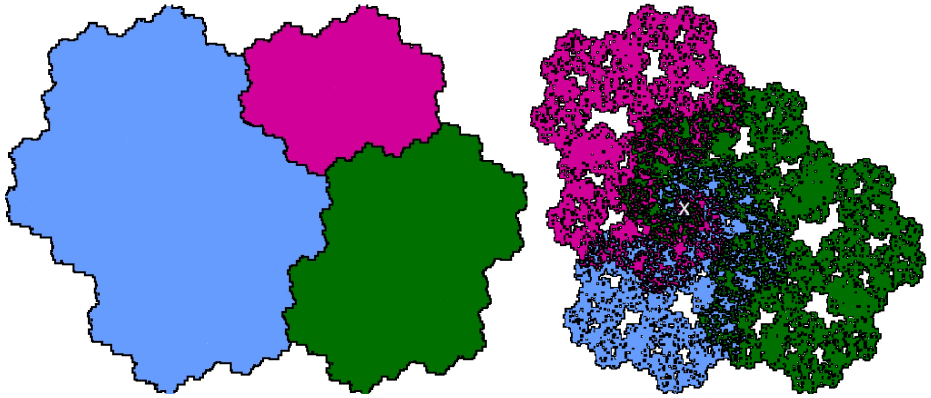
First consequence : dependency to the fixed point ?

Theorem A Rauzy fractal is compact and it is independant from the periodic point that was chosen.

Proof Same GIFS equation whenever the periodic point.

Second consequence : condition for disjointness

Are the subtiles disjoint ?



Strong coincidence condition For every pair of letters $(j_1, j_2) \in A^2$, we shall check that $\sigma^k(j_1) = p_1 i s_1$ and $\sigma^k(j_2) = p_2 i s_2$ with $\mathbf{P}(p_1) = \mathbf{P}(p_2)$ or $\mathbf{P}(s_1) = \mathbf{P}(s_2)$.

Theorem(Arnoux&Ito'01) If σ satisfies the strong coincidence condition, then the subtiles $\mathcal{T}(i)$ of the central tile $\mathcal{T}$ have disjoint interiors.

Proof : $\mathbf{h}\mathcal{T}(j_1) + \mathbf{P}(p_1)$ and $\mathbf{h}\mathcal{T}(j_2) + \mathbf{P}(p_2)$ both appear in the GIFS decomposition of $\mathcal{T}(i)$. Therefore they are disjoint.

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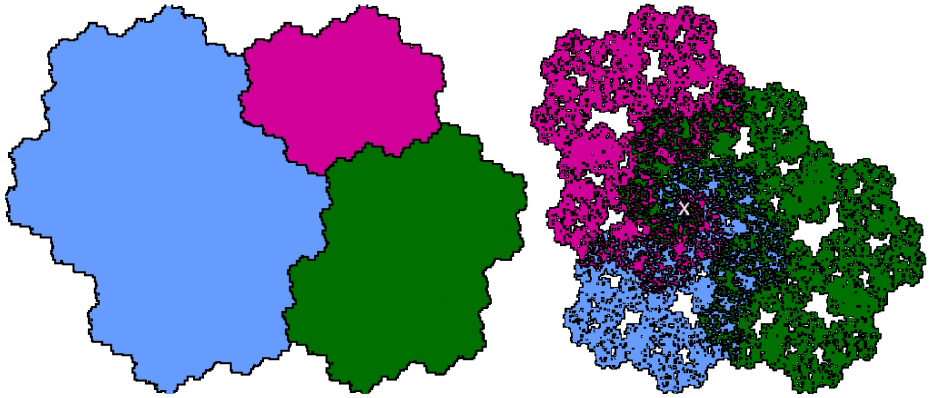
Are the subtiles disjoint ?



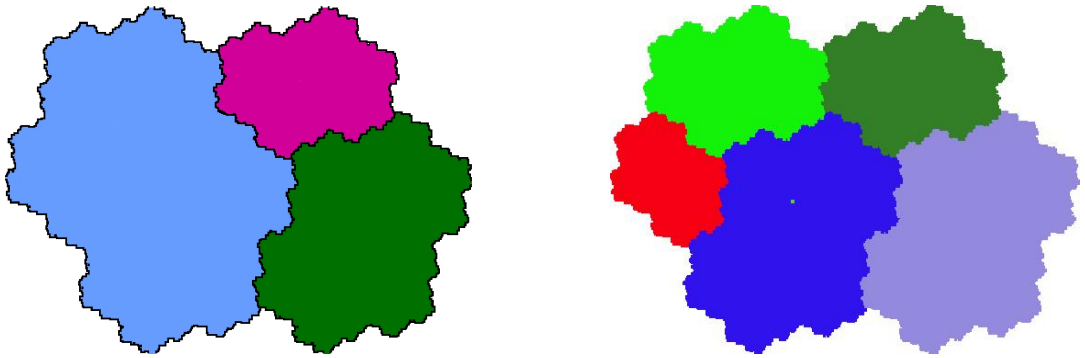
Strong coincidence condition For every pair of letters $(j_1, j_2) \in A^2$, we shall check that $\sigma^k(j_1) = p_1 i s_1$ and $\sigma^k(j_2) = p_2 i s_2$ with $\mathbf{P}(p_1) = \mathbf{P}(p_2)$ or $\mathbf{P}(s_1) = \mathbf{P}(s_2)$.

Theorem(Arnoux&Ito'01) If σ satisfies the strong coincidence condition, then the subtiles $\mathcal{T}(i)$ of the central tile $\mathcal{T}$ have disjoint interiors.

Proof : $\mathbf{h}\mathcal{T}(j_1) + \mathbf{P}(p_1)$ and $\mathbf{h}\mathcal{T}(j_2) + \mathbf{P}(p_2)$ both appear in the GIFS decomposition of $\mathcal{T}(i)$. Therefore they are disjoint.

Inflating the fractal ?

We have a Rauzy fractal and a decomposition rule.



Can we produce a tiling from these pieces ?

Formal definition

Multiple tilings Let K_i, $i \in A$ be a finite collection of compact sets of a Euclidean space $\mathbb{H}$.

A *multiple tiling* of the space $\mathbb{H}$ by the compact sets K_i is given by a *translation set* $\Gamma \subset \mathbb{H} \times A$ such that

(A) Covering property

$$\mathbb{H} = \bigcup_{(\gamma,i)\in\Gamma} K_i + \gamma$$

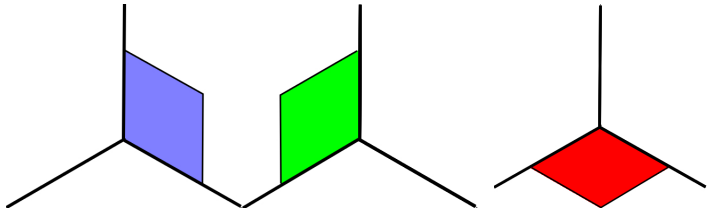
(B) Uniform covering property almost all points in $\mathbb{H}$ are covered exactly p times for some positive integer p.

(C) Local "sparsity" Each compact subset of $\mathbb{H}$ intersects a finite number of tiles.

Tiling property If $p = 1$, then the multiple tiling is called a *tiling*.

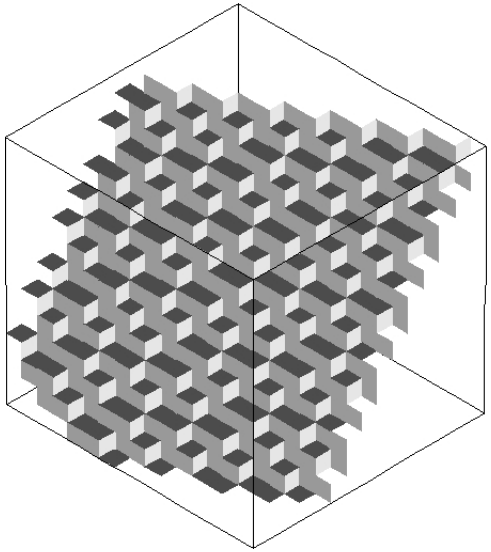
How to define location of tiles ?

Face of type i located in $\mathbf{x}$ orthogonal to the i-th axis of a translate of the unit cube located at $\mathbf{x}$.



Stepped hyperplane (Arnoux&Berthe&Ito'02) Union of faces of unit cubes which cross the contracting plane

$$\mathbf{x} \textit{ is above } \mathbb{H}_c \textit{ and } \mathbf{x} - \mathbf{e}_i \textit{ is below } \mathbb{H}_c.$$



How to define location of tiles ?

$$\textit{Translation set } \Gamma \subset \underbrace{\mathbb{H}}_{\substack{\text{location}\\ \text{of a tile}}} \times \underbrace{\mathcal{A}}_{\substack{\text{name of}\\ \text{the tile}\\ \text{to draw}}} \ ?$$

Definition

The Self-replicating translation set is the projection of the stepped surface on the contracting plane.

$$\Gamma_{sr} = \{[\gamma, i^*] \in \pi(\mathbb{Z}^n) \times \mathcal{A} \mid \underbrace{\gamma = \pi(\mathbf{x}),\ \mathbf{x} \in \mathbb{Z}^n}_{\text{projection on } \mathbb{H}_c},\ \underbrace{\mathbf{x} \text{ above } \mathbb{H}_c,\ \mathbf{x} - \mathbf{e}_i \text{ below } \mathbb{H}_c}_{\text{stepped surface}}\}.$$

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From covering to Rauzy fractals

Covering property The tiles $\mathcal{T}(i) + \gamma$ for all $[\gamma, i]$ in the translation set cover the full plane $\mathbb{R}^{n-1}$.

Proof

- $\mathbb{Z}^n$ is covered by translated copies of the stair projection along the stepped surface.
- From Kronecker's theorem, every point of the plane is approximated by projection of points in $\mathbb{Z}^n$.

Application to Rauzy fractal **The Rauzy fractal has a nonempty interior**.

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Self-replicating ?

Substitution on faces $\mathbf{E}_1^*$ is defined on the subsets of $\pi(\mathbb{Z}^n) \times A$:

$$\mathbf{E}_1{}^*\{[\gamma, i^*]\} = \bigcup_{j \in A,\ \sigma(j)=pis} \{[\mathbf{h}^{-1}(\gamma + \pi\mathbf{P}(p)), j^*]\},$$

Similar to the rules of decomposition of the Rauzy fractal !

$\mathbf{h}^{-1}\mathcal{T}(1) = \mathcal{T}(1) \cup \mathcal{T}(2) \cup \mathcal{T}(3)$
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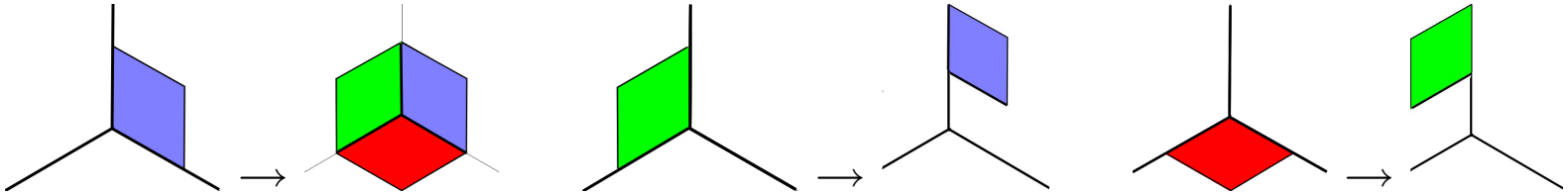

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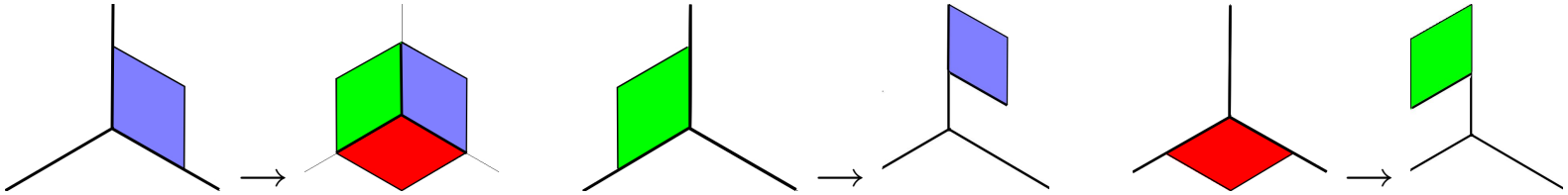

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Application to the boundary of the Rauzy fractal

Proposition The **boundary** of the Rauzy fractal has a **zero measure** and the Rauzy fractal is the **closure of its interior**.

- From the GIFS equation, either all subtiles have zero measure boundaries or none.
- Play with the Perron-Frobenius formula
- Deeply use that patches in iterations of $\mathbf{E}_1^*$ are disjoint and corresponding Rauzy pieces are measure disjoint.

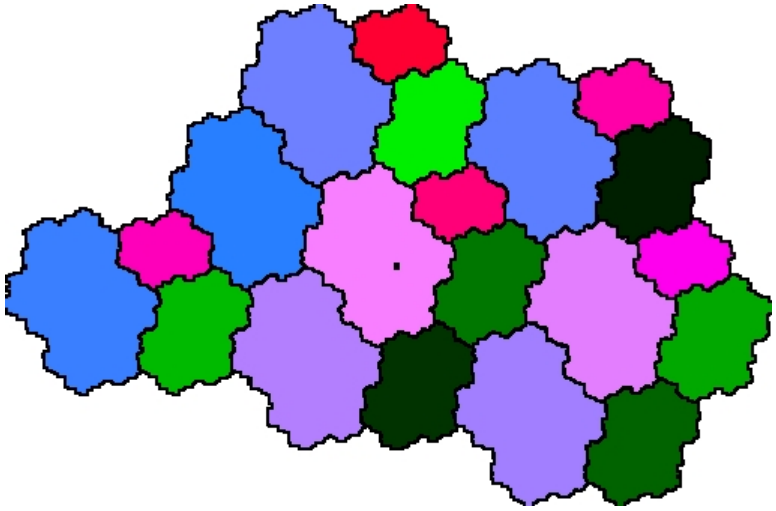
Consequence : multiple tiling

Proposition The self-replicating translation set Γ_{sr} is **repetitive** and satisfies

$$\mathbb{H}_c = \bigcup_{[\gamma, i^*] \in \Gamma_{sr}} (\mathcal{T}(i) + \gamma)$$

where almost all points of $\mathbb{H}_c$ are covered p times (p is a positive integer).

In other words : we have defined a
Self-replicating multiple tiling



Proof for multiple tiling

Locally finite deduced from compactness.

Repetitivity Kronecker's theorem + relatively dense.

Multiple tiling Boundary with zero measure + repetitivity.

Conclusion : quite general proofs !

Framework

- GIFS equation with a primitive graph
- information on location of pieces (=properties of $\mathbf{E}_1^*$)

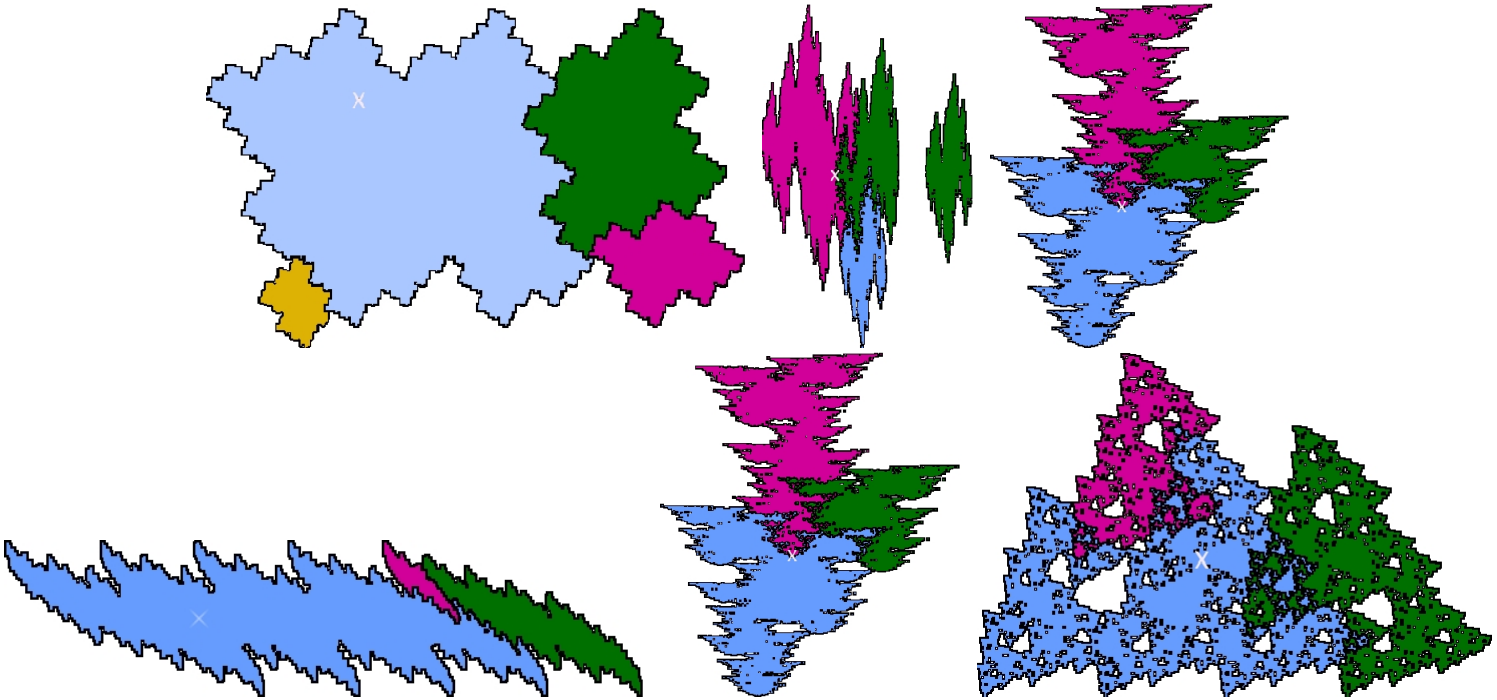
Conclusions

- Compact set with nonempty interior.
- Closure of its interior.
- **Boundary with zero measure**.
- Covering with a almost constant degree.

How valid are these proofs for all GIFS's ? ?

No general topological properties

Zero inner point, connectivity, disklikeness are not always satisfied.



Relations to other talks in the conference

- Arnoux : Geometric representation of a symbolic dynamical system.
- Adamczewski : Rauzy fractal represent integers in non-real expansions bases.
- Berthe : Example of local rules to generate combinatorial tilings.
- Akiyama : The tiling property is very close to the Pisot conjecture.
- Harriss, Bressaud : can we exhibit matching rules for the Rauzy tiling ?
- Sellami, Jolivet : properties of families of Rauzy fractals....
- Other talks : good chance that Rauzy fractals are hidden somewhere

To be continued... **Rauzy fractal represent self-induced actions, therefore they lie in many mathematical objects**