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Gap sequence, Lipschitz equivalence and box dimension of fractal sets

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Abstract
We introduce a notion of gap sequences for compact sets $E \subset \mathbb{R}^d$, which is a generalization of the gap sequences of compact sets on the real line. We show that if the gap sequences of two fractal sets are not equivalent, then these two sets cannot be Lipschitz equivalent, where the latter fact is usually very hard to verify. Finally, we show that for some typical fractal sets, the gap sequences characterize the upper box dimension.

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1. Introduction

The definition of gap sequence of a compact set $E \subset \mathbb{R}$ is nature. An open interval $]a, b[$ is said to be a *gap* of E if $a, b \in E$ but $]a, b[\cap E = \emptyset$. The set of gaps of E is thus a collection of open intervals and it is at most countable. We are interested in the lengths of these intervals; let us list them in a non-increasing order and call this (finite or infinite) sequence of positive reals the *gap sequence* of E.

For example, the gap sequence of Cantor middle-third set is

$$1/3, 1/9, 1/9, 1/27, 1/27, 1/27, 1/27, 1/81, \ldots.$$

While the notion of gap sequence is firstly introduced in this paper as we know, the idea of gap sequence in one dimension has already been widely used to characterize fractal properties of E, especially when E has zero Lebesgue measure, for example, Besicovitch and Taylor [1]

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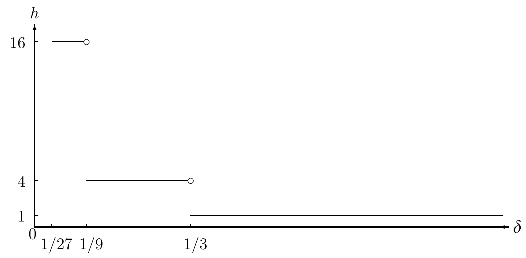

Figure 1. The function h of $\mathcal{C} \times \mathcal{C}$.

on the Hausdorff dimension, Tricot [11] on twelve definitions of fractal dimensions, Lapidus and Pomerance [8], Lapidus and Maier [7] and Falconer [3] on the Minkowski measurability.

In this paper, we will generalize the notion of gap sequence to higher dimensions and apply it in the following two things: one is Lipschitz equivalence and the other is box dimension.

Definition of gap sequence in $\mathbb{R}^d$. A compact subset E of $\mathbb{R}^d$ is said to be *δ-connected* if for any $x, y \in E$, there is a δ-chain connecting x and y. That is, there is a sequence $\{x_1 = x, x_2, \ldots, x_{n-1}, x_n = y\} \subset E$ such that $|x_{i+1} - x_i| \leqslant \delta$ holds for $1 \leqslant i \leqslant n-1$.

We call $F \subset E$ a *δ-connected component* of E if F is δ-connected, but for any $F \subsetneqq F' \subset E$, F' is not δ-connected. Let us denote by $h(\delta)$ the number of δ-connected components of E, which is finite by the compactness of E.

It can be shown that $h(\delta) : \mathbb{R}^+ \to \mathbb{N}$ is a non-increasing function, is locally constant except at the neighbourhoods of discontinuous points, and is right continuous (see lemma 1). Let us denote by $\{a_k\}_{k\geqslant 1}$ the discontinuous points (or jump points) of h in decreasing order. We call $i_k = h(a_k-) - h(a_k)$ the multiple of a_k and define the gap sequence of E to be the sequence (also see definiton 2.1):

$$\underbrace{a_1, \ldots, a_1}_{i_1}, \underbrace{a_2, \ldots, a_2}_{i_2}, \ldots, \underbrace{a_m, \ldots, a_m}_{i_m}, \ldots.$$

It is not hard to verify that the above definition is a generalization of the one-dimensional case.

Example 1.1. The function h of $\mathcal{C} \times \mathcal{C}$ is depicted by figure 1 where $\mathcal{C}$ is the Cantor middle-third set, and the gap sequence of $\mathcal{C} \times \mathcal{C}$ is

$$\underbrace{1/3, 1/3, 1/3}_{3}, \underbrace{1/9, \ldots, 1/9}_{3\times 4}, \underbrace{1/27, \ldots, 1/27}_{3\times 4^2}, \ldots.$$

Gap sequence and Lipschitz equivalence. Let E and F be two compact subsets of $\mathbb{R}^d$. Let f be a function from E to F. The *Lipschitz constant* of f is defined by

$$M(f) = \sup_{x_1, x_2 \in E;\ x_1 \neq x_2} \left| \frac{f(x_1) - f(x_2)}{x_1 - x_2} \right|.$$

The map f is said to be *Lipschitz* if $M(f) < +\infty$. Moreover, f is said to be *bi-Lipschitz* if f is a bijection and f, f^{-1} are both Lipschitz. Two sets E and F are said to be *Lipschitz equivalent* if there exists a bi-Lipschitz function from E to F.

Recently, there are many works devoted to the Lipschitz equivalence of fractal sets, Falconer and Marsh [5] on the Lipschitz equivalence of quasi-self-similar circles, Falconer and Marsh [6] on dust-like self-similar sets, Wen and Xi [12] on self-similar arcs, Xi [13] on dust-like self-conformal sets, and Rao, Ruan and Xi [10] on just touching self-similar sets and answering an open question of David and Semmes [2].

In general, it is hard to assert that two sets are not Lipschitz equivalent. This is often done by constructing some Lipschitz invariant, for instance, the dimensions of fractal sets. In [6], Falconer and Marsh constructed a free group associated with a dust-like self-similar set, and proved that it is a Lipschitz invariant.

We will show that the gap sequence defined above is a new type of Lipschitz invariant.

In the following, we will always assume the sets under consideration have infinite gap sequence since typical fractal sets have this property.

Theorem 1. *Let E and F be compact subsets of* $\mathbb{R}^d$ *with gap sequences* $\{\alpha_m\}_{m\geqslant 1}$ *and* $\{\beta_m\}_{m\geqslant 1}$*, respectively. If f is a bijection from E to F, then*

$$M(f) \geqslant \limsup_{m\to\infty} \frac{\beta_m}{\alpha_m}.$$

Consequently, if E and F are Lipschitz equivalent, then there exist two constants $0 < c < C < +\infty$ *such that* $c < \beta_m/\alpha_m < C$ *holds for all m.*

If the last condition is satisfied, then we say two infinite real sequences $\{\alpha_m\}$ and $\{\beta_m\}$ are *equivalent*. Apparently this is an equivalence relation on the set of positive sequences. Any Cantor set is associated with an *equivalence class* containing its gap sequence, which is a Lipschitz invariant according to theorem 1.

Gap sequence and box dimension. It is well known that if E is a compact set of $\mathbb{R}$ with Lebesgue measure 0, then the upper box dimension $\overline{\dim}_B$ is determined by the gap sequence of E.

Theorem A (see [11]). Let E be a compact subset of $\mathbb{R}$ with Lebesgue measure 0 and $\{\alpha_m\}_{m\geqslant 1}$ be the gap sequence of E, then

$$\overline{\dim}_B(E) = \limsup_{m\to\infty} \frac{\log m}{-\log \alpha_m}. \tag{1}$$

Unfortunately, the above property is not true in general for higher dimensional cases. Some counter-examples are listed in section 3 (example 3.1). However we could show that for a certain class of fractal sets in $\mathbb{R}^d$, formula (1) still holds.

Let us denote by $E(\delta) = \{x \in \mathbb{R}^d : \inf_{y\in E} |x - y| \leqslant \delta\}$, the δ-parallel body of E.

Theorem 2. *Let E be a compact subset of* $\mathbb{R}^d$*. If there exist a constant C and a sequence* $\{\delta_k\}_{k\geqslant 1}$ *tending to* 0 *such that* $\liminf_{k\to\infty} \frac{\log \delta_k}{\log \delta_{k+1}} = 1$*, and every connected component of* $E(\delta_k)$ *has Lebesgue measure less than* $C\delta_k^d$*, then formula (1) holds.*

Using this theorem, we show that formula (1) holds if either E is a self-similar set satisfying strong separation condition (example 3.2) or E belongs to a class of McMullen self-affine sets (example 3.3).

Remark 1. It is easy to check that if formula (1) holds for two fractal sets E, F, then gap sequences of E and F are equivalent implies $\overline{\dim}_B(E) = \overline{\dim}_B(F)$. Thus for typical Cantor sets of $\mathbb{R}$ and some typical fractal sets of $\mathbb{R}^d$, the new invariant gap sequence is stronger than the upper box dimension.

This paper is arranged as follows. In section 2, we prove theorem 1. Theorem 2 is proved in section 3.

2. Gap sequence and Lipschitz equivalence

In this section, we will study the relation between gap sequence and Lipschitz equivalence. Recall that $h(\delta)$ is the number of δ-connected components of E.

Lemma 1. *Let E be a compact subset of $\mathbb{R}^d$. Then, the function $h : \mathbb{R}^+ \to \mathbb{N}$ satisfies the following properties.*

(i) $h(\delta)$ is equal to the number of connected components of the set $E(\delta/2)$.
(ii) $h(\delta)$ is a non-increasing function of δ.
(iii) The set of discontinuous points of $h(\delta)$ is at most countable; let us denote them by $\{a_1, a_2, \cdots\}$ in decreasing order, then $h(\delta)$ is a constant on the interval $[a_{k+1}, a_k)$ and $h(\delta) = 1$ on the interval $[a_1, +\infty)$.

Proof.

(i) For any $\delta > 0$, let $E_1, \cdots, E_m$ be the δ-connected components of E, where $m = h(\delta)$. First, $E = E_1 \cup \cdots \cup E_m$ implies that $E(\delta/2) = E_1(\delta/2) \cup \cdots \cup E_m(\delta/2)$ and the union is disjoint since $d(E_i, E_j) > \delta$ for $i \neq j$.

Second, we show that $E_i(\delta/2)$ is connected, $1 \leqslant i \leqslant m$. Take $x, y \in E_i(\delta/2)$, then there are $x_1, y_1 \in E_i$ such that $|x_1 - x| \leqslant \delta/2$, $|y_1 - y| \leqslant \delta/2$. Let $\{x_1, \cdots, x_n = y_1\} \subset E_i$ be a δ-chain connecting x_1 and y_1; clearly the broken line $\overline{xx_1}, \overline{x_1x_2}, \ldots, \overline{x_{n-1}x_n}, \overline{x_ny}$ is a path from x to y in $E_i(\delta/2)$.

(ii) For $0 < \delta_1 < \delta_2$, noting that every δ_1-connected set is δ_2-connected, we have $h(\delta_1) \geqslant h(\delta_2)$.

(iii) The function $h(\delta)$ has at most numerable discontinuous points (jump points) follows from its monotonicity. Let us denote these jump points of h by $a_1, a_2, \cdots, a_k, \cdots$, where $a_k > a_{k+1}$.

Pick $k \geqslant 1$. Let us denote $m = h(a_{k+1})$ and denote by $E_1, \cdots, E_m$ the (a_{k+1})-components of E. Then $d(E_i, E_j) > a_{k+1}$ for any $i \neq j$. Set

$$\alpha = \min\{d(E_i, E_j) : 1 \leqslant i < j \leqslant m\}.$$

For $a_{k+1} \leqslant \delta < \alpha$, $\{E_i(\delta/2)\}_{1 \leqslant i \leqslant m}$ are disjoint by the minimality of α; on the other hand, $E_i(\delta/2)$ are connected since $E_i(a_{k+1}/2)$ are connected. This proves that $h(\delta) = m = h(a_{k+1})$.

If $\delta = \alpha$, then $d(E_i, E_j) = \alpha$ for some $1 \leqslant i < j \leqslant m$. Hence $E_i(\delta/2)$ and $E_j(\delta/2)$ forms one connected component, so that $h(\delta) < h(a_{k+1})$. Therefore α is a discontinuous point and $\alpha = a_k$.

Finally, it is clear that $h(\delta) = 1$ on $[a_1, +\infty)$. □

Definition 2.1. *Let $E \subset \mathbb{R}^d$ be a compact set and let $\{a_k\}_{k \geqslant 1}$ be the discontinuous points of h. Set*

$$\alpha_m = a_k \qquad \text{if } h(a_k) \leqslant m \leqslant h(a_{k+1}) - 1,$$

and we call $\{\alpha_m\}_{m \geqslant 1}$ the **gap sequence** *of E.*

Now we show that the gap sequence is closely related to the Lipschitz equivalent property.

Proof of theorem 1. Let E, F be two compact subsets of $\mathbb{R}^d$ with gap sequences $\{\alpha_m\}_{m \geqslant 1}$ and $\{\beta_m\}_{m \geqslant 1}$, respectively. Pick any $m \geqslant 1$. Let k be the integer satisfying $h(a_k) \leqslant m < h(a_{k+1})$, then $\alpha_m = a_k$ by definition.

Let $L = h(a_{k+1})$ be the number of (a_{k+1})-connected components of E; let us denote them by $E_1, \cdots, E_L$. Note that

(i) $d(E_i, E_j) \geqslant a_k$ and the value a_k is attainable. (See the last paragraph of the proof of lemma 1.)

We define a graph G as follows. The vertex set is $\{E_1, \ldots, E_L\}$. There is an edge connecting E_i and E_j if and only if $d(E_i, E_j) = a_k$. It is clear that

(ii) The graph G contains exactly $h(a_k)$ connected components.

Now we delete the edges of G one by one in any order. Then the number of connected components of the remaining graph will increase. We stop as soon as the number of connected components becomes $m+1$. This is possible since the number of vertices of G is $L = h(a_{k+1}) \geqslant m+1$. Let us denote the connected components of the resulting graph by $G_1, \ldots, G_{m+1}$.

Let $I_j = \bigcup\{E_s : E_s \in G_j\}$, $1 \leqslant j \leqslant m+1$. Then $E = I_1 \cup \cdots \cup I_{m+1}$ is a partition of E. We claim that

(iii) I_j is a_k-connected. (So it is α_m-connected.)

Actually, we will show that for any connected subgraph G' of G, $\bigcup_{E_s \in G'}$ is (a_k)-connected. First, E_s is (a_{k+1})-connected and thus (a_k)-connected. Second, for $E_s, E_{s'} \in G'$, if there is an edge between E_s and $E_{s'}$, then $d(E_s, E_{s'}) = a_k$. It follows that $E_s \bigcup E_{s'}$ is (a_k)-connected. Therefore one can show that $\bigcup_{E_s \in G'} E_s$ is (a_k)-connected inductively.

(iv) $d(I_i, I_j) \geqslant a_k$ and there exists at least one pair I_i and I_j such that $d(I_i, I_j) = a_k = \alpha_m$.

It follows from (i) that $d(I_i, I_j) \geqslant a_k$. If the second assertion is false, then $d(I_i, I_j) > a_k$ for any $i \neq j$. So every G_j is not only a connected subgraph but also a connected component of G. So G has $m+1 \geqslant h(a_k)+1$ components, which contradicts (ii).

In the same way, let k' be the integer such that $h(b_{k'}) \leqslant m < h(b_{k'+1})$, we can obtain a partition $F = J_1 \cup \cdots \cup J_{m+1}$ having the same properties as the partition $E = I_1 \cup \cdots \cup I_{m+1}$, but here we only need the property

(v) $d(J_i, J_j) \geqslant b_{k'} = \beta_m$ for any $1 \leqslant i, j \leqslant m+1$ with $i \neq j$.

Let f be a bijection from E to F. Now we show that $M(f) \geqslant \beta_m/\alpha_m$. According to f and the partitions of E and F, we define a map $T : E \to \{1, 2, \cdots, m+1\}$ as

$$T(x) = i \qquad \text{if } f(x) \in J_i.$$

Case 1. There exist $1 \leqslant \ell \leqslant m+1$ and $x_1, x_2 \in I_\ell$ such that $T(x_1) \neq T(x_2)$.

Set $H_j = \{x \in I_\ell : T(x) = j\}$. Then $I_\ell = \bigcup_{j=1}^{m+1} H_j$ is a partition of I_ℓ and at least two parts of the partition are not empty. Since I_ℓ is a_k-connected by (iii), we can choose two non-empty sets H_{j_1} and H_{j_2} satisfying $d(H_{j_1}, H_{j_2}) \leqslant a_k = \alpha_m$. Then there exist $x^* \in H_{j_1}$ and $y^* \in H_{j_2}$ such that $|x^* - y^*| \leqslant \alpha_m$.

On the other hand, $T(x^*) \in J_{j_1}$ and $T(y^*) \in J_{j_2}$ implies that $|f(x^*) - f(y^*)| \geqslant \beta_m$ by (v). Hence

$$M(f) \geqslant |f(x^*) - f(y^*)|/|x^* - y^*| = \beta_m/\alpha_m.$$

Case 2. $T(x)$ is constant on I_ℓ for each $1 \leqslant \ell \leqslant m+1$.

Under this assumption, f maps I_i to one set J_j. Different I_i maps to different J_j since f is bijective (and it implies that $f(I_i) = J_j$). By (iv), there exists a pair I_i and I_j such that $d(I_i, I_j) = \alpha_m$. Pick $x \in I_i$ and $y \in I_j$ such that $|x - y| = \alpha_m$. Then $|f(x) - f(y)| \geqslant \beta_m$ since $f(x)$ and $f(y)$ belong to different J_ℓ. Again $M(f) \geqslant \beta_m/\alpha_m$.

The theorem is proved. □

In the following, we briefly describe how to construct a Cantor set $E \subset [0, 1]$ with diameter 1 and with gap sequence $\{\alpha_m\}_{m \geqslant 1}$, where $\sum_m \alpha_m \leqslant 1$.

Let $c_1 = \alpha_2 + \alpha_4 + \cdots + \alpha_{2m} + \cdots$, $d_1 = \alpha_3 + \alpha_5 + \cdots + \alpha_{2m+1} + \cdots$, then $c_1 + \alpha_1 + d_1 \leqslant 1$. Thus c_1, d_1 must belong to one of the following three cases: $\frac{1-\alpha_1}{2} \geqslant \max\{c_1, d_1\}$; $c_1 \geqslant \frac{1-\alpha_1}{2} > d_1$; $d_1 \geqslant \frac{1-\alpha_1}{2} > c_1$. We remove an open interval $]\frac{1-\alpha_1}{2}, \frac{1+\alpha_1}{2}[$, $]c_1, c_1+\alpha_1[$ and $]1-d_1-\alpha_1, 1-d_1[$ in corresponding cases, respectively. Denote the remaining two intervals by $I_1^{(1)}$ and $I_2^{(1)}$. It is clear that in all three cases, we have

$$\max\{1/2, c_1\} \geqslant |I_1^{(1)}| \geqslant c_1, \qquad \max\{1/2, d_1\} \geqslant |I_1^{(2)}| \geqslant d_1.$$

Define $c_2 = \sum_{m=1}^{\infty} \alpha_{4m}$, $d_2 = \sum_{m=1}^{\infty} \alpha_{4m+2}$, $c_3 = \sum_{m=1}^{\infty} \alpha_{4m+1}$ and $d_3 = \sum_{m=1}^{\infty} \alpha_{4m+3}$. Then c_2, d_2 must belong to one of the following three cases: $\frac{|I_1^{(1)}|-\alpha_2}{2} \geqslant \max\{c_2, d_2\}$; $c_2 \geqslant \frac{|I_1^{(1)}|-\alpha_2}{2} > d_2$; $d_2 \geqslant \frac{|I_1^{(1)}|-\alpha_2}{2} > c_2$. Thus, we can remove an open interval in $I_1^{(1)}$ with length α_2 as above such that the remaining two subintervals $I_1^{(2)}, I_2^{(2)}$ of $I_1^{(1)}$ satisfy

$$\max\{1/2^2, c_2\} \geqslant |I_1^{(2)}| \geqslant c_2, \qquad \max\{1/2^2, d_2\} \geqslant |I_2^{(2)}| \geqslant d_2.$$

Similarly, we can remove an open interval in $I_2^{(1)}$ with length α_3 such that the remaining two subintervals $I_3^{(2)}, I_4^{(2)}$ of $I_2^{(1)}$ satisfy

$$\max\{1/2^2, c_3\} \geqslant |I_3^{(2)}| \geqslant c_3, \qquad \max\{1/2^2, d_3\} \geqslant |I_4^{(2)}| \geqslant d_3.$$

Continue the above construction inductively; finally we obtain a Cantor set $E \subset [0, 1]$ with gap sequence $\{\alpha_m\}$.

3. Gap sequence and box dimension

Let E be a compact subset of $\mathbb{R}^d$; the following inequality always holds.

Proposition 1. *Let E be a compact set of $\mathbb{R}^d$, then*

$$\overline{\dim}_B(E) \geqslant \limsup_{m\to\infty} \frac{\log m}{-\log \alpha_m}, \tag{2}$$

where $\{\alpha_m\}_{m\geqslant 1}$ is the gap sequence of E.

Proof. For any $\delta > 0$, take $m \in \mathbb{N}$ such that $\alpha_{m+1} \leqslant 2\delta < \alpha_m$. Then $h(2\delta) = h(\alpha_{m+1}) = m+1$. Let us denote $\mathcal{L}(A)$ the Lebesgue measure of the set $A \subset \mathbb{R}^d$ and denote C_0 the Lebesgue measure of the unit ball of $\mathbb{R}^d$. Then $\mathcal{L}(E(\delta)) \geqslant h(2\delta)C_0\delta^d$ and from [4],

$$\overline{\dim}_B(E) = d + \limsup_{\delta\to 0} \frac{\log \mathcal{L}(E(\delta))}{-\log \delta}. \tag{3}$$

Thus,

$$\begin{aligned}\overline{\dim}_B(E) &\geqslant d + \limsup_{\delta\to 0} \frac{\log h(2\delta)C_0\delta^d}{-\log \delta} = \limsup_{\delta\to 0} \frac{\log(m+1)C_0}{-\log \delta}\\ &\geqslant \limsup_{m\to\infty} \frac{\log(m+1)C_0}{-\log(\alpha_{m+1}/2)} = \limsup_{m\to\infty} \frac{\log(m+1)}{-\log \alpha_{m+1}}.\end{aligned}$$

□

In the case of $d = 1$ and Lebesgue measure of E equals 0, formula (2) is actually an equality (see [4, 11]). But for $d \geqslant 2$, the situation is much more delicate.

Examples 3.1.

1. Let $E = \mathcal{C} \times [0, 1]$, where $\mathcal{C}$ is the Cantor middle-third set. It is easy to check that $\overline{\dim}_B(E) = 1 + \frac{\log 2}{\log 3}$, while $\limsup_{m\to\infty} \frac{\log m}{-\log \alpha_m} = \frac{\log 2}{\log 3}$.

2. Even if E is totally disconnected and has Lebesgue measure 0, the inequality in (2) can be strict. For example, let E be a Cantor set in $[0, 1]$ with gap sequence $\{2^{-(k+1)} : k \geqslant 1\}$. Then the one-dimensional Lebesgue measure of E is $1/2$. Let us embed E into $\mathbb{R}^2$, then $\overline{\dim}_B(E) = 1$, while $\limsup_{m\to\infty} \frac{\log m}{-\log \alpha_m} = 0$.

Recalling that theorem 2 is: *Let E be a compact subset of $\mathbb{R}^d$. If there exist a constant C and a sequence $\{\delta_k\}$ tending to 0 such that $\liminf_{k\to\infty} \dfrac{\log \delta_k}{\log \delta_{k+1}} = 1$, and every connected component of $E(\delta_k)$ has Lebesgue measure less than $C\delta_k^d$, then*

$$\overline{\dim}_B(E) = \limsup_{m\to\infty} \frac{\log m}{-\log \alpha_m}.$$

Basically, it requires δ_k-components of E are small. It is clear that all examples in example 3.1 violate this requirement.

Proof of theorem 2. For any δ, let k be the integer such that $\delta_{k+1} \leqslant \delta < \delta_k$. Then

$$\frac{\log \mathcal{L}(E(\delta))}{-\log \delta} \leqslant \frac{\log \mathcal{L}(E(\delta_k))}{-\log \delta_{k+1}} = \frac{\log \mathcal{L}(E(\delta_k))}{-\log \delta_k} \frac{\log \delta_k}{\log \delta_{k+1}}.$$

Let m be the integer such that $\alpha_{m+1} \leqslant 2\delta_k < \alpha_m$, then $h(2\delta_k) = h(\alpha_{m+1}) = m + 1$. Therefore by (3),

$$\begin{aligned}
\overline{\dim}_B(E) &\leqslant d + \limsup_{k\to\infty} \frac{\log \mathcal{L}(E(\delta_k))}{-\log \delta_k} \cdot \liminf_{k\to\infty} \frac{\log \delta_k}{\log \delta_{k+1}} \\
&\leqslant d + \limsup_{k\to\infty} \frac{\log(m+1)C\delta_k^d}{-\log \delta_k} \cdot 1 \\
&= \limsup_{k\to\infty} \frac{\log C(m+1)}{-\log \delta_k} \\
&\leqslant \limsup_{m\to\infty} \frac{\log C(m+1)}{-\log \alpha_m/2} = \limsup_{m\to\infty} \frac{\log(m+1)}{-\log \alpha_m}.
\end{aligned}$$

This, together with proposition 1, implies the desired equality. □

The following examples show that formula (1) holds for some typical fractal sets.

Example 3.2. Let E be the self-similar set defined by the iterated function system $\{S_j\}_{j=1}^L$, where S_j are contractive similitudes on $\mathbb{R}^d$ with contraction ratios c_j. If the IFS satisfies a strong separation condition, i.e. $S_i(E) \cap S_j(E) = \emptyset$ wherever $i \neq j$, then the conditions of theorem 2 are satisfied and formula (1) holds for E.

Let $\Sigma = \{1, 2, \ldots, L\}$ and $\Sigma^n = \{i_1 i_2 \cdots i_n : i_j \in \Sigma \text{ for all } 1 \leqslant j \leqslant n\}$. Define $\Sigma^* = \cup_{n=1}^\infty \Sigma^n$. For $I = i_1 i_2 \ldots i_k \in \Sigma^*$, we denote $c_I = c_{i_1} \circ \cdots \circ c_{i_k}$, $S_I = S_{i_1} \circ \cdots \circ S_{i_k}$ and $E_I = S_I(E)$. Denote $|E|$ the diameter of the set E, $d(E_i, E_j)$ the distance of the sets E_i and E_j. Without loss of generality, let us assume $|E| = 1$. Let

$$r = \min\{d(E_i, E_j) : 1 \leqslant i < j \leqslant m\} > 0.$$

Denote $c_* = \min\{c_i : i \in \Sigma\}$. For any $0 < \delta \leqslant c_*$, let $\mathcal{I}_\delta = \{i_1 \cdots i_n \in \Sigma^* : c_{i_1\cdots i_n} < \delta \leqslant c_{i_1\cdots i_{n-1}}\}$. Then $E = \bigcup_{I\in\mathcal{I}_\delta} E_I$ is a finite disjoint union.

We first prove that $d(E_I, E_J) \geqslant r\delta$ holds for any $I \neq J \in \mathcal{I}_\delta$. Suppose $I = i_1 \cdots i_n$, $J = j_1 \cdots j_m$. Let k be the integer satisfying $i_1 \cdots i_k = j_1 \cdots j_k$ but $i_{k+1} \neq j_{k+1}$. Since $E_I \subset E_{i_1\cdots i_k i_{k+1}}$ and $E_J \subset E_{j_1\cdots j_k j_{k+1}}$, we have

$$d(E_I, E_J) \geqslant d(E_{i_1\cdots i_k i_{k+1}}, E_{j_1\cdots j_k j_{k+1}}) = d(E_{i_1\cdots i_k}) \cdot d(E_{i_{k+1}}, E_{j_{k+1}}) \geqslant r\delta.$$

Set $\delta' = r\delta/3$, then the δ'-parallel bodies of E_I and E_J are disjoint for $I \neq J$, hence the diameter of any connected component of $E(\delta')$ is less than $\delta + 2\delta'$ and its Lebesgue measure is bounded by $C(\delta')^d$ with $C = C_0(\delta + 2\delta')^d/(\delta')^d = C_0(3/r + 2)^d$, where C_0 is the volume of the unit ball of $\mathbb{R}^d$.

Let $\delta_k = 3^{-k}(k \geqslant 0)$, then $\{\delta_k' = \frac{r\delta_k}{3}\}_{k\geqslant 0}$ and $C = C_0(3/r + 2)^d$ are the sequence and constant satisfying theorem 2.

Example 3.3. McMullen [9] has studied the fractal dimensions of the following sets. Let $N \geqslant M \geqslant 2$ be positive integers. Denote $R(N, M) = \{(i, j) : i = 0, 1, \ldots, N-1;\ j = 0, 1, \ldots, M-1\}$. Let R_0 be a subset of $R(N, M)$ with $\sharp\, R_0 \geqslant 2$. For $(i, j) \in R_0$, let us define $S_{ij} : \mathcal{K} \times \mathcal{K} \to \mathcal{K} \times \mathcal{K}$ as $S_{ij}(x, y) = (\frac{x}{N}, \frac{y}{M}) + (\frac{i}{N}, \frac{j}{M})$, where $\mathcal{K} = [0, 1]$. The maps S_{ij} are affine contractions. Let E be the invariant set of the IFS $\{S_{ij}\}_{(i,j)\in R_0}$ and we call it a McMullen set.

We will show that if E satisfies

$$\begin{aligned} &S_{ij}(E) \cap S_{i'j'}(E) = \emptyset \textit{ for any } (i, j) \neq (i', j') \in R_0 \textit{ and} \\ &\textit{the projection of } E \textit{ to the } y\textit{-axis is not an interval,} \end{aligned} \tag{4}$$

then the conditions of theorem 2 are satisfied and formula (1) holds.

Take any integer $k > 0$, let l_k be the integer satisfying

$$M^{-(k+l_k)-1} \leqslant N^{-k} < M^{-(k+l_k)}.$$

Assume $p = \sharp\, R_0, q = \sharp\,\{j$: there exists i such that $(i, j) \in R_0\}$. The kth approximation of E is

$$\{S_\sigma(\mathcal{K} \times \mathcal{K}) : \sigma = (i_1 j_1) \cdots (i_k j_k) \text{ and } (i_t, j_t) \in R_0 \text{ for } 1 \leqslant t \leqslant k\},$$

which consists of p^k rectangles of the same size.

Pick any of these rectangles, we divide it into M^{l_k} equal parts along the y-direction. By the definition of q, we know that among these M^{l_k} equal rectangles, there are exactly q^{l_k} rectangles which contain the points of the set E. Totally we get $p^k q^{l_k}$ rectangles with size $N^{-k} \times M^{-(k+l_k)}$ which intersect E.

By (4), the projection of the sets $\{S_{ij}(\mathcal{K} \times \mathcal{K})\}_{(i,j)\in R_0}$ on y-axis is a Cantor set as follows: divide the unit interval $\mathcal{K}$ into M equal parts labelled by $0, 1, \ldots, M-1$, and the interval with label j is selected provided $(i, j) \in R_0$ for some i. Suppose these q intervals form s connected components, containing n_i, $1 \leqslant i \leqslant s$ intervals, respectively. Set $T = \max\{n_2, \cdots, n_{s-1}, n_1 + n_s\}$ and let D_{k,l_k} be the union of the above $p^k q^{l_k}$ rectangles. By $S_{ij}(E) \cap S_{i'j'}(E) = \emptyset$ for any $(i, j) \neq (i', j')$, any connected component P of D_{k,l_k} is a rectangle of size $N^{-k} \times (\alpha_P M^{-(k+l_k)})$, where $\alpha_P \leqslant T$.

Let $\delta_k = N^{-k}/3, k \geqslant 1$; then every connected component of $E(\delta_k)$ contains at most T small rectangles and has a diameter of less than $\sqrt{T^2M^2 + 1}\ N^{-k} + 2\delta_k$, and its Lebesgue measure is bounded by $C\delta_k^2$, where $C = (3\sqrt{T^2M^2 + 1} + 2)^2$. Therefore formula (1) holds by theorem 2.

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