BOUNDARY OF CENTRAL TILES ASSOCIATED WITH PISOT BETA-NUMERATION AND PURELY PERIODIC EXPANSIONS

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Abstract. This paper studies tilings and representation sapces related to the β-transformation when β is a Pisot number (that is not supposed to be a unit). The obtained results are applied to study the set of rational numbers having a purely periodic β-expansion. We indeed make use of the connection between pure periodicity and a compact self-similar representation of numbers having no fractional part in their β-expansion, called central tile: for elements x of the ring $\mathbb{Z}[1/\beta]$, so-called x-tiles are introduced, so the central tile is a finite union of x-tiles up to translation. These x-tiles provide a covering (and even in some cases a tiling) of the space we are working in. This space, called complete representation space, is based on Archimedean as well as non-Archimedean completions of the number field $\mathbb{Q}(\beta)$ for primes dividing the norm of β. This representation space has numerous potential implications.

We focus here on one application concerning the gamma function $\gamma(\beta)$ defined as the supremum of the set of elements v in $[0,1]$ such that every positive rational number p/q, with $p/q \le v$ and q coprime with the norm of β, has a purely periodic β-expansion. Our study relies on the description of the topological properties of central tiles in terms of boundary graphs. Special focus is given to some quadratic examples, showing that the behaviour of $\gamma(\beta)$ in the non-unit case is slightly different from its behaviour in the unit case.

1. Introduction

Beta-numeration generalises usual binary and decimal numeration. Taking any real number $\beta > 1$, it consists in expanding numbers $x \in [0,1]$ as power series in base β^{-1} with digits in $\mathcal{D} = \{0, \ldots, \lceil \beta \rceil - 1\}$. As for $\beta \in \mathbb{N}$, the digits are obtained with the so-called *greedy algorithm*: the β-transformation $T_\beta : x \mapsto \beta x \pmod 1$ computes the digits $u_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$, which yield the expansion $x = \sum_{i \ge 1} u_i \beta^{-i}$. The sequence of digits is denoted by $d_\beta(x) = (u_i)_{i \ge 1}$.

The set of expansions $(u_i)_{i\ge1}$ was characterised by Parry in [Par60]) (see Theorem 2.1 below). When β is a Pisot number, Bertrand [Ber77] and Schmidt [Sch80] independently proved that the β-expansion $d_\beta(x)$ of a real number $x \in [0,1]$ is ultimately periodic if and only if x belongs to $\mathbb{Q}(\beta) \cap [0,1]$. A further natural question was to identify the set of numbers with purely periodic expansions. For $\beta \in \mathbb{N}$, it has long been known that rational numbers a/b with a purely periodic β-expansion are exactly those such that b and β are coprime, with the length of the period being the order of β in $(\mathbb{Z}/b\mathbb{Z})^*$. Using an approximation and renormalisation technique, Schmidt proved in [Sch80] that when $\beta^2 = n\beta + 1$ and $n \in \mathbb{N}^*$, then all rational numbers less than 1 have a purely periodic β-expansion. This result was completed in [HI97], with $\beta^2 = n\beta - 1$, $n \ge 3$, for which no rational number has a purely periodic β-expansion. More generally, the latter result is satisfied by all β's admitting at least one positive real Galois conjugate in $[0,1]$ [Aki98][Proposition 5]. Ito and

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Rao characterised real numbers having a purely periodic β-expansion in terms of the associated Rauzy fractal for any Pisot unit β [IR04], whereas the non-unit case was handled in [BS07]. The length of the periodic expansions with respect to quadratic Pisot units were investigated in [QRY05].

Another natural issue is to determine real numbers with finite expansion. According to [FS92], we say that β satisfies the finiteness property (F) if the positive elements of $\mathbb{Z}[1/\beta]$ all have a finite β-expansion (the converse is clear). A complete characterisation of β satisfying the finiteness property (F) is known when β is a Pisot number of degree 2 or 3 [Aki00]. It turns out that these numbers β also play a role in purely periodic expansions issues. Indeed, if β is a unit Pisot number and satisfies the finiteness property (F), then there is a neighbourhood of 0 in $\mathbb{Q}_+$ whose elements all have a purely periodic β-expansion [Aki98]. This is quite unexpected since there is no reason *a priori* why only purely periodic expansions around zero would be obtained.

The present paper investigates the case when β is still a Pisot number, but not necessarily a unit. We make use of the connection between pure periodicity and a compact self-similar representation of numbers having no fractional part in their β-expansion, as similarly described in [IR04, BS07]. This representation is called the *central tile* associated with β (*Rauzy fractal*, or *atomic surface* may also be encountered in the literature, see e.g. the survey [BS05]). For elements x of the ring $\mathbb{Z}[1/\beta]$, so-called x-tiles are introduced, so the central tile is a finite union of x-tiles up to translation. These x-tiles provide a covering of the space we are working in. We first discuss the topological and metric properties of the central tile in flavor of [Aki02, Pra99, Sie03] and the relations between the tiles.

In the unit case, the covering by x-tiles is defined in a Euclidean space $\mathbb{K}_\infty \simeq \mathbb{R}^{r-1} \times \mathbb{C}^s$, where $d = r + 2s$ is the degree of the extension $[\mathbb{Q}(\beta) : \mathbb{Q}]$ and r is the number of real roots of the minimal polynomial of β. The space $\mathbb{K}_\infty$ can be interpreted as the product of all Archimedean completions of $\mathbb{Q}(\beta)$ distinct from the usual one. It turns out that this is generally not enough: in order to have suitable measure-preserving properties, one has to take the non-Archimedean completions associated with the principal ideal (β) into account. Therefore, everything takes place in the product $\mathbb{K}_\beta = \mathbb{K}_\infty \times \mathbb{K}_f$, where the latter is a finite product of local fields. In the substitution framework, this approach has been already used in [Sie03], and was inspired by [Rau88]. See also [Sin06]. Completions and (complete) tiles are introduced in Section 3. We discuss why taking non-Archimedean completions into account is suitable from a tiling point of view: when the finiteness property (F) holds, we prove that the x-tiles are disjoint if the non-Archimedean completions are considered, which was not the case when only taking Archimedean completions into account. Our principal result in this context is Theorem 3.18.

Let us stress the fact that the complete representation spaces introduced here have numerous potential implications, such as e.g. Markov partitions for toral endomorphisms in the flavour of [Sch00, LS05]. Our main goal here is to study the set of rational numbers having a purely periodic beta-expansion, for which we introduce the following notation.

Notation 1.1. Π_β *denotes the set of real numbers* $x \in [0, 1)$ *having a purely periodic beta-expansion. We also note* $\Pi_\beta^{(r)} = \Pi_\beta \cap \mathbb{Q}$.

The study of these sets begins in Section 4. After reviewing the characterisation of purely periodic expansions in terms of the complete tiles due to [BS07] (see [IR04] for the unit case), we apply it to obtain results on periodic expansions of rational integers.

Theorem 1.2. *Let* β *be a Pisot number that satisfies the property (F). Then there exist* ε *and* D *such that for every* $x = \frac{p}{q} \in \mathbb{Q} \cap [0, 1)$*, if* $x \le \varepsilon$, $\gcd(N(\beta), q) = 1$ *and* $N(\beta)^D$ *divides* p*, then* x *has a purely periodic expansion in base* β.

Definition 1.3 (Function gamma)**.** *The function γ is defined on the set of Pisot numbers and takes its values in $[0,1]$. Let β be a Pisot number. Let $N(\beta)$ denote the norm of β. Then, $\gamma(\beta)$ is defined as*

$$\gamma(\beta) = \sup\left\{ v \in [0,1];\ \forall x = \frac{p}{q} \in \mathbb{Q} \cap]0,v] \ \textit{with} \gcd(q, N(\beta)) = 1,\ \textit{then}\ x \in \Pi_\beta^{(r)} \right\}.$$

The reasons for condition $\gcd(q, N(\beta)) = 1$ will be given in Lemma 4.1. We also use the central tile and its tiling properties to obtain, in Section 5, an explicit computation of the quantity $\gamma(\beta)$ for two quadratic Pisot numbers, i.e.,

Theorem 1.4. $\gamma(2+\sqrt{7}) = 0$ *and* $\gamma(5+2\sqrt{7}) = (7-\sqrt{7})/12$.

The second example shows that the behaviour of $\gamma(\beta)$ in the non-unit case is slightly different from its behaviour in the unit case.

This paper is organised as follows. Section 2 recalls the terminology and results necessary to state and prove the results, including Euclidean tiles and the unit case. Section 3 goes beyond the unit case and extends the previous concepts including non-Archimedean components. This section starts with a short compendium on what we need from algebraic number theory. Section 4 studies purely periodic expansions and Section 5 is devoted to examples in quadratic fields.

Since we work with Pisot numbers and in order to avoid using plethoric vocabulary, we will always assume in this section that β is a Pisot number, even if the result is more general. Readers interested in generalities concerning beta-numeration could have a look at [Bla89, BS05, BBLT06].

2. Beta-numeration, automata, and tiles

2.1. **Beta-numeration.** We recall in the present section some well-known facts about beta-numeration. We assume that β is a Pisot number. Since $1 \in \mathbb{Q}$, $d_\beta(1)$ is ultimately periodic by [Ber77, Sch80] and we have the following (see e.g. [Par60, Bla89, Fro00, Lot02]):

Theorem and Definition 2.1. *Let β be a Pisot number. Let $\mathcal{D} = \{0, 1, \cdots, \lceil\beta\rceil - 1\}$. Let $d^*_\beta(1) = d_\beta(1)$ if $d_\beta(1)$ is infinite, and $d^*_\beta(1) = (t_1 \dots t_{n-1} t_n)^\infty$, if $d_\beta(1) = t_1 \dots t_{n-1}(t_n+1)0^\infty$, with $t_i \in \mathcal{D}$ for all i. Then the set of β-expansions of real numbers in $[0,1)$ is exactly the set of sequences $(u_i)_{i\geq 1}$ in $\mathcal{D}^{\mathbb{N}}$ that satisfy the so-called* admissibility condition

$$\forall k \geq 1,\ (u_i)_{i\geq k} <_{lex} d^*_\beta(1). \tag{2.1}$$

A finite string w is said to be admissible if the sequence $w \cdot 0^\infty$ satisfies the condition (2.1), where $A \cdot B$ denotes the concatenation of words A and B. The set of admissible strings is denoted by $\mathcal{L}_\beta$; the set of admissible sequences by $\mathcal{L}^\infty_\beta$. The map $x \mapsto d_\beta(x)$ is an increasing bijection from $[0,1)$ onto $\mathcal{L}_\beta$, endowed with the lexicographical order.

Notation 2.2. *Hereafter, β will be a Pisot number of degree d, with*

$$d^*_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_n)^\infty,$$

that is, n is the sum of the lengths of the preperiod and of the period; in particular, $m = 0$ if and only if $d^*_\beta(1)$ is purely periodic.

The Pisot number β is said to be a *simple Parry number* if $d_\beta(1)$ is finite, otherwise it is said to be a *non-simple Parry number.* One has $m = 0$ if and only if β is a simple Parry number: indeed, $d_\beta(1)$ is never purely periodic according to Remark 7.2.5 in [Lot02]). The set $\mathcal{A}$ denotes the alphabet $\{1, \dots, n\}$.

Expansion of the non-negative real numbers. The β-expansion of any $x \in \mathbb{R}^+$ is deduced by rescaling from the expansion of $\beta^{-p}x$, where p is the smallest integer such that $\beta^{-p}x \in [0,1)$:
(2.2)

$$\forall x \in \mathbb{R}^+,\ x = \underbrace{w_p\beta^p + \cdots + w_0}_{\text{integer part}} + \underbrace{u_1\beta^{-1} + \cdots + u_i^i\beta^{-i} + \cdots}_{\text{fractional part}}, \quad w_p \cdots w_0 u_1 \cdots u_i \cdots \text{ satisfies (2.1).}$$

In this case, we call $[x]_\beta = w_p\beta^p + \cdots + w_0$ the *integer part* of x and $\{x\}_\beta = u_1\beta^{-1} + \cdots + u_i^i\beta^{-i} + \cdots$ the *fractional part* of x. We extend the notation d_β to $\mathbb{R}^+$ and write $d_\beta(x) = w_p \cdots w_0.u_1 \cdots u_i \cdots$.

Integers in base β. We define the set of integers in base β as the set of positive real numbers with no fractional part:

$$\begin{aligned} \text{Int}(\beta) &= \{w_p\beta^p + \cdots + w_0;\ w_p \ldots w_0 \in \mathcal{L}_\beta\} \\ &= \{[x]_\beta;\ x \in \mathbb{R}_+\} \subset \mathbb{Z}[\beta]. \end{aligned} \tag{2.3}$$

The set $\text{Int}(\beta)$ builds a discrete subset of $\mathbb{R}_+$. It has some regularity: two consecutive points in $\text{Int}(\beta)$ differ by a finite number of values, i.e., the positive numbers $T_\beta^{a-1}(1)$, $a \in \{1, \cdots, n\}$ (see [Thu89, Aki07]). It can even be shown that it is a Meyer set [BFGK98].

2.2. **Admissibility graph.** The set of admissible sequences described by (2.1) is the set of infinite labellings of an explicit finite graph with nodes in $\mathcal{A} = \{1, \ldots, n\}$ and edges $b \xrightarrow{\varepsilon} a$, with $a, b \in \mathcal{A}$ labelled by digits $\varepsilon \in \mathcal{D} = \{0, 1, \cdots, \lceil\beta\rceil - 1\}$. This so-called *admissibility graph* is depicted in Figure 1.

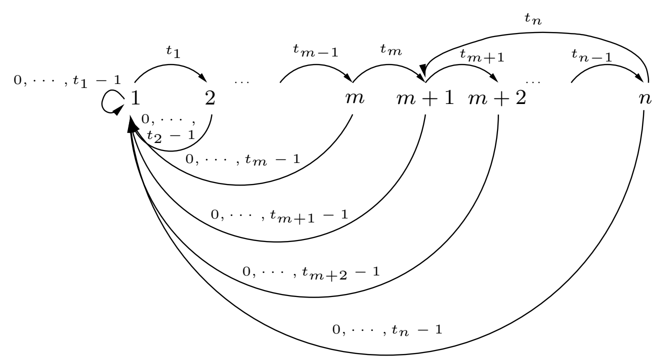

FIGURE 1. The graph describes admissible sequences for the β-shift. The number n of nodes is given by the sum of the preperiod and the period of $d^*_\beta(1) = t_1 \cdots t_m(t_{m+1} \cdots t_n)^\infty$. From each node a to node 1, there are t_a edges labelled by $0, \ldots, t_a - 1$. From each node a to node $a+1$, there is one edge labelled by t_a. Let m denote the length of the preperiod of $d^*_\beta(1)$ (it can possibly be zero). From node n to node $m+1$ there is an edge labelled by t_n.

For $a \in \mathcal{A}$, define $\mathcal{L}_\beta^{(a)}$ as the set of admissible strings w (see Definition 2.1) that the admissibility graph conducts from the initial node 1 to node a. In other words, for $a \neq 1$, $\mathcal{L}_\beta^{(a)}$ is the set of admissible strings having $t_1 \cdots t_{a-1}$ as a suffix. Clearly, according to the form of the admissibility graph, one has $\mathcal{L}_\beta = \bigcup_{a \in \mathcal{A}} \mathcal{L}_\beta^{(a)}$.

Let S denote the shift operator on the set of sequences in the set of digits $\{0, \ldots, \lceil\beta\rceil - 1\}^{\mathbb{N}} = \mathcal{D}^{\mathbb{N}}$. The beta-expansion of $T_\beta^k(1)$ is $d_\beta(T_\beta^k(1)) = S^k(d_\beta(1))$. By increasingness of the map d_β, it follows that for any $x \in [0,1)$:

$$t_1 t_2 \cdots t_{a-1} d_\beta(x) \in \mathcal{L}_\beta^\infty \iff d_\beta(x) <_{\text{lex}} S^{a-1}(d_\beta^*(1)) \iff x \in [0, T^{a-1}(1)). \tag{2.4}$$

Note that if β is a simple Parry number (that is, if $m = 0$) and $k \in \mathbb{N}$, then the sequence $S^k(d_\beta^*(1))$ is not admissible.

2.3. **Central tiles.** The *central tile* associated with a Pisot number is a compact geometric representation of the set $\mathrm{Int}(\beta)$ of integers in base β. It is defined as follows.

Galois conjugates of β and Euclidean completions. Let $\beta_2, \ldots, \beta_r$ be the real conjugates of $\beta = \beta_1$, which all have a modulus strictly smaller than 1 since β is a Pisot number. Let β_{r+1}, $\overline{\beta_{r+1}}, \ldots, \beta_{r+s}, \overline{\beta_{r+s}}$ stand for its complex conjugates. For $2 \le i \le r$, let $\mathbb{K}_{\beta_i}$ be equal to $\mathbb{R}$, and for $r+1 \le i \le r+s$, let $\mathbb{K}_{\beta_i}$ be equal to $\mathbb{C}$. The fields $\mathbb{R}$ and $\mathbb{C}$ are endowed with the normalised absolute value $|x|_{\mathbb{K}_{\beta_i}} = |x|$ if $\mathbb{K}_{\beta_i} = \mathbb{R}$ and $|x|_{\mathbb{K}_{\beta_i}} = |x|^2$ if $\mathbb{K}_{\beta_i} = \mathbb{C}$. These absolute values induce the usual topologies on $\mathbb{R}$ (resp. $\mathbb{C}$). For any $i = 2$ to $r+s$, the $\mathbb{Q}$-homomorphism defined on $\mathbb{Q}(\beta)$ by $\tau_i(\beta) = \beta_i$ realises a $\mathbb{Q}$-isomorphism between $\mathbb{Q}(\beta)$ and $\mathbb{K}_{\beta_i} = \mathbb{Q}(\beta_i) \hookrightarrow \mathbb{R}, \mathbb{C}$.

Euclidean β-representation space. We obtain a Euclidean representation $\mathbb{Q}$-vector space $\mathbb{K}_\infty$ by gathering the fields $\mathbb{K}_{\beta_i}$:

$$\mathbb{K}_\infty = \mathbb{K}_{\beta_2} \times \cdots \times \mathbb{K}_{\beta_r} \times \mathbb{K}_{\beta_{r+1}} \times \mathbb{K}_{\beta_{r+2}} \times \cdots \times \mathbb{K}_{\beta_{r+s}} \simeq \mathbb{R}^{r-1} \times \mathbb{C}^s.$$

We denote by $\|\cdot\|_\infty$ the maximum norm on $\mathbb{K}_\infty$. We have a natural embedding

$$\begin{array}{rccc} \phi_\infty \colon & \mathbb{Q}(\beta) & \longrightarrow & \mathbb{K}_\infty \\ & x & \longmapsto & (\tau_i(x))_{2 \le i \le r+s} \end{array}$$

Euclidean central tile. We are now able to define the central tile:

Definition 2.3 (Central tile)**.** *Let β be a Pisot number with degree d. The* Euclidean central tile *of β is the representation of the set of integers in base β:*

$$\mathcal{T} = \overline{\phi_\infty(\mathrm{Int}(\beta))} \subset \overline{\mathbb{Q}(\beta_2) \times \cdots \times \mathbb{Q}(\beta_{r+s})} \subset \mathbb{K}_\infty.$$

Since the roots β_i have a modulus smaller than one, $\mathcal{T}$ is a compact subset of $\mathbb{K}_\infty$.

2.4. **Property (F) and tilings.** More generally, to each $x \in \mathbb{Z}[1/\beta] \cap [0,1)$, we associate a geometric representation of points that admit w as a fractional part.

Definition 2.4 (x-tile)**.** *Let $x \in \mathbb{Z}[1/\beta] \cap [0,1)$. The tile associated with x is*

$$\mathcal{T}(x) = \overline{\phi_\infty(\{y \in \mathbb{R}^+;\ \{y\}_\beta = x\})} \subset \phi_\infty(x) + \mathcal{T}.$$

It is proved in [Aki02] that tiles $\mathcal{T}(x)$ provide a covering of $\mathbb{K}_\infty$, i.e.,

$$\mathbb{K}_\infty = \bigcup_{x \in \mathbb{Z}[1/\beta] \cap [0,1)} \mathcal{T}(x). \tag{2.5}$$

Since we know that tiles $\mathcal{T}(x)$ cover the space $\mathbb{K}_\infty$, a natural question is whether this covering is a tiling (up to sets of zero measure).

Definition 2.5 (Exclusive points)**.** *We say that a point $z \in \mathbb{K}_\infty$ is* exclusive *in tile $\mathcal{T}(x)$ if z is contained in no other tile $\mathcal{T}(x')$ with $x' \in \mathbb{Z}[1/\beta] \cap [0,1)$, and $x' \neq x$.*

Definition 2.6 (Finiteness property)**.** *The Pisot number β satisfies the* finiteness property (F) *if and only if every $x \in \mathbb{Z}[1/\beta] \cap [0,1)$ has a finite β-expansion.*

If the finiteness property is satisfied, a sufficient tiling condition is known when β is a unit.

Theorem 2.7 (Tiling property)**.** *Let β be a unit Pisot number. The number β satisfies the finiteness property (F) if and only if* 0 *is an exclusive inner point of the central tile of β. In this latter case, every tile $\mathcal{T}(x)$, for $x \in \mathbb{Z}[1/\beta] \cap [0,1]$ has a non-empty interior, and all its inner points are exclusive. In other words, tiles $\mathcal{T}(x)$ provide a tiling of $\mathbb{K}_\infty$.*

Proof. The proof is obtained in [Aki02]. In [ST07], this property is restated in a discrete geometry framework. □

2.5. **Purely periodic points.** In [IR04], Ito and Rao establish a relation between the central tile and purely periodic β-expansions. For this purpose, a geometric realisation of the natural extension of the beta-transformation is built using the central tile. More precisely, the central tile represents, by construction (up to closure), the strings $w_n \dots w_0$ that can be read in the admissibility graph shown in Figure 1. We gather strings $w_m \dots w_0$ depending on the nodes of the graph the string $w_m \dots w_0$ reaches in order to divide the central tile into subtiles.

Definition 2.8 (Central subtiles)**.** *Let $a \in \mathcal{A} = \{1, \cdots, n\}$. The* central a-subtile *is defined as*

$$\mathcal{T}^{(a)} = \overline{\phi_\infty\left(\left\{x \in \mathrm{Int}(\beta);\ d_\beta(x) \in \mathcal{L}_\beta^{(a)}\right\}\right)}.$$

Theorem 2.9 ([IR04])**.** *Let β be a Pisot unit. We recall that $\mathcal{A} = \{1, \cdots, n\}$. Let $x \in \mathbb{Q}(\beta) \cap [0,1)$. The β-expansion of x is purely periodic if and only if*

$$(-\phi_\infty(x), x) \in \bigcup_{a \in \mathcal{A}} \mathcal{T}^{(a)} \times [0, T_\beta^{a-1}(1)).$$

As soon as 0 is an inner point of the central tile, we deduce that small rational numbers have a purely periodic expansion.

Corollary 2.10 ([Aki98])**.** *Let β be a Pisot unit. If β satisfies the finiteness property (F), then there exists a constant $c > 0$ such that every $x \in \mathbb{Q} \cap [0, c)$ has a purely periodic expansion in base β.*

Proof. Since 0 is an inner point of $\mathcal{T}$ and $\mathcal{A}$ is finite, there exists $c > 0$ such that $0 < c \leq \min\{T_\beta^{(a-1)}(1);\ a \in \mathcal{A}\}$ and $B_\infty(0,c) \subset \mathcal{T}$. For $x \in [0,c)$, we have $\phi_\infty(x) = (x, x, \dots, x)$ and

$$(-\phi_\infty(x), x) \in \mathcal{T} \times [0, c) \subset \bigcup_{a \in \mathcal{A}} \left(\mathcal{T}^{(a)} \times \left[0, T_\beta^{(a-1)}(1)\right)\right).$$

Then the periodicity follows from Theorem 2.9. □

This result was first proved directly by Akiyama [Aki98]. Recall that $\gamma(\beta)$ is the supremum of such c's according to Definition 1.3. Once one of the conjugates of β is positive, then $\gamma(\beta) = 0$. The quadratic unit case is completely understood: in this case, Ito and Rao proved that $\gamma(\beta)$ equals 0 or 1 ([IR04]). Examples of computations of $\gamma(\beta)$ for higher degrees are also performed by Akiyama in the unit case in [Aki98].

Algebraic natural extension. By abuse of language, one may say that Theorem 2.9 implies that $\bigcup_{a \in \mathcal{A}} (\mathcal{T}^{(a)} \times \left[0, T_\beta^{a-1}(1)\right)$ is a fundamental domain for an *algebraic* realisation of the natural extension of the β-transformation T_β, although it does not satisfy Rohklin's minimality condition for natural extensions (see [Roh61] and also [CFS82]). We will briefly explain this in the sequel.

In [DKS96], Dajani *et al.* provide an explicit construction of the natural extension of the β-transformation for any $\beta > 1$ in dimension three, with third dimension given by the height in a stacking structure. This construction is minimal in the above sense. As a by-product, one can retrieve the invariant measure of the system as an induced measure. However, this natural extension provides no information on the purely periodic orbits under the action of the β-transformation T_β. The main reason is that the geometric realisation map which plays the role of our ϕ_∞ is not an additive homomorphism. Therefore, this embedding destroys the algebraic structure of the β-transformation. Our construction, which was derived from Thurston in the Pisot unit case [Thu89], only works for restricted cases but it has the advantage that we can use conjugate maps that are additive homomorphisms. This is the clue used by Ito and Rao in [IR04] for the description of purely periodic orbits. In summary, we need a more geometric natural extension than that of Rohklin to answer several number theoretical questions like periodicity issues.

Let us note that in the non-unit case, measure-preserving properties are no longer satisfied by the embedding ϕ_∞. Indeed, it is clear that T_β is an expanding map with ratio β. When only Archimedean embeddings as in the unit case, we will only take into account ϕ_∞ which is a contracting map with ratio $N(\beta)/\beta$ and we will not be able to get a measure-preserving natural extension. This is the key reason why we now introduce non-Archimedean embeddings.

3. Complete tilings

Thanks to the non-Archimedean part, we will show that we obtain a map ϕ_β, which is a contracting map with ratio $1/\beta$. Let us recall that T_β is an expanding map with ratio β. We thus recover a realisation of the natural extension via a measure-preserving map. Moreover, the extended map acting on the fundamental domain of the natural extension will be almost one-to-one (being a kind of variant of Baker's transform). Therefore there is good chances of getting a one-to-one map onto a suitable *lattice* for this algebraic natural extension. Considering that a bijection onto a finite set yields purely periodic expansions, we will obtain a description of purely periodic elements of this system. This heuristic (see e.g. [IR04]) will be achieved in Proposition 3.15 and Theorem 3.18 below.

3.1. **Algebraic framework.** In order to extend the above results to the case where β is not a unit, we follow the idea of [Sie03] and embed the central tile in a larger space including local components. To avoid confusion, the central tile $\mathcal{T} \subset \mathbb{R}^{r-1} \times \mathbb{C}^s$ will be called the *Euclidean central tile*. The large tile will be called *complete tile* and denoted as $\widetilde{\mathcal{T}}$.

Let us briefly recall some facts and set notation. The results can be found, for instance, in the first two chapters of [CF86]. Let $\mathfrak{O}$ be the ring of integers of the field $\mathbb{Q}(\beta)$. If $\mathfrak{P}$ is a prime ideal in $\mathfrak{O}$ such that $\mathfrak{P} \cap \mathbb{Z} = p\mathbb{Z}$, with relative degree $f(\mathfrak{P}) = [\mathfrak{O}/\mathfrak{P} : \mathbb{Z}/p\mathbb{Z}]$ and ramification index $e(\mathfrak{P})$, then $\mathbb{K}_\mathfrak{P}$ stands for the completion of $\mathbb{Q}(\beta)$ with respect to the $\mathfrak{P}$-adic topology. It is an extension of $\mathbb{Q}_p$ of degree $e(\mathfrak{P})f(\mathfrak{P})$. The corresponding normalised absolute value is given by $|x|_\mathfrak{P} = \left| N_{\mathbb{K}_\mathfrak{P}/\mathbb{Q}_p}(y) \right|_p^{1/e(\mathfrak{P})f(\mathfrak{P})} = p^{-f(\mathfrak{P})v_\mathfrak{P}(y)}$. Let $\mathfrak{O}_\mathfrak{P}$ denote its ring of integers and $\mathfrak{p}_\mathfrak{P}$ its maximal ideal; then

$$\mathfrak{O}_\mathfrak{P} = \{y \in \mathbb{K}_\mathfrak{P}\,;\, v_\mathfrak{P}(y) \geq 0\} = \{y \in \mathbb{K}_\mathfrak{P}\,;\, |y|_\mathfrak{P} \leq 1\}.$$
$$\mathfrak{p}_\mathfrak{P} = \{y \in \mathbb{K}_\mathfrak{P}\,;\, v_\mathfrak{P}(y) \geq 1\} = \{y \in \mathbb{K}_\mathfrak{P}\,;\, |y|_\mathfrak{P} < 1\}.$$

The normalised Haar measure on $\mathbb{K}_\mathfrak{P}$ is $\mu_\mathfrak{P}(a + \mathfrak{p}_\mathfrak{P}^m) = p^{mf(\mathfrak{P})}$. In particular: $\mu_\mathfrak{P}(\mathfrak{O}_\mathfrak{P}) = 1$.

Lemma 3.1. *Let $\mathcal{V}$ be the set of places in $\mathbb{Q}(\beta)$. For any place $v \in \mathcal{V}$, the associated normalised absolute value is denoted $|\cdot|_v$. If v is Archimedean, we make the usual convention $\mathfrak{O}_v = \mathbb{K}_v$.*
(1) *Let $\mathcal{S} \subset \mathcal{V}$ be a finite set of places. Let $(a_v)_{v\in\mathcal{S}} \in \prod_{v\in\mathcal{S}} \mathbb{K}_v$. Then, for any $\varepsilon > 0$, there exists $x \in \mathbb{K}$ such that $|x - a_v|_v \leq \varepsilon$ for all $v \in \mathcal{S}$.*
(2) *Let $\mathcal{S} \subset \mathcal{V}$ be a finite set of places and $v_0 \in \mathcal{V} \setminus \mathcal{S}$. Let $(a_v)_{v\in\mathcal{S}} \in \prod_{v\in\mathcal{S}} \mathbb{K}_v$. Then, for any $\varepsilon > 0$, there exists $x \in \mathbb{K}$ such that $|x - a_v|_v \leq \varepsilon$ for all $v \in \mathcal{S}$ and $v \in \mathfrak{O}_s$ for all $v \notin \mathcal{S} \cup \{v_0\}$. Furthermore, if v_0 is an Archimedean place and $(a_v)_{v\in\mathcal{S}} \in \prod_{v\in\mathcal{S}} \mathfrak{O}_v$, then $x \in \mathfrak{O}$.*

Proof. (1) (resp. the first part of (2)) are widely known as the weak (resp. strong) approximation theorems. Concerning the last sentence, let $x \in \mathbb{Q}(\beta)$ given by (2). By assumption, $x \in \mathfrak{O}_v$ for all v, therefore $x \in \mathfrak{O}$, since $\mathfrak{O}$ is the intersection of the local rings $\mathfrak{O}_v$, where v runs along the non-Archimedean places. □

3.2. **Complete representation space.**

Notation 3.2. *Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_\nu$ be the prime ideals in the ring of integers $\mathfrak{O}$ that contain β, that is,*

$$(\beta) = \beta\mathfrak{O} = \prod_{i=1}^{\nu} \mathfrak{P}_i^{n_i}.$$

For $x \in \mathbb{Q}(\beta)$, $N(x)$ shortly denotes the norm $N_{\mathbb{Q}(\beta)/\mathbb{Q}}(x)$. We have $N(\beta\mathfrak{O}) = |N(\beta)|$; the prime numbers p arising from $\mathfrak{P}_i \cap \mathbb{Z} = p\mathbb{Z}$ are the prime factors of $N(\beta)$. Let $\mathcal{S}_\beta$ be the set consisting of Archimedean places corresponding to β_i, $2 \le i \le r+s$ and ν non-Archimedean places corresponding to $\mathfrak{P}_i$.

The *complete representation space* $\mathbb{K}_\beta$ is obtained by adjoining to the Euclidean representation the product of local fields $\mathbb{K}_f = \prod_{i=1}^{\nu} \mathbb{K}_{\mathfrak{P}_i}$, that is, $\mathbb{K}_\beta = \mathbb{K}_\infty \times \mathbb{K}_f = \prod_{v \in \mathcal{S}_\beta} \mathbb{K}_v$. The field $\mathbb{Q}(\beta)$ naturally embeds in $\mathbb{K}_\beta$:

$$\begin{array}{rcl} \phi_\beta \colon \mathbb{Q}(\beta) & \longrightarrow & \mathbb{K}_\infty \times \prod_{i=1}^{\nu} \mathbb{K}_{\mathfrak{P}_i} \\ x & \longmapsto & (\phi_\infty(x), x, \ldots, x). \end{array}$$

The complete representation space is endowed with the product topology, and with coordinatewise addition and multiplication. This makes it a locally compact abelian ring. Then the approximation theorems yield the following:

Lemma 3.3. *With the previous notation, it is established that $\phi_\beta(\mathbb{Q}(\beta))$ is dense in $\mathbb{K}_\beta$, and that $\phi_\beta(\mathfrak{O})$ is dense in $\prod_{v \in \mathcal{S}_\beta} \mathfrak{O}_v$.*

Proof. The first assertion follows from the first part of Lemma 3.1 with $\mathcal{S} = \mathcal{S}_\beta$. The second assertion follows from its second part with $\mathcal{S} = \mathcal{S}_\beta$ and v_0 being the Archimedean valuation corresponding to the trivial embedding $\tau(\beta) = \beta$. □

The normalised Haar measure μ_β of the additive group $(\mathbb{K}_\beta, +)$ is the product measure of the normalised Haar measures on the complete fields $\mathbb{K}_{\beta_i}$ (Lebesgue measure) and $\mathbb{K}_{\mathfrak{P}_i}$ (Haar measure $\mu_{\mathfrak{P}_i}$). By a standard measure-theoretical argument, if $\alpha \in \mathbb{Q}(\beta)$ and if B is a borelian subset of $\mathbb{K}_\beta$, then

$$\mu_\beta(\alpha \cdot B) = \mu_\beta(B) \prod_{v \in \mathcal{S}_\beta} |\alpha|_v. \tag{3.1}$$

Consequently, if $\alpha \in \mathbb{Q}(\beta)$ is a $\mathcal{S}_\beta$-unit (i.e., if $|\alpha|_v = 1$ for all $v \notin \mathcal{S}_\beta$), then $\mu_\beta(\alpha \cdot B) = |\alpha|^{-1}\mu_\beta(B)$ by the product formula ($|\cdot|$ is there the usual real absolute value). This holds in particular for $\alpha = \beta$.

Finally, we also denote by $\|\cdot\|$ the maximum norm on $\mathbb{K}_\beta$, that is $\|x\| = \max_{v \in \mathcal{S}_\beta} |x|_v$. The following finiteness remark will be used several times.

Lemma 3.4. *If $B \subset \mathbb{K}_\beta$ is bounded with respect to $\|\cdot\|$, then $\phi_\beta^{-1}(B) \cap \mathbb{Z}[1/\beta]$ is locally finite.*

Proof. Let B be a bounded subset of $\mathbb{K}_\beta$, and $x \in \mathbb{Q}(\beta)$ such that $\phi_\beta(x) \in B$. In particular, for every i, $1 \le i \le \nu$, there exists a rational integer m_i, such that the embedding of x in $\mathbb{K}_{\mathfrak{P}_i}$ has valuation at most m_i. For $m = \max_{1 \le i \le \nu} m_i$, we get $\beta^m x \in \prod_{v \in \mathcal{S}_\beta} \mathfrak{O}_v$. On the other hand, β is a $\mathcal{S}_\beta$-unit, so that $\beta^m\mathbb{Z}[1/\beta] = \mathbb{Z}[1/\beta] \subset \mathfrak{O}_\mathfrak{P}$ for any $\mathfrak{P}$ coprime with (β). Therefore, $\beta^m x \in \mathfrak{O}$. Furthermore, the Archimedean absolute values $|\beta^m x|_{\beta_i}$ are also bounded for $i = 2, \dots, r+s$. If we further assume that x belongs to some bounded subset of $\mathbb{Q}(\beta)$ (w.r.t. the usual metric), then all conjugates of $\beta^m x$ are bounded. Since these numbers belong to $\mathfrak{O}$, there are only a finite number of them. □

3.3. **Complete tiles and an Iterated Function system.**

Definition 3.5 (Complete tiles)**.** *The following omplete tiles are analogues of Euclidean tiles in $\mathbb{K}_\beta$.*

- *Complete central tile. One has*
$$\widetilde{\mathcal{T}} = \overline{\phi_\beta(\mathrm{Int}(\beta))} \subset \prod_{v \in \mathcal{S}_\beta} \mathfrak{O}_v.$$
- *Complete x-tiles. For every $x \in \mathbb{Z}[1/\beta] \cap [0,1)$,*
$$\widetilde{\mathcal{T}}(x) = \overline{\phi_\beta(\{y \in \mathbb{R}^+;\ \{y\}_\beta = x\})} \subset \phi_\beta(x) + \widetilde{\mathcal{T}}.\ \textit{In particular, } \widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}(0).$$
- *Complete central subtiles. For every $a \in \{1, \dots, n\}$,*
$$\widetilde{\mathcal{T}}^{(a)} = \overline{\phi_\beta\left(\left\{x \in \mathrm{Int}(\beta);\ d_\beta(x) \in \mathcal{L}_\beta^{(a)}\right\}\right)}.$$

Using (2.4), we get:

$$\begin{aligned}
\widetilde{\mathcal{T}}(x) &= \phi_\beta(x) + \overline{\phi_\beta\left(\left\{y \in \mathrm{Int}(\beta); d_\beta(y)\cdot d_\beta(x) \in \mathcal{L}_\beta^\infty\right\}\right)} \\
&= \phi_\beta(x) + \overline{\bigcup_{a;\, t_1 \cdots t_{a-1}\cdot d_\beta(x) \in \mathcal{L}_\beta^\infty} \phi_\beta\left(\left\{y \in \mathrm{Int}(\beta);\ d_\beta(y) \in \mathcal{L}_\beta^{(a)}\right\}\right)} \\
&= \phi_\beta(x) + \bigcup_{a;\, x < T_\beta^{(a-1)}(1)} \widetilde{\mathcal{T}}^{(a)}.
\end{aligned} \tag{3.2}$$

Hence, any complete x-tile is a finite union of translates of complete central subtiles.
We now consider the following self-similarity property satisfied by the complete central subtiles:

Proposition 3.6. *Let β be a Pisot number. The complete central subtiles satisfy an Iterated Function System equation (IFS) directed by the admissibility graph (drawn in Figure 1) in which the edge direction is reversed:*

$$\widetilde{\mathcal{T}}^{(a)} = \bigcup_{b \xrightarrow{\varepsilon} a} \left(\phi_\beta(\beta)\widetilde{\mathcal{T}}^{(b)} + \phi_\beta(\varepsilon)\right). \tag{3.3}$$

We use here Notation 2.2 and recall that the digits ε belong to $\mathcal{D} = \{0, \dots, \lceil\beta\rceil - 1\}$, and that the nodes a, b belong to $\mathcal{A} = \{0, 1, \cdots, n\}$.

Proof. The following decomposition of the languages $\mathcal{L}_\beta^{(a)}$ can be read off from the admissibility graph 1

$$\mathcal{L}_\beta^{(a)} = \bigcup_{b \xrightarrow{\varepsilon} a} \mathcal{L}_\beta^{(b)} \cdot \{\varepsilon\}. \tag{3.4}$$

This decomposition yields a similar IFS as in (3.3) where the complete central subtiles $\widetilde{\mathcal{T}}^{(a)}$ are replaced by the images $\phi_\beta(\{x \in \mathrm{Int}(\beta);\ d_\beta(x) \in \mathcal{L}_\beta^{(a)}\})$ of the languages $\mathcal{L}_\beta^{(a)}$ into $\mathbb{K}_\beta$. Lastly, one gets (3.3) by taking the closure (the unions are finite). It should be noted that this argument does not depend on the embedding; it is therefore the same as in the unit case, which can be found e.g. in [SW02, Sie03, BS05]. □

Remark 3.7. If one details the IFS given by (3.3), this gives (with m defined in Notation 2.2):

$$\left\{\begin{array}{rcl} \widetilde{\mathcal{T}}^{(1)} & = & \bigcup_{a\in\mathcal{A}}\bigcup_{\varepsilon<t_a}\left(\phi_\beta(\beta)\widetilde{\mathcal{T}}^{(a)}+\phi_\beta(\varepsilon)\right) \\ \widetilde{\mathcal{T}}^{(r+1)} & = & \left(\phi_\beta(\beta)\widetilde{\mathcal{T}}^{(m)}+\phi_\beta(t_m)\right)\bigcup\left(\phi_\beta(\beta)\widetilde{\mathcal{T}}^{(n)}+\phi_\beta(t_n)\right) \\ \widetilde{\mathcal{T}}^{(k+1)} & = & \phi_\beta(\beta)\widetilde{\mathcal{T}}^{(k)}+\phi_\beta(t_k),\ k\in\{1,\ldots,n-1\}\setminus\{m\}. \end{array}\right. \tag{3.5}$$

3.4. **Boundary graph.** The aim of this section is to introduce the notion of boundary graph which will be a crucial tool for our estimations of the function γ in Section 5. This graph is based on the self-similarity properties of the boundary of the central tile, in line with those defined in [Sie03, Thu06, ST07]. The idea is as follows: in order to better understand the covering (2.5), we need to exhibit which points belong to the intersections between the central tile $\widetilde{\mathcal{T}}$ and the x-tiles $\widetilde{\mathcal{T}}(x)$. To do this, we first decompose $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{T}}(x)$ into subtiles: we know that $\widetilde{\mathcal{T}}=\cup_{a\in\mathcal{A}}\widetilde{\mathcal{T}}^{(a)}$ and Eq. (3.2) gives

$$\widetilde{\mathcal{T}}(x)=\cup_{b\in\mathcal{A},T_\beta^{b-1}(1)>x}\ \widetilde{\mathcal{T}}^{(b)}+\phi_\beta(x).$$

Then the intersection between $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{T}}(x)$ is the union of intersections between $\widetilde{\mathcal{T}}^{(a)}$ and $\widetilde{\mathcal{T}}^{(b)}+\phi_\beta(x)$ for $T_\beta^{b-1}(1)>x$. We build a graph (see Definition 3.8 below) whose *nodes* stand for each intersection of this type, hence the nodes are labelled by triplets $[a,x,b]$. To avoid the non-significant intersection $\widetilde{\mathcal{T}}^{(a)}\cap\widetilde{\mathcal{T}}^{(a)}$, we have to exclude the case $x=0$ and $a=b$. Then we use the IFS equation Eq. (3.3) to decompose the intersection $\widetilde{\mathcal{T}}^{(a)}\cap(\widetilde{\mathcal{T}}^{(b)}+\phi_\beta(x))$ into new intersections of the same nature (Eq. (3.6)). An *edge* is labelled with a couple of digits (p_1,q_1) in $\mathcal{D}$, so that an edge from one node to another one acts as $\beta^{-1}(x+q_1-p_1)$ on x.

By applying this process, we show below that we obtain a graph that describes the intersections $\widetilde{\mathcal{T}}^{(a)}\cap(\widetilde{\mathcal{T}}^{(b)}+\phi_\beta(x))$ (Theorem 3.11). It can be used to check whether the covering (2.5) is a tiling, as was done in [Sie03, ST07] but this is not the purpose of the present paper. Indeed, in the last section, we use this graph to deduce information on purely periodic expansions.

Definition 3.8. *The nodes of the* boundary graph *are the triplets* $[a,x,b]\in\mathcal{A}\times\mathbb{Z}[1/\beta]\times\mathcal{A}$ *such that:*

(N1) $-T_\beta^{(a-1)}(1)<x<T_\beta^{(b-1)}(1)$ *and* $a\neq b$ *if* $x=0$.
(N2) $\phi_\beta(x)\in\widetilde{\mathcal{T}}^{(a)}-\widetilde{\mathcal{T}}^{(b)}$.

The labels of the edges of the boundary graph belong to $\mathcal{D}^2$. *There exists an edge* $[a,x,b]\xrightarrow{(p_1,q_1)}[a_1,x_1,b_1]$ *if and only if:*

(E1) $x_1=\beta^{-1}(x+q_1-p_1)$,
(E2) $a_1\xrightarrow{p_1}a$ *and* $b_1\xrightarrow{q_1}b$ *are edges of the admissibility graph.*

We first deduce from the definition that the boundary graph is finite and the Archimedean norms of its nodes are explicitly bounded:

Proposition 3.9. *The boundary graph is finite. If* $[a,x,b]$ *is a node of the boundary graph, then we have:*

(N3) $x\in\mathfrak{O}$;
(N4) *for every conjugate* β_i *of* β, $|\tau_i(x)|\leq\frac{[\beta]}{1-|\beta_i|}$.

Proof. Let $[a,x,b]$ be a node of the graph. By (N2), $\phi_\beta(x)\in\widetilde{\mathcal{T}}^{(a)}-\widetilde{\mathcal{T}}^{(b)}$, which implies $|\tau_i(x)|\leq\frac{[\beta]}{1-|\beta_i|}$.

Let $\mathfrak{P}$ be a prime ideal in $\mathfrak{O}$. If $\mathfrak{P} \mid (\beta)$, then $x \in \mathfrak{O}_{\mathfrak{P}}$ - since $\phi_\beta(x) \in \widetilde{\mathcal{T}}^{(a)} - \widetilde{\mathcal{T}}^{(b)}$. Otherwise, if $\mathfrak{P}$ is coprime with β, we use the fact that $x \in \mathbb{Z}[1/\beta]$ to deduce that $x \in \mathfrak{O}_{\mathfrak{P}}$. We thus have $x \in \mathfrak{O}$. It directly follows from Lemma 3.4 that the boundary graph is finite. □

Proposition 3.9 will be used in Section 5 to compute explicitly the boundary graph in some specific cases. Let us stress the fact that condition (N2) in Definition 3.8 cannot be directly checked algorithmically, whereas numbers satisfying condition (N3) and (N4) are explicitly computable. Nevertheless, conditions (N3) and (N4) are only necessary conditions for a triplet to belong to the graph. Theorem 3.11 below has two aims: it first details how the boundary graph actually describes the boundary of the graph, as intersections between the central tile and its neighbours. Secondly, we will decuce from this theorem an explicit way of computation for the boundary graph in Corollary 3.13.

The following lemma shows that Condition (N1) in Definition 3.8 automatically holds for a node $[a_1, x_1, b_1] \in \mathcal{A} \times \mathbb{Z}[1/\beta] \times \mathcal{A}$ as soon as the edge conditions (E1) and (E2) hold between $[a, x, b] \in \mathcal{A} \times \mathbb{Z}[1/\beta] \times \mathcal{A}$ and $[a_1, x_1, b_1]$.

Lemma 3.10. *Let* $x \in \left(-T_\beta^{(a-1)}(1), T_\beta^{(b-1)}(1)\right) \cap \mathbb{Z}[1/\beta]$. *Let* $a_1 \xrightarrow{p_1} a$ *and* $b_1 \xrightarrow{q_1} b$ *be two edges in the admissibility graph. Let* $x_1 = \beta^{-1}(x + q_1 - p_1)$. *One has* $x_1 \in \left(-T_\beta^{(a_1-1)}(1), T_\beta^{(b_1-1)}(1)\right)$.

Proof. Assume that x is non-negative (otherwise, the same argument applies to $-x$). We thus have $-\frac{p_1}{\beta} \leq x_1 \leq \frac{x+q_1}{\beta}$. Since $a_1 \xrightarrow{p_1} a$, then $p_1 \leq t_{a_1}$, hence $p_1\, 0^\infty <_{\text{lex}} S^{a_1-1}(d_\beta^*(1))$ (the strict inequality comes from the fact that $d_\beta^*(1))$ does not ultimately end in 0^∞). Therefore, $x_1 \geq -\frac{p_1}{\beta} > -T_\beta^{(a_1-1)}(1)$ by (2.4).

On the other hand, since $x < T_\beta^{(b-1)}(1)$, then the sequence $t_1 \cdots t_{b-1}\, d_\beta(x)$ is admissible, again by (2.4). We thus deduce from $b_1 \xrightarrow{q_1} b$ that $t_1 \cdots t_{b_1-1}\,(q_1\, d_\beta(x))$ is admissible. We thus get $x_1 \leq \frac{x+q_1}{\beta} < T_\beta^{(b_1-1)}(1)$. □

However, if β is not a unit, it does not follow from Lemma 3.10 that if $[a, x, b]$ is a node of the boundary graph, $a_1 \xrightarrow{p_1} a$ and $b_1 \xrightarrow{q_1} b$ are edges of the admissibility graph, and $x_1 = \beta^{-1}(x+q_1-p_1)$, then $[a_1, x_1, b_1]$ is a node (we also have to check Condition (N2) or (N3)): for instance, consider the two edges of the admissibility graph $1 \xrightarrow{0} 1$ and $1 \xrightarrow{t_1} 2$. Starting from the node $[1, 0, 2]$, the above edges would yield $x_1 = -\frac{t_1}{\beta} \notin \mathfrak{O}$. Hence $[1, x_1, 1]$ is not a node of the boundary graph by Proposition 3.9.

We now prove that the boundary graph is indeed a good description of the boundary of the central tile, by relating it with intersections between translates of the complete central subtiles.

Theorem 3.11. *Let* $z \in \mathbb{K}_\beta$. *The point* z *belongs to the intersection* $\widetilde{\mathcal{T}}^{(a)} \cap (\widetilde{\mathcal{T}}^{(b)} + \phi_\beta(x))$, *for* $x \in \mathbb{Z}[1/\beta]$, *with* $a \neq b$ *if* $x = 0$, *if and only if* $[a, x, b]$ *is a node of the graph and there exists an infinite path in the boundary graph, starting from the node* $[a, x, b]$ *and labeled by* $(p_i, q_i)_{i \geq 0}$ *such that*

$$z = \sum_{i=0}^{\infty} \phi_\beta(p_i \beta^i).$$

Proof. Let $x \in (-T_\beta^{(a-1)}(1), T_\beta^{(b-1)}(1)) \cap \mathbb{Z}[1/\beta]$. The complete central subtiles satisfy a graph-directed IFS equation detailed in Proposition 3.6 that yields the decomposition

$$\widetilde{\mathcal{T}}^{(a)} \cap \left(\widetilde{\mathcal{T}}^{(b)} + \phi_\beta(x)\right) = \bigcup_{\substack{a_1 \xrightarrow{p_1} a \\ b_1 \xrightarrow{q_1} b}} \left[\left(\phi_\beta(\beta)\widetilde{\mathcal{T}}^{(a_1)} + \phi_\beta(p_1)\right) \cap \left(\phi_\beta(\beta)\widetilde{\mathcal{T}}^{(b_1)} + \phi_\beta(q_1) + \phi_\beta(x)\right)\right]. \tag{3.6}$$

Let $z \in \widetilde{\mathcal{T}}^{(a)} \cap (\widetilde{\mathcal{T}}^{(b)} + \phi_\beta(x))$. Then there exist two edges $a_1 \xrightarrow{p_1} a$ and $b_1 \xrightarrow{q_1} b$ such that the corresponding intersection on the right-hand side of (3.6) contains z. Setting $x_1 = \beta^{-1}(x + q_1 - p_1)$ and $z_1 = \phi_\beta(\beta)^{-1}(z - \phi_\beta(p_1))$, we get $z_1 \in \widetilde{\mathcal{T}}^{(a_1)} \cap \left(\widetilde{\mathcal{T}}^{(b_1)} + \phi_\beta(x_1)\right)$. By construction, $x_1 \in \mathbb{Z}[1/\beta]$ and belongs to the interval $(-T_\beta^{(a_1-1)}(1), T_\beta^{(b_1-1)}(1))$ by Lemma 3.10. Then, by definition, $[a_1, x_1, b_1]$ is a node of the boundary graph, and we may iterate the above procedure. After n steps, we have

$$\frac{z - \phi_\beta\left(\sum_{i=1}^n p_i\beta^{i-1}\right)}{\phi_\beta(\beta^n)} \in \widetilde{\mathcal{T}}^{(a_n)} \cap \left(\widetilde{\mathcal{T}}^{(b_n)} + \phi_\beta(x_n)\right).$$

It follows that $\|z - \phi_\beta\left(\sum_{i=1}^n p_i\beta^{i-1}\right)\| \ll \|\phi_\beta(\beta)\|^n$ for n tending to infinity; therefore $z = \sum_{i=1}^\infty \phi_\beta(p_i\beta^{i-1})$.

Conversely, let z such that $z = \sum_{i\geq 1} \phi_\beta(\beta^{i-1}p_i)$ with $(p_i, q_i)_{i\geq 1}$ being the labeling of a path on the boundary graph starting from $[a, x, b]$. By the definition of the edges of the graph, one checks that $t_1 \cdots t_{a-1}$ is a suffix of $t_1 \cdots t_{a_1-1}p_1$, which is itself a suffix of $t_1 \cdots t_{a_2-1}p_2p_1$, and so on. Hence $z \in \widetilde{\mathcal{T}}^{(a)}$. Let $y = \sum_{i\geq 1} \phi_\beta(\beta^{i-1}q_i)$. By construction, we also have $y \in \widetilde{\mathcal{T}}^{(b)}$. Furthermore, the recursive definition of x_i gives

$$x + \sum_{i=1}^n q_i\beta^{i-1} = \sum_{i=1}^n p_i\beta^{i-1} + \beta^n x_n.$$

The sequence $(x_n)_n$ takes only finitely many values by Proposition 3.9, hence $\phi_\beta(\beta^n x_n)$ tends to 0, which yields $\phi_\beta(x) + y = z$. Therefore $z \in \widetilde{\mathcal{T}}^{(a)} \cap \left(\widetilde{\mathcal{T}}^{(b)} + \phi_\beta(x)\right)$. □

Corollary 3.12. *Let $x \in \mathbb{Z}[1/\beta]$ and let a, b in $\mathcal{A}$. We assume that $a \neq b$ if $x = 0$. The intersection $\widetilde{\mathcal{T}}^{(a)} \cap (\widetilde{\mathcal{T}}^{(b)} + \phi_\beta(x))$ is non-empty if and only if $[a, x, b]$ is a node of the boudary graph and there exists at least one infinite path in the boundary graph starting from $[a, x, b]$.*

We deduce a procedure for computation of the boundary graph.

Corollary 3.13. *The boundary graph can be obtained as follows:*

- *Compute the set of triplets $[a, x, b]$ that satisfy conditions (N1), (N3) and (N4);*
- *Put edges between two triplets if conditions (E1) and (E2) are satisfied;*
- *Recursively remove nodes that have no outgoing edges.*

Proof. We consider the (possiby infinite) graph G obtained according to the above procedure. The particularity of this graph is that any node belongs to an infinite path. Proposition 3.9 and Theorem 3.11 show that this graph is bigger than (or equal to) the boundary graph. Nevertheless, the converse part of the proof of Theorem 3.11 ensures that if an infinite path of the graph G starts from $[a, x, b]$, then this path produces an element z in $\widetilde{\mathcal{T}}^{(a)} \cap (\widetilde{\mathcal{T}}^{(b)} + \phi_\beta(x))$. Therefore, $\phi_\beta(x) \in \widetilde{\mathcal{T}}^{(a)} - \widetilde{\mathcal{T}}^{(b)}$, and $[a, x, b]$ is indeed a node of the boundary graph. Finally, even if the recursive procedure described in the statement of the corollary mentions infinite paths, it only requires finitely many operations, since the number of nodes is finite: it has been proved in Proposition 3.9 for the boundary graph; and it is an immediate consequence of Lemma 3.4 for triplets satisfying (N1), (N3) and (N4). □

3.5. **Covering of the complete representation space.** In order to generalise the tiling property stated in Theorem 2.7 to the non-unit case, we need to better understand how $\mathbb{Z}[1/\beta]\cap\mathbb{R}^+$ embeds in the complete representation space. We first prove the following lemma, that makes Lemma 3.3 more precise.

Lemma 3.14. *The set $\phi_\beta(\mathfrak{O}\cap\mathbb{R}^+)$ is dense in $\prod_{v\in\mathcal{S}_\beta}\mathfrak{O}_v$ and $\phi_\beta(\mathbb{Z}[1/\beta]\cap\mathbb{R}^+)$ is dense in $\mathbb{K}_\beta$. These density properties remain true if one replaces $\mathbb{R}_+$ by any neighbourhood of $+\infty$.*

Proof. We already know by Lemma 3.3 that $\phi_\beta(\mathfrak{O})$ is dense in $\prod_{v\in\mathcal{S}_\beta}\mathfrak{O}_v$. Let $U\geq 0$. For any $x\in\mathfrak{O}$, we have $x+\beta^n>U$ if n is sufficiently large. Since β^n tends to 0 in $\mathbb{K}_\beta$, $\phi_\beta(x+\beta^n)$ tends to $\phi(x)$; hence $\phi_\beta(\mathfrak{O}\cap[U,+\infty))$ is dense in $\prod_{v\in\mathcal{S}_\beta}\mathfrak{O}_v$.

Let $Z=(z,y_1,\dots,y_\nu)\in\mathbb{K}_\beta$. Since $\mathbb{K}_\beta$ is built from the prime divisors of β, there exists a natural integer n such that $\beta^n y_i\in\mathfrak{O}_{\mathfrak{P}_i}$ for every $i=1,...,\nu$. Moreover, there exists an integer A such that $A\mathfrak{O}\subset\mathbb{Z}[\beta]$ (e.g., the discriminant of $(1,\beta,\dots,\beta^{d-1})$). Split A into $A=A_1A_2$, such that A_1 is coprime with β and the prime divisors of A_2 are also divisors of $N(\beta)$. Then A_1 is a unit in each $\mathfrak{O}_{\mathfrak{P}_i}$ such that $y_i/A_1\in\mathfrak{O}_{\mathfrak{P}_i}$ for $1\leq i\leq\nu$. By the definition of A_2, there exists m such that $\beta^m/A_2\in\mathfrak{O}$. Therefore, $\beta^{\max(n,m)}Z/A_2\in\prod_{v\in\mathcal{S}_\beta}\mathfrak{O}_v$. Applying the first part of the lemma, there exists a sequence $(x_l)_l$ in $\mathbb{Z}[\beta^{-1}]\cap[U\beta^{\max(n,m)},+\infty)$ such that $(\phi_\beta(x_l))_l$ tends to $\beta^{\max(n,m)}Z/A_2$. Then, $(\phi_\beta(\beta^{-\max(n,m)}A_2x_m))_l$ tends to Z. Since $\beta^{-\max(n,m)}A_2x_l\in\mathbb{Z}[1/\beta]\cap[U,+\infty)$, the proof is complete. □

Proposition 3.15. *The complete central tile $\widetilde{\mathcal{T}}$ is compact. The x-tiles $\widetilde{\mathcal{T}}(x)$ provide a covering of the β-representation space:*

$$\bigcup_{x\in\mathbb{Z}[1/\beta]\cap[0,1)}\widetilde{\mathcal{T}}(x)=\mathbb{K}_\beta. \tag{3.7}$$

Moreover, this covering is uniformly locally finite: for any $R>0$, there exists $\kappa(R)\in\mathbb{R}_+$ such that, for all $z\in\mathbb{K}_\beta$, one has

$$\#\left\{x\in\mathbb{Z}[1/\beta]\cap[0,1);\ \widetilde{\mathcal{T}}(x)\cap B(z,R)\neq\emptyset\right\}\leq\kappa(R).$$

Proof. The projection of $\widetilde{\mathcal{T}}$ on $\mathbb{K}_f$ is compact since the local rings $\mathfrak{O}_v$ are compact. Its projection on $\mathbb{K}_\infty$ is bounded because β is a Pisot number. Since $\widetilde{\mathcal{T}}$ is obviously closed, it is therefore compact. Explicitly, we have, by construction, that $\|\phi_\beta(\beta)\|<1$. Since $\|n\|=n$ for each $n\in\mathbb{Z}$, it follows that $\widetilde{\mathcal{T}}\subset B(0,M_1)$ with $M_1=(\lfloor\beta\rfloor)/(1-\|\phi_\beta(\beta)\|)$.

Since β is an integer, we have $\mathrm{Int}(\beta)\subset\mathbb{Z}[1/\beta]$. Therefore, for $y\in\mathbb{R}_+$, y belongs to $\mathbb{Z}[1/\beta]$ if and only if $\{y\}_\beta$ belongs to $\mathbb{Z}[1/\beta]$. In other words,

$$\bigcup_{x\in\mathbb{Z}[1/\beta]\cap[0,1)}\left\{y\in\mathbb{R}_+;\ \{y\}_\beta=x\right\}=\mathbb{Z}[1/\beta]\cap\mathbb{R}_+,$$

and, by Lemma 3.14, we have that

$$\mathbb{K}_\beta=\overline{\phi_\beta(\mathbb{Z}[1/\beta]\cap\mathbb{R}^+)}=\overline{\bigcup_{x\in\mathbb{Z}[1/\beta]\cap[0,1)}\phi_\beta\left(\left\{y\in\mathbb{R}_+;\ \{y\}_\beta=x\right\}\right)}. \tag{3.8}$$

Let us fix $z\in\mathbb{K}_\beta$ and $R>0$. We consider the ball $B(z,R)$ in $\mathbb{K}_\beta$. Assume that $x\in\mathbb{Z}[1/\beta]\cap[0,1)$ is such that $\widetilde{\mathcal{T}}(x)\cap B(z,R)\neq\emptyset$. By $\widetilde{\mathcal{T}}(x)\subset\phi_\beta(x)+\widetilde{\mathcal{T}}\subset\phi_\beta(x)+B(0,M_1)$. Hence $\phi_\beta(x)\in B(z,R+M_1)$. Then Lemma 3.4 ensures that there exist only finitely many such x.

It certainly remains to prove that the number of these x is bounded independently of z, but it already shows that the union on the right-hand side of (3.8) is finite, which allows us to permute the union and closure operations and proves (3.7).

We then use (3.7) to prove the existence of some $x_0 \in \mathbb{Z}[1/\beta] \cap [0,1]$, such that $\|\phi_\beta(x_0) - z\| < 1$. Therefore, any $x \in \mathbb{Z}[1/\beta] \cap [0,1)$ satisfying $\widetilde{\mathcal{T}}(x) \cap B(z,R) \neq \emptyset$ can be written as $x = x_0 + x_1$, where $x_1 \in \mathbb{Z}[1/\beta] \cap [-2,1)$ and $\phi_\beta(x_1) \in B(0, R + M_1 + 1)$. Lemma 3.4 gives an upper bound $\kappa(R)$ for the number of such x_1, and the lemma is proved. □

Corollary 3.16. *The complete central tile $\widetilde{\mathcal{T}}$ has non-empty interior in the representation space $\mathbb{K}_\beta$, hence a non-zero Haar measure.*

Proof. The property concerning the complete central tile has already been proved in [BS07], Theorem 2-(2), by geometrical considerations. However, most of this proposition is now an immediate consequence of (3.7): since $\mathbb{K}_\beta$ is locally compact, it is a Baire space. Therefore, some $\widetilde{\mathcal{T}}(x)$ must have a non-empty interior, hence the central tile itself, by $\widetilde{\mathcal{T}}(x) \subset \phi_\beta(x) + \widetilde{\mathcal{T}}$. Thus it has a positive measure. Moreover, (3.7) also gives a direct proof of that fact without any topological consideration, by using the σ-additivity of the measure and $\mu_\beta(\widetilde{\mathcal{T}}(x)) \leq \mu_\beta(\widetilde{\mathcal{T}})$. □

3.6. **Inner points.** We use the covering property to express the complete central tile as the closure of its exclusive inner points (see Definition 2.5). Since we will use it extensively, we introduce the notation $c_\beta = \|\phi_\beta(\beta)\|$. We have seen that $0 < c_\beta < 1$.

Proposition 3.17. *Let β be a Pisot number. If β satisfies property (F), then 0 is an inner point of the complete central subtile $\widetilde{\mathcal{T}}^{(1)}$ and an exclusive inner point of the complete central tile $\widetilde{\mathcal{T}}$.*

Proof. By Lemma 3.4, there exist finitely many $x \in \mathbb{Z}[1/\beta] \cap [0,1)$ such that $\|\phi_\beta(x)\| \leq 2M_1$, where the constant M_1 is taken from the proof of Proposition 3.15. According to property (F), all these x have a finite β-expansion. Let p be the maximal length of these expansions.

Let m be a non-negative integer and $x \in (\mathbb{Z}[1/\beta] \cap \mathbb{R}_+) \setminus \beta^m \mathrm{Int}(\beta)$. Set $x_1 = \lfloor \beta^{-p-m} x \rfloor_\beta$ and $x_2 = \{\beta^{-p-m} x\}_\beta$. By construction, we have $\|\phi_\beta(x_1)\| \leq M_1$ and $\|\phi_\beta(x_2)\| > 2M_1$, the latter because $d_\beta(x_2)$ has a length greater than p. Set $M_2 = M_1 c_\beta^p$.Therefore

$$\|\phi_\beta(x)\| = c_\beta^{p+m} \|\phi_\beta(x_1) + \phi_\beta(x_2)\| > M_1 c_\beta^{p+m} = c_\beta^m M_2.$$

Hence, we have $\phi_\beta^{-1}\big(B(0, c_\beta^m M_2)\big) \cap \mathbb{Z}[1/\beta] \cap \mathbb{R}_+ \subset \beta^m \mathrm{Int}(\beta)$. Taking $m = 0$, this shows that the origin is exclusive. Moreover, since $B(0, c_\beta^m M_2)$ is open, and since $\phi_\beta(\beta^m \mathrm{Int}(\beta)) \subset \widetilde{\mathcal{T}}^{(1)}$ for m sufficiently large, Lemma 3.14 ensures that $B(0, c_\beta^m M_2) \subset \widetilde{\mathcal{T}}^{(1)}$. □

Theorem 3.18. *Let β be a Pisot number. Assume that β satisfies the finiteness property (F). Then each tile $\widetilde{\mathcal{T}}(x)$, for $x \in \mathbb{Z}[1/\beta] \cap [0,1)$, is the closure of its interior, and each inner point of $\widetilde{\mathcal{T}}(x)$ is exclusive. Hence, for every $x \neq x' \in \mathbb{Z}[1/\beta] \cap [0,1)$, $\widetilde{\mathcal{T}}(x')$ does not intersect the interior of $\widetilde{\mathcal{T}}(x)$. The tiles $\widetilde{\mathcal{T}}(x)$ are measurably disjoint in $\mathbb{K}_\beta$. Moreover, their boundary has zero measure.*

The same properties hold for the translates of complete central subtiles $\widetilde{\mathcal{T}}^{(a)} + \phi_\beta(x)$, for $a \in \mathcal{A}$ and $x \in \mathbb{Z}[1/\beta] \cap [0,1)$.

Proof. The proof of the unit case can be found in [Aki02](Theorem 2, Corollary 1) and could have been adapted. We follow a slightly different approach here. For $x \in \mathbb{Z}[1/\beta] \cap \mathbb{R}_+$, let $Y(x) = \{y \in \mathbb{R}_+;\ \{y\}_\beta = x\} \subset \mathbb{Z}[1/\beta] \cap \mathbb{R}_+$. By definition, $\widetilde{\mathcal{T}}(x) = \overline{\phi_\beta(Y(x))}$. According to the proof of Proposition 3.17, we have

$$(3.9) \qquad {\phi_\beta}^{-1}(B(0, c_\beta^m M_2)) \cap \mathbb{Z}[1/\beta] \cap \mathbb{R}_+ \subset \beta^m \mathrm{Int}(\beta) \quad \& \quad B(0, c_\beta^m M_2) \subset \overline{\phi_\beta(\beta^m \mathrm{Int}(\beta))}.$$

Recall that n is the length of $d_\beta(1)$. Therefore, if w_1 and w_2 are admissible, so is $w_1 \cdot 0^n \cdot w_2$. Now, for any given $y \in Y(x)$, we have $y + \beta^m \mathrm{Int}(\beta) \subset Y(x)$ for any $m \geq m(y) = n + \lceil (\log y)/(\log \beta) \rceil$.

Therefore,

$$\widetilde{\mathcal{T}}(x) = \overline{\bigcup_{y\in Y(x)} \phi_\beta\big(y + \beta^{m(y)}\mathrm{Int}(\beta)\big)} = \overline{\bigcup_{y\in Y(x)} B\big(\phi_\beta(y), c_\beta^{m(y)} M_2\big)}, \tag{3.10}$$

and $\widetilde{\mathcal{T}}(x)$ is the closure of an open set, hence of its interior.

Therefore, in order to prove the exclusivity, we only have to show that two different x-tiles have disjoint interiors. Let x, x' in $\mathbb{Z}[1/\beta] \cap [0,1)$. According to (3.10), any non-empty open subset of $\widetilde{\mathcal{T}}(x) \cap \widetilde{\mathcal{T}}(x')$ contains some ball $B = B(\phi_\beta(y), c_\beta^m M_2)$, with $y \in Y(x)$ and $m = m(y)$ chosen as above. Since ϕ_β is a ring homomorphism, the first part of (3.9) implies that ${\phi_\beta}^{-1}(B) \cap \mathbb{Z}[1/\beta] \cap [y, +\infty) \subset y + \beta^m \mathrm{Int}(\beta)$. But there also exist $y' \in Y(x')$ and a natural integer m' such that $\phi_\beta(y' + \beta^{m'}\mathrm{Int}(\beta)) \subset B$. Since $y' + \beta^{m'}\mathrm{Int}(\beta)$ contains arbitrary large real numbers, this shows that $Y(x) \cap Y(x') \neq \emptyset$. Hence $x = x'$ and the exclusivity follows.

The proof for the subtiles $\widetilde{\mathcal{T}}^{(a)}$ works exactly in the same way, because of the key property

$$d_\beta(x) \in \mathcal{L}_\beta^{(a)} \Longrightarrow \forall y \in \mathrm{Int}(\beta):\ d_\beta(x + \beta^m y) \in \mathcal{L}_\beta^{(a)}$$

for m sufficiently large (depending on x).

It is possible to prove directly that the subtiles $\widetilde{\mathcal{T}}^{(a)}$ are measurably disjoint (for an efficient proof based on the IFS (3.3) and Perron-Frobenius Theorem, see [SW02, BS05][Theorem 2]). However, it follows directly from the fact that the boundary of the subtiles have zero-measure, since two different subtiles have disjoint interiors.

To prove the latter, we follow [Pra99] [Proposition 1.1]. Since $\mathcal{A}$ is finite, there exist δ and $a \in \mathcal{A}$ such that $\mu_\beta(\partial\widetilde{\mathcal{T}}^{(a)}) = \delta\mu_\beta(\widetilde{\mathcal{T}}^{(a)})$ and $\mu_\beta(\partial\widetilde{\mathcal{T}}^{(b)}) \leq \delta\mu_\beta(\widetilde{\mathcal{T}}^{(b)})$ for all $b \in \mathcal{A}$. Let $k \geq n$ be a rational integer. Then, by (3.2), we have

$$\begin{aligned} \phi_\beta(\beta)^{-k}\widetilde{\mathcal{T}}^{(a)} &= \overline{\{\phi_\beta(\beta^{-k}x);\ x \in \mathrm{Int}(\beta), d_\beta(x) \in \mathcal{L}_\beta^{(1)}\}} \\ &= \bigcup_{x\in\Lambda_k} \widetilde{\mathcal{T}}(x) = \bigcup_{x\in\Lambda_k} \bigcup_{b;\, x<T^{(b-1)}(1)} \left(\phi_\beta(x) + \widetilde{\mathcal{T}}^{(b)}\right), \end{aligned} \tag{3.11}$$

where $\Lambda_k = \left\{\sum_{i=1}^{k} \omega_i\beta^{-i};\ \omega_1\cdots\omega_k \in \mathcal{L}_\beta^{(a)}\right\}$. As the x-tiles (resp. the subtiles) have disjoint interiors, the family of tiles $\phi_\beta(x) + \widetilde{\mathcal{T}}^{(b)}$ occurring in (3.11) has the same property. Then, for a subfamily $(\mathcal{T}_i)_i$ of these tiles, we have $\mathcal{T}_i \cap \mathcal{T}_j = \partial\mathcal{T}_i \cap \partial\mathcal{T}_j$, and a simple argument gives $\mu_\beta(\partial(\cup\mathcal{T}_i)) \leq \delta\mu_\beta(\cup\mathcal{T}_i)$. Let us split the union (3.11) as $\phi_\beta(\beta)^{-k}\widetilde{\mathcal{T}}^{(a)} = \mathcal{U}_1 \cup \mathcal{U}_2$, where $\mathcal{U}_1$ is the union of tiles intersecting the boundary of $\phi_\beta(\beta)^{-k}\widetilde{\mathcal{T}}^{(a)}$ and $\mathcal{U}_2$ the union of tiles included in its interior. If k is large, $\phi_\beta(\beta)^{-k}\widetilde{\mathcal{T}}^{(a)}$ contains open balls of sufficiently large size to contain some of the tiles, whose diameter are at most $\max_{b\in\mathcal{A}} \mathrm{diam}(\widetilde{\mathcal{T}}^{(b)})$. Hence $\mathcal{U}_2$ is not empty, and has actually positive measure. Finally, since the multiplication by $\phi_\beta(\beta)$ preserves the boundary, we have

$$\delta\mu_\beta\left(\phi_\beta(\beta)^{-k}\widetilde{\mathcal{T}}^{(a)}\right) = \mu_\beta\left(\partial(\phi_\beta(\beta)^{-k}\widetilde{\mathcal{T}}^{(a)})\right) \leq \mu_\beta(\mathcal{U}_2) < \delta\mu_\beta\left(\phi_\beta(\beta)^{-k}\widetilde{\mathcal{T}}^{(a)}\right),$$

if $\delta \neq 0$, which would yield a contradiction. The metric disjointness follows for the tiles $\widetilde{\mathcal{T}}^{(a)}$, hence also for the $\widetilde{\mathcal{T}}(x)$ by (3.2). □

We can project this relation on the Euclidean space.

Corollary 3.19. *Let β be a Pisot number. If β satisfies the finiteness property (F), then* 0 *is an inner point of the central tile $\mathcal{T}$ and each tile $\mathcal{T}(x)$ is the closure of its interior.*

Proof. If 0 in an inner point of $\widetilde{\mathcal{T}}$ in the field $\mathbb{K}_\beta$, then 0 is also an inner point in its projection on $\mathbb{K}_\infty$. □

This corollary is the most extended generalisation of Theorem 2.7 to the non-unit case: if we only consider Archimedean embeddings to build the central tile, the finiteness property still implies that 0 is an inner point of the central tile. Nevertheless, inner points are no longer exclusive, hence the tiling property is not satisfied.

Choosing the suitable non-Archimedean embedding. We have already explained that the Archimedean embedding was not suitable for building a measure-preserving algebraic extension. We shall now comment why the beta-adic representation space $\mathbb{K}_\beta$ is suitable from a tiling viewpoint. It is a general fact based on the admissibility graph that the (complete) subtiles $\widetilde{\mathcal{T}}^{(a)}$ satisfy an Iterated Function System (IFS). Thanks to the introduction of the beta-representation space, the action of the multiplication by $\phi_\beta(\beta)$ in $\mathbb{K}_\beta$ affects the measure as a multiplication by a ratio $1/\beta$ according to (3.1). This property allows us to deduce from the IFS that the (complete) subtiles are measurably disjoint in $\mathbb{K}_\beta$ - whereas their projection $\mathcal{T}^{(a)}$ on $\mathbb{K}_\infty$ are not (Theorem 3.18 below). More geometrically, the space $\mathbb{K}_\beta$ is chosen so that:

- the tiles are big enough to cover it (covering property, first part of Proposition 3.15), and
- they are small enough, so they do not overlap much - neither combinatorically (locally finitely many overlaps - second part of Proposition 3.15) nor topologically (disjoint interiors), nor metrically (measurable disjointness) - as Theorem 3.18 (tiling property) shows.

If β is not a unit, the space $\mathbb{K}_\infty$ is too small to ensure the tiling property. Conversely, the restricted topological product of the $\mathbb{K}_v$ with respect to the $\mathfrak{O}_v$ for all places v but the Archimedean one given by the identity embedding (i.e., the projection of the adèle group $\mathbb{A}_{\mathbb{Q}(\beta)}$ obtained by canceling the coordinate corresponding to that Archimedean valuation) would have satisfied the tiling property and given an interesting algebraical framework, but would have been too big for the covering property since the principal adèles build a discrete subset in the adèle group.

4. Purely periodic expansions

Elements with a purely periodic expansion, denoted by Π_β (see Notation 1.1), belong to $\mathbb{Q}(\beta)$ and, as explained in the introduction, there are numbers β for which $\Pi_\beta^{(r)} = [0,1) \cap \mathbb{Q}$. However, Lemma 4.1 below shows that if β is a Pisot number, but not a unit, there exist arbitrary small rational numbers that do not belong to $\Pi_\beta^{(r)}$. This justifies the restriction in the definition of $\gamma(\beta)$, that only takes rational numbers whose denominator is coprime with the norm of β into account.

Lemma 4.1. *Let β be a non-unit Pisot number. Let $x = \frac{a}{b} \in \mathbb{Q} \cap [0,1)$ with $\gcd(b, N(\beta)) > 1$. Then $d_\beta(x)$ is not purely periodic.*

Proof. Suppose that the β-expansion of $x \in \mathbb{Q} \cap [0,1)$ is purely periodic with period l. Then we can write:

$$x = \frac{a}{b} = \sum_{k \geq 0} \beta^{-k\ell}(a_1\beta^{-1} + \cdots + a_\ell\beta^{-\ell}) = \frac{a_1\beta^{\ell-1} + \cdots + a_\ell}{\beta^\ell - 1}.$$

Hence $x = \frac{A}{\beta^\ell - 1}$ with $A \in \mathfrak{O}$. Since the principal ideals (β) and $(\beta^\ell - 1)$ are coprime, we get $\phi_\beta(x) \in \prod_v \mathfrak{O}_v$. On the other hand, if $p \mid \gcd(b, N(\beta)$, then $\phi_\beta(a/b)$ contains a component in $\mathbb{Q}_p \setminus \mathbb{Z}_p$. Hence $a/b \neq A/(\beta^\ell - 1)$. □

4.1. **Pure periodicity and complete tiles.** Using and adapting ideas from [Pra99, IR04, San02], one obtains the following characterisation of real numbers having a purely periodic β-expansion; this result can be considered as a first step towards the realisation of an algebraic natural extension of the β-transformation. Note that Theorem 4.2 is stated in [BS07] with compact intervals, which obliges to take periodic points into account and to determine whenever $d_\beta(1)$ is finite or infinite. Our point of view simplifies the proof, so we thus give it. By abuse of language, we call hereafter *natural extension* the set $\bigcup_{a\in\mathcal{A}}(\widetilde{\mathcal{T}}^{(a)}) \times \big[0, T_\beta^{a-1}(1)\big)$ introduced below.

Theorem 4.2 ([BS07], Theorem 3)**.** *Let $x \in [0,1)$. Then, x belongs to Π_β if and only if*

$$(-\phi_\beta(x), x) \in \bigcup_{a\in\mathcal{A}} (\widetilde{\mathcal{T}}^{(a)}) \times \big[0, T_\beta^{a-1}(1)\big). \tag{4.1}$$

Proof. Let $x \in [0,1) \cap \Pi_\beta$ with purely periodic beta-expansion $d_\beta(x) = (a_1 \cdots a_\ell)^\omega$. Obviously, $x \in \mathbb{Q}(\beta)$. A geometric summation gives

$$x = \frac{1}{1-\beta^{-\ell}} \sum_{k=1}^{\ell} a_k \beta^{-k} = -\frac{1}{1-\beta^{\ell}} \sum_{j=0}^{\ell-1} a_{\ell-j}\beta^j.$$

Applying $-\phi_\beta$ to the latter and doing the geometric summation backwards yields

$$-\phi_\beta(x) = \sum_{j=0}^{\infty} \tilde{a}_j \phi_\beta(\beta)^j = \lim_{n\to\infty} \phi_\beta\left(\sum_{j=0}^{n} \tilde{a}_j \beta^j\right), \text{ with } \tilde{a}_j = a_{\ell-j} \pmod{\ell}. \tag{4.2}$$

Clearly the sum $\sum_{j=0}^{n} \tilde{a}_j\beta^j$ is a beta-expansion, since we have by construction $\tilde{a}_{\ell-1}\cdots\tilde{a}_0 = a_1 \cdots a_\ell$. Therefore, $-\phi_\beta(x) \in \widetilde{\mathcal{T}}$. Moreover, the admissibility of the concatenation $\tilde{a}_n \cdots \tilde{a}_0\, d_\beta(x) = \tilde{a}_n \cdots \tilde{a}_0 \tilde{a}_{-1}\tilde{a}_{-2}\cdots$ is the exact translation of the condition $(-\phi_\beta(x), x) \in \widetilde{\mathcal{T}}^{(a)} \times [0, T^{a-1}(1))$. Hence the condition is necessary.

Let us prove that the condition is sufficient, and let $z \in \mathbb{Q}(\beta) \cap [0,1)$ such that $(-\phi_\beta(z), z) \in \widetilde{\mathcal{T}}^{(a)} \times [0, T^{a-1}(1))$ for some $a \in \mathcal{A}$. By compactness, there exists a sequence of digits $(w_n)_n$ such that $\phi_\beta(z) = \lim_{n\to\infty} \phi_\beta(\sum_{j=0}^{n} w_j\beta^j)$, with the latter sums being beta-expansions for all n. Moreover, the bi-infinite word $\cdots w_n w_{n-1} \cdots w_0 \cdot d_\beta(z)$ is admissible. Define a sequence $(z_k)_k$ by $d_\beta(z_k) = w_{k-1}w_{k-2}\cdots w_0 \cdot d_\beta(z)$. Write $z_0 = z = a/b$, with $b \in \mathbb{N}^*$ and $a \in \mathbb{Z}[1/\beta]$. Then,

$$z_k = \beta^{-k}\left(z + \sum_{j=0}^{k-1} w_j\beta^j\right) \in b^{-1}\mathbb{Z}[1/\beta]. \tag{4.3}$$

Applying $-\phi_\beta$ to (4.3) gives $-\phi_\beta(z_k) = \lim_{n\to\infty} \phi_\beta\left(\sum_{j=0}^{n} w_{k+j}\beta^j\right)$. In particular, $-\phi_\beta(z_k) \in \widetilde{\mathcal{T}}$ for any k, which ensures that the sequence $(\phi_\beta(z_k))_k$ is bounded, as well as the sequence $(\phi_\beta(bz_k))_k$, which is hence finite by Lemma 3.4. Thus $z_j = z_{j+s}$ for some j and $s \neq 0$. This shows that $d_\beta(z) = (w_{s-1}w_{s-2}\cdots w_0)^\infty$ and concludes the proof. □

As Theorem [BS07] shows, the points of the orbit of 1 under the action of T_β play a special role. They have to be treated separately.

Lemma 4.3. *We have either $T_\beta^k(1) = 0$, or $T_\beta^k(1) \in \mathbb{Q}(\beta) \setminus \mathbb{Q}$ (with $T_\beta^k(1) \neq 0$). Moreover, $T_\beta^k(1) \in \Pi_\beta$ if and only if β is a non-simple Parry number (i.e., $m \neq 0$) and $k \geq m$.*

Proof. The T_β transformation preserves $\mathfrak{O}$. Hence $T_\beta^k(1) \in \mathfrak{O}$ for all k. Since $\mathbb{Q}$ is integrally closed, if $T_\beta^k(1) \in \mathbb{Q}$, then $T_\beta^k(1) \in \mathbb{Z}$. Therefore, the only possibility for $T_\beta^k(1) \in \mathbb{Q}$ is $T_\beta^k(1) = 0$. This specifically happens exactly when β is a simple Parry number (i.e., if $m = 0$) and $k \geq n$. We mentioned in Section 2.2 that $d_\beta(T_\beta^k(1)) = S^k(d_\beta(1))$. Therefore, $T_\beta^k(1) \in \Pi_\beta$ if and only if β is a non-simple Parry number and $k \geq m$. According to $d_\beta^*(1) = d_\beta(1) = (t_1 \cdots t_m)(t_{m+1} \cdots t_n)^\infty$, the orbit possesses n elements, m of them having purely periodic beta-expansion. □

Application to the γ function. We use Theorem 4.2 to deduce several conditions for pure periodicity in $\mathbb{Q}(\beta)$. The fact that 0 is an inner point of the complete central tile $\widetilde{\mathcal{T}}$ yields a first sufficient condition for a rational number to have a purely periodic expansion. We can see this property as a generalisation of Corollary 2.10.

Corollary 4.4. *Let β be a Pisot number that satisfies the finiteness property (F). There exist m and v such that for every $x = \frac{N(\beta)^m p}{q} \in \mathbb{Q}$, with $\gcd(N(\beta), q) = 1$, and $0 \leq x \leq v$, then one has $x \in \Pi_\beta^{(r)}$.*

Proof. Let M be the maximum of the $|N(\beta)|_v$, for $v \in \mathcal{S}_\beta$, v non-Archimedean. We have $M < 1$. Therefore, for $x = \frac{N(\beta)^m p}{q} \in \mathbb{Q}$, with $\gcd(N(\beta), q) = 1$ and $x \leq v$, we have $\|x\| \leq \max(v, M^m)$. Since 0 is an inner point of $\widetilde{\mathcal{T}}$ - actually an inner point of $\widetilde{\mathcal{T}}^{(1)}$ by (3.9) - it follows that $(-\phi_\beta(x), x) \in \widetilde{\mathcal{T}}^{(1)} \times [0, 1)$ if m is big enough, and v small enough. □

4.2. **From the topology of the central tile to that of $\Pi_\beta^{(r)}$.** We begin by completing the notation introduced in Sections 3.1 and 3.2.

Notation 4.5. *Recall that $(\beta) = \prod_{i=1}^{\nu} \mathfrak{P}_i^{n_i}$, and that $\mathbb{K}_f$ is the product of the corresponding local fields. Let ϕ_f denote the associated embedding, such that*

$$\phi_\beta(x) = (\phi_\infty(x), \phi_f(x)) \in \mathbb{K}_\infty \times \mathbb{K}_f.$$

We also write $\mathfrak{O}_f = \prod_{i=1}^{\nu} \mathfrak{O}_{\mathfrak{P}_i}$ and $\mathfrak{O}_{(\beta)}$ denotes its reciprocal image by ϕ_f, that is,

$$\mathfrak{O}_{(\beta)} = \{x \in \mathbb{Q}(\beta);\ \forall i,\ 1 \leq i \leq \nu,\ v_{\mathfrak{P}_i}(x) \geq 0\} \supset \mathfrak{O}.$$

There are primes p_i such that $\mathfrak{P}_i \cap \mathbb{Z} = p_i\mathbb{Z}$. We write $\mathbf{k}_f = \prod_{i=1}^{\nu} \mathbb{Q}_{p_i}$ and $\mathfrak{o}_f = \prod_{i=1}^{\nu} \mathbb{Z}_{p_i}$. We also introduce

$$\mathbb{Z}_{(N(\beta))} = \left\{ \frac{p}{q};\ \gcd(q, N(\beta)) = 1 \right\}.$$

If $N(\beta)$ is prime, this notation coincides with the usual one concerning localisation. Let us finally introduce the canonical projections

$$\pi_\infty \colon \mathbb{K}_\beta \to \mathbb{K}_\infty \text{ and } \pi_f \colon \mathbb{K}_\beta \to \mathbb{K}_f.$$

Most of this notation is summarised in the commutative diagrams below:

$$\text{(4.4)} \qquad \begin{array}{ccc} \mathbb{Q}(\beta) & \xrightarrow{\phi_f} & \prod_{i=1}^{\nu} \mathbb{K}_{\mathfrak{P}_i} = \mathbb{K}_f \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle i_2} \\ \mathbb{Q} & \xrightarrow{\phi_f} & \prod_{i=1}^{\nu} \mathbb{Q}_{p_i} = \mathbf{k}_f \end{array} \qquad \begin{array}{ccc} \mathfrak{O}_{(\beta)} & \xrightarrow{\phi_f} & \prod_{i=1}^{\nu} \mathfrak{O}_{\mathfrak{P}_i} = \mathfrak{O}_f \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle i_2} \\ \mathbb{Z}_{(N(\beta))} & \xrightarrow{\phi_f} & \prod_{i=1}^{\nu} \mathbb{Z}_{p_i} = \mathfrak{o}_f \end{array}$$

We want to generalise the idea of the proof of Corollary 2.10. Since we are hereafter interested in the beta-expansion of rational integers, our first goal is to understand how they imbed into $\mathbb{K}_\beta$. The Archimedean embedding is trivial: $\phi_\infty(x) = (x, x, \ldots, x)$.

Notation 4.6 (Diagonal sets). *Let $A \subset \mathbb{R}$. The notation*

$$\Delta_\infty(A) := \{(x, \ldots, x);\ x \in A\} \subset \mathbb{K}_\infty$$

stands for the set of $(r+s-1)$-uples of elements of A whose coordinates are all equal.

By an abuse of language, when A is reduced to a single point $A = \{a\}$, we use the notation $\Delta_\infty(a)$ for the point $\phi_\infty(a, \ldots, a) \in \mathbb{K}_\infty$.

We now need to understand the non-Archimedean embedding of $\mathbb{Z}_{(N(\beta))}$: this is the aim of Lemma 4.7 and Proposition 4.9 below.

Lemma 4.7. *Let V be a non-empty open subset of $\mathbb{Z}_{(N(\beta))}$. Then*

$$\overline{\phi_f(V)} = \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} \quad \textit{and} \quad \overline{\phi_\beta(V)} = \Delta_\infty(\overline{V}) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}.$$

Furthermore, for any non-empty interval I in $[0,1]$, we have

$$\overline{\phi_\beta(I \cap \mathbb{Z}_{(N(\beta))})} = \Delta_\infty(\bar{I}) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}. \tag{4.5}$$

The same results hold if $\mathbb{Z}_{(N(\beta))}$ is replaced by $\mathbb{Q}$, $\mathbb{Q}(\beta)$, or $\mathfrak{O}_{(\beta)}$.

Proof. We only prove the result for $\mathbb{Z}_{(N(\beta))}$, with the other cases being similar. Let V be a non-empty open subset of $\mathbb{Z}_{(N(\beta))}$ and $u \in V$. For $y \in \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}$, there exists a sequence $(x_n)_n$ in $\mathbb{Z}_{(N(\beta))}$ such that $\lim \phi_f(x_n) = y - \phi_f(u)$ (using the fact that ϕ_f is an additive group homomorphism). Let us introduce $\vartheta_n = (1 + N(\beta)^n)^{-1}$. Then $\vartheta_n \in \mathbb{Z}_{(N(\beta))}$, and we have both $\lim \vartheta_n = 0$ and $\lim \phi_f(\vartheta_n) = 1$. Then we can choose a subsequence $(\sigma(n))_n$, such that $u + x_n \vartheta_{\sigma(n)} \in V$ and $\lim \phi_f(u + x_n \vartheta_{\sigma(n)}) = y$. Finally, $\phi_\beta(x_n)$ converges to $(\phi_\infty(u), y)$.

This means that $\Delta_\infty(V) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} \subset \overline{\phi_\beta(V)}$; taking the closure yields $\Delta_\infty(\overline{V}) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} \subset \overline{\phi_\beta(V)}$. We conclude by noting that the definition of ϕ_β directly ensures that $\overline{\phi_\beta(V)} \subset \Delta_\infty(\overline{V}) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}$. Hence we have proved $\overline{\phi_\beta(V)} = \Delta_\infty(\overline{V}) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}$. The equality $\overline{\phi_f(V)} = \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}$ follows by applying the projection π_f.

Equation (4.5) is clearly satisfied if I is open. In general, it follows from

$$\Delta_\infty(\bar{I}) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} = \overline{\phi_\beta(\mathring{I} \cap \mathbb{Z}_{(N(\beta))})} \subset \overline{\phi_\beta(I \cap \mathbb{Z}_{(N(\beta))})} \subset \Delta_\infty(\bar{I}) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}.$$

□

We deduce from Lemma 4.7 and Theorem 4.2:

Corollary 4.8. *Let $0 < \varepsilon \leq \min\{T_\beta^{a-1}(1);\ a \in \{1, \ldots, n\}\}$. Then $\gamma(\beta) \geq \varepsilon$ if $\Delta_\infty([0,\varepsilon]) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} \subset -\widetilde{\mathcal{T}}$.*

Note that there is no reason here for $\phi_\beta(\mathbb{Q})$ to be dense in $\mathbb{K}_\beta$, contrary to what happens for the Archimedean part.

Proposition 4.9. *The following propositions are equivalent:*

1. *Let $0 \leq u < v < \infty$. Then $\overline{\phi_f(\mathbb{Z}_{(N(\beta))} \cap (u,v))} = \mathfrak{O}_f$;*
2. *The set $\phi_f(\mathbb{Q})$ is dense in $\mathbb{K}_f$;*
3. *For all i, $1 \leq i \leq \nu$, we have $e(\mathfrak{P}_i) = f(\mathfrak{P}_i) = 1$, and the prime numbers p_i are all distinct;*
4. *The norm $N(\beta)$ is square-free and none of its prime divisors ramifies.*

Proof. For given i, one has $[\mathbb{K}_{\mathfrak{P}_i} : \mathbb{Q}_{p_i}] = e(\mathfrak{P}_i) f(\mathfrak{P}_i)$. By completeness of the p-adic fields (resp. p-adic rings), the image by i_2 of $\mathbf{k}_f$ (resp. $\mathfrak{o}_f$) in the commutative diagram (4.4) is closed in $\mathbb{K}_f$ (resp. $\mathfrak{O}_f$). Hence, these images are dense if and only if these products are equal, that is if $\mathbb{K}_{\mathfrak{P}_i} = \mathbb{Q}_{p_i}$ for all i, i.e., $e(\mathfrak{P}_i) = f(\mathfrak{P}_i) = 1$.

Moreover, by the Chinese remainder theorem, the image by ϕ_f of $\mathbb{Q}$ (resp. $\mathbb{Z}_{(N(\beta))}$) is dense in $\mathbf{k}_f$ (resp. $\mathfrak{o}$) if and only if the p_i are distinct. Hence we have proved that (2), as (1) for $\kappa = +\infty$, are equivalent to (3). The equivalence with (1) with an arbitrary non-empty open interval (u, v) is given by Lemma 4.7.

Finally, the equivalence of (3) and (4) follows from

$$N(\beta) = N((\beta)) = \prod_{i=1}^{\nu} N(\mathfrak{P}_i) = \prod_{i=1}^{\nu} p_i^{f(\mathfrak{P}_i)}.$$

$\square$

Remark 4.10. If the prime numbers p_i are not distinct, there is a partition of ν, $\nu = \mu_1 + \cdots + \mu_\ell$ and a suitable reordering of the prime ideals $\mathfrak{P}_1, \ldots, \mathfrak{P}_\nu$ containing β, such that one has the equality of multisets

$$\{p_1, p_2, \ldots, p_\nu\} = \{\underbrace{p_1, \ldots, p_1}_{\#\mu_1}, \underbrace{p_2, \ldots, p_2}_{\#\mu_2}, \ldots, \underbrace{p_\ell, \ldots, p_\ell}_{\#\mu_\ell}\}.$$

Then, $\overline{\phi_f(\mathbb{Q})}$ (resp. $\overline{\phi_f(\mathbb{Z}_{(N(\beta))})}$) is equal to $\prod_{j=1}^{\ell} \Delta(\mathbb{Q}_{p_j}^{\mu_j})$ (resp. $\prod_{j=1}^{\ell} \Delta(\mathbb{Z}_{p_j}^{\mu_j})$), where $\Delta(M^\mu)$ denotes the set of μ-uples of elements of M. We will try hereafter to avoid to deal with these p-adic diagonal components for effectiveness issues.

4.3. **Topological properties of $\Pi_\beta^{(r)}$.** Before being able to deduce bounds on $\gamma(\beta)$ from Corollary 4.8, we need to first investigate the topological structure of $\Pi_\beta^{(r)}$.

We already know that $\Pi_\beta^{(r)} \subset \mathbb{Z}_{(N(\beta))}$ by Lemma 4.1. We endow $\Pi_\beta^{(r)}$ with the induced topology of $\mathbb{R}$ on $\mathbb{Z}_{(N(\beta))}$. The following proposition investigates the extremities of the connected components of Π_β. An example of such a component is of course $[0, \gamma(\beta)]$ (or $[0, \gamma(\beta))$). An illustration of Theorem 4.11 is given in Fig. 2.

Theorem 4.11. *Let (u, v) be a non-empty open interval with $(u, v) \cap \mathbb{Z}_{(N(\beta))} \subset \Pi_\beta^{(r)}$.*

If $v \in \mathbb{Z}_{(N(\beta))}$, then $v \in \Pi_\beta^{(r)}$. If the assumptions of Proposition 4.9 are satisfied and $v \in \mathbb{Q}$, then the same conclusion holds, namely $v \in \Pi_\beta^{(r)}$. The same results also hold for u.

If (u, v), as above, is maximal and $v < 1$, then there are three possibilities for v, namely:

(A) *There exists $a \in \mathcal{A}$ such that*

$$\Delta_\infty(v) \in \pi_\infty\left(-\widetilde{\mathcal{T}}^{(a)} \cap \left(\Delta_\infty(T_\beta^{(a-1)}(1)) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}\right)\right).$$

In particular, $v = T_\beta^{(a-1)}(1)$.

(B) *There exist a and b in $\mathcal{A}$ such that*

$$\Delta_\infty(v) \in \pi_\infty\left(-\widetilde{\mathcal{T}}^{(a)} \cap -\widetilde{\mathcal{T}}^{(b)} \cap \left(\Delta_\infty([T_\beta^{(b-1)}(1), T_\beta^{(a-1)}(1))) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}\right)\right).$$

In particular, $T_\beta^{(b-1)}(1) \le v < T_\beta^{(a-1)}(1)$.

(C) *There exist $a \in \mathcal{A}$ and $x \in \mathbb{Z}[1/\beta] \cap (0, 1)$ such that*

$$\Delta_\infty(v) \in \pi_\infty\left(-\widetilde{\mathcal{T}}^{(a)} \cap -\widetilde{\mathcal{T}}(x) \cap \left(\Delta_\infty((0, T_\beta^{(a-1)}(1))) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}\right)\right).$$

In particular, $v < T_\beta^{(a-1)}(1)$.

Cases (B) and (C) are not exclusive of each other. The same results hold for u if (u, v) is maximal and $u > 0$.

Proof. Let (u, v) be a non-empty open interval with $(u, v) \cap \mathbb{Z}_{(N(\beta))} \subset \Pi_\beta^{(r)}$. Assume that $v \in \mathbb{Z}_{(N(\beta))}$. Then, by Lemma 4.7, one can construct a sequence $(z_n)_n$ in (u, v) such that $\lim z_n = v$ and $\lim \phi_f(z_n) = \phi_f(v)$. Furthermore, $\lim z_n = v$ is equivalent to $\lim \phi_\infty(z_n) = \phi_\infty(v)$. Hence, we have $\lim \phi_\beta(z_n) = \phi_\beta(v)$. Moreover, by taking a subsequence, we may assume that for some $a \in \mathcal{A}$, one has $(-\phi_\beta(z_n), z_n) \in \widetilde{\mathcal{T}}^{(a)} \times [0, T_\beta^{(a-1)}(1))$ for all n. Then $(-\phi_\beta(v), v) \in \widetilde{\mathcal{T}}^{(a)} \times [0, T_\beta^{(a-1)}(1)]$. By Lemma 4.3, the assumption $v \in \mathbb{Z}_{(N(\beta))} \subset \mathbb{Q}$ guarantees that $v \neq T_\beta^{(a-1)}(1)$. Therefore, we have that $(-\phi_\beta(v), v) \in \widetilde{\mathcal{T}}^{(a)} \times [0, T_\beta^{(a-1)}(1))$ and $v \in \Pi_\beta$. The same argument applies to $v \in \mathbb{Q}$ under the assumptions of Proposition 4.9. The case of u is similar.

We now assume that the interval (u, v) is maximal and $v \neq 1$. We first claim that there exists a sequence $(y_n)_n$ in $\mathbb{Z}_{(N(\beta))} \setminus \Pi_\beta^{(r)}$ with $\lim y_n = v$. Indeed, by maximality of (u, v), if there exists $w > v$ such that $(u, w) \cap \mathbb{Z}_{(N(\beta))} \subset \Pi_\beta^{(r)}$, this implies that $v \in \mathbb{Z}_{(N(\beta))} \setminus \Pi_\beta^{(r)}$. By the first part of the theorem, this cannot happen, and our claim is proved.

Let us then start with a sequence $(y_n)_n$ with $v < y_n$, $\lim y_n = v$ and $y_n \notin \Pi_\beta$. By compacity, one may assume that $(\phi_\beta(y_n))_n$ converges, to $(\Delta_\infty(v), z)$, say, with $z \in \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}$. By Lemma 4.7, there exists a sequence $(z_n)_n$ with $u < z_n < v$, $\lim z_n = v$ and $\lim \phi_\beta(z_n) = (\Delta_\infty(v), z)$. By extracting a subsequence, we may also assume that there exists $a \in \mathcal{A}$ with $(\phi_\beta(z_n), z_n) \in -\widetilde{\mathcal{T}}^{(a)} \times [0, T_\beta^{(a-1)}(1))$ for all n. The first possibility to take into account is that $v = T_\beta^{(a-1)}(1)$. Since $\widetilde{\mathcal{T}}^{(a)}$ is closed, we then have $(\Delta_\infty(v), z, v) \in -\widetilde{\mathcal{T}}^{(a)} \times \{T_\beta^{(a-1)}(1)\}$. In other words, one gets the possibility (A) of the theorem:

$$\Delta_\infty(v) \in \pi_\infty \left(-\widetilde{\mathcal{T}}^{(a)} \cap \left(\Delta_\infty(T_\beta^{(a-1)}(1)) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} \right) \right).$$

We may assume from now on that $v \neq T_\beta^{(a-1)}(1)$ (which does not mean that v could not be equal to another element of the T_β-orbit of 1). We then have $(\Delta_\infty(v), y) \in -\widetilde{\mathcal{T}}^{(a)} \times [0, T_\beta^{(a-1)}(1))$. Since $y_n \notin \Pi_\beta$, we get $\phi_\beta(y_n) \notin -\widetilde{\mathcal{T}}^{(a)}$. For fixed n, there are two possibilities:

(i) $\phi_\beta(z_n) \in -\widetilde{\mathcal{T}}$. Since $z_n \notin \Pi_\beta$, we have $\phi_\beta(z_n) \in -\widetilde{\mathcal{T}}^{(b)}$ for some $b \in \mathcal{A}$ such that $T_\beta^{(b-1)}(1) \leq v$.

(ii) $\phi_\beta(z_n) \notin -\widetilde{\mathcal{T}}$. Then, by Proposition 3.15, there exists $x_n \in \mathbb{Z}[1/\beta] \cap (0, 1)$ such that $\phi_\beta(z_n) \in -\widetilde{\mathcal{T}}(x_n)$.

At least one of the properties (i) of (ii) has to be satisfied for infinitely many n's.

If this is the case for (i), since $\mathcal{A}$ is finite, there is a b corresponding to a further subsequence of $(z_n)_n$. Taking the limit, we get $\phi_\beta(v) \in -\widetilde{\mathcal{T}}^{(b)}$, and we get thereferore case (B) of the theorem:

$$\Delta_\infty(v) \in \pi_\infty \left(-\widetilde{\mathcal{T}}^{(a)} \cap -\widetilde{\mathcal{T}}^{(b)} \cap \left(\Delta_\infty([T_\beta^{(b-1)}(1), T_\beta^{(a-1)}(1))) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} \right) \right).$$

If there are infinitely many n's satisfying (ii), Proposition 3.15 shows that the family $\{x_n, n \in \mathbb{N}\}$ is finite. Hence, by extracting a subsequence, there is some $x \neq 0$ with $\phi_\beta(z_n) \in -\widetilde{\mathcal{T}}(x)$. Taking the limit, we get case (C):

$$\Delta_\infty(v) \in \pi_\infty \left(-\widetilde{\mathcal{T}}^{(a)} \cap -\widetilde{\mathcal{T}}(x) \cap \left(\Delta_\infty((0, T_\beta^{(a-1)}(1))) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} \right) \right).$$

□

Proposition 4.12. *If the finitess property (F) is satisfied, then* Π_β *is dense in* $\mathbb{Z}_{(N(\beta))}$.

Proof. If the property (F) is satisfied, then $\widetilde{\mathcal{T}}^{(1)}$ contains a neighbourhood of the origin, hence $\overline{\phi_f(\mathrm{Int}(\beta))}$ contains $N(\beta)^m\overline{\phi_f(\mathbb{Z}_{(N(\beta))})}$ for some $m \geq 0$. Then

$$\phi_\beta^{-1}\left(\Delta_\infty([0,1)) \times N(\beta)^m\overline{\phi_f(\mathbb{Z}_{(N(\beta))})}\right) \subset \Pi_\beta,$$

and is dense by Lemma 4.7. □

4.4. **Upper and lower bounds for $\gamma(\beta)$.** We now have collected all the required material to be able to deduce the upper and lower bounds for $\gamma(\beta)$. The present section collects results that may be of some interest in every dimension, whereas Section 4.5 is devoted to the quadratic case.

A first upper bound for $\gamma(\beta)$ can be directly deduced from Theorem 4.2. We consider the intersection between the complete central subtiles and the set of points whose canonical Archimedean projection by π_∞ belong to the diagonal sets of the form $\Delta_\infty([0, T_\beta^{(a-1)}(1)))$.

Proposition 4.13. *Let β be a Pisot number. One has:*

$$\gamma(\beta) \leq \max\left\{T_\beta^{(a-1)}(1);\ a \in \mathcal{A},\ (-\widetilde{\mathcal{T}}^{(a)}) \cap \pi_\infty^{-1}\Delta_\infty([0, T_\beta^{(a-1)}(1))) \neq \emptyset\right\}.$$

Proof. Let $x \in \mathbb{Q} \cap [0,1)$. If $(\phi_\beta(x), x)$ belongs to $\bigcup_{a\in\mathcal{A}}(-\widetilde{\mathcal{T}}^{(a)}) \times [0, T_\beta^{a-1}(1))$, then there exists $a \in \mathcal{A}$ such that $\pi_\infty \circ \phi_\beta(x) \in -\widetilde{\mathcal{T}}^{(a)} \cap \Delta_\infty([0, T_\beta^{(a-1)}(1))$. Hence if

$$x > \max\left\{T_\beta^{a-1}(1);\ a \in \mathcal{A},\ (-\widetilde{\mathcal{T}}^{(a)}) \cap \pi_\infty^{-1}\Delta_\infty([0, T_\beta^{(a-1)}(1))) \neq \emptyset\right\},$$

then $(\phi_\beta(x), x)$ does not belong to $\bigcup_{a\in\mathcal{A}}(-\widetilde{\mathcal{T}}^{(a)}) \times [0, T_\beta^{a-1}(1))$. We deduce from Theorem 4.2 that its β-expansion is not purely periodic. □

Let us stress that this upper bound is quite rough: if the finiteness property (F) is satisfied, then the inequality yields the trivial bound $\gamma(\beta) \leq 1$. Indeed Proposition 3.17 says that $\widetilde{\mathcal{T}}^{(1)}$ contains a neighbourhood of the origin. Hence the intersection $(-\widetilde{\mathcal{T}}^{(1)}) \cap \pi_\infty^{-1}\Delta_\infty([0, T_\beta(1)))$ is not empty, which yields $\gamma(\beta) \leq 1$.

However, Theorem 4.2 states that real numbers have a purely periodic expansion if their embedding is included in the representation $\bigcup_{a\in\mathcal{A}}(-\widetilde{\mathcal{T}}^{(a)}) \times [0, T_\beta^{a-1}(1))$ of the natural extension of T_β. From Lemma 4.7, we know that an interval of rationals $(\eta, \nu) \cap \mathbb{Z}_{(N(\beta))}$ embeds in $\mathbb{K}_\beta$ as the product of a diagonal set with a local part whose closure is independant of (η, ν). We deduce below a recursive characterisation for $\gamma(\beta)$.

Notation 4.14. *Let us order and relabel the elements in $\mathcal{A}$ as follows: we set $\mathcal{A} = \{a_1, \ldots, a_n\}$ with*

$$T_\beta^{a_1-1}(1) < T_\beta^{a_2-1}(1) < \cdots < T_\beta^{a_{n-1}-1}(1) < T_\beta^{a_n-1}(1) = 1.$$

Clearly, $a_n = 1$. For notational convenience, we state $T_\beta^{a_0-1}(1) = 0$.

Proposition 4.15. *Let β be a Pisot number.*

- $\gamma(\beta) \geq T_\beta^{a_k-1}(1)$ *if and only if:*

$$\gamma(\beta) \geq T_\beta^{a_{k-1}-1}(1) \quad \textit{and} \quad \Delta_\infty([T_\beta^{a_{k-1}-1}(1), T_\beta^{a_k-1}(1)]) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} \subset \bigcup_{j=k}^{n}(-\widetilde{\mathcal{T}}^{(a_j)}).$$

- *If $T_\beta^{a_{k-1}-1}(1) < \gamma(\beta) \le T_\beta^{a_k-1}(1)$, then*

$$\gamma(\beta) = \sup\left\{ \eta \ge T_\beta^{a_{k-1}-1}(1);\ \Delta_\infty([T_\beta^{a_{k-1}-1}(1), \eta]) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} \subset \bigcup_{j=k}^{n} (-\widetilde{\mathcal{T}}^{(a_j)}) \right\}. \tag{4.6}$$

In particular, if $\widetilde{\mathcal{T}}$ does not contain $\Delta_\infty([0,\eta]) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}$ for any positive η, then $\gamma(\beta) = 0$.

Proof. Let I be a non-empty open interval in $[0,1]$. By Lemma 4.7, $I \cap \mathbb{Z}_{(N(\beta))} \subset \Pi_\beta$ if and only if

$$\Delta_\infty(\bar{I}) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} \subset \bigcup_{j=k}^{n} (-\widetilde{\mathcal{T}}^{(a_j)}).$$

Equation (4.6) follows from (4.5) and Theorem 4.2 too. The last assertion is a particular case of (4.6) when $k = 1$ and can be deduced from the observation that $\phi_f(\mathbb{Z}_{(N(\beta))}) = -\phi_f(\mathbb{Z}_{(N(\beta))})$. □

This result has a geometric interpretation related to the natural extension of T_β. Let Δ denote the diagonal line in $\mathbb{K}_\infty \times \mathbb{R}$, i.e., the Euclidean component of the natural extension. Proposition 4.15 means that $\gamma(\beta)$ is the largest part of Δ starting from 0 such that its product with the full non-Archimedean component $\overline{\phi_f(\mathbb{Z}_{(N(\beta))})}$ is totally included in the natural extension $\bigcup_{a\in\mathcal{A}}(-\widetilde{\mathcal{T}}^{(a)}) \times [0, T_\beta^{a-1}(1))$.

In the unit case, since the representation contains only Archimedean components, Proposition 4.15 simply means that $\gamma(\beta)$ is the length of the largest diagonal interval that is fully included in the natural extension (see an illustration in Fig. 2).

Theorem 4.11 yields lower and upper bounds for $\gamma(\beta)$.

Proposition 4.16. *We introduce some local notation. For a and b in $\mathcal{A}$ such that $T_\beta^{(b-1)}(1) \le T_\beta^{(a-1)}(1)$, let*

$$A_{a,b} = \pi_\infty\left(-\widetilde{\mathcal{T}}^{(a)} \cap -\widetilde{\mathcal{T}}^{(b)} \cap \left(\Delta_\infty([T_\beta^{(b-1)}(1), T_\beta^{(a-1)}(1)]) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}\right)\right) \subset \mathbb{K}_\infty.$$

For $a \in \mathcal{A}$ and $x \in \mathbb{Z}[1/\beta]$, let

$$B_{a,x} = \pi_\infty\left(-\widetilde{\mathcal{T}}^{(a)} \cap -\widetilde{\mathcal{T}}(x) \cap \left(\Delta_\infty((0, T_\beta^{(a-1)}(1))) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}\right)\right) \subset \mathbb{K}_\infty.$$

Finally, let

$$A = \bigcup_{\substack{(a,b)\in\mathcal{A}^2 \\ T_\beta^{(b-1)}(1) \le T_\beta^{(a-1)}(1)}} A_{a,b} \quad \text{and} \quad B = \bigcup_{\substack{a\in\mathcal{A} \\ x\in\mathbb{Z}[1/\beta]\cap(0,1)}} B_{a,x}.$$

Then, a lower bound for $\gamma(\beta)$ is given by

$$\gamma(\beta) \ge \min\left(\min_{\substack{(a,b)\in\mathcal{A}^2 \\ T_\beta^{(b-1)}(1) \le T_\beta^{(a-1)}(1) \\ A_{a,b}\ne\emptyset}} \min_{x\in A_{a,b}} \|\pi_\infty(x)\|_\infty,\ \min_{\substack{a\in\mathcal{A} \\ x\in\mathbb{Z}[1/\beta]\cap(0,1) \\ B_{a,x}\ne\emptyset}} \inf_{x\in B_{a,x}} \|\pi_\infty(x)\|_\infty \right). \tag{4.7}$$

An upper bound for $\gamma(\beta)$ is as follows:

$$\gamma(\beta) \le \max\{\eta;\ [0,\eta] \subset A \cup B\}. \tag{4.8}$$

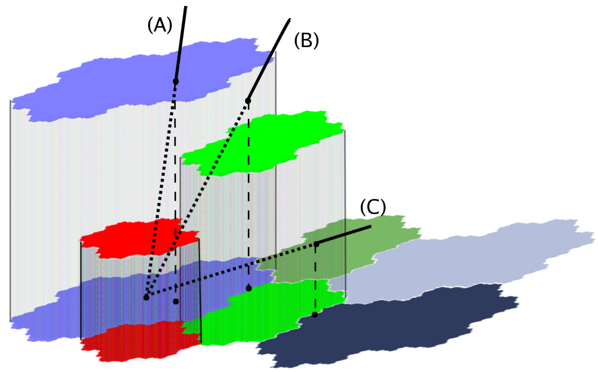

Figure 2. Illustration of the three cases of Theorem 4.11 and Proposition 4.15. We have chosen a unit Pisot number for the illustration of these three cases for the sake of clarity. By Proposition 4.15, $\gamma(\beta)$ is given by the largest part of the diagonal line to be fully included in the natural extension $\bigcup_{a\in\mathcal{A}}(-\widetilde{\mathcal{T}}^{(a)}) \times [0, T_\beta^{(a-1)}(1))$. The natural extension is represented with subtiles $-\widetilde{\mathcal{T}}^{(a)}$ in the horizontal direction, and the interval $[0,1)$ on the vertical axis. Then, the natural extension involves a union of cylinders with fractal horizontal base and vertical height. The height of the cylinder with basis $-\widetilde{\mathcal{T}}^{(a)}$ is $T_\beta^{(a-1)}(1)$. Depending on the location of the point where the diagonal first goes out from the natural extension, we recover the different situations which are highlighted in Theorem 4.11.
Situation (A) means that both $\gamma(\beta)$ belongs to the orbit of 1 under the action of T_β and that its Euclidean embedding $\phi_\infty(\gamma(\beta))$ is the Euclidean part of a point of the corresponding subtile. Then the diagonal starts from 0 and exits from the natural extension on a plateau with height $T_\beta^{(a-1)}(1)$.
Situation (B) involves the intersection between two complete central subtiles $(-\widetilde{\mathcal{T}}^{(a)}) \cap (-\widetilde{\mathcal{T}}^{(b)})$. The diagonal line goes from the natural extension on a vertical line above the intersection between two subtiles. The main point is that the plateau of the lowest cylinder $(T^{(b-1)}(1))$ lies below the diagonal line whereas the plateau of the upper cylinder $(T^{(a-1)}(1))$ lies above it.
Situation (C) means that the diagonal line completely crosses the natural extension and exits above a new x-tile.

Proof. First note that the infimum in (4.7) is due to the fact that $B_{a,x}$ does not need to be compact. We use Theorem 4.11 and the fact, that, by definition, $\gamma(\beta)$ is the largest number $\daleth$ such that $(0,\daleth) \cap \mathbb{Z}_{(N(\beta))} \subset \Pi_\beta^{(r)}$. Situation (A) in Theorem 4.11 implies that there exists $a \in \mathcal{A}$ such that $\gamma(\beta) = T_\beta^{(a-1)}(1)$ and $\gamma(\beta) \in \pi_\infty(-\widetilde{\mathcal{T}}^{(a)})$; it follows that $\gamma(\beta) \in A_{a,a}$. Situation (B) implies that there exist $a, b \in \mathcal{A}$ with $T_\beta^{b-1}(1) < T_\beta^{a-1}(a)$ such that $\gamma(\beta) \in A_{a,b}$. However, the interval is closed in the present proposition, and it is half-closed in Theorem 4.11. Nevertheless, by continuity of π_∞, taking open or closed intervals in $B_{a,x}$ or $A_{a,b}$ has no influence on the infimum we are interested in. From Situation (C) it follows that there exist $a \in \mathcal{A}$ and $x \in \mathbb{Z}[1/\beta] \cap (0,1)$ such that $\gamma(\beta) \in B_{a,x}$. Since one of the three situations must occur, we deduce that $\gamma(\beta)$ is greater than the smallest of the infimum of all these sets. Formulas (4.8) hold for the same reasons.

We recall that the three cases are illustrated in Fig. 2. □

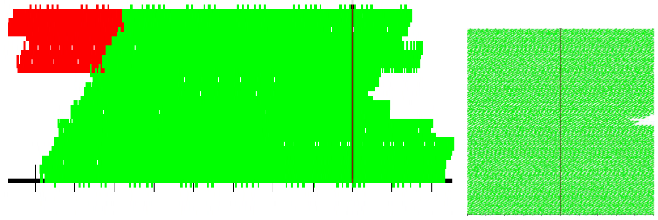

FIGURE 3. A representation of the complete central tile for $\beta = 5 + 2\sqrt{7}$. Then β has the minimum polynomial $X^2 - 10X - 3$ and $N((\beta)) = -3$. The quadratic field $\mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{7})$ has discriminant 28. By $\left(\frac{28}{3}\right) = \left(\frac{1}{3}\right) = 1$, the discriminant is a quadratic residue modulo 3 and Corollary 4.17 shows that the complete central tile is a subset of $\mathbb{R} \times \mathbb{Z}_3$. The vertical axis stands for a representation of $\mathbb{Z}_3$ as embedded in $[0,1)$. The horizontal axis stands for the real line. Since $d_\beta(1) = 1030^\infty$, there are two complete central subtiles (grey and lightgrey). The right figure depicts a zoom along the vertical axis. This zoom seems to suggest that the inverse $-\widetilde{\mathcal{T}}$ of the complete central tile contains a full stripe of the form $[0,\varepsilon] \times \mathbb{Z}_3$, yielding $\gamma(\beta) > 0$.

4.5. **Quadratic Pisot numbers.** Let us now consider the particular case of quadratic Pisot numbers of degree 2, for which many things can be done in an effective way. For instance, $\mathbb{Q}(\beta)$ is an extension of degree two, and then the algebraic conditions (3) or (4) of Proposition 4.9 can be easily tested. Indeed, let d be the square-free positive rational integer such that $\mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{d})$. Then the discriminant $\delta_{\mathbb{Q}(\beta)}$ of the quadratic field is d if $d \equiv 1 \pmod 4$ and $4d$ if $d \equiv 2, 3 \pmod 4$.

Corollary 4.17. *If $\beta^2 = a\beta + b$, with $(a,b) \in \mathbb{Z}^2$, $b \neq 0$, the equivalent conditions of Proposition 4.9 are satisfied if and only if:*

(1) *b is square free,*
(2) *b is coprime with $\delta_{\mathbb{Q}(\beta)}$,*
(3) *d is a quadratic residue with respect to all odd prime divisors of b,*
(4) *$d \equiv 1 \pmod 8$ if b is even.*

The Euclidean representation space $\mathbb{K}_\infty$ is a one-dimensional line. Consequently, the diagonal $\Delta_\infty([0,\varepsilon])$ is indeed the interval $[0,\varepsilon] \subset \mathbb{K}_\infty = \mathbb{R}$. This allows us to use graphical representation of the complete central tile to conjecture lower bounds for $\gamma(\beta)$ as illustrated by Fig. 3 and Fig. 4.

A particularly manageable case is as follows: $(\beta) = \beta\mathfrak{O}$ is a prime ideal overlying a prime number p that splits. Hence (β) has inertia degree 1, we have $N((\beta)) = |N(\beta)| = p$, and $\mathbb{K}_\beta = \mathbb{R} \times \mathbb{Q}_p$ (which is a special case of Corollary 4.17). We can represent $\mathbb{Z}_p$ by the Mona map defined for $x \in \mathbb{Z}_p$ as $x = \sum a_i p^i \mapsto \sum a_i p^{-i} \in [0,1]$. This mapping is onto, continuous and preserves the Haar measure, but is it not a morphism for the addition. Corollary 4.8 implies that $\gamma(\beta) \geq \varepsilon$ if and only if a stripe of length ε is totally included in the representation of the central tile, as depicted on Fig. 3 and Fig. 4 below.

According to [FS92] (Corollary, Theorem 2 and Lemma 3), quadratic Pisot numbers β are exactly the dominant roots of the polynomials $X^2 - aX - b$ with $a \geq b \geq 1$ or $-a + 2 \leq b \leq -1$.

The first family of Pisot numbers (the case $a \geq b \geq 1$) consists of simple Parry numbers which satisfy the finiteness property (F). Consequently, we may apply them Theorem 3.18, and intersections between complete x-tiles determine their boundary, which have zero measure. The same property holds for the subtiles. We then use the fact that inner points of x-tiles and subtiles are

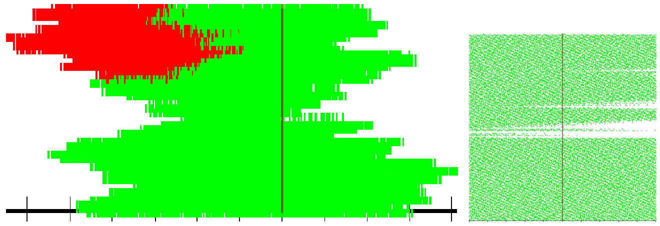

FIGURE 4. A representation of the complete central tile for the Pisot number satisfying $\beta^2 = 4\beta+3$. As in the previous case, we have $\mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{7})$ and $N((\beta)) = -3$. Thus the complete central tile is again a subset of $\mathbb{R}\times\mathbb{Z}_3$. Since $d_\beta(1) = 430^\infty$, there are two complete central subtiles (grey and lightgrey). The zoom suggests that $-\widetilde{\mathcal{T}}$ contains no stripe of the form $[0,\varepsilon] \times \mathbb{Z}_3$, so that $\gamma(\beta) = 0$.

exclusive to deduce an explicit formula for $\gamma(\beta)$. As we will see in Section 5, $\gamma(\beta)$ can be zero in that case, as it also can be non-zero.

The numbers β issued from the second family ($-a+2 \le b \le -1$) satisfy a weaker finiteness property that (F), called (F_2) in [FS92]: all elements of $\mathbb{N}[1/\beta]$ have a finite beta-expansion. However, no one satisfies (F) itself. These are non-simple Parry numbers, since $d_\beta(1) = (a-1)(a+B-1)^\infty$ and verify $\gamma(\beta) = 0$.

Theorem 4.18. *If β is the dominant root of $X^2 - aX - b$ with $a \ge b \ge 1$, then $\gamma(\beta)$ is given by Formula* (4.7)*, which in this case is an equality.*

Proof. First recall that since $\mathbb{K}_\infty$ is one-dimensional, one has $\Delta(x) = x$ for all $x \in [0,1]$. We use the notation introduced in Proposition 4.16. We have to show that the lower bound is an upper bound too. We will show the following:
If $x \in \mathbb{Z}[1/\beta] \cap (0,1)$ with $-\widetilde{\mathcal{T}}(x) \cap (0,1) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} \neq \emptyset$, then

$$\gamma(\beta) \le \inf\left\{\pi_\infty\left(-\widetilde{\mathcal{T}}(x) \cap (0,1) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}\right)\right\}. \tag{4.9a}$$

If $a \in \mathcal{A}$ with $-\widetilde{\mathcal{T}}^{(a)} \cap [T_\beta^{(a-1)}, 1) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} \neq \emptyset$, then

$$\gamma(\beta) \le \inf\left\{\pi_\infty\left(-\widetilde{\mathcal{T}}^{(a)} \cap (T_\beta^{(a-1)}, 1) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}\right)\right\}. \tag{4.9b}$$

Since $-\widetilde{\mathcal{T}}(x) \cap (0,1) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} \supset B_{a,x}$ for every $a \in \mathcal{A}$ and $-\widetilde{\mathcal{T}}^{(a)} \cap [T_\beta^{(a-1)}, 1) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})} \supset A_{a,b}$ for every b with $T_\beta^{(b-1)}(1) \le T_\beta^{(a-1)}(1)$, the theorem will follow from (4.9). Note that by continuity of π_∞, taking open or closed intervals in $B_{a,x}$ or $A_{a,b}$ has no influence on the infimum we are interested in.

We begin with (4.9a). Let $x \in \mathbb{Z}[1/\beta] \cap (0,1)$. Let $z \in -\widetilde{\mathcal{T}}(x) \cap (0,1) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}$. Since β has degree 2, the property (F) is satisfied, and $\widetilde{\mathcal{T}}(x)$ is the closure of its subset of exclusive inner points by Proposition 3.17.

Let us fix $\varepsilon > 0$. There exists an exclusive inner point $y \in -\widetilde{\mathcal{T}}(x) \setminus (-\widetilde{\mathcal{T}})$ such that $\|y - z\| \le \varepsilon/2$. Since y is an inner point and all inner points are exclusive, there exists $\nu < \varepsilon/2$ such that the ball $B(y,\nu)$ is contained in $-\widetilde{\mathcal{T}}(x) \setminus (-\widetilde{\mathcal{T}})$. By Lemma 4.7, the set $\phi_\beta((\pi_\infty(y) - \nu, \pi_\infty(y) + \nu) \cap \mathbb{Z}_{(N(\beta))})$ is dense in $[\pi_\infty(y) - \nu, \pi_\infty(y) + \nu] \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}$. Therefore, it intersects $B(y,\nu)$, and there exists $w \in (\pi_\infty(y) - \nu, \pi_\infty(y) + \nu) \cap \mathbb{Z}_{(N(\beta))}$ such that $\phi_\beta(w) \in -\widetilde{\mathcal{T}}(x) \setminus (-\widetilde{\mathcal{T}})$. For $w \le \pi_\infty(z) + \varepsilon$, we know by Theorem 4.2 that the β-expansion of w is not purely periodic. Hence $\gamma(\beta) \le \pi_\infty(z) + \varepsilon$.

Finally, $\gamma(\beta) \leq \pi_\infty(z)$ and (4.9a) is proved.

The proof for the upper bound (4.9b) follows the same lines. Let $z \in -\widetilde{\mathcal{T}}^{(a)} \cap (T_\beta^{(a-1)}, 1) \times \overline{\phi_f(\mathbb{Z}_{(N(\beta))})}$. Then $\widetilde{\mathcal{T}}^{(a)}$ is the closure of its set of exclusive inner points (with respect to $\widetilde{\mathcal{T}}^{(b)}$, $b \neq a$). For $\varepsilon > 0$, there exists an exclusive inner point y and $\nu > 0$ such that $B(y, \nu) \subset -\widetilde{\mathcal{T}}^{(a)} \setminus \bigcup_{b \in \mathcal{A}\setminus\{a\}} \widetilde{\mathcal{T}}^{(b)}$ and $(\pi_\infty(y) - \nu, \pi_\infty(y) + \nu) \subset (T_\beta^{(a-1)}, 1)$ (this second condition is the reason for which we take an open interval in (4.9b)). By Lemma 4.7, there exists $w \in (\pi_\infty(y) - \nu, \pi_\infty(y) + \nu) \cap \mathbb{Z}_{(N(\beta))}$ such that $\phi_\beta(w) \in -\widetilde{\mathcal{T}}^{(a)} \setminus \bigcup_{b \in \mathcal{A}\setminus\{a\}} \widetilde{\mathcal{T}}^{(b)}$. Since $\pi_\infty(w) > T_\beta^{(a-1)}$, $w \notin \Pi_\beta^{(r)}$. Therefore, $\gamma(\beta) \leq \pi_\infty(z) + \varepsilon$. Finally, $\gamma(\beta) \leq \pi_\infty(z)$ and (4.9b) is proved. $\square$

Suppose that the degree of β is larger than 2. We know that $\pi_\infty(\phi_\beta(\mathbb{Q} \cap [0,1])) \subset \Delta_\infty([0,1])$. However, the diagonal set $\Delta_\infty([0,\infty))$ has an empty interior in $\mathbb{K}_\infty$. Consequently, it may happen that $\pi_\infty(-\widetilde{\mathcal{T}}(x))$ is tangent to the diagonal $\Delta_\infty([0,\infty))$; in this latter case, $-\widetilde{\mathcal{T}}(x)$ provides no point with a non-periodic beta-expansion and the conclusion of Theorem 4.18 may fail.

5. Two quadratic examples

In the previous section, we proved that $\gamma(\beta)$ is deeply related with the intersections between subtiles and x-tiles. In this section, we use two examples to describe how $\gamma(\beta)$ can be explicitly computed. To achieve this task, we will use the boundary graph defined in Section 3.4. In Corollary 3.13, we have proved that the boundary graph can be computed by using the three conditions (N1), (N3) and (N4). Conditions (N1) and (N4) are simple numerical conditions. On the contrary, a structure theorem for the ring $\mathfrak{O}$ is required to verify Condition (N3): we need to find an explicit basis of $\mathfrak{O} \cap \mathbb{Z}[1/\beta]$. We thus introduce below a sufficient condition that reduces $\mathfrak{O} \cap \mathbb{Z}[1/\beta]$ to $\mathbb{Z}[\beta]$.

Lemma 5.1. *Let β be such that $\beta\mathfrak{O}$ has only divisors of degree 1, and with inertia degree 1. Let $x \in \mathbb{Z}[1/\beta]$. For any $k \in \mathbb{N}$, if $\beta^k x \in \mathfrak{O}$, then $\beta^k x \in \mathbb{Z}[\beta]$. In particular, $\mathfrak{O} \cap \mathbb{Z}[1/\beta] = \mathbb{Z}[\beta]$.*

Proof. We fix $k \in \mathbb{N}$. Let us expand x as $x = a_{d-1}\beta^{d-1} + \cdots + a_0 + \cdots + a_{-N}\beta^{-N}$, with $a_i \in \mathbb{Z}$ (it is not the β-expansion). If $N > k$, then $\beta^N x = \beta^{N-k}(\beta^k x) \in \beta^{N-k}\mathfrak{O}$. We deduce that $a_{-N} \in \beta^{N-k}\mathfrak{O} + \beta\mathbb{Z}[\beta] \subset \beta\mathfrak{O}$. Hence $a_{-N} \in \beta\mathfrak{O} \cap \mathbb{Z}$. Since $\beta\mathfrak{O}$ has only divisors of degree 1 and with inertia degree 1, $N(\beta)$ divides a_{-N}. From $N(\beta)/\beta \in \mathbb{Z}[\beta]$, we deduce that $a_{-N}/\beta \in \mathbb{Z}[\beta]$. Then x admits an expansion of size at most β^{-N+1}: $x = b_{d-1}\beta^{d-1} + \cdots b_0 + \cdots + b_{-N+1}\beta^{-N+1}$. We conclude by induction that $\beta^k x \in \mathbb{Z}[\beta]$. $\square$

Let us stress the fact that Lemma 5.1 if β is a quadratic number that satisfies the conditions of Proposition 4.9. In this case, Corollary 3.13 reads as follows to compute the boundary graph.

Corollary 5.2. *Suppose that β is a quadratic number such that $\beta\mathfrak{O}$ has only divisors of degree 1 and inertia degree 1. Let $\beta^2 = a\beta + b$ be its minimal polynomial. We assume that $a \geq b \geq 1$. The boundary graph of β can be explicitly computed as follows.*

1. *Consider all triplets $[a, x, b]$ such that $x = K + L\beta$, $(K, L) \in \mathbb{Z}^2$, with*
   - $K \leq \frac{\beta - a + 3a\beta - \beta^2 - a^2}{(2\beta - a)(1 + a - \beta)}$ *and* $L \leq \frac{1 + 2a - \beta}{(2\beta - a)(1 + a - \beta)}$.
   - $-T_\beta^{(a-1)}(1) < x < T_\beta^{(b-1)}(1)$ *and $a \neq b$ if $x = 0$.*
2. *Put an edge between two triplets $[a, x, b]$ and $[a_1, x_1, b_1]$ if there exist q_1 and p_1 such that*
   - $x_1 = \beta^{-1}(x + q_1 - p_1)$,
   - $a_1 \xrightarrow{p_1} a$ *and* $b_1 \xrightarrow{q_1} b$ *are edges of the admissibility graph.*
3. *Recursively remove edges that have no outgoing edge.*

Proof. We deduce from the proof of Corollary 3.13 that it is sufficient to exhibit a set that contains all triplets $[a, x, b]$ satisfying conditions (N1), (N3) and (N4). Then the recursive deletion of edges reduces the graph to the exact boundary graph. In this case, condition (N3) implies that $x \in \mathbb{Z}[\beta]$. We thus are looking for all x of the form$x = K + L\beta$, with $K, L \in \mathbb{Z}$ such that conditions (N1) and (N4) are satisfied. Let $\beta_2 = a - \beta$ denote the conjugate of β and $x_2 = K + L\beta_2$ denote the conjugate of x. We obtain $K = (-\beta_2 x + \beta x_2)/(\beta - \beta_2)$ and $L = (x - x_2)/(\beta - \beta_2)$. Condition (N1) implies that $|x| \leq 1$, and condition (N4) implies that $|x_2| \leq \frac{\lfloor\beta\rfloor}{1-|\beta_2|} = \frac{a}{1+a-\beta}$. Indeed, one has $\lfloor\beta\rfloor = a$, since $d_\beta\beta = ab$, and $\beta_2 < 0$. We deduce that if $[a, x, b]$ satisfies the three conditions (N1), (N3) and (N4), then $x = K + L\beta$ with $K \leq \frac{\beta - a + 3a\beta - \beta^2 - a^2}{(2\beta - a)(1+a-\beta)}$ and $L \leq \frac{1+2a-\beta}{(2\beta-a)(1+a-\beta)}$. □

When $\beta^2 = 4\beta + 3$, the bounds are $K \leq 11$ and $L \leq 3$. We deduce that the boundary graph contains eighteen nodes (Fig. 5). If $[a, x, b]$ is a node of the boundary graph, we have $x \in \pm\{0, \beta - 4, 5 - \beta, 2\beta - 10, 2\beta - 9\}$. When $\beta^2 = 10\beta + 3$, the bounds are $K \leq 14$ and $L \leq 2$. The boundary graph contains eight nodes and is depicted in Fig. 6. If $[a, x, b]$ is a node of the boundary graph, we have $x \in \pm\{0, 11 - \beta, \beta - 10\}$.

Proposition 5.3. *Let $\beta > 1$ defined by $\beta^2 = 4\beta + 3$. There are nine non-empty intersections between the central subtiles and the neighbouring x-tiles, namely $\widetilde{\mathcal{T}}^{(1)} \cap \widetilde{\mathcal{T}}^{(2)}$, $\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(2\beta - 9))$, $\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(2)} + \phi_\beta(2\beta - 9))$, $\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(\beta - 4))$, $\widetilde{\mathcal{T}}^{(2)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(\beta - 4))$, $\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(5 - \beta))$, $\widetilde{\mathcal{T}}^{(2)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(5 - \beta))$, $\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(2)} + \phi_\beta(5 - \beta))$ $\widetilde{\mathcal{T}}^{(2)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(10 - 2\beta))$.*

Proof. Suitable intersections correspond to nodes of the graph of the form $[a, x, b]$ with $x \in [0, T_\beta^{(b-1)}(1))$ and $a < b$ if $x = 0$. There are nine such nodes in the graph: $[1, 0, 2]$, $[1, 2\beta - 9, 1]$, $[1, 2\beta - 9, 2]$, $[1, \beta - 4, 1]$, $[2, \beta - 4, 1]$, $[1, -\beta + 5, 1]$, $[1, -\beta + 5, 2]$, $[2, -\beta + 5, 1]$, $[2, -2\beta + 10, 1]$. □

We obtain another graph for $\beta^2 = 10\beta + 3$.

Proposition 5.4. *Let $\beta > 1$ be defined by $\beta^2 = 10\beta + 3$. There are exactly four non-empty intersections between the central subtiles and x-tiles, namely $\widetilde{\mathcal{T}}^{(1)} \cap \widetilde{\mathcal{T}}^{(2)}$, $\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(\beta - 10))$, $\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(-\beta + 11))$, $\widetilde{\mathcal{T}}^{(2)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(-\beta + 11))$.*

Proof. In the boundary graph, nodes $[a, x, b]$ that satisfy the condition $x \in [0, T_\beta^{(b-1)}(1)[$, with $a < b$ if $x = 0$, are $[1, 0, 2]$, $[1, \beta - 10, 1]$, $[1, 11 - \beta, 1]$ and $[2, 11 - \beta, 1]$. □

We now have the tools to compute $\gamma(\beta)$ in these specific cases.

Lemma 5.5. *Let $\beta^2 = 4\beta + 3$. Let β_2 stand for the algebraic conjugate of β. We recall that π_∞ stands for the projection from $\mathbb{K}_\infty$ to $\mathbb{R}$. Then*

$$\left\{\sum a_i(\beta_2)^i \mid a_{2i} \in \{0, 1, 2\}, a_{2i+1} \in \{2, 3, 4\}\right\} \subset \pi_\infty\left(\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(\beta - 4))\right).$$

Proof. We construct as follows the graph depicted in Fig. 7 to describe the expansions of points lying at the intersection between $\widetilde{\mathcal{T}}^{(1)}$ and $\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(\beta - 4)$. By construction, any point of the intersection $\widetilde{\mathcal{T}}^{(1)} \cap \left(\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(\beta - 4)\right)$ can be expanded as $z = \sum_{i \geq 0} p_i \phi_\beta(\beta^i)$, where (p_i) is the first coordinate of the labelling of a path starting from the node $[1, \beta - 4, 1]$ in the boundary graph shown in Fig. 5 These paths are exhibited in the projection graph depicted in Fig. 7. One checks that there exists a subgraph which produces paths whose labels alternatively belong to the digit sets $\{0, 1, 2\}$ and $\{2, 3, 4\}$, which yields:

$$\left\{\sum_{i \geq 0}\{0, 1, 2\}(\beta_2)^{2i} + \sum_{i \geq 0}\{2, 3, 4\}(\beta_2)^{2i+1}\right\} \subset \pi_\infty(\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(\beta - 4))).$$

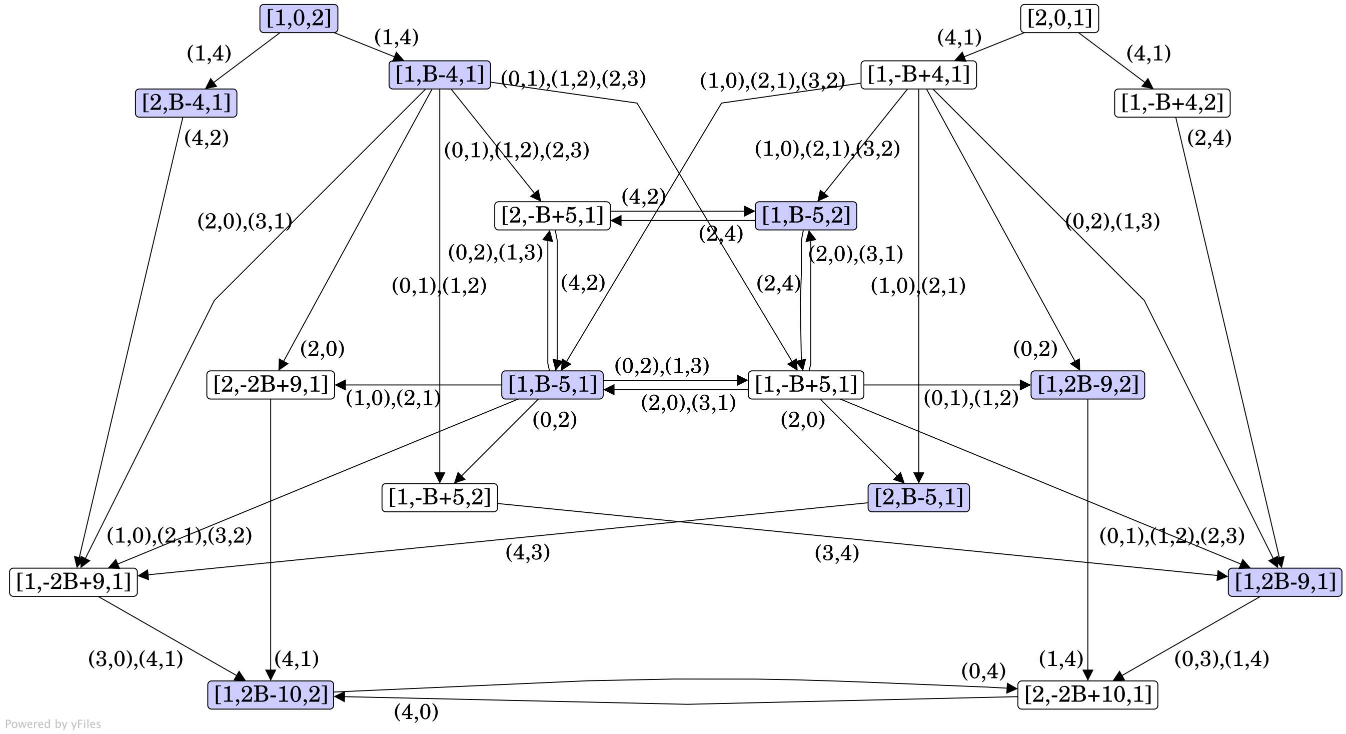

FIGURE 5. Boundary graph for $\beta^2 = 4\beta + 3$. Notation B stands for β. Grey nodes correspond to intersections between a central subtile and a neighbouring x-tile, that is, intersections that contribute to the computation of $\gamma(\beta)$ in Proposition 4.16.

□

In order to compute $\gamma(\beta)$, we use the following folklore lemma.

Lemma 5.6 (Cantor Cookie-cutter set Lemma)**.** *Let $\alpha < 1$ be an algebraic integer, and let*

$$X(\alpha, n) := \left\{ \sum_{i \geq 0} a_i \alpha^i \,\middle|\, a_i \in \{0, 1, \ldots, n-1\} \right\} \subset \left[0, \frac{n-1}{1-\alpha}\right].$$

The two end points $\{0, \frac{n-1}{1-\alpha}\}$ belong to $X(\alpha, n)$. Furthermore, if $\alpha > 1/n$, then it is a Cantor cookie-cutter set and if $\alpha \in [1/n, 1)$, then $X(\alpha, n)$ coincides with the interval $[0, (n-1)/(1-\alpha)]$.

Proof. The set $X(\alpha, n)$ is the attractor of the IFS: $X = \bigcup_{i=0}^{n-1} \alpha X + i$ which has a unique non-empty compact solution. It is easy to see that the right-hand side is a solution if $\alpha \in [1/n, 1)$. For more details, see e.g. [Bed91]. □

Theorem 5.7. *One has*

$$\gamma(2 + \sqrt{7}) = 0.$$

Proof. The Pisot number β satisfies $\beta^2 = 4\beta + 3$, hence $\beta = 2 + \sqrt{7}$. We also check that β satisfies the conditions of Corollary 4.17, hence $\overline{\phi_f(\mathbb{Z}_{(N(\beta))})} = \mathbb{Z}_3$. In this case, the sets $A_{a,b}$ and $B_{a,x}$ in Proposition 4.16 simply correspond to intersections between tiles, with no more diagonal set: $A_{a,b} = \pi_\infty(-\widetilde{\mathcal{T}}^{(a)} \cap -\widetilde{\mathcal{T}}^{(b)} \cap [T_\beta^{(b-1)}(1), T_\beta^{(a-1)}(1)]) \times \mathbb{Z}_3)$ and $B_{a,x} = \pi_\infty(-\widetilde{\mathcal{T}}^{(a)} \cap -\widetilde{\mathcal{T}}(x) \cap (0, T_\beta^{(a-1)}(1)) \times \mathbb{Z}_3)$. Then computing $\gamma(\beta)$ simply involves understanding intersections between tiles.

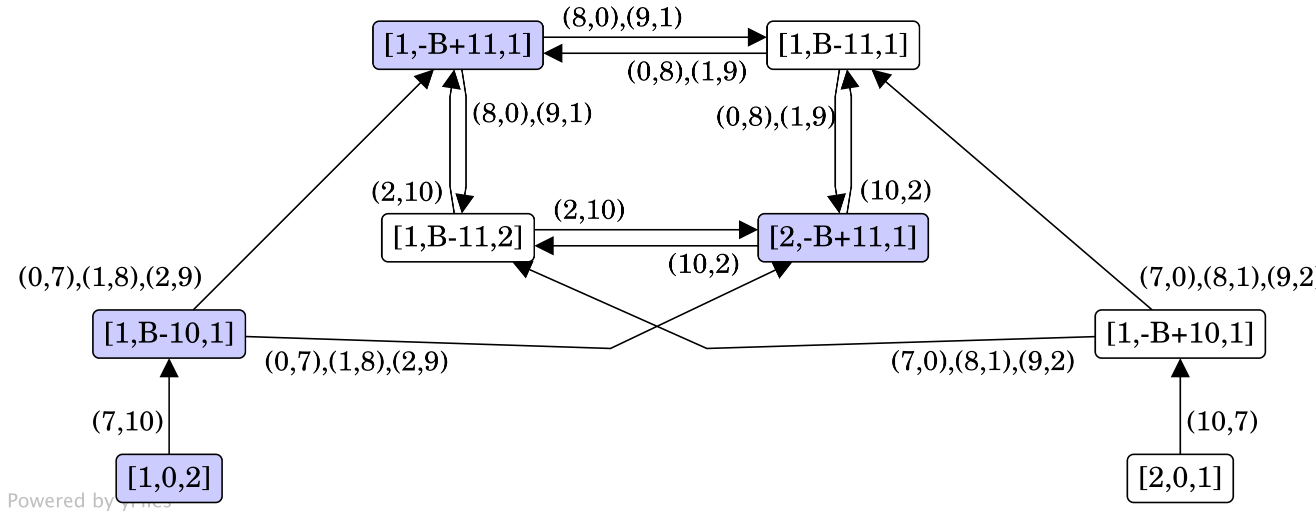

FIGURE 6. Boundary graph for $\beta^2 = 10\beta + 3$. Notation B stands for β. Grey nodes correspond to intersections between a central subtile and a neighbouring x-tile, that is, intersections that contribute to the computation of $\gamma(\beta)$ in Proposition 4.16.

Let $-\alpha$ denote the conjugate of β, i.e., $\alpha = \sqrt{7} - 2 > 1/3$. Lemma 5.5 exhibits a set that we need to explicitly compute.

$$\begin{aligned}
\pi_\infty\left(\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(\beta-4))\right) &\supset \left\{\sum a_i\alpha^{2i} - \sum b_i\alpha^{2i+1} \mid a_i \in \{0,1,2\}, b_i \in \{2,3,4\}\right\} \\
&= \left\{\sum a_i\alpha^{2i} + \sum c_i\alpha^{2i+1} \mid a_i \in \{0,1,2\}, c_i \in \{-2,-3,-4\}\right\} \\
&= X(\alpha,3) - \alpha(4 + 4\alpha^2 + 4\alpha^4 + \dots) \\
&= \left[-\frac{4\alpha}{1-\alpha^2}, -\frac{4\alpha}{1-\alpha^2} + \frac{2}{1-\alpha}\right] \\
&= \left[-\frac{4\alpha}{1-\alpha^2}, \frac{2-2\alpha}{1-\alpha^2}\right] \ni 0
\end{aligned}$$

Hence zero is the minimum of $[0, T_\beta(1)] \cap \pi_\infty(-\widetilde{\mathcal{T}}^{(1)} \cap (-\widetilde{\mathcal{T}}(\beta-4))$ and Theorem 4.18 implies that $\gamma(\beta) = 0$. □

A completely different behaviour occurs when modifying only one digit in the quadratic equation satisfied by β.

Theorem 5.8. *One has*

$$\gamma(5 + 2\sqrt{7}) = \frac{7 - \sqrt{7}}{12}$$

Proof. The number $5 + 2\sqrt{7}$ is the positive root of $\beta^2 - 10\beta - 3 = 0$. As before, the conditions of Corollary 4.17 are satisfied, hence $\overline{\phi_f(\mathbb{Z}_{(N(\beta))})} = \mathbb{Z}_3$, and studying intersections between tiles is enough to compute $\gamma(\beta)$.

We deduce from the graph depicted in Fig. 6 that non-empty intersections in the numeration tiling are given by $\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(2)}$, $\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(11-\beta))$, $\widetilde{\mathcal{T}}^{(2)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(11-\beta))$, and $\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(+10+\beta))$.

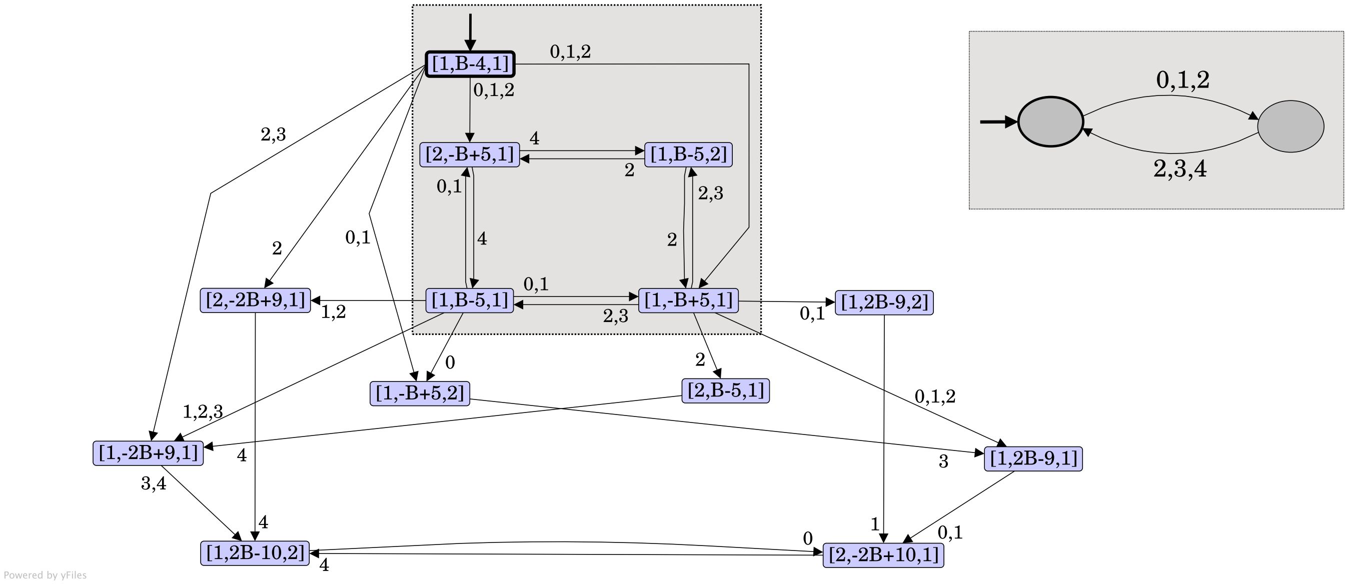

FIGURE 7. (Left) Notation B stands for β. Expansion graph for the real projection of points lying at the intersection of $\widetilde{\mathcal{T}}^{(1)}$ with $\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(\beta - 4)$, where $\beta^2 = 4\beta + 3$. Let β_2 be the Galois conjugate of β. We have $z \in \pi_\infty\left(\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(\beta-4))\right)$ if and only if $z = \sum_{i\geq 0} p_i \beta_2^i$ where $(p_i)_{i\geq 0}$ is the labelling of an infinite path in the graph issued from $[1, \beta - 4, 1]$.
(Right) Minimal automaton that describes the langage recognized by the subgraph colored in grey in the left-hand side graph.

We can deduce from the graph depicted in Fig. 8 the expansion of the real projection of the last three sets.

$$\pi_\infty(\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(11-\beta))) = \{8,9\} + \beta_2 \sum_{i\geq 0} \{0,1,2\}\beta_2^{2i} + \{8,9,10\}\beta_2^{2i+1}$$

$$\pi_\infty(\widetilde{\mathcal{T}}^{(2)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(11-\beta))) = 10 + \beta_2 \sum_{i\geq 0} \{0,1,2\}\beta_2^{2i} + \{8,9,10\}\beta_2^{2i+1}$$

$$\pi_\infty(\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(-10+\beta))) = \{0,1,2\} + \beta_2 \sum_{i\geq 0} \{8,9,10\}\beta_2^{2i} + \{0,1,2\}\beta_2^{2i+1}$$

We use the Cantor Cookie-cutter set Lemma stated above with $\alpha := -\beta_2 = 3\beta^{-1}$ and $n = 3 > \alpha^{-1}$ to compute the sum that is involved at each intersection.

$$\begin{aligned} \sum_{i\geq 1} \{0,1,2\}\beta_2^{2i} + \{8,9,10\}\beta_2^{2i+1} &= \sum_{i\geq 0} \{0,1,2\}\alpha^{2i} - \{8,9,10\}\alpha^{2i+1} \\ &= -10\sum_{i\geq 0} \alpha^{2i+1} + \sum_{j\geq 0}\{0,1,2\}\alpha^j \\ &= \frac{-10\alpha}{1-\alpha^2} + \left[0, \frac{2}{1-\alpha}\right] = \left[\frac{-10\alpha}{1-\alpha^2}, \frac{-8\alpha+2}{1-\alpha^2}\right] \end{aligned}$$

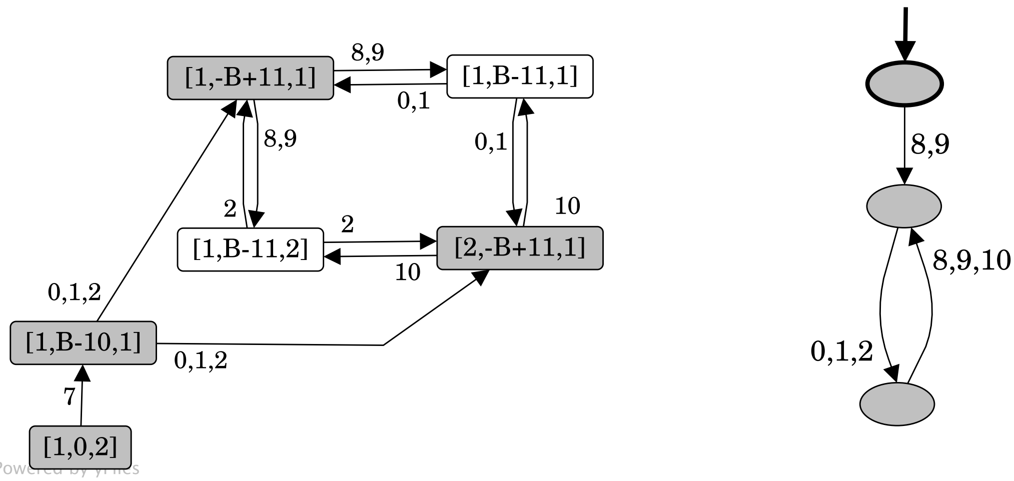

FIGURE 8. (Left) Notation B stands for β. Expansion graph for the real projection of points lying at the intersection of two tiles $\widetilde{\mathcal{T}}^{(a)}$ and $\widetilde{\mathcal{T}}^{(b)} + \phi_\beta(x)$, where $\beta^2 = 10\beta+3$. Let β_2 be the Galois conjugate of β. We have $z \in \pi_\infty\left(\widetilde{\mathcal{T}}^{(a)} \cap (\widetilde{\mathcal{T}}^{(b)} + \phi_\beta(x)\right)$ if and only if $z = \sum_{i\geq 0} p_i \beta_2^i$ where $(p_i)_{i\geq 0}$ is the labelling of an infinite path in the graph issued from $[a, x, b]$. Admissible $[a, x, b]$ are colored in grey in the graph. (Right) Minimal automaton that describes infinite paths starting in $[1, -\beta + 11, 1]$ in the left-hand side graph.

Similarly, we have

$$\begin{aligned}\sum_{i\geq 1}\{8,9,10\}\beta_2^{2i} + \{0,1,2\}\beta_2^{2i+1} &= 10\sum_{i\geq 0}\alpha^{2i} - \sum_{j\geq 0}\{0,1,2\}\alpha^j \\ &= \frac{10}{1-\alpha^2} - \left[0, \frac{2}{1-\alpha}\right] = \left[\frac{8-2\alpha}{1-\alpha^2}, \frac{10}{1-\alpha^2}\right].\end{aligned}$$

We deduce that

$$\pi_\infty(\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(11-\beta))) = \left[8 - \alpha\frac{-8\alpha+2}{1-\alpha^2}, 8 - \alpha\frac{-10\alpha}{1-\alpha^2}\right] \subset]0, \infty[\,.$$

Hence $-\pi_\infty(\widetilde{\mathcal{T}} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(11-\beta))) \cap [0,1] = \emptyset$. Similarly, we have $-\pi_\infty(\widetilde{\mathcal{T}} \cap (\widetilde{\mathcal{T}^{(2)}} + \phi_\beta(11-\beta))) \cap [0,1] = \emptyset$, therefore both intersections cannot be taken into account in the computation of $\gamma(\beta)$. This implies that $-\pi_\infty(\widetilde{\mathcal{T}}^{(2)}) \cap [0,\infty]$ does not intersect the projection of any tile $-\pi_\infty(\widetilde{\mathcal{T}}(x))$.

We also have

$$\pi_\infty(\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(\beta-10))) = \left[\frac{-10\alpha}{1-\alpha^2}, \frac{-8\alpha+2}{1-\alpha^2}\right].$$

Hence, the minimum of $-\pi_\infty(\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(\beta-10)))$ is $\frac{8\alpha-2}{1-\alpha^2}$.

In order to apply Theorem 4.18, we prove that the infimum of intersections of the form $\pi_\infty(A_{a,b})$ (situation (A) or (B)) is strictly larger than the infimum of intersections $\pi_\infty(B_{x,a})$ (situation (C)). By definition, we have $\pi_\infty(\widetilde{\mathcal{T}}^{(2)}) = \{\sum_{i\geq 0} a_{2i}\alpha^{2i} - \sum_{i\geq 0} a_{2i+1}\alpha^{2i+1}\}$ where sequences $a_1 \dots a_i \dots$ are sequences starting from 2 in reverse direction on the admissibility graph. We deduce that

$a_0 = 10$, $a_1 \le 9$, $a_2 \ge 0$, and then, $a_{2i+2} \ge 0$ and $a_{2i+3} \le 10$. Hence

$$\min \pi_\infty(\widetilde{\mathcal{T}}^{(2)}) \ge 10 - 9\alpha + 0\alpha^2 - 10\alpha^3 + \cdots = 10 - 9\alpha + 10\frac{\alpha^3}{1-\alpha^2} > 0.$$

Consequently, $-\pi_\infty(\widetilde{\mathcal{T}}^{(2)}) \cap [0,\infty] = \emptyset$ and situations (A) or (B) do not contribute to $\gamma(\beta)$.

We deduce from Theorem 4.18 that $\gamma(\beta) = \min -\pi_\infty(\widetilde{\mathcal{T}}^{(1)} \cap (\widetilde{\mathcal{T}}^{(1)} + \phi_\beta(11-\beta))) = \frac{8\alpha-2}{1-\alpha^2} = \frac{7-\sqrt{7}}{12}$. □

6. Perspectives

At least two main directions now deserve to be discussed. In the quadratic case, what is the structure of the intersection graph required to compute $\gamma(\beta)$? The first question is whether we can obtain an algorithmic way to compute $\gamma(\beta)$ for every quadratic β. Then, can we deduce a general formula for $\gamma(\beta)$ for whole families of β? The first step would be to accurately describe the structure of the boundary graph, at least for the $\beta^2 = n\beta + 3$ family.

Another direction involves the application of these methods in the three (or more dimensional case), including the unit case. We currently cannot give an explicit formula for $\gamma(\beta)$. In order to generalise the results to higher degrees, an approximation of exclusive inner points by the diagonal line of $\mathbb{K}_\beta$ is needed. This seems reasonable at least in the unit case, but requires a precise study of the topology of the central tile. Examples of computations of intersections between line and fractals are obtained by numeric approximations in [AS05]. As an example, an intersection is proved to be approximated by 0.66666666608644067488. Then it is not equal to 2/3, though very close to it. Theorem 5.8 is an example in which we were able to explicitly compute the value of $\gamma(\beta)$ and it turned out that $\gamma(\beta) \in \mathbb{Q}(\beta)$. The general question of the algebraic nature of $\gamma(\beta)$ is interesting.

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References

[Aki98] S. Akiyama. Pisot numbers and greedy algorithm. In *Number theory (Eger, 1996)*, pages 9–21. de Gruyter, Berlin, 1998.

[Aki00] S. Akiyama. Cubic Pisot units with finite beta expansions. In *Algebraic number theory and Diophantine analysis (Graz, 1998)*, pages 11–26. de Gruyter, Berlin, 2000.

[Aki02] S. Akiyama. On the boundary of self affine tilings generated by Pisot numbers. *J. Math. Soc. Japan*, 54(2):283–308, 2002.

[Aki07] S. Akiyama. Pisot number system and its dual tiling. In *Physics and Theoretical Computer Science (Cargese, 2006)*, pages 133–154. IOS Press, 2007.

[AS05] S. Akiyama and K. Scheicher. Intersecting two dimensional fractals and lines. *Acta Sci. Math. (Szeged)*, 3-4:555–580, 2005.

[BBLT06] G. Barat, V. Berthé, P. Liardet, and J. M. Thuswaldner. Dynamical directions in numeration. *Ann. Inst. Fourier (Grenoble)*, 56(7):1987–2092, 2006.

[Bed91] T. Bedford. Applications of dynamical systems theory to fractals—a study of cookie-cutter Cantor sets. In *Fractal geometry and analysis (Montreal, PQ, 1989)*, volume 346 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 1–44. Kluwer Acad. Publ., Dordrecht, 1991.

[Ber77] A. Bertrand. Développements en base de Pisot et répartition modulo 1. *C. R. Acad. Sci. Paris Sér. A-B*, 285(6):A419–A421, 1977.

[BFGK98] Č. Burdík, Ch. Frougny, J. P. Gazeau, and R. Krejcar. Beta-integers as natural counting systems for quasicrystals. *J. Phys. A*, 31(30):6449–6472, 1998.

[Bla89] F. Blanchard. β-expansions and symbolic dynamics. *Theoret. Comput. Sci.*, 65(2):131–141, 1989.

[BS05] V. Berthé and A. Siegel. Tilings associated with beta-numeration and substitutions. *Integers*, 5(3):A2, 46 pp. (electronic), 2005.

[BS07] V. Berthé and A. Siegel. Purely periodic β-expansions in the Pisot non-unit case. *J. Number Theory*, 127:153–172, 2007.

[CF86] J. W. S. Cassels and A. Fröhlich, editors. *Algebraic number theory*, London, 1986. Academic Press Inc.

[CFS82] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinaĭ. *Ergodic theory*, volume 245 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982. Translated from the Russian by A. B. Sosinskiĭ.

[DKS96] K. Dajani, C. Kraaikamp, and B. Solomyak. The natural extension of the β-transformation. *Acta Math. Hungar.*, 73(1-2):97–109, 1996.

[Fro00] C. Frougny. Number representation and finite automata. In *Topics in symbolic dynamics and applications (Temuco, 1997)*, volume 279 of *London Math. Soc. Lecture Note Ser.*, pages 207–228. Cambridge Univ. Press, 2000.

[FS92] C. Frougny and B. Solomyak. Finite beta-expansions. *Ergodic Theory Dynamical Systems*, 12:45–82, 1992.

[HI97] M. Hama and T. Imahashi. Periodic β-expansions for certain classes of Pisot numbers. *Comment. Math. Univ. St. Paul.*, 46(2):103–116, 1997.

[IR04] S. Ito and H. Rao. Purely periodic β-expansions with Pisot unit base. *Proc. Amer. Math. Soc.*, 133(4):953–964 (electronic), 2004.

[Lot02] M. Lothaire. *Algebraic combinatorics on words*, volume 90 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2002. (Chapter 7, written by C. Frougny).

[LS05] E. Lindenstrauss and K. Schmidt. Symbolic representations of nonexpansive group automorphisms. *Israel J. Math.*, 149:227–266, 2005.

[Par60] W. Parry. On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.*, 11:401–416, 1960.

[Pra99] B. Praggastis. Numeration systems and Markov partitions from self-similar tilings. *Trans. Amer. Math. Soc.*, 351(8):3315–3349, 1999.

[QRY05] Y.-H Qu, H. Rao, and Y.-M. Yang. Periods of β-expansions and linear recurrent sequences. *Acta Arith.*, 120(1):27–37, 2005.

[Rau88] G. Rauzy. Rotations sur les groupes, nombres algébriques, et substitutions. In *Séminaire de Théorie des Nombres, 1987–1988 (Talence, 1987–1988)*. Univ. Bordeaux I, Talence., 1988. Exp. No. 21, 12.

[Roh61] V. A. Rohlin. Exact endomorphisms of a Lebesgue space. *Izv. Akad. Nauk SSSR Ser. Mat.*, 25:499–530, 1961.

[San02] Y. Sano. On purely periodic beta-expansions of Pisot numbers. *Nagoya Math. J.*, 166:183–207, 2002.

[Sch80] K. Schmidt. On periodic expansions of Pisot numbers and Salem numbers. *Bull. London Math. Soc.*, 12(4):269–278, 1980.

[Sch00] K. Schmidt. Algebraic coding of expansive group automorphisms and two-sided beta-shifts. *Monatsh. Math.*, 129(1):37–61, 2000.

[Sie03] A. Siegel. Représentation des systèmes dynamiques substitutifs non unimodulaires. *Ergodic Theory Dynam. Systems*, 23(4):1247–1273, 2003.

[Sin06] B. Sing. Iterated function systems in mixed Euclidean and p-adic spaces. In *Complexus mundi*, pages 267–276. World Sci. Publ., Hackensack, NJ, 2006.

[ST07] A. Siegel and J. M. Thuswaldner. Topological properties of rauzy fractals. preprint, 2007.

[SW02] V. F. Sirvent and Y. Wang. Self-affine tiling via substitution dynamical systems and Rauzy fractals. *Pacific J. Math.*, 206(2):465–485, 2002.

[Thu89] W. P. Thurston. Groups, tilings and finite state automata. In *AMS Colloquium lectures*. AMS Colloquium lectures, 1989.

[Thu06] J. M. Thuswaldner. Unimodular Pisot substitutions and their associated tiles. *J. Théor. Nombres Bordeaux*, 18(2):487–536, 2006.

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