Some progress on Lipschitz equivalence of self-similar sets

Huo-Jun Ruan, Zhejiang University

(Joint work with Hui Rao, Yang Wang and Ya-Min Yang)

Sun Yat Sen University – July 4-11, 2010

Outline

- Gap sequence of fractal sets *(based on the joint work with Hui Rao and Ya-Ming Yang)*
  - Definition of gap sequence in $\mathbb{R}^d$
  - Gap sequence is a Lipschitz invariant
  - Gap sequence and box dimension
- Algebraic properties of contractive ratios of dust-like self-similar sets *(based on the joint work with Hui Rao and Yang Wang)*
  - Fundamental work of Falconer-Marsh'1992
  - Dust-like self-similar sets with two branches, and related results

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Part I

Gap sequence of fractal sets

Gap sequence of a compact set $E \subset \mathbb{R}$

- An open interval $]a, b[$ is said to be a gap of E if $a, b \in E$ but $]a, b[\cap E = \emptyset$.
- The set of gaps of E is a collection of open intervals and it is at most countable.
- We are interested in the **lengths of these intervals**, let us list them in a **non-increasing order**, we shall call this (finite or infinite) sequence of positive reals the gap sequence of E.

For example, the gap sequence of Cantor middle-third set $\mathcal{C}$ is

$$1/3, 1/9, 1/9, 1/27, 1/27, 1/27, 1/27, 1/81, \dots$$

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Figure: Cantor middle-third set $\mathcal{C}$

The idea of gap sequence in one dimension has already been used widely to characterize fractal properties of E, especially when E has zero Lebesgue measure.

- Besicovitch-Tayor (1954): Hausdorff dimension
- Tricot (1981): twelve definitions of fractal dimensions
- Lapidus-Pomerance (1993), Lapidus-Maier (1995), Falconer (1995): Minkowski measurability

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- Apply it: Lipschitz equivalence, box dimension.

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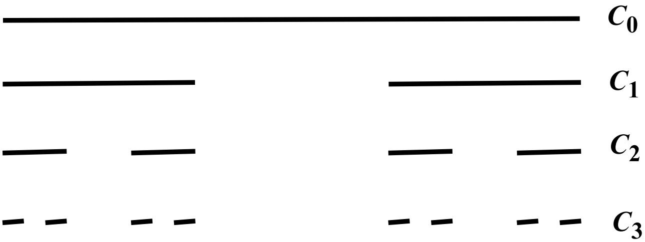

- $C_0 = [0,1]$ is an interval.
- The number of intervals of C_1: 2.
  Distance of two intervals: $1/3$.
- The number of intervals of C_2: 4.
  Shortest distance of two distinct intervals: $1/9$.
- The number of intervals of C_3: 8.
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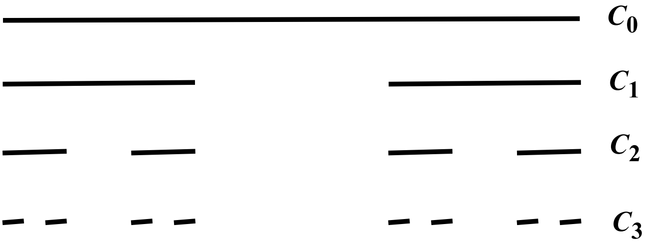

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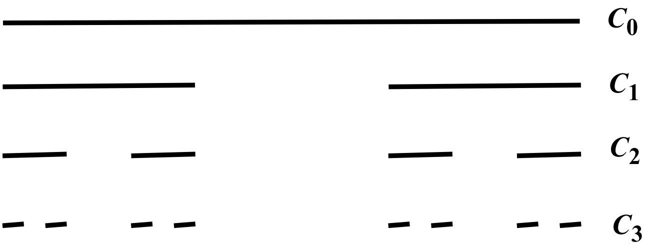

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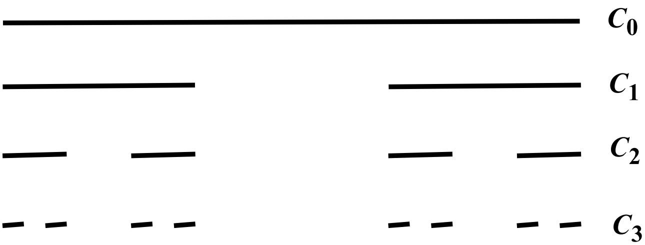

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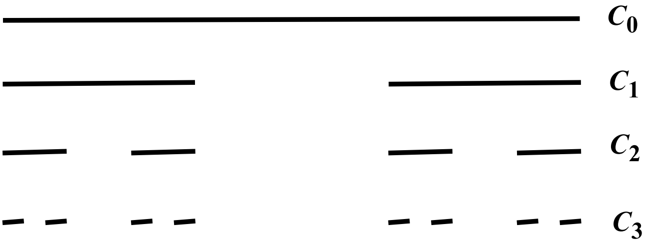

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- A compact subset E of $\mathbb{R}^d$ is said to be δ-connected if for any $x, y \in E$, there is a δ-chain connecting x and y. That is, there is a sequence $\{x_1 = x, x_2, \dots, x_{n-1}, x_n = y\} \subset E$, such that $|x_{j+1} - x_j| \le \delta$ holds for $1 \le j \le n - 1$.
- We call $F \subset E$ a δ-connected component of E if F is δ-connected, but for any $F \subsetneq F' \subset E$, F' is not δ-connected.
- Let us denote by $h_E(\delta)$, or $h(\delta)$ for short, the number of δ-connected components of E, which is finite by the compactness of E.

Definition of gap sequence in $\mathbb{R}^d$

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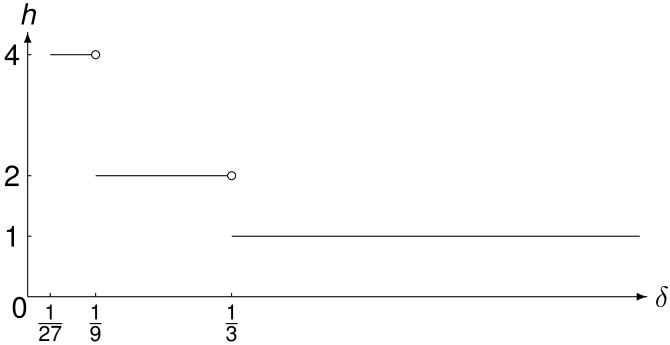
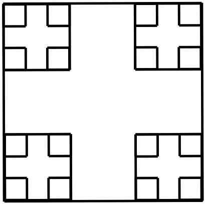
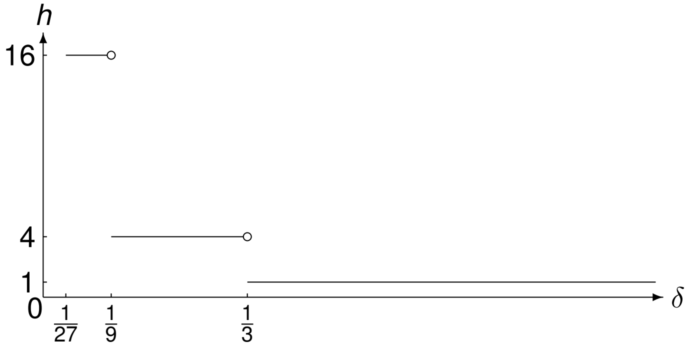


Figure: The function h of $\mathcal{C}$.



The structure of $\mathcal{C} \times \mathcal{C}$



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Definition of gap sequence in $\mathbb{R}^d$

- It can be shown that $h(\delta) : \mathbb{R}^+ \to \mathbb{Z}^+$ is a non-increasing function, is locally constant except at the neighborhoods of discontinuous points, and is right continuous.
- Let us denote by $\{a_k\}_{k\geq 1}$ the discontinuous points (or jump points) of h in decreasing order.
- We call $j_k = h(a_{k+1}) - h(a_k)$ the multiplicity of a_k and define the gap sequence of E to be the sequence:

$$\underbrace{a_1, \dots, a_1}_{j_1}, \underbrace{a_2, \dots, a_2}_{j_2}, \dots, \underbrace{a_m, \dots, a_m}_{j_m}, \dots$$

In the following, we will always assume:

- The sets in consideration have infinite gap sequence.

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Definition of Lipschitz equivalence

Let E and F be two compact subsets of $\mathbb{R}^d$. Let f be a function from E to F.

- The Lipschitz constant of f is defined by

$$M(f) = \sup_{x_1, x_2 \in E;\ x_1 \neq x_2} \left| \frac{f(x_1) - f(x_2)}{x_1 - x_2} \right|.$$

- The map f is said to be Lipschitz if $M(f) < +\infty$.
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Recently, there are some works devoted to the Lipschitz equivalence of fractal sets.

- Falconer-Marsh'1989: Lipschitz equivalence of quasi-self-similar circles.
- Falconer-Marsh'1992: dust-like self-similar sets.
- Wen-Xi'2003: self-similar arcs.
- Xi'2004: dust-like self-conformal sets.
- Rao-R-Xi'2006: just touching self-similar sets and answering an open question of David-Semmes'1997.

In general, it is hard to assert two sets are not Lipschitz equivalent. This is often done by constructing some Lipschitz invariant.

- The dimensions of fractal sets.
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Our result: gap sequence is a Lipschitz invariant.

Theorem (Rao-R-Yang'2008)

Let E and F be compact subsets of $\mathbb{R}^d$ with gap sequences $\{\alpha_m\}_{m\geq 1}$ and $\{\beta_m\}_{m\geq 1}$ respectively. If f is a bijection from E to F, then

$$M(f) \geq \sup_m \frac{\beta_m}{\alpha_m}.$$

Consequently, if E and F are Lipschitz equivalent, then $\exists 0 < c_1 < c_2 < +\infty$ such that $c_1 < \beta_m/\alpha_m < c_2$ holds for all m.

- If the last condition is satisfied, then we say two infinite real sequence $\{\alpha_m\}$ and $\{\beta_m\}$ are equivalent.
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Gap sequence and box dimension

Theorem (Tricot'1981)

Let E be a compact subset of $\mathbb{R}$ with Lebesgue measure 0 and $\{\alpha_m\}_{m\geq 1}$ be the gap sequence of E, then

$$\overline{\dim}_B(E) = \limsup_{m\to\infty} \frac{\log m}{-\log \alpha_m}. \tag{1}$$

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Proposition (Rao-R-Yang'2008)

Let E be a compact set of $\mathbb{R}^d$, then

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The inequality in formula (2) cannot be replaced by equality for higher dimensional case, if we only require that the Lebesgue measure of E equals 0.

Example

1. Let $E = \mathcal{C} \times [0,1]$. Then $\overline{\dim}_B(E) = 1 + \frac{\log 2}{\log 3}$, while $\limsup_{m\to\infty} \frac{\log m}{-\log \alpha_m} = \frac{\log 2}{\log 3}$.
2. Let E be a Cantor set in $[0,1]$ with gap sequence $\{2^{-(m+1)} : \ m \geq 1\}$. Then $\mathcal{L}^1(E) = 1/2$. Let us embed E into $\mathbb{R}^2$, then $\overline{\dim}_B(E) = 1$, while $\limsup_{m\to\infty} \frac{\log m}{-\log \alpha_m} = 0$.

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1. Let $E = \mathcal{C} \times [0,1]$. Then $\overline{\dim}_B(E) = 1 + \frac{\log 2}{\log 3}$, while $\limsup_{m\to\infty} \frac{\log m}{-\log \alpha_m} = \frac{\log 2}{\log 3}$.
2. Let E be a Cantor set in $[0,1]$ with gap sequence $\{2^{-(m+1)} : \ m \geq 1\}$. Then $\mathcal{L}^1(E) = 1/2$. Let us embed E into $\mathbb{R}^2$, then $\overline{\dim}_B(E) = 1$, while $\limsup_{m\to\infty} \frac{\log m}{-\log \alpha_m} = 0$.

Gap sequence and box dimension

- We will show that for a certain class of fractal sets in $\mathbb{R}^d$, dimension formula **(1)** still holds.
- $E(\delta) = \{x \in \mathbb{R}^d : \inf_{y \in E} |x - y| \leq \delta\}$, the δ-parallel body of E.

Theorem (Rao-R-Yang'2008)

Let E be a compact subset of $\mathbb{R}^d$. If there exist a constant C and a sequence $\{\delta_k\}_{k \geq 1}$ tending to 0 such that

(a). $\liminf_{k \to \infty} \frac{\log \delta_k}{\log \delta_{k+1}} = 1$,

(b). every connected component of $E(\delta_k)$ has Lebesgue measure less than $C\delta_k^d$,

then dimension formula **(1)** holds.

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Gap sequence and box dimension

Some fractals such that dimension formula **(1)** holds:

- *E* is a self-similar set satisfying strong separation condition.
- *E* belongs to a certain class of McMullen self-affine sets.

Part II

Algebraic properties of contraction ratios of dust-like self-similar sets

Dust-like self-similar sets

- Let K be a self-similar set determined by the IFS $\{f_1, \ldots, f_m\}$, where each f_j is a similarity on $\mathbb{R}^d$ with contractive ratio ρ_j. We call K a self-similar set with contraction vector $(\rho_1, \ldots, \rho_m)$.
- We write c.v. for contraction vector.
- Example. Cantor middle-third set is a self-similar set with c.v. $(1/3, 1/3)$.
- We say K is dust-like if $f_i(K) \cap f_j(K) = \emptyset$ for any $i \neq j$.
- For dust-like self-similar set K with c.v. $(\rho_1, \ldots, \rho_m)$, we have $\dim_H K = s$, where $\sum \rho_j^s = 1$.

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Dust-like self-similar sets

- For any c.v. $\overrightarrow{\rho} = (\rho_1, \ldots, \rho_m)$ with $\sum \rho_j^d < 1$, we define $\mathcal{D}_d(\overrightarrow{\rho}) := \mathcal{D}_d(\rho_1, \ldots, \rho_m)$ to be all dust-like self-similar sets with c.v. $(\rho_1, \ldots, \rho_m)$ in $\mathbb{R}^d$.
- Throughout the talk, the dimension d will be implicit.
- We write $\mathcal{D}(\overrightarrow{\rho})$ for $\mathcal{D}_d(\overrightarrow{\rho})$.
- We define $\dim_H \mathcal{D}(\overrightarrow{\rho}) = \dim_H E$, for some (then for all) $E \in \mathcal{D}(\overrightarrow{\rho})$.

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One well-known result on Lipschitz equivalence of dust-like self-similar sets:

Proposition

$E \sim F$ for any $E, F \in \mathcal{D}(\overrightarrow{\rho})$.

- Define $\mathcal{D}(\overrightarrow{\rho}) \sim \mathcal{D}(\overrightarrow{\tau})$ if $E \sim F$ for some $E \in \mathcal{D}(\overrightarrow{\rho})$ and $F \in \mathcal{D}(\overrightarrow{\tau})$.

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Fundamental result by Falconer and Marsh

Theorem (Falconer-Marsh'1992)

Let $\overrightarrow{\rho} = (\rho_1, \dots, \rho_m)$, $\overrightarrow{\tau} = (\tau_1, \dots, \tau_n)$ be two c.v.s with $\mathcal{D}(\overrightarrow{\rho}) \sim \mathcal{D}(\overrightarrow{\tau})$. Let $s = \dim_H \mathcal{D}(\overrightarrow{\rho}) = \dim_H \mathcal{D}(\overrightarrow{\tau})$. Denote

- $\mathbb{Q}(a_1, \dots, a_m)$: subfield of $\mathbb{R}$ generated by $\mathbb{Q}$ and $a_1, \dots, a_m$.
- $\mathrm{sgp}(a_1, \dots, a_m)$: subsemigroup of $(\mathbb{R}^+, \times)$ generated by $a_1, \dots, a_m$.

Then

(1) $\mathbb{Q}(\rho_1^s, \dots, \rho_m^s) = \mathbb{Q}(\tau_1^s, \dots, \tau_n^s)$;

(2) $\exists p, q \in \mathbb{Z}^+$, s.t. $\mathrm{sgp}(\rho_1^p, \dots, \rho_m^p) \subset \mathrm{sgp}(\tau_1, \dots, \tau_n)$ and $\mathrm{sgp}(\tau_1^q, \dots, \tau_n^q) \subset \mathrm{sgp}(\rho_1, \dots, \rho_m)$.

- (2)$\Longleftrightarrow$ $\exists p, q \in \mathbb{Z}^+$, s.t. $\mathrm{sgp}(\rho_1^{sp}, \dots, \rho_m^{sp}) \subset \mathrm{sgp}(\tau_1^s, \dots, \tau_n^s)$ and $\mathrm{sgp}(\tau_1^{sq}, \dots, \tau_n^{sq}) \subset \mathrm{sgp}(\rho_1^s, \dots, \rho_m^s)$.

Example

$\mathcal{D}(1/3, 1/3) \not\sim \mathcal{D}(r, r, r)$, where $2 \cdot (1/3)^s = 1 = 3 \cdot r^s$.

Proof.

- Denote $\rho_1 = \rho_2 = 1/3$, $\tau_1 = \tau_2 = \tau_3 = r$. Then $\rho_j^s = 1/2, \quad \tau_j^s = 1/3, \forall j$.
- $\mathrm{sgp}(\rho_1^{sp}, \rho_2^{sp}) = \{(1/2)^{np} : n \in \mathbb{N}\}$, $\mathrm{sgp}(\tau_1^s, \tau_2^s, \tau_3^s) = \{(1/3)^n : n \in \mathbb{N}\}$.
- $\mathrm{sgp}(\rho_1^{sp}, \rho_2^{sp}) \not\subset \mathrm{sgp}(\tau_1^s, \tau_2^s, \tau_3^s)$ for any $p \in \mathbb{Z}^+$.

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What's the necessary and sufficient condition for $\mathcal{D}(\rho_1, \rho_2) \sim \mathcal{D}(\tau_1, \tau_2)$?

- We assume that $\rho_1 \leq \rho_2$, $\tau_1 \leq \tau_2$ and $\rho_1 \leq \tau_1$.
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Assume that $\mathcal{D}(\rho_1, \rho_2) \sim \mathcal{D}(\tau_1, \tau_2)$. By Fal-Mar' theorem, one of followings must happen:

Case 1. $\exists \lambda \in (0,1)$, and $p_1, p_2, q_1, q_2 \in \mathbb{Z}^+$ such that

$$\rho_1 = \lambda^{p_1}, \quad \rho_2 = \lambda^{p_2}, \quad \tau_1 = \lambda^{q_1}, \quad \tau_2 = \lambda^{q_2}.$$

Case 2. $\exists \lambda, \mu \in (0,1)$ with $\log \lambda / \log \mu \notin \mathbb{Q}$, and $p_1, q_1, p_2, q_2 \in \mathbb{Z}^+$ such that

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Let's study Case 1 first.

- From $s = \dim_H \mathcal{D}(\rho_1, \rho_2) = \dim_H \mathcal{D}(\tau_1, \tau_2)$, we have

$$(\lambda^{p_1})^s + (\lambda^{p_2})^s = (\lambda^{q_1})^s + (\lambda^{q_2})^s = 1.$$

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- That is,

$$x^{p_1} + x^{p_2} - 1 = 0 \quad \text{and} \quad x^{q_1} + x^{q_2} - 1 = 0$$

have same root in $(0, 1)$, where $p_1 \geq p_2, q_1 \geq q_2, p_1 \geq q_1$.
- Using Ljunggren's result on the irreducibility of trinomials $x^n \pm x^m \pm 1$, we proved that the above happen iff
  - $(p_1, p_2) = (q_1, q_2)$ or
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$$(\rho_1, \rho_2, \tau_1, \tau_2) = (\lambda^5, \lambda, \lambda^3, \lambda^2). \tag{3}$$

We will check that if Eqn (3) holds, then $\mathcal{D}(\lambda_1, \lambda_2) \sim \mathcal{D}(\tau_1, \tau_2)$.

- Consider IFSs $\{f_1, f_2\}$ and $\{f_1, f_2 \circ f_1, f_2 \circ f_2\}$, we have

$$\mathcal{D}(\lambda^5, \lambda) \sim \mathcal{D}(\lambda^5, \lambda^6, \lambda^2).$$

- Consider IFSs $\{f_1, f_2\}$ and $\{f_1 \circ f_1, f_1 \circ f_2, f_2\}$, we have

$$\mathcal{D}(\lambda^3, \lambda^2) \sim \mathcal{D}(\lambda^6, \lambda^5, \lambda^2).$$

Step 2 to solve the Question

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We will check that if Eqn (3) holds, then $\mathcal{D}(\lambda_1, \lambda_2) \sim \mathcal{D}(\tau_1, \tau_2)$.

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Step 2 to solve the Question

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Let's study Case 2 now.

- Given a c.v. $\overrightarrow{\rho} = (\rho_1, \ldots, \rho_m)$. Define

$$\langle \overrightarrow{\rho} \rangle := \langle \rho_1, \ldots, \rho_m \rangle := \{\rho_1^{\alpha_1} \cdots \rho_m^{\alpha_m} : \ \alpha_1, \ldots, \alpha_m \in \mathbb{Z}\}.$$

- $\langle \overrightarrow{\rho} \rangle$ is a free abelian group and has a nonempty basis.
- Define $\mathrm{rank}\langle \overrightarrow{\rho} \rangle$ to be the cardinality of the basis.
- Clearly, $1 \leq \mathrm{rank}\langle \overrightarrow{\rho} \rangle \leq \mathrm{m}$.
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Assume that both $\overrightarrow{\rho}$ and $\overrightarrow{\tau}$ have full rank m. Then $\mathcal{D}(\overrightarrow{\rho}) \sim \mathcal{D}(\overrightarrow{\tau})$ if and only if $\overrightarrow{\tau}$ is a permutation of $\overrightarrow{\rho}$.

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Thank you!