

Fractal Families and Basic Properties

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- Suppose that we have some experience of
 - Hausdorff measure and Hausdorff dimension
 - Iterated Function System - IFS
 - Graph-Directed IFS - GIFS
 - Open Set Condition - OSC
- We will consider a few typical types of fractal sets
 - specified by IFS or GIFS and
 - related to tiling theory & substitutions,
- We aim to consider the structure of such a set, trying to understand particular aspects of its structure, especially fractal and/or topological aspects, or ...
- Based on recent studies of the above fractal-topology nature, we will
 - review some ideas that use graphs in the study of fractal sets
 - discuss the possibility of further extending those ideas from graphs to complexes

- According to Hutchinson [1981], an IFS is a finite family of contractions $\{f_1, \dots, f_q\}$ on a complete metric space (X, d) , which then specifies a unique compact set $T \subset X$ satisfying the set equation

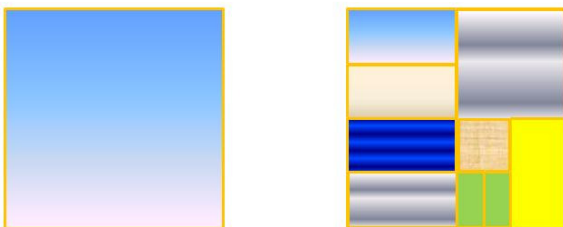
$$T = \bigcup_{i=1}^q f_i(T)$$

- Roughly speaking, the set T has some self-similar structure, in the sense that T can be partitioned into several copies $f_i(T)$ of itself, each of which is the image of T under a contraction f_i from the IFS
- For the sake of brevity, T is also called a self-similar fractal.
- when each f_i is a similarity contraction, with ratio $c_i < 1$, we say T is a self-similar set and call the number s determined by $\sum_i c_i^s = 1$ the similarity dimension of T .
- We will confine ourselves to basic properties concerning the fractal structure or topology of T .
- To feel self-similarity arising from IFS ?

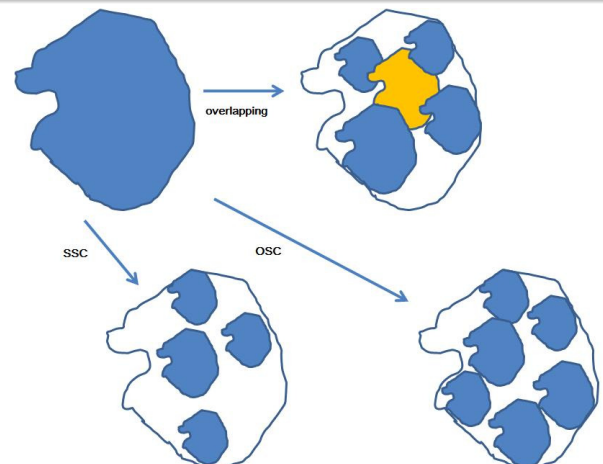
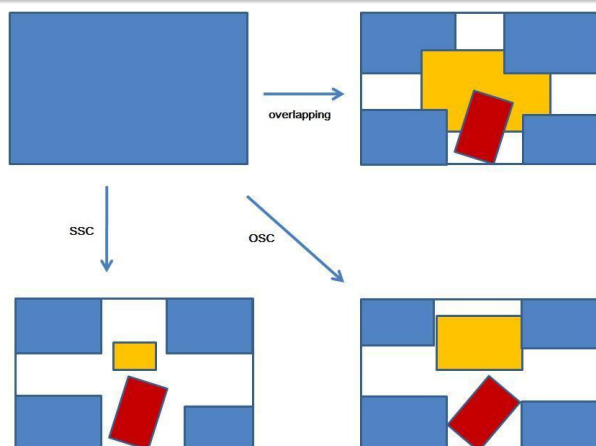
A square can be considered as a self-similar set:



The copies $f_i(T)$ may have different shapes and different sizes:



- Among others, there is a very special type of IFS's, which satisfy Moran's open set condition.
- First, we may fix a bounded set, represented as a rectangle.
- Then, we may have three distinct situations



- As a generalization of IFS, a GIFS is a family of contractions $\{f_e : e \in \mathcal{E}\}$ on a complete metric space (X, d) (usually, it is \mathbb{R}^n)
- where each f_e is a contraction labeled by an edge from a directed graph with finite many vertices, say q vertices
- which then specifies a unique q -tuple of compact sets $T_1, \dots, T_q \subset X$ satisfying the set equation

$$T_j = \bigcup_i [\cup_{e \in \mathcal{E}(i,j)} f_e(T_i)] \quad 1 \leq j \leq q$$

- Here, $\mathcal{E}(i, j)$ is the set of edges starting from i and ending at j .
- Again, the sets T_1, \dots, T_q have some self-similar structure, in the sense that each T_j can be partitioned into copies of the sets T_i ,
- The OSC for IFS can be generalized to the version for GIFS,
- Dimension estimates can be obtained for the sets T_1, \dots, T_q
- Many other fractal and/or topological properties are also shared by IFS fractals and GIFS, which make a large part of our discussion.

- Fix an IFS $\{f_i : 1 \leq i \leq k\}$ by contractions on \mathbb{R}^n with ratio r_i . Then, the map $\Phi(K) := \cup_i f_i(K)$ on the space \mathcal{C} of nonempty compact subsets of \mathbb{R}^n is a contraction (under Hausdorff distance), and the attractor T is the unique fixed point of $\Phi : \mathcal{C} \rightarrow \mathcal{C}$.
- Furthermore, T is the global attractor of the system (\mathcal{C}, Φ) : for any nonempty compact set K , the sequence $\Phi^j(K)$ converges to T (exponentially)

- Fix a GIFS system and the set of attractors T_1, \dots, T_q , we may define a map Φ^* on the space \mathcal{C}^q by

$$(K_1, \dots, K_q) \mapsto \left(\bigcup_i [\cup_{e \in \mathcal{E}(i,1)} f_e(T_i)], \dots, \bigcup_i [\cup_{e \in \mathcal{E}(i,q)} f_e(T_i)] \right)$$

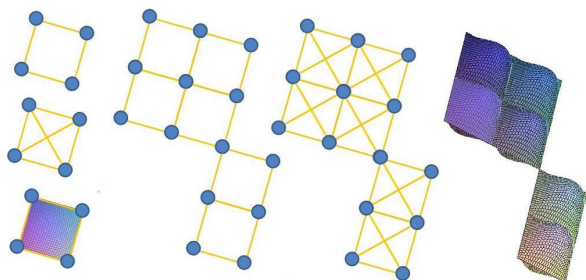
- Then (T_1, \dots, T_q) is the unique fixed point of Φ^* , and is also the global attractor of the system (\mathcal{C}^q, Φ^*) .
- If we start from a carefully chosen $K \in \mathcal{C}$ or $(K_1, \dots, K_q) \in \mathcal{C}^q$, we may use the converging sequences $\Phi^j(K)$ or $\Phi^{*j}(K_1, \dots, K_q)$ as approximations of T or (T_1, \dots, T_q) , to investigate fractal and/or topological structure of the limiting set.

- Fix an IFS $\{f_i : 1 \leq i \leq k\}$ by similarity contractions with ratio r_i and similarity dimension s , and its attractor T with $T = \cup_i f_i(T)$.
- For any integer $n \geq 1$,

$$\mathcal{V}_n := \{f_w(T) : w = i_1 i_2 \dots i_n \in \{1, 2, \dots, k\}^n\}$$

is a cover of T , let \mathcal{G}_n be the graph with set of states \mathcal{V}_n for which two states $f_u(T)$ and $f_v(T)$ are coincident $\Leftrightarrow f_u(T) \cap f_v(T) \neq \emptyset$.

- We call \mathcal{G}_n the n -th Hata graph of T , with respect to the IFS $\{f_i : 1 \leq i \leq k\}$.
- Furthermore, the nerve \mathcal{K}_n of \mathcal{V}_n is called the n -th Hata complex of T , with respect to the IFS $\{f_i : 1 \leq i \leq k\}$.
- Hata graphs \mathcal{G}_n and Hata complexes \mathcal{K}_n provide a combinatoric viewpoint to explore the topology of T and/or its boundary, such as
- connectivity, arc-like, parametrization
- circle-like, disk-like, sphere-like, ball-like (Schönflies Theorem)
- How to construct a Hata complex ?



Given a fractal set $T \subset \mathbb{R}^2$, say, attractor of IFS or GIFS, we may pose some questions of a topological nature:

1. How to describe connectivity of T ?

This question is nicely solved by Hata, with several applications to specific family of fractal sets. Recall that when T is an attractor of IFS or GIFS its connectivity implies that it is locally connected. By Hahn-Mazurkiewicz Theorem, there is a continuous onto mapping $f : [0, 1] \Rightarrow T$, which is called a parametrization of T . In this case, we may find interests in a special parametrization f .

- If T contains interior points, can we find reasonable conditions for T to be disk-like, i.e. homeomorphic with a closed disk ?
- If T is connected and non-disk-like, can we find a mathematical characterizing of cut-points, nontrivial loops, etc. ?
- In the special case T is a tile, or one of the substitution tiles, we may ask similar questions for its boundary ∂T .
- Questions posed on $T \subset \mathbb{R}^n$ with $n \geq 3$ and its boundary ∂T .

- Fix an IFS $\{f_i : 1 \leq i \leq k\}$ by similarity contractions with ratio r_i and similarity dimension s , determined by $\sum_i r_i^s = 1$.
- For any word $w = i_1 i_2 \dots i_n$ in $S^n = \cup_n \{1, 2, \dots, k\}^n$ let

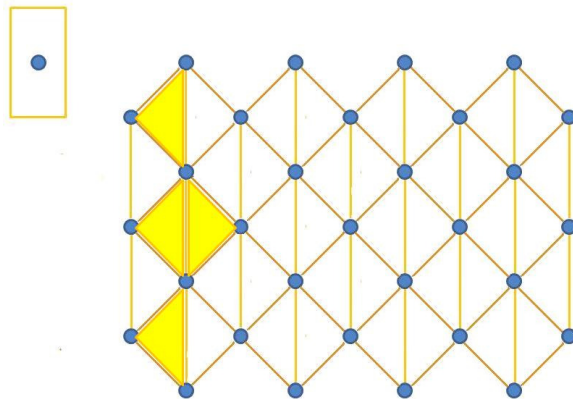
$$f_w := f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_n}$$

Then, the map π assigning the unique point $\pi(\alpha)$ in

$$\bigcap_{n=1}^{\infty} f_{i_1 i_2 \dots i_n}(T)$$

to a sequence $\alpha = i_1 i_2 \dots i_n \dots$ in $\sum_k := \{1, 2, \dots, k\}^\infty$ is a continuous surjection. We also say π is the natural projection of \sum_k onto T , determined by the IFS $\{f_i\}$.

- The projection π connects the dynamics of \sum_k to fractal properties of T and specific subsets of T , including
- dimension problems
- multi-fractal analysis

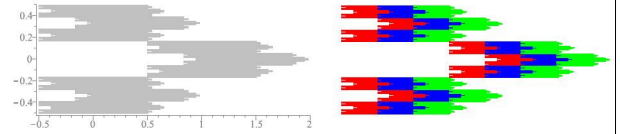


- Fix an IFS $\{f_i : 1 \leq i \leq k\}$ by similarity contractions with ratio r_i and similarity dimension s , determined by $\sum_i r_i^s = 1$.
- If there is a bounded open set U so that the disjoint union $\cup_i f_i(U)$ is entirely contained in U , we say that $\{f_i\}$ satisfies Moran's open set condition [Moran 1946], or simply OSC
- In this case, Hausdorff dimension = the similarity dimension s .
- Among others, due to [Bandt-Graf 1992] and [Shief 1994] and [Bandt-Rao 2007], it is well known that OSC is equivalent to:
- The s -dimensional Hausdorff measure $H^s(T) > 0$
- $\exists N$ s.t. for every piece $f_w(T) = f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_n}(T)$ there are at most N incomparable pieces $f_u(T)$ of diameter $\geq |f_w(T)|$ and with distance $< |f_w(T)|$ from $f_w(T)$.
- The identity map is not an accumulation point of the set of neighbor maps $f_w^{-1} \circ f_u$ of T
- When $T \subset \mathbb{R}^n$ and $s = n$: OSC $\Leftrightarrow \mathcal{L}^n(T) > 0 \Leftrightarrow T^\circ \neq \emptyset$ [Schief 1994]

- If an IFS $\{f_i : 1 \leq i \leq k\}$ on \mathbb{R}^n consists of affine contractions with the same linear part, we have a self-affine IFS.
- Special case: there exist an expanding $n \times n$ matrix A and a *digit set* $\mathcal{D} = \{d_1, \dots, d_k\} \subset \mathbb{R}^n$ with $f_i(x) = A^{-1}(x + d_i)$
 - the attractor T is called a *self-affine tile* whenever $T^\circ \neq \emptyset$, this is equivalent to non-zero Lebesgue measure of dimension n [Largarias-Wang 1996].
- For such a self-affine tile T , determined by matrix A and a digit set \mathcal{D} , we can induce a *self-replicating tiling* of \mathbb{R}^n (based on periodic sequence from the symbolic space \sum_k)
 - [Hata 1985]: T connected \Leftrightarrow Hata graph \mathcal{G}_1 connected
 - [ALT 2004]: T connected $\Leftrightarrow \partial T$ connected ($n > 1$)
 - [Tang 2005]: T connected $\Rightarrow \partial T$ locally connected ($n > 1$)
 - [Luo-Thuswaldner 2006]: if T is a \mathbb{Z}^2 -tile, the complement $\mathbb{R}^2 \setminus T$ either is connected or contains infinitely many components ($n \geq 3$)?
 - The above result does not hold for T° . See picture on the next slide

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\mathcal{D} = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix} \right\}$$



- T° has three components and T is the union of three disk-like pieces which intersect at single points.

- If A is an expanding $n \times n$ matrix of integers a complete set of coset representatives $\mathcal{D} \subset \mathbb{Z}^n$, which necessarily has cardinality $|\mathcal{D}| = |\det A|$, is called a *standard digit set*
- A self-affine tile T determined by such an integral matrix A and a standard digit set \mathcal{D} is called a \mathbb{Z}^n -tile if $T + \mathbb{Z}^n$ is a tiling of \mathbb{R}^n .
 - Hata graphs and Hata complexes can be determined completely, providing a nice starting point to investigate the structure of such a tile.
- Connectivity of T : $|\mathcal{D}| = 2 \Rightarrow T$ is connected
 - Algebraically checkable criterion based on characteristic polynomial of A is given. In particular, in the plane, for any given integral expanding matrix A there exists a digit set \mathcal{D} so that the corresponding tile T is connected [Kirat-Lau 1999]
 - For dimension 2 and 3, *Consecutive Collinear Digit & Height Reducing Property* $\Rightarrow T$ is connected [Kirat-Lau-Rao 2004]
 - Connectivity of T in the context of *canonical number systems*, check papers by [Akiyama](#), [Thuswaldner](#) and others.

- OSC & injective contractions & in \mathbb{R}^2 : connected interior $T^\circ \Leftrightarrow$ disk-like T ([Luo-Rao-Tan])
- OSC & affine contraction & $T^\circ \neq \emptyset$ & non-vanishing Jacobian: the boundary has dimension strictly less than dimension of the underlying \mathbb{R}^n [Lau-Xu 2000]
 - * Here, $H^n(\partial T) = 0$ is guaranteed by OSC. And, boundary of a self-affine tile = special case discussed by [Largarias-Wang 1996]
- T is \mathbb{Z}^n -tile: the boundary ∂T is covered by the attractor system of a GIFS
 - Moreover, the set V_2 of triple points, or V_3 for quadruple points, and so on are also covered by the attractor system of a GIFS
 - Then, the study of fractal structure or topology of V_k with $k \geq 1$ based on GIFS theory, including dimension estimates and arc-like property and so on

- [Bandt-Gelbrich 1994], [Bandt-Wang 2001]: *connected T + 6-neighbor or 8-neighbor & adjacent-connected digit* \Rightarrow disk-like T . This has been used to discuss disk-like T arising from canonical number systems and general consecutive collinear digit sets [Akiyama-Thuswaldner 2005], [Lau-Leung 2007]
- [Luo-Rao-Tan 2002]: T° connected \Rightarrow disk-like T
- [Loridant-Luo-Thuswaldner 2007]: Characterizing disk-like crystallographic reptile T based on *complete right coset representatives*
- [Loridant-Luo 2009]: complete classification for *$p2$ reptiles* T that are disk-like which necessarily have 6,7,8, or 12 neighbors in the resulted crystallographic tiling
 - * The universal lower bound of neighbor numbers is 6. What is the corresponding bound for the case 3D ?
 - ** Can we design a general enough frame to discuss disk-like property for general self-replicating 2D tiling ? (There are some publications concerning disk-like Rauzy fractals)
 - *** An interesting special case of this question concerns disk-like property of substitutive tiles