Spectral Structure of tile digit sets

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Content

Introduction

Consider in $\mathbb{R}^n$, let $A \in M_n(\mathbb{Z})$ be an expanding matrix with $b = |\det A|$ is an integer, and $\mathcal{D} = \{d_0, ..., d_{b-1}\}$ be a subset of $\mathbb{Z}^n$ which we will call it a *digit set*. Consider the IFS $\{\phi_i\}_{i=0}^{b-1}$ defined by

$$\phi_i(x) = A^{-1}(x + d_i), \;\; 0 \le i \le b-1.$$

Then there exists a unique compact set $T = T(A, \mathcal{D})$ satisfying $T = \bigcup_{i=0}^{b-1} \phi_i(T)$,

$$AT = \bigcup_{i=0}^{b-1} (T + d_i) = T + \mathcal{D}.$$

Introduction

Theorem

Suppose $T = T(A, \mathcal{D})$ satisfies $T^o \neq \phi$, then T is a translational tile in $\mathbb{R}^n$.

We call it a *self-affine tile* and $\mathcal{D}$ a *tile digit set* (w.r.t. A).
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Recall that a compact set T is a *translational tile* if there exists $\mathcal{J}$ such that

$$Leb((T+t)\cap(T+t')) = 0. \ (T \oplus \mathcal{J} = \mathbb{R}^n).$$

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For a self-affine tile, it has a *self-replicating tiling set* $\mathcal{J}$, there exists $k \geq 1$ such that

$$A^k \mathcal{J} \oplus \mathcal{D}_{A,k} = \mathcal{J}.$$

$\mathcal{D}_{A,k} = \{\sum_{j=0}^{k-1} A^j d_j : d_j \in \mathcal{D}\}$.

Introduction

Main Question:

1. Given a matrix A, classify the tile digit sets $\mathcal{D}$?
2. What is the structure of the zeros of the mask function $m_{\mathcal{D}}$ of $\mathcal{D}$?

$$m_{\mathcal{D}}(\xi) = \sum_{d\in\mathcal{D}} e^{2\pi i\langle d,\xi\rangle}$$

or the mask polynomial?

$$P_{\mathcal{D}}(\mathbf{x}) = \sum_{d\in\mathcal{D}} \mathbf{x}^d?$$

(Spectral Structure of tile digit sets)

Introduction

Remark

1. General classification is very difficult,
2. It is known that if $A = [b]$ $b = \#\mathcal{D}$ is prime, $\mathcal{D}$ is tile digit set if and only if $\mathcal{D}$ is complete residue (mod b).
3. The same holds in high dimension under additional hypothesis [HL] and [LW].
4. There are classification of tile digit sets when $b = p^\alpha$ and $b = pq$.

Relation with integer tiling

A finite set $\mathcal{A} \subset \mathbb{Z}^n$ is called an *integer tile* if there exists $\mathcal{T}$ such that $\mathcal{A} \oplus \mathcal{T} = \mathbb{Z}^n$. It is clear that $\mathcal{A}$ is an integer tile if and only if $\mathcal{A} + [0,1]^n$ is a translational tile of $\mathbb{R}^n$.

Theorem

Let A be an integral expanding matrix, and let $\mathcal{D} \subset \mathbb{Z}^n$ be a digit set. Suppose $T(A, \mathcal{D})$ is a self-affine tile, then there exists $\mathcal{T} \subset \mathbb{Z}^n$ such that $\mathcal{D} \oplus \mathcal{T} = \mathbb{Z}^n$.

Relation with integer tiling

Sketch of Proof. Translate digit $\mathcal{D}_{A,k}$ by $d^* \in \mathcal{D}_{A,k}$, so that the boundary of the tile $T = T(A^k, \mathcal{D} - d^*)$ hits no integer points. Let

$$\mathcal{R} = \mathbb{Z}^n \cap T(A^k, \mathcal{D} - d^*).$$

Let $\mathcal{J}$ be the self-replicating tiling set of T, then

$$\mathcal{R} \oplus \mathcal{J} = \mathbb{Z}^n.$$

Hence the theorem follows from self-replicating property of $\mathcal{J}$.
□

Relation with integer tiling

There is a vast literature studying the integer tiling, Minokowski, Hajo, de Bruijn, Sands, Szabo. The theorem connects tile digit sets to integer tiling theory. We shall focus on one dimensional case. It connects closely with the cyclotomic polynomials.
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Recall that $\Phi_s(x)$, the s^{th} order cyclotomic polynomial is the minimal polynomial for the primitive s^{th} root of unity. e.g. $\Phi_3(x) = 1 + x + x^2$.
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A tiling set $\mathcal{J}$ is periodic if $\mathcal{J} + t = \mathcal{J}$. Every one dimensional tiles admits a periodic tiling set.

Relation with integer tiling

Lemma

(translation of language of tiling into algebra)
Let n be an integer and let A,B be finite set of non-negative integers. The following are equivalent:

1. $\mathcal{A} \oplus \mathcal{B} \oplus n\mathbf{Z} = \mathbf{Z}$.
2. $\mathcal{A} \oplus \mathcal{B} = \mathbf{Z_n}$.
3. $P_{\mathcal{A}}(x)P_{\mathcal{B}}(x) \equiv 1 + x + x^2 + \ldots + x^{n-1} (mod x^n - 1)$.
4. $n = P_{\mathcal{A}}(1)P_{\mathcal{B}}(1) = (\#A)(\#B)$ *and for every factor* $t > 1$ *of* n*, the cyclotomic polynomial* $\Phi_t(x)$ *is either a divisor of* $P_{\mathcal{A}}(x)$ *or a divisor of* $P_{\mathcal{B}}(x)$.

Relation with integer tiling

For a finite set $\mathcal{A} \subset \mathbb{Z}_+$, we let

$$S_{\mathcal{A}} = \{p^\alpha > 1 : \ p \text{ prime}, \ \Phi_{p^\alpha}(x) | P_{\mathcal{A}}(x)\} \qquad (1)$$

the *prime-power spectrum* of $\mathcal{A}$, and $\widetilde{S}_{\mathcal{A}} = \{s > 1 : \ \Phi_s(x) | P_{\mathcal{A}}(x)\}$ the *spectrum* of $\mathcal{A}$. In [CM], Coven and Meyerowitz introduced the following two conditions to study the integer tiles:

(T1) $\#\mathcal{A} = P_{\mathcal{A}}(1) = \prod_{s \in S_{\mathcal{A}}} \Phi_s(1)$
(T2) *For any distinct prime powers* $s_1, \ldots, s_n \in S_{\mathcal{A}}$ *, then* $s_1 \ldots s_n \in \widetilde{S}_{\mathcal{A}}$.

Theorem

Let $\mathcal{A} \subset \mathbb{Z}^+ \cup \{0\}$ be a finite set. Suppose $(T1)$ and $(T2)$ hold, then $\mathcal{A}$ tiles $\mathbb{Z}$ with period $n = l.c.m.(S_{\mathcal{A}})$.
Conversely, if $\mathcal{A}$ is an integer tile, then $(T1)$ holds; if in addition $\#\mathcal{A}$ has at most two prime factors, then $(T2)$ holds.

Higher order product forms

For T to be a self-similar tile in $\mathbb{R}^1$,

$$\chi_T(x) = \sum_{d\in\mathcal{D}} \chi_T(bx-d), \text{ for a.e. } x\in\mathbb{R}$$

and the Fourier transform is $\widehat{\chi}_T(\xi) = \prod_{k=1}^{\infty} P_{\mathcal{D}}(e^{2\pi i\xi/b^k})$. By using the Riemann Lebesgue lemma, Kenyon gave a basic criterion on $\mathcal{D}$ for χ_T to be an L^1-function, which is equivalent to $\mathcal{D}$ is a tile digit set.

Theorem

(Kenyon criterion) $T(b,\mathcal{D})$ *is a self-similar tile if and only if for each integer* $m>0$*, there exists* $k\geq 1$ *(depending on* m*) such that*

$$P_{\mathcal{D}}(e^{2\pi i m/b^k}) = 0.$$

Higher order product forms

Different kind of tile digit sets

1. If $\mathcal{D} = \{0, 1, ..., b-1\}$, then $T(b, \mathcal{D}) = [0, 1]$. So

$$P_{\mathcal{D}}(x) = \prod_{d|b, d>1} \Phi_d(x)$$

satisfies the Kenyon criterion (We can also check it directly).

2. (Lagarias and Wang) $\mathcal{D}$ is called a product-form digit set (with respect to b). if

$$\mathcal{D} = \mathcal{E}_0 + b^{\ell_1}\mathcal{E}_1 + \ldots + b^{\ell_k}\mathcal{E}_k$$

where $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_k \equiv \mathbb{Z}_b$, and $0 \le \ell_1 \le \ell_2 \le \ldots \le \ell_k$; if $\mathcal{E} = \{0, 1, 2, \ldots, b-1\}$, then $\mathcal{D}$ is called the *strict product-form*

Higher order product forms

3. (Lau and Rao) If $\mathcal{D} \equiv \mathcal{D}' \pmod{b^{\ell_k+1}}$ and $\mathcal{D}'$ is the product form above, then $\mathcal{D}$ is called a *weak product-form* .
4. (Lai, Lau and Rao) We say that $\mathcal{D}$ is a *modulo product-form* if $\mathcal{D}$ is defined by a product-form $\mathcal{D}' = \mathcal{E}_0 + b^{\ell_1}\mathcal{E}_1 + \ldots + b^{\ell_k}\mathcal{E}_k,\ 0 \le \ell_1 \le \ldots \le \ell_k$ through the following relation:

$$\begin{cases} \mathcal{D}^{(0)} = \mathcal{E}_0 \ (\text{mod } n_0), \\ \vdots \\ \mathcal{D}^{(i)} = \mathcal{D}^{(i-1)} \oplus b^{l_i}\mathcal{E}_i \ (\text{mod } n_i), \\ \vdots \\ \mathcal{D} = \mathcal{D}^{(k-1)} \oplus b^{l_k}\mathcal{E}_k \ (\text{mod } n_k) \end{cases} \tag{2}$$

for some suitably chosen n_i depending on $\mathcal{E}_0,\ldots,\mathcal{E}_i$.

Higher order product forms

Actually all 1-4 satisfies the following condition which is stronger than the Kenyon criterion.
$(\mathbf{P_1})$ *for any $d > 1$ and $d|b$, there exists $j \geq 0$ such that $\Phi_d(x^{b^j})|P_{\mathcal{D}}(x)$.*

We can actually improve reprove the classification of tile digit sets of $b = p^\alpha$ by Lagarias and Wang.

Theorem

Suppose $\mathcal{D}$ is a tile digit set and $\#\mathcal{D} = p^\alpha$ for some $\alpha \geq 1$, then $\mathcal{D}$ is a modulo product-form as described above.

The case for $b = pq$ are also classified by Lau and Rao in terms of weak Product form.

Higher order product forms

How about other cases? like p^2q..
The product forms described are not enough. We propose another conditions.

$$j_1 = j_1(d) := \min\{j : \exists t \text{ such that } \Phi_t(x)|\Phi_d(x^{b^j}) \text{ and } \Phi_t(x)|P_{\mathcal{D}}(x)\}.$$

$(\mathbf{P}_2)$ *For each* $d|b, d > 1$, $j_1(d) < \infty$ *and for any factor* $\Phi_{t_1}(x)$ *of* $\Phi_d(x^{b^{j_1}})$*, there exists* $j_2 \geq 0$ *(depends on* t_1*) with*
$\Phi_{t_1}(x^{b^{j_2}}) \mid P_{\mathcal{D}}(x)$.
Define similarly the condition (P_k).

Proposition

Suppose $\mathcal{D}$ *satisfies* (P_k)*, then* $\mathcal{D}$ *is a tile digit set.*

Higher order product forms

Example

Let $b = 12 = 2^2 \cdot 3$ and

$$\begin{aligned}\mathcal{D} =& \{0,1\} \oplus 2^4 \cdot 3^2\{0,6\} \oplus 2^4 \cdot 3^2\{0,2,4\} \\ =& \{0,1\} \oplus (12)^2\{0,6\} \oplus (12)^2\{0,2,4\}\end{aligned}$$

Then condition (P) is satisfied and $\mathcal{D}$ is a self-similar tile digit set (indeed Product form).

$$\begin{aligned}P_{\mathcal{D}}(x) =& \Phi_2(x) \cdot \Phi_{2^6}(x)\Phi_{2^6 \cdot 3}(x)\Phi_{2^6 3^2}(x) \cdot \Phi_{3^3}(x)\Phi_{2 \cdot 3^3}(x) \dots \Phi_{2^6 \cdot 3^3}(x) \\ =& \Phi_2(x) \cdot \Phi_{2^6}(x^{3^2}) \cdot \Phi_{3^3}(x^{2^6}).\end{aligned}$$

Higher order product forms

Example

Let $b = 12 = 2^2 \cdot 3$ and

$$\mathcal{D} = \{0,1\} \oplus 2^4\{0,6\} \oplus 2^7 \cdot 3^2\{0,2,4\}.$$

Then condition (P) fails, but (P_2) is satisfied and hence $\mathcal{D}$ is still a self-similar tile digit set.

$$\begin{aligned} P_{\mathcal{D}}(x) =& \Phi_2(x) \cdot \Phi_{2^6}(x)\Phi_{2^6 \cdot 3}(x) \cdot \Phi_{3^3}(x)\Phi_{2\cdot 3^3}(x) \ldots \Phi_{2^7 3^3}(x)\Phi_{2^8 \cdot 3^3}(x) \\ =& \Phi_2(x) \cdot \Phi_{2^6}(x^3) \cdot \Phi_{3^3}(x^{2^8}). \end{aligned}$$

Higher order product forms

Definition

$\mathcal{D}$ is called a 2^{nd} order product-form digit set (with respect to b) if

$$\mathcal{D} = \mathcal{E}_0 + b^{\ell_1}\mathcal{E}_1 + \ldots + b^{\ell_k}\mathcal{E}_k$$

where $0 \leq \ell_1 \leq \ell_2 \leq \ldots \leq \ell_k$ and $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_k$ is a modulo product form.
Similarly, we can also define the 2^{nd} order modulo product-form.
By a similar procedure, we can define k^{th} order product form and modulo product form.

Example (continue). *The set $\mathcal{D}$ in Example is a 2^{nd} order product-form digit set.*
Indeed by noting that

$$\mathcal{D} = \{0,1\} \oplus 2^4\{0,6\} \oplus 2^7 \cdot 3^2\{0,2,4\} = \{0,1\} \oplus 12\{0,8\} \oplus (12)^2\{0,16,32\},$$

Higher order product forms

and $\{0,1\} \oplus \{0,8\} \oplus \{0,16,32\} = \{0,1,8,9,16,17\} \oplus 12\{0,2\}$ is a 1^{st} order product-form.

We use the following figure to illustrate the notations and results we have introduced. For a fixed integer $b \geq 2$ and $\#\mathcal{D} = b$, we have

$$\begin{array}{ccccccccc} 1^{st}\ order\ mpf & \Rightarrow & 2^{nd}\ order\ mpf & \Rightarrow & \ldots & & \\ \Downarrow & & \Downarrow & & & & \\ (P_1) & \Rightarrow & (P_2) & \Rightarrow & \ldots & \Rightarrow & \mathcal{D}\ is\ a\ tile\ digit\ set. \end{array}$$

(*mpf* means modulo product form.)

Kernel Polynomials

Kenyon criterion requires the checking on the integer u such that $u \neq b^k v$.
This observation leads us define the Protasov tree
Let $V_0 = \{\emptyset\}$ be the root, and let

$$V_k = \{\mathbf{j} = j_k \dots j_1 : \ j_\ell \in \{0, 1, \dots, b-1\}, \ j_1 \neq 0\}, \quad k \geq 1,$$

and let $V = \bigcup_{k\geq 0} V_k$, it defines a tree.
For any $\mathbf{j} \in V_k$, define $m_{\mathbf{j}} = j_k b^{k-1} + \cdots + j_2 b + j_1$. and define a map τ

$$\tau(\mathbf{j}) = d_{\mathbf{j}} = b^k / g.c.d.\{m_{\mathbf{j}}, b^k\}.$$

Kernel Polynomials

We can actually show that the mapping preserve two tree structures.

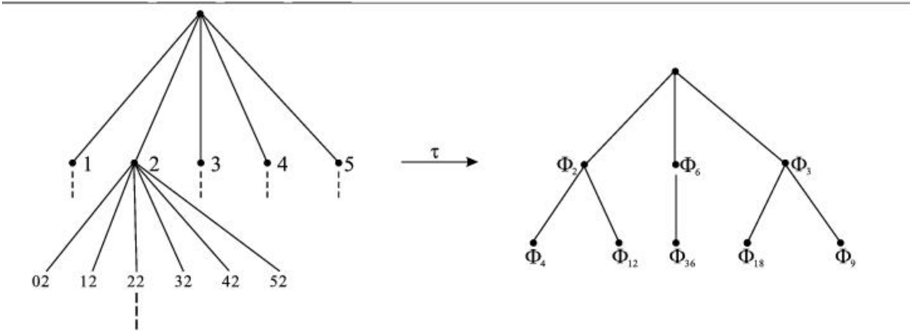

Figure: $b = 6$

Kernel Polynomials

We call $B \subset V \setminus \{\emptyset\}$ a *blocking* if it is a finite set and every infinite path starting from ϕ must intersect exactly one element of B.

Theorem

(Protasov) $\mathcal{D}$ is a tile digit set if and only if there is a blocking B such that for any $\mathbf{j} \in B$,

$$P_{\mathcal{D}}(e(m_{\mathbf{j}})) = 0. \tag{3}$$

It is called a *symmetric blocking* if for each $\mathbf{j} \in B$, we have $\tau^{-1}(\tau(\mathbf{j})) \subset B$.

$$\mathcal{N}_B := \tau(B) = \{\Phi_d : \tau(\mathbf{j}) = \Phi_d,\ \mathbf{j} \in B\}.$$

Kernel Polynomials

Lemma

If B is a symmetric blocking in the Protasov tree, then $\mathcal{N}_B$ is a blocking in the Φ-tree.

Definition

Let $\mathcal{N}$ be a blocking of the Φ-tree, we call

$$K(x) = \prod_{d \in \mathcal{N}} \Phi_d(x)$$

a kernel polynomial (with respect to b).

Kernel Polynomials

Theorem

Let $\mathcal{D}$ be a set of non-negative integers with $\#\mathcal{D} = b$. Then the following are equivalent.

(i) $\mathcal{D}$ is a tile digit set;

(ii) there is a symmetric $P_{\mathcal{D}}$-blocking in the Protasov tree;

(iii) there is a blocking $\mathcal{N}$ in the Φ-tree such that

$$K(x) := \prod_{\Phi_d \in \mathcal{N}} \Phi_d(x)$$

is a kernel polynomial and $K(x) | P_{\mathcal{D}}(x)$ (i.e. the spectrum of $\mathcal{D}$ contains a kernel polynomial);

(iv) $P_{\mathcal{D}}(x)$ satisfies condition (P_k) for some $k \geq 1$.

The case of $p^\alpha q^\beta$

Note that not all kernel polynomial can generate a tile digit set. We study the case when $b = p^\alpha q^\beta$, using a similar idea of proving $(T2)$ in Coven and Meyerowitz, we obtain another structure of the spectrum an integer tile.

Theorem

Let $\mathcal{A}$ be an integer tile such that $\#\mathcal{A} = p^\alpha q^\beta$. If $\Phi_{p^\lambda q^\mu}(x)|P_{\mathcal{A}}(x)$, then either

$$\Phi_{p^\lambda}(x)|P_{\mathcal{A}}(x) \quad \text{or} \quad \Phi_{q^\mu}(x)|P_{\mathcal{A}}(x).$$

The case of $p^\alpha q^\beta$

Example

If

$$K(x) = \Phi_2(x)\Phi_{2^4\cdot 3}(x)\Phi_{2^6}(x^3)\Phi_{3^3}(x^{2^6}),$$

this is a polynomial which can satisfy the Kenyon criterion, however, it cannot generate a tile digit set.

Question: Classify the kernel polynomials which can generate a tile digit set?
Does the theorem above hold when b contains more than 2 prime factors?

The case of $p^\alpha q^\beta$

Theorem

Let $\mathcal{D}$ be a tile digit set with $\#\mathcal{D} = p^2q$ and g.c.d.$\mathcal{D} = 1$.
(a) Suppose $S_{\mathcal{D}} = \{p, p^{2m}, q^n\}$, then $P_{\mathcal{D}}(x)$ must contain the polynomial of the form

$$(I)\Phi_p(x)\Phi_{q^n}(x^{p^{2(n-1)+1}})\Phi_{p^{2m}}(x^{q^m}),$$
$$(II)\ \Phi_p(x)\Phi_{q^n}(x^{p^{2(m+n-\ell)}})\Phi_{p^{2m}}(x^{q^{\ell-1}}),\ \ell = 1, ..., m..$$

(b) Suppose that $S_{\mathcal{D}} = \{q, p^{2m}, p^{2n+1}\}$, then $P_{\mathcal{D}}(x)$ must contain the polynomial of the form

$$(III)\ \Phi_q(x)\Phi_{p^{2m}}(x^{q^m})\Phi_{p^{2n+1}}(x^{q^{n+1}}).$$

Consequently, $\mathcal{D}$ must be a k^{th} order modulo product form of the digit set determined by the polynomial of the form (I) to (III).

The case of $p^\alpha q^\beta$

Sketch of Proof. Certainly, the Theorem of integer tile is not enough. One needs to "connect" the cyclotomic polynomials.

Conjecture

Suppose $\mathcal{A}$ is an integer tile with $p^\alpha q^\beta$ If $\Phi_{p^\lambda q^\mu}(x)|P_{\mathcal{A}}(x)$ and $\Phi_{p^\lambda}(x) \nmid P_{\mathcal{A}}(x)$, then $\Phi_{q^\mu}(x^{p^\lambda})|P_{\mathcal{A}}(x)$.

This holds for $b = p^2q$, with some argument, we can prove the Theorem.

Other open questions

1. Given a tile digit sets $\mathcal{D}$, does it satisfy $(T2)$?
2. Given a finite set $\mathcal{A}$. A set $\{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda}$ is an orthonormal basis for $\ell^2(A)$ if

$$P_{\mathcal{A}}(e^{2\pi i(\lambda-\lambda')}) = \delta_{\lambda,\lambda'}.$$

Theorem

(Laba) If $\mathcal{A}$ satisfies $(T1)$ and $(T2)$, then we can find such orthonormal basis. (This is directly related to the Fuglede's Conjecture)

Can we find orthonormal basis for a tile digit set $\mathcal{D}$ from the kernel polynomial?

Thank You !!