Pisot conjecture and Tilings

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A **Pisot number** is an algebraic integer > 1 such that all conjugates other than itself has modulus strictly less than 1.

A well known property: if β is a Pisot number, then $d(\beta^n, \mathbb{Z}) \to 0$ as $n \to \infty$.

A partial converse is shown by Hardy:

Let $\beta > 1$ be an algebraic number and $x \neq 0$ is a real number. If $d(x\beta^n, \mathbb{Z}) \to 0$ then β is a Pisot number.

Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure preserving transformation. Then $(X, \mathcal{B}, \mu, T)$ forms a measure theoretical dynamical system. By Poincaré's recurrence theorem, for a set $Y \in \mathcal{B}$ with $\mu(Y) > 0$, almost all T-orbit from Y is **recurrent**. The first return map on Y is defined by:

$$\hat{T} = T^{m(x)}(x)$$

where $m(x) = \min\{m \in \mathbb{Z}_{>0} \mid T^m(x) \in Y\}$. This gives the **induced system**:

$$(Y, \mathcal{B} \cap Y, \frac{1}{\mu(Y)}\mu, \hat{T}).$$

From now on let $X \subset \mathbb{R}^d$. The system $(X, \mathcal{B}, \mu, T)$ is **self-inducing** if there is a Y such that $(Y, \mathcal{B} \cap Y, \frac{1}{\mu(Y)}\mu, \hat{T})$ is isomorphic to the original dynamics by the affine isomorphism map ϕ:

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \phi \big\downarrow & & \big\downarrow \phi \\ Y & \xrightarrow[\hat{T}]{} & Y \end{array}$$

Motivation:

The self-inducing structure corresponds to pure periodic expansion in arithmetic algorithms. The **scaling constant** (the maximal eigenvalue of the matrix of ϕ^{-1}) often becomes a Pisot number, moreover a Pisot unit.

Many examples: irrational rotation and continued fraction, interval exchange, piecewise isometry, outer billiard, etc.

We wish to know why the Pisot number plays the role. Self-inducing structure is modeled by **Substitutive dynamical system**.

Pisot conjecture

Put $\mathcal{A} = \{0, 1, \ldots, k-1\}$ and take a substitution, for e.g.: $\sigma(0) = 01$, $\sigma(1) = 02$, $\sigma(2) = 0$ having a fixed point $x = 0102010010201010201001 02\ldots$. Let s be the shift: $s(a_1 a_2 \ldots) = a_2 a_3 \ldots$. We have a topological dynamics $X_\sigma = \overline{\{s^n(x) \mid n = 0, 1, \ldots\}}$ where s acts. The incidence matrix M_σ is the $k \times k$ matrix whose (i, j) entry is the number of i occurs in $\sigma(j)$: for e.g.,

$$M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

If this is primitive, (X_σ, s) is minimal and uniquely ergodic. The cylinder set for a word w is $[w] = \{x \in X_\sigma \mid x$ has prefix $w\}$. The unique invariant measure is given by

$$\mu([w]) = \text{Frequency of the word } w \text{ in the fixed point } x$$

We are interested in the spectral property of the isometry $U_s : f \to f \circ s$ on $L^2(X_\sigma, \mu)$.

I start with an important fundamental result (under some mild conditions):

Theorem 1 (Host [9]). Eigenfunctions of U_s are chosen to be continuous.

The dynamical system has **pure discrete spectrum** if measurable eigenfunctions of U_s forms a complete orthonormal basis. **von Neumann's theorem** asserts that a dynamical system is pure discrete iff it is conjugate to the translation action $x \to x + a$ on a compact abelian group.

Let Φ_σ be the characteristic polynomial of M_σ. The substitution is **irreducible** if Φ_σ is irreducible. We say σ is a **Pisot substitution** if the Perron-Frobenius root of M_σ is a Pisot number.

Conjecture 1 (Pisot Conjecture). If σ is a irreducible Pisot substitution then (X_σ, s) has purely discrete spectrum.

One way to attack this conjecture is the construction of **Rauzy fractal** (Rauzy, Arnoux-Ito, Siegel and many others). This makes explicit the group action on the torus and has a lot of applications in number theory and discrete geometry.

Another way is to use **Harmonic analysis** (Solomyak, Moody, Lee, Baake and many others). A motivation comes from structure analysis of quasi-crystals. I discuss from this side today, and try to explain the necessity of above two assumptions, i.e.. Pisot property and Irreducibility.

Substitution Tiling

The fix point x of σ naturally gives rise to a 1-dimensional self-affine tiling $\mathcal{T}$ by giving length to each letter. The lengths are coordinates of a left eigenvector of M_σ. Taking closure of translations of $\mathbb{R}$ we get a different dynamical system $(X_\mathcal{T}, \mathbb{R})$ with continuous time, which is called a suspension.

Now we have two systems: (X_σ, s) and $(X_\mathcal{T}, \mathbb{R})$. Spectral property of (X_σ, s) is rather intricate. Clark-Sadun [5] showed that if we ask whether pure discrete or not, two dynamics behave the same.

Why Pisot?

A reason is hidden in the proof of

Theorem 2 (Dekking-Keane [8]). (X_σ, s) is not mixing.

In the tribonacci fixed point $x = 0102010010201\ldots$, take a word $w = 010$. The return word of is 0102. Considering

$$x = \sigma^n(010)\sigma^n(2)\sigma^n(010)\ldots,$$

we find too many patterns of the shape $[w] \cap s^{-|\sigma^n(0102)|}[w]$. We can show that

$$\mu([w] \cap s^{-|\sigma^n(0102)|}[w]) > \text{const} \cdot \mu([w])$$

which shows that the system is not mixing. Pursuing this discussion we arrive at:

Theorem 3 (Bombieri-Taylor [4], Solomyak [14])**.** $(X_{\mathcal{T}}, \mathbb{R})$ is not weakly mixing if and only if the Perron Frobenius root of M_σ is a Pisot number.

This is a dynamical version of Hardy's theorem. Let $\ell(w)$ the suspension length of a word w. Then $f(x - \beta^n \ell(0102)) = \lambda^{\beta^n \ell(0102)} f$ must be close to f for $f = \chi_{[\text{tile for } 010]}$, in the above example. Thus we must have $\lambda^{\beta^n \ell(w)} \to 1$, for a return word w. Thus β must be a Pisot number.

Solomyak showed an impressive result: $(X_{\mathcal{T}}, \mathbb{R})$ is pure discrete iff

$$\text{density } \mathcal{T} \cap (\mathcal{T} - \beta^n v) \to 1,$$

where β is the Perron-Frobenius root of M_σ and v be a return vector. We easily see from where the shape $\beta^n v$ comes! This is equivalent to a combinatorial condition : **Overlap coincidence**.

We can define self affine tiling (with FLC and repetitivity) dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ in higher dimensional case with an expanding matrix Q. If it satisfies **Pisot family condition**:

Every conjugate γ *with* $|\gamma| \geq 1$ *of an eigenvalue* Q *is again an eigenvalue of* Q,

the dynamical system has relatively dense eigenvalues. It is pure discrete iff

$$\text{density } \mathcal{T} \cap (\mathcal{T} - Q^n v_i) \to 1,$$

for d-linearly independent return vectors $v_1, \ldots, v_d$ in $\mathbb{R}^d$.

Algorithm for pure discreteness

Overlap coincidence gives an algorithm to determine pure discreteness. However this is difficult to compute, because it depends on topology of tiles. Recently with Jeong-Yup Lee, we invented an easy practical algorithm [1]. We also implemented a Mathematica program.

`http://mathweb.sc.niigata-u.ac.jp/~akiyama/Research1.html`

It works for all self-affine tilings, including non-unit scaling and non lattice-based tilings. Just give us a GIFS data of a self-affine tiling!

Why irreducible ?: An example by Bernd Sing.

Tiling dynamical system of substitution on 4 letters:

$$0 \to 0\bar{1},\ 1 \to 0,\ \bar{0} \to \bar{0}1,\ \bar{1} \to \bar{0}$$

is not purely discrete (c.f. [13]). This substitution is a skew product of Fibonacci substitution by a finite cocycle over $\mathbb{Z}/2\mathbb{Z}$. This example somewhat shows the necessity of irreducibility. However, we may say more.

Naive Pisot Conjecture in Higher dimension

Assume two conditions:

1. Q satisfies Pisot family condition.

2. Congruent tiles have the same color.

Then the tiling dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ may be pure discrete. Lee-Solomyak [12, 11] showed:

Pisot-family condition $\Longleftrightarrow$ Meyer property

In 1-dim, irreducibility implies the color condition.

Example by the endomorphism of free group (Dekking [6, 7], Kenyon [10])

We consider a self similar tiling is generated a boundary substitution:

$$\begin{aligned}\theta(a) &= b\\ \theta(b) &= c\\ \theta(c) &= a^{-1}b^{-1}\end{aligned}$$

acting on the boundary word $aba^{-1}b^{-1}, aca^{-1}c^{-1}, bcb^{-1}c^{-1}$, representing three fundamental parallelogram. The associated tile equation is

$$\begin{aligned} \alpha A_1 &= A_2 \\ \alpha A_2 &= (A_2 - 1 - \alpha) \cup (A_3 - 1) \\ \alpha A_3 &= A_1 - 1 \end{aligned}$$

with $\alpha \approx 0.341164 + i1.16154$ which is a root of the polynomial $x^3 + x + 1$.

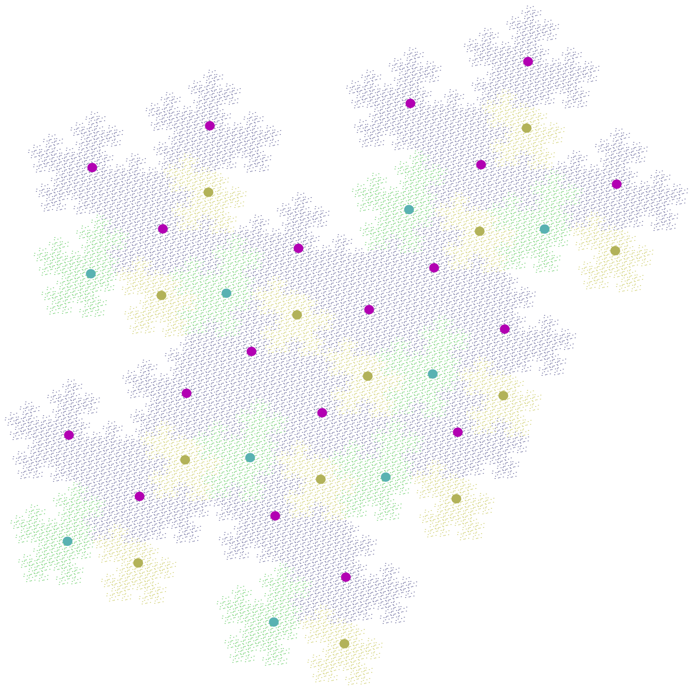

Figure 2: Tiling by boundary endomorphism

The tiling dynamical system has pure discrete spectrum.

Example: Arnoux-Furukado-Harriss-Ito tiling

Arnoux-Furukado-Harriss-Ito [2] recently gave an explicit Markov partition of the toral automorphism for the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

which has two dim expanding and two dim contractive planes. They defined 2-dim substitution of 6 polygons. Let $\alpha = -0.518913 - 0.66661\sqrt{-1}$ a root of $x^4 - x^3 + 1$. The multi colour Delone set is given by 6×6 matrix:

$$\begin{pmatrix} \{\} & \{z/\alpha\} & \{z/\alpha\} & \{\} & \{\} & \{\} \\ \{\} & \{\} & \{\} & \{z/\alpha\} & \{z/\alpha\} & \{\} \\ \{\} & \{\} & \{\} & \{\} & \{\} & \{z/\alpha\} \\ \{z/\alpha\} & \{\} & \{\} & \{\} & \{\} & \{\} \\ \{\} & \{z/\alpha + 1 - \alpha\} & \{\} & \{\} & \{\} & \{\} \\ \{\} & \{\} & \{\} & \{(z-1)/\alpha + \alpha\} & \{\} & \{\} \end{pmatrix}$$

and the associated tiling for contractive plane is:

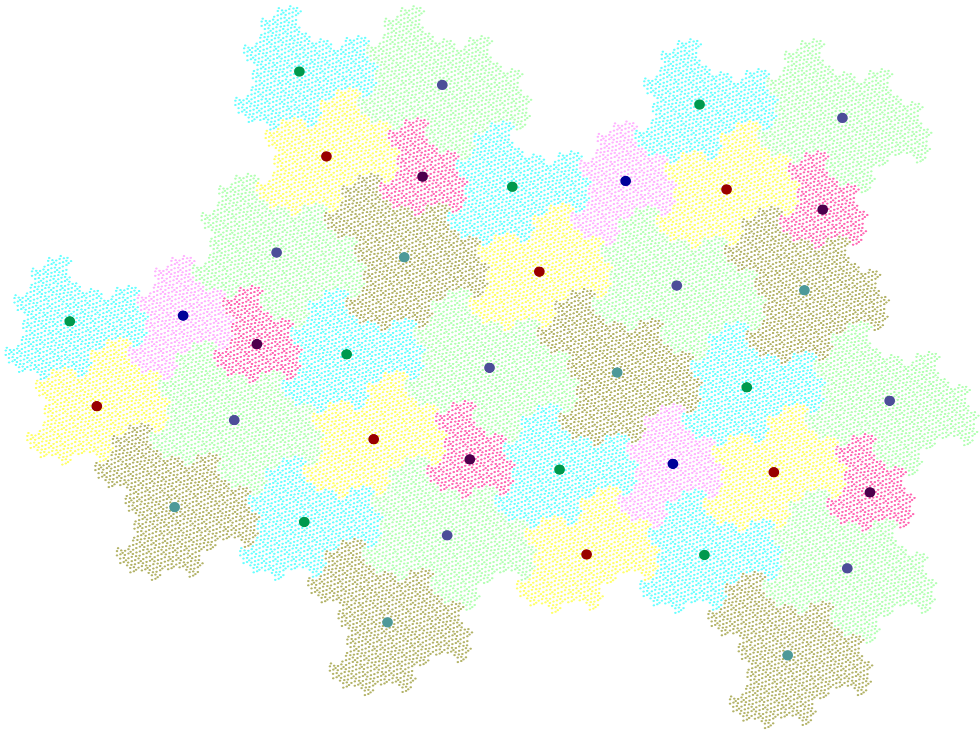

Figure 3: AFHI Tiling

Our program saids it is purely discrete.

A counter example !?

Christoph Bandt discovered a non-periodic tiling in [3] whose setting comes from crystallographic tiles. This is a 3-reptile defined by:

$$-I\omega\sqrt{3}A = A \cup (A+1) \cup (\omega A + \omega)$$

where $\omega = (1+\sqrt{-3})/2$ is the 6-th root of unity.

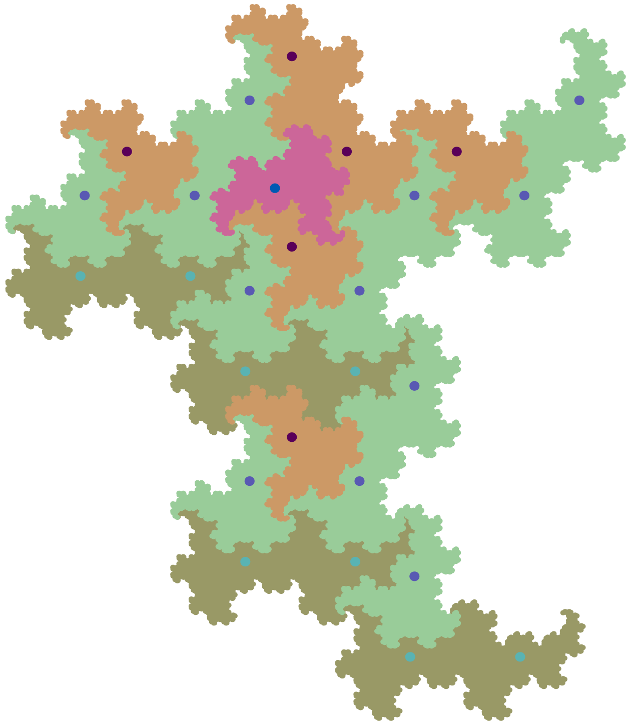

Figure 4: Fractal chair tiling

Fractal chair tiling is **not purely discrete**! An overlap creates new overlaps without any coincidence. One can draw an overlap graph.

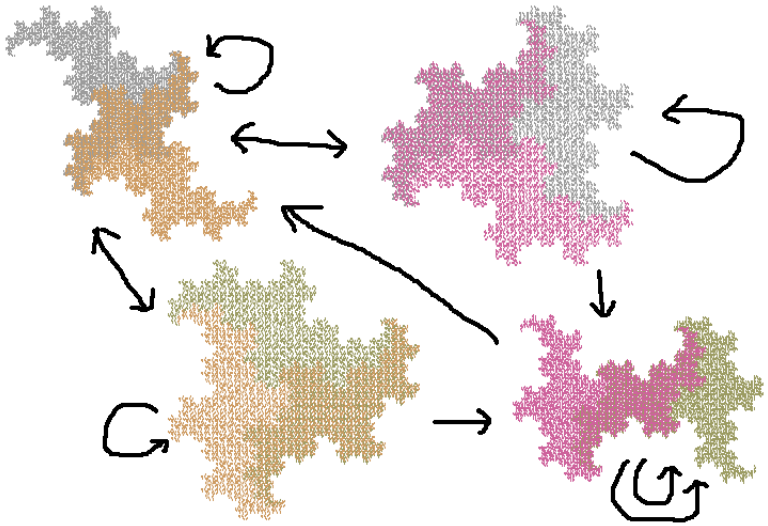

Figure 5: Overlap graph of fractal chair

However, from this overlap graph, one can construct another tiling which explains well this non pureness.

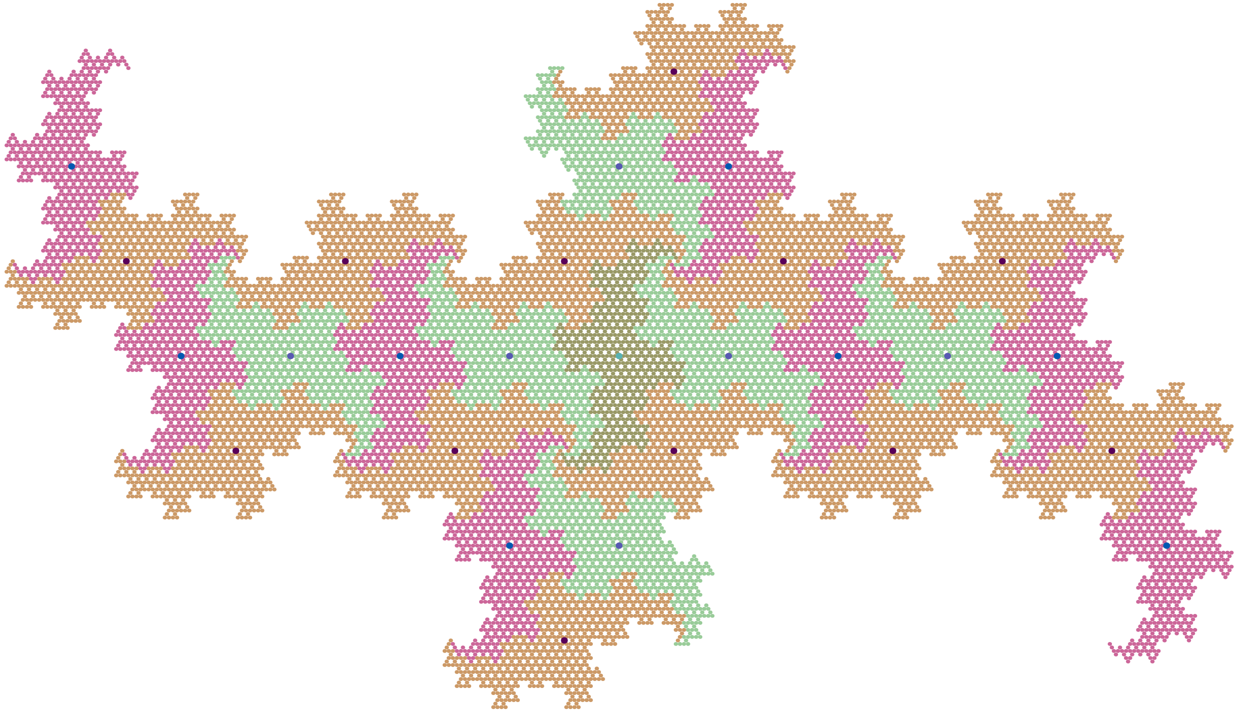

Figure 6: Tiling from overlaps

The 2nd tiling associates different colors to translationally equivalent tiles. Forgetting colors of tiles, it is periodic. The situation is similar to Sing's example. Our conjecture still alive in this very naive sense.

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