Fractal Families and Basic Properties

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- Suppose that we have some experience of
  - Hausdorff measure and Hausdorff dimension
  - Iterated Function System - IFS
  - Graph-Directed IFS - GIFS
  - Open Set Condition - OSC
- We will consider a few typical types of fractal sets
  - specified by IFS or GIFS and
  - related to tiling theory & substitutions,
- We aim to consider the structure of such a set,

  trying to understand particular aspects of its structure, especially fractal and/or topological aspects, or $\cdots\cdots$
- Based on recent studies of the above *fractal-topology* nature, we will
  - review some ideas that use graphs in the study of fractal sets
  - discuss the possibility of further extending those ideas

  from graphs to complexes

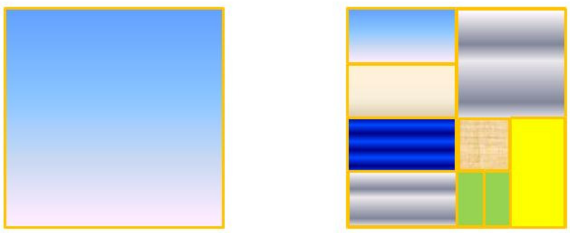
- According to Hutchinson [1981], an IFS is a finite family of contractions $\{f_1, \ldots, f_q\}$ on a complete metric space (X, d),
  - which then specifies a unique compact set $T \subset X$ satisfying the set equation

$$T = \bigcup_{i=1}^{q} f_i(T)$$

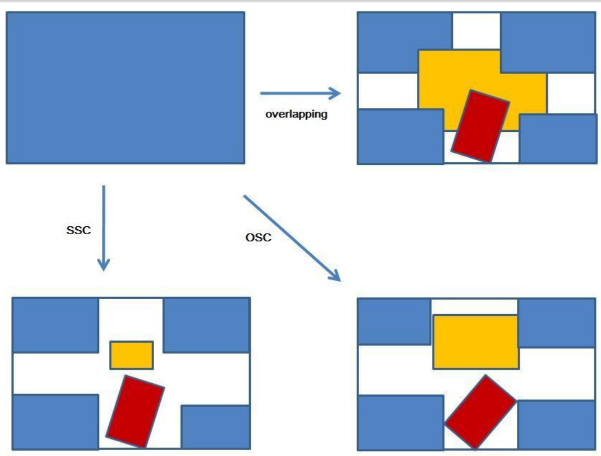
- Roughly speaking, the set T has some *self-similar structure*, in the sense that T can be *partitioned into* several *copies $f_i(T)$ of itself*, each of which is the image of T under a contraction f_i from the IFS
  - For the sake of briefness, T is also called a *self-similar fractal*.
  - when each f_i is a similarity contraction, with ratio $c_i < 1$, we say T is a *self-similar set* and call the number s determined by $\sum_i c_i^s = 1$ the similarity dimension of T.
- We will confine ourselves to basic properties concerning the fractal structure or topology of T.
- To feel *self-similarity* arising from IFS ?

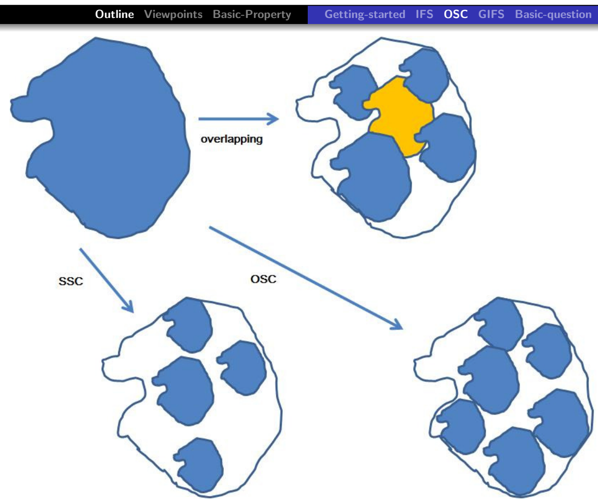
A square can be considered as a self-similar set:

The copies $f_i(T)$ may have different shapes and different sizes:



- Among others, there is a very special type of IFS's, which satisfy Moran's open set condition.
- First, we may fix a bounded set, represented as a rectangle.
- Then, we may have three distinct situations





- As a generalization of IFS, a GIFS is a family of contractions $\{f_e : e \in \mathcal{E}\}$ on a complete metric space (X, d) (usually, it is $\mathbb{R}^n$)
- where each f_e is a contraction labeled by an edge from a directed graph with finite many vertices, say q vertices
- which then specifies a unique q-tuple of compact sets $T_1, \ldots, T_q \subset X$ satisfying the set equation

$$T_j = \bigcup_i \left[\cup_{e \in \mathcal{E}(i,j)} f_e(T_i)\right] \qquad 1 \le j \le q$$

* Here, $\mathcal{E}(i,j)$ is the set of edges starting from i and ending at j.

- Again, the sets $T_1, \ldots, T_q$ have some *self-similar structure*, in the sense that each T_j can be *partitioned into copies of the sets* T_i,
- The OSC for IFS can be generalized to the version for GIFS,
- Dimension estimates can be obtained for the sets $T_1, \ldots, T_q$
- Many other fractal and/or topological properties are also shared by IFS fractals and GIFS, which make a large part of our discussion.

Given a fractal set $T \subset \mathbb{R}^2$, say, attractor of IFS or GIFS, we may pose some questions of a topological nature:

1. *How to describe connectivity of T ?*

   This question is nicely solved by Hata, with several applications to specific family of fractal sets. Recall that when T is an attractor of IFS or GIFS its connectivity implies that it is locally connected. By Hahn-Mazurkiewicz Theorem, there is a continuous onto mapping $f : [0,1] \Rightarrow T$, which is called a parametrization of T. In this case, we may find interests in a special parametrization f.

2. *If T contains interior points, can we find reasonable conditions for T to be disk-like,* *i.e.* homeomorphic with a closed disk ?
3. *If T is connected and non-disk-like, can we find a mathematical characterizing of cut-points, nontrivial loops, etc. ?*
4. In the special case T is a tile, or one of the substitution tiles, we may ask similar questions for its boundary ∂T.
5. Questions posed on $T \subset \mathbb{R}^n$ with $n \ge 3$ and its boundary ∂T.

- Fix an IFS $\{f_i : 1 \le i \le k\}$ by contractions on $\mathbb{R}^n$ with ratio r_i. Then, the map $\Phi(K) := \cup_i f_i(K)$ on the space $\mathcal{C}$ of nonempty compact subsets of $\mathbb{R}^n$ is a contraction (under Hausdorff distance), and the attractor T is the unique fixed point of $\Phi : \mathcal{C} \to \mathcal{C}$.
- Furthermore, T is the *global attractor* of the system $(\mathcal{C}, \Phi)$: for any nonempty compact set K, the sequence $\Phi^j(K)$ converges to T (exponentially)
- Fix a GIFS system and the set of attractors $T_1, \ldots, T_q$, we may define a map Φ^* on the space $\mathcal{C}^q$ by

$$(K_1, \ldots, K_q) \mapsto \left(\bigcup_i \left[\cup_{e \in \mathcal{E}(i,1)} f_e(T_i)\right], \ldots, \bigcup_i \left[\cup_{e \in \mathcal{E}(i,q)} f_e(T_i)\right] \right)$$

- Then $(T_1, \ldots, T_q)$ is the unique fixed point of Φ^*, and is also the *global attractor* of the system $(\mathcal{C}^q, \Phi^*)$.
- If we start from a carefully chosen $K \in \mathcal{C}$ or $(K_1, \ldots, K_q) \in \mathcal{C}^q$, we may use the converging sequences $\Phi^j(K)$ or $\Phi^{*j}(K_1, \ldots, K_q)$ as *approximations* of T or $(T_1, \ldots, T_q)$, to investigate fractal and/or topological structure of the limiting set.

- Fix an IFS $\{f_i : 1 \le i \le k\}$ by similarity contractions with ratio r_i and similarity dimension s, determined by $\sum_i r_i^s = 1$.
- For any word $w = i_1 i_2 \cdots i_n$ in $S^* = \bigcup_n \{1, 2, \ldots, k\}^n$ let

$$f_w := f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}$$

Then, the map π assigning the unique point $\pi(\alpha)$ in

$$\bigcap_{n=1}^{\infty} f_{i_1 i_2 \cdots i_n}(T)$$

to a sequence $\alpha = i_1 i_2 \cdots i_n \cdots$ in $\sum_k := \{1, 2, \ldots, k\}^\infty$ is a continuous surjection. We also say π is the *natural projection* of $\sum_k$ onto T, determined by the IFS $\{f_i\}$.

- The projection π connects the dynamics of $\sum_k$ to fractal properties of T and specific subsets of T, including
- dimension problems
- multi-fractal analysis

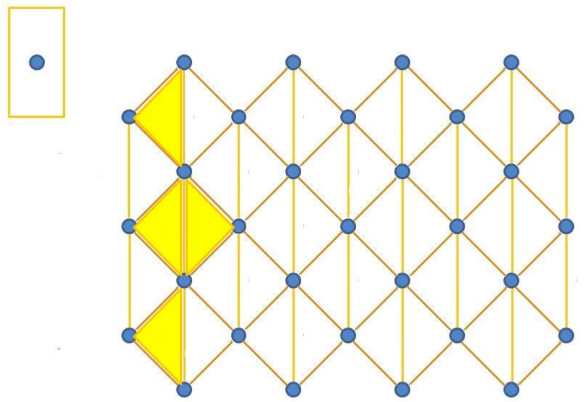
- Fix an IFS $\{f_i : 1 \le i \le k\}$ by similarity contractions with ratio r_i and similarity dimension s, and its attractor T with $T = \cup_i f_i(T)$.
- For any integer $n \ge 1$,

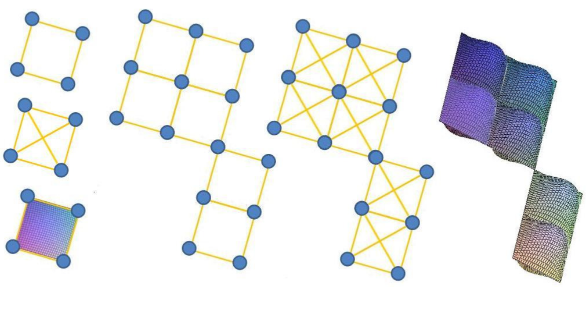
$$\mathcal{V}_n := \{f_w(T) : w = i_1 i_2 \cdots i_n \in \{1, 2, \ldots, k\}^n\}$$

is a cover of T, let $\mathcal{G}_n$ be the graph with set of states $\mathcal{V}_n$ for which two states $f_u(T)$ and $f_v(T)$ are coincident $\Leftrightarrow$ $f_u(T) \cap f_v(T) \neq \emptyset$.

- We call $\mathcal{G}_n$ the n-th Hata graph of T, with respect to the IFS $\{f_i : 1 \le i \le k\}$.
- Furthermore, the nerve $\mathcal{K}_n$ of $\mathcal{V}_n$ is called the n-th Hata complex of T, with respect to the IFS $\{f_i : 1 \le i \le k\}$.
- Hata graphs $\mathcal{G}_n$ and Hata complexes $\mathcal{K}_n$ provide a combinatoric viewpoint to explore the topology of T and/or its boundary, such as
- connectivity, arc-like, parametrization
- circle-like, disk-like, sphere-like, ball-like (Schönflies Theorem)

* How to construct a Hata complex ?





- Fix an IFS $\{f_i : 1 \le i \le k\}$ by similarity contractions with ratio r_i and similarity dimension s, determined by $\sum_i r_i^s = 1$.
- If there is a bounded open set U so that the disjoint union $\bigcup_i f_i(U)$ is entirely contained in U, we say that $\{f_i\}$ satisfies Moran's open set condition [Moran 1946], or simply OSC
- In this case, Hausdorff dimension $=$ the similarity dimension s.
- Among others, due to [Bandt-Graf 1992] and [Shief 1994] and [Bandt-Rao 2007], it is well known that OSC is equivalent to:
- The s-dimensional Hausdorff measure $H^s(T) > 0$
- $\exists\ N$ s.t. for every piece $f_w(T) = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(T)$ there are at most N incomparable pieces $f_u(T)$ of diameter $\ge |f_w(T)|$ and with distance $< |f_w(T)|$ from $f_w(T)$.
- The identity map is not an accumulation point of the set of neighbor maps $f_w^{-1} \circ f_u$ of T
- When $T \subset \mathbb{R}^n$ and $s = n$: OSC $\Leftrightarrow$ $\mathcal{L}^n(T) > 0$ $\Leftrightarrow$ $T^o \neq \emptyset$

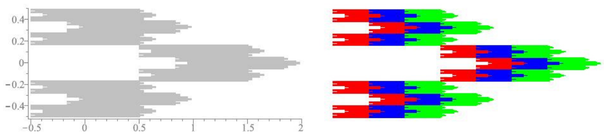
[Schief 1994]

- If an IFS $\{f_i : 1 \le i \le k\}$ on $\mathbb{R}^n$ consists of affine contractions with the same linear part, we have a self-affine IFS.
- Special case: there exist an expanding $n \times n$ matrix A and a *digit set* $\mathcal{D} = \{d_1, \dots, d_k\} \subset \mathbb{R}^n$ with $f_i(x) = A^{-1}(x + d_i)$
  - the attractor T is called a self-affine tile whenever $T^o \neq \emptyset$, this is equivalent to non-zero Lebesgue measure of dimension n [Largarias-Wang 1996].
- For such a self-affine tile T, determined by matrix A an digit set $\mathcal{D}$, we can induce a *self-replicating tiling* of $\mathbb{R}^n$

  (based on periodic sequence from the symbolic space $\sum_k$)
  - [Hata 1985]: T connected $\Leftrightarrow$ Hata graph $\mathcal{G}_1$ connected
  - [ALT 2004]: T connected $\Leftrightarrow$ ∂T connected $(n > 1)$
  - [Tang 2005]: T connected $\Rightarrow$ ∂T locally connected $(n > 1)$
  - [Luo-Thuswaldner 2006]: if T is a $\mathbb{Z}^2$-tile, the complement $\mathbb{R}^2 \setminus T$ either is connected or contains infinitely many components $(n \ge 3?)$
  - The above result does not hold for T^o. See picture on the next slide

- $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$\mathcal{D} = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix} \right\}$$



- T^o has three components and T is the union of three disk-like pieces which intersect at single points.

- If A is an expanding $n \times n$ matrix of integers a complete set of coset representatives $\mathcal{D} \subset \mathbb{Z}^n$, which necessarily has cardinality $|\mathcal{D}| = |detA|$, is called a *standard digit set*
- A self-affine tile T determined by such an integral matrix A and a standard digit set $\mathcal{D}$ is called a $\mathbb{Z}^n$-tile if $T + \mathbb{Z}^n$ is a tiling of $\mathbb{R}^n$.
  - Hata graphs and Hata complexes can be determined completely, providing a nice starting point to investigate the structure of such a tile.
- Connectivity of T: $|\mathcal{D}| = 2 \;\Rightarrow\; T$ is connected
  - Algebraically checkable criterion based on characteristic polynomial of A is given. In particular, in the plane, for any given integral expanding matrix A there exists a digit set $\mathcal{D}$ so that the corresponding tile T is connected [Kirat-Lau 1999]
  - For dimension 2 and 3, *Consecutive Collinear Digit & Height Reducing Property* $\Rightarrow$ T is connected [Kirat-Lau-Rao 2004]
  - Connectivity of T in the context of *canonical number systems*, check papers by Akiyama, Thuswaldner and others.

- OSC & injective contractions & in $\mathbb{R}^2$: connected interior $T^o \Leftrightarrow$ disk-like T ([Luo-Rao-Tan])
- OSC & affine contraction & $T^o \neq \emptyset$ & non-vanishing Jacobian: the boundary has dimension strictly less than dimension of the underlying $\mathbb{R}^n$ [Lau-Xu 2000]
  - * Here, $H^n(\partial T) = 0$ is guaranteed by OSC. And, boundary of a self-affine tile = special case discussed by [Largarias-Wang 1996]
- T is $\mathbb{Z}^n$-tile: the boundary ∂T is covered by the attractor system of a GIFS
  - Moreover, the set V_2 of triple points, or V_3 for quadruple points, and so on are also covered by the attractor system of a GIFS
  - Then, the study of fractal structure or topology of V_k with $k \ge 1$ based on GIFS theory, including dimension estimates and arc-like property and so on

1. [Bandt-Gelbrich 1994], [Bandt-Wang 2001]: connected $T+$ 6-neighbor or 8-neighbor & adjacent-connected digit $\Rightarrow$ disk-like T. This has been used to discuss disk-like T arising from canonical number systems and general consecutive collinear digit sets [Akiyama-Thuswaldner 2005], [Lau-Leung 2007]
2. [Luo-Rao-Tan 2002]: T^o connected $\Rightarrow$ disk-like T
3. [Loridant-Luo-Thuswaldner 2007]: Characterizing disk-like crystallographic reptile T based on complete right coset representatives
4. [Loridant-Luo 2009]: complete classification for $p2$ reptiles T that are disk-like which necessarily have 6,7,8, or 12 neighbors in the resulted crystallographic tiling

* The universal lower bound of neighbor numbers is 6. What is the corresponding bound for the case 3D ?

** Can we design a general enough frame to discuss disk-like property for general self-replicating 2D tiling ?
(There are some publications concerning disk-like Rauzy fractals)

*** An interesting special case of this question concerns disk-like property of substitutive tiles