Volatility of daily stock returns estimation by means of particle filter: The IBEX 35 case

M. P. Muñoz¹, D. Marquez², M. Martí-Recober¹, C. Villazón², L. Acosta¹

¹ Statistics and Operational Research Dept. (UPC)
² Business Economic Dept. (UAB)

Person contact: pilar.munyoz@upc.es

Index

1. Introduction
2. Models
3. The IBEX 35 case
4. Conclusions
1. Introduction

Volatility is an important characteristic of financial markets

Financial time series present:
• Excess kurtosis
• Asymmetric effects of positive or negative shocks
• Small first-order autocorrelation of squared observations
• Slow decay towards zero of the autocorrelation coefficients of squared observations

Variance responds asymmetrically to past returns (Harvey and Shepard, 1996): variance tends to be higher under influence of bad news than under influence of good news

Our approach in this work is:
• To estimate the Spanish IBEX volatility by means of a model to capture simultaneously the mean and variance asymmetries in time series
• Particle filter is adopted for parameter estimation
2. Models (i)

**Stochastic Volatility model** *(Taylor 1982):*
- flexible in capturing the excess kurtosis observed
- formulated with leverage effects, captures the asymmetric behaviour in stock returns

**GARCH model** *(Bollerslev 1986):*
- It describes volatility clustering and excess kurtosis (although not entirely)

**TAR: Threshold Autoregressive model** *(Tong, 1990):*
- It is a piecewise AR model
- The switching mechanism is controlled by the delayed process variable (threshold variable)
- It captures the asymmetric patterns of volatility

2. Models (ii)

Previous models can be combined to produce “second-generation” models:

- **Threshold GARCH (TGARCH)** *(Rabemananjara and Zakoïan, 1993):*
  - Threshold non-linearity is incorporated into the GARCH variance specifications

- **Threshold Stochastic Volatility Model** *(So, Li and Lam, 2002)*
  - Threshold non-linearity is incorporated into the Stochastic Volatility:
    - in the mean
    - in the variance

- And so on ...
2.1 Basic Stochastic Volatility model (i)

The known model is:
\[ y_t = \sigma_t \varepsilon_t \quad (1a) \]

- \( y_t \): the observed variable, in general the return on an asset
- \( \sigma_t \): the unobserved volatility of \( y_t \)

Volatility evolution supposed to be governed by:
\[ \log \sigma_t^2 = \alpha + \beta \log \sigma_{t-1}^2 + \sigma_{\nu_t} \quad (1b) \]

- \( \varepsilon_t \) and \( \nu_t \): Gaussian white sequences, independent, with mean 0 and variance 1
- \( \beta \): model persistence

In state space representation:
- The observation equation
  \[ y_t = \exp(0.5x_t)\varepsilon_t \quad (2a) \]
- The system equation:
  \[ x_t = \alpha + \beta x_{t-1} + \sigma_{\nu_t} \quad (2b) \]
  - \( x_t = \log \sigma_t^2 \), a latent variable
  - It describes a discrete time, non-linear and Gaussian dynamic system
  - The non-linearity appears only in the observation equation
  - It evolves as a first-order Markov process
2.1 Basic Stochastic Volatility model (iii)

- **Main goal:** To estimate the system state:
  \[ x_t = \log \sigma_t^2 \quad \leftrightarrow \quad \text{Volatility} \]
- **Implicitly:** To estimate the unknown parameters
  \[ \theta = (\alpha, \beta, \sigma_v) \]

- Different estimation methods have been developed (Broto and Ruiz, 2002):
  - Method of moments
  - Quasi-maximum likelihood
  - Bayesian methods

2.2 Threshold Stochastic Volatility model (i)

THSV model (So, Li and Lam, 2002):

- Define a set of Bernoulli random variables \( s_t \) by:
  \[ s_t = \begin{cases} 
  0 & \text{if } r_t < 0 \\
  1 & \text{if } r_t \geq 0 
  \end{cases} \quad r_t: \text{returns} \]

- Model:
  \[
  \begin{align*}
  r_t &= \phi_1 x_t + \phi_2 r_{t-1} + \epsilon_t \\
  y_t &= \sigma \epsilon_t, \quad \epsilon_t \sim N(0,1) \\
  \log \sigma_t^2 &= \alpha_x + \beta_x \log \sigma_{t-1}^2 + \sigma_v \nu_t, \quad \nu_t \sim N(0,1) \\
  \epsilon_t \text{ and } \nu_t &\text{: Stochastically independent}
  \end{align*}
  \]

The unknown parameters \( \theta = (\phi_x, \phi_1, \alpha_x, \beta_x, \sigma_v) \)

switch between the two regimes corresponding to the rise and fall in the asset prices.

This model can also be formulated into state space form
2.3 Bayesian State space approach (i)

- **Bayesian approach:** To include the parameters as part of the state vector

\[
\begin{bmatrix}
  x_t \\
  \theta
\end{bmatrix}
\]

- **From a bayesian point of view:** To obtain the a posteriori PDF

\[
p(x_t, \theta \mid D_t)
\]

Where \(D_t = \{y_1, \ldots, y_t\}\) represents the available information until time \(t\).

It is well known that using recursive formula based on Bayes’ rule is used assuming that the a priori PDF at time \(t-1\) is knowing.

\[p(x_{t-1}, \theta \mid Y_{t-1})\]

- **State PDF prediction:**

\[
p(x_t, \theta \mid Y_{t-1}) = \int p(x_t \mid x_{t-1}, \theta) p(x_{t-1}, \theta \mid Y_{t-1}) dx_{t-1}
\]

\[p(x_t \mid x_{t-1}, \theta) : \text{State evolution density obtained from (2b)}\]

- **Filtering PDF**

\[
p(x_t, \theta \mid Y_t) \propto p(y_t \mid x_t, \theta) p(x_t, \theta \mid Y_{t-1})
\]

\[p(y_t \mid x_t, \theta) : \text{Likelihood obtained from (2a)}\]
2.4 Particle filtering

- Filtering PDF is approximated by an empirical distribution formed from particles (point masses) in two steps (Liu and West, 2001):

**Prediction PDF approximation:**

\[ p(x_t, \theta \mid Y_{t-1}) \sim \sum_{j=1}^{M} p(x_t \mid x_{t-1}^{(j)}, \theta) \tilde{w}_{t-1}^{(j)} \]  

(5)

**Filtering PDF approximation:**

\[ p(x_t, \theta \mid Y_{t-1}) \propto p(y_t \mid x_t, \theta) \sum_{j=1}^{M} p(x_t \mid x_{t-1}^{(j)}, \theta) \tilde{w}_{t-1}^{(j)} \]  

(6)

In this case:

\[ \tilde{w}_{t}^{(j)} \propto p(y_t \mid x_{t}^{(j)}, \theta) \]  

(7)

These recursions can be implemented using Sampling Importance Resampling Algorithm (SIR) (Arulampalam et al., 2002)

2.5 SIRJ Algorithm (i)

Using SIR, after few iterations, there is an impoverishment problem for the parameters.

- **Our approach SIRJ:** Modifies the Sampling Importance Resampling (SIR) procedure adding at the end of each iteration, the jitter proposed by Liu and West (2001). (Muñoz et al. 2004)

- We have compare SIRJ with the approach of Liu and West (2001). The SIRJ gives almost the same precision as de second one for the parameters estimation and it is computationally less expensive

- The algorithm is implemented using the R language for Statistical Computing (http://cran.r-project.org/)
2.5 SIRJ Algorithm (ii)

Pseudo-code Sequential Importance Resampling with Jitter (SIRJ) Algorithm

\[ \{x_{i0}^{(j)}, \theta_{i0}^{(j)}\} \rightarrow \text{SIRJ}\{x_{t0}^{(j)}, \theta_{t0}^{(j)}\} \]

1. Initialization \( t = 0 \)
   - FOR \( j = 1:M \)
     - Sample \( x_{i0}^{(j)} \sim p(x_i) \)
     - Sample \( \theta_{i0}^{(j)} \sim p(\theta_i) \)
     - END FOR
   - FOR \( t = 1:N \)

2. Importance sampling
   - FOR \( j = 1:M \)
     - Prediction: Sample \( x_{i0}^{(j)} \sim q(x_i | x_{i0}^{(j)}, \theta_{i0}^{(j)}) \)
     - Filtering: Assign to each particle \( x_{i0}^{(j)} \) the new weight \( w_{i0}^{(j)} \) according to (7)
       - This means that \( w_{i0}^{(j)} \sim N(y_i | 0, \exp(x_{i0}^{(j)})) \)
     - END FOR
   - FOR \( j = 1:M \)
     - Resampling with replacement the particles \( \{x_{i0}^{(j)}, \theta_{i0}^{(j)}\}, j = 1, ..., M \) with the importance weights \( \{w_{i0}^{(j)}\}, j = 1, ..., M \)
     - END FOR

3. Jitter
   - FOR \( j = 1:M \)
     - Sample a new parameter vector \( \theta_{i0}^{(j)} \sim N(\theta_{i0}^{(j)}, \vartheta^2) \)
     - \( m^{(j)} = \alpha \theta_{i0}^{(j)} + (1 - \alpha) \theta_{i0}^{(j)} \)
     - \( \vartheta^2, \vartheta \) mean and variance of the \( p(\theta | D) \) Monte Carlo approximation
     - \( \alpha \approx 0.95 - 0.995 \)
     - \( \vartheta^2 \approx 1 - \alpha^2 \)
     - END FOR

END FOR

Volatility estimation with particle filter  
Particle and MC Methods  
Barcelona 2004
3. The IBEX 35 case (i)

IBEX 35 is the Spanish stock-exchange index

The prices series is observed daily from 02/01/1990 to 17/03/04 (3539 observations) (Fig. 1):

![IBEX 35 Stock Index prices](image1)

Fig.1: IBEX 35 Stock Index prices observed daily (02/01/1990 to 17/03/2004)

3. The IBEX 35 case (ii)

The prices \( p_t \) are transformed to returns \( r_t \) by

\[
    r_t = 100 \cdot \log \left( \frac{p_t}{p_{t-1}} \right)
\]

- Main features of returns series (i):

  - It exhibits volatility clusters
  - The squared observations are correlated

![IBEX 35 Stock Index returns](image2)

![ACF of squared returns](image3)

Fig.2: IBEX 35 Stock Index returns observed daily (02/01/1990 to 17/03/2004)

Fig.3: ACF of squaredreturns
3. The IBEX 35 case (iv)

• Main features of returns series (ii):

![Returns Histogram and Normal Q-Q Plot]

Skewness = -0.237
Kurtosis = 6.339

3. The IBEX 35 case (v)

Models fitted:

3.1. Basic stochastic volatility (i)

\[ y_t = \sigma_t \varepsilon_t \]
\[ \log \sigma_t^2 = \alpha + \beta \log \sigma_{t-1}^2 + \sigma_v v_t \]

Estimated parameters by means of the SIRJ procedure:

<table>
<thead>
<tr>
<th>( \hat{\alpha} )</th>
<th>( \hat{\beta} )</th>
<th>( \hat{\sigma}_v )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.010</td>
<td>0.970</td>
<td>0.192</td>
</tr>
<tr>
<td>(0.003)</td>
<td>(0.004)</td>
<td>(0.020)</td>
</tr>
</tbody>
</table>

( ) St. dev
3. The IBEX 35 case (vi)

3.1. Basic stochastic volatility (ii)

Model Checking

The model diagnostics were based on the standardized observations, defined as \( \hat{\epsilon}_t = \hat{y}_t / \hat{\sigma}_t \) called "residuals"

\( \hat{\sigma}_t \) is the volatility estimated, obtained by substituting the estimated parameters in the model equation (1b)

Descriptive statistics of standardized observations:

<table>
<thead>
<tr>
<th>( \hat{\epsilon}_t ) = ( \hat{y}_t / \hat{\sigma}_t )</th>
<th>Mean</th>
<th>S. Dev.</th>
<th>Skew</th>
<th>Kurto</th>
<th>Q(20)</th>
<th>Q2(20)</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.028</td>
<td>0.976</td>
<td>-0.179</td>
<td>2.852</td>
<td>55.870*</td>
<td>27.322</td>
<td>3539</td>
<td></td>
</tr>
</tbody>
</table>

Q(20) and Q2(20): Box-Ljung statistics for observations and squared observations

* Significant at the 95% level

3. The IBEX 35 case (vii)

3.1. Basic stochastic volatility (iii)

Conclusion

Stochastic volatility model captures the Kurtosis of the IBEX returns but:

- Does not capture completely the Skewness
- There is autocorrelation between observations
3. The IBEX 35 case (viii)

3.2. Threshold stochastic volatility model (i)

Our approach (i):

A) Identify and estimate a threshold model (SETAR) for the mean following the methodology proposed by Tsay (1989) and use the algorithm designed by Márquez (2002)

• The model obtained is the following:

\[
\begin{align*}
\begin{cases}
\kappa_1 + \phi_1 r_{t-1} + \cdots + \phi_{15} r_{t-15} + y_t, & r_{t-1} < 0 \\
\kappa_2 + \phi_1^2 r_{t-1} + \cdots + \phi_{15}^2 r_{t-15} + y_t, & r_{t-1} \geq 0
\end{cases}
\end{align*}
\]

<table>
<thead>
<tr>
<th>SETAR (2,15,13)</th>
<th>$\hat{\phi}_1$</th>
<th>$\hat{\phi}_2$</th>
<th>$\hat{\phi}_3$</th>
<th>$\hat{\phi}_4$</th>
<th>$\hat{\phi}_5$</th>
<th>$\hat{\phi}_6$</th>
<th>$\hat{\phi}_7$</th>
<th>$\hat{\phi}_8$</th>
<th>$\hat{\phi}_9$</th>
<th>AIC</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>1r. reg.</td>
<td>-0.0507</td>
<td>0</td>
<td>0</td>
<td>0.0602</td>
<td>0</td>
<td>0</td>
<td>0.0977</td>
<td>2.13</td>
<td>1288.96</td>
<td>1662</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0259)</td>
<td></td>
<td></td>
<td>(0.0258)</td>
<td></td>
<td></td>
<td>(0.0265)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2r. Reg.</td>
<td>0.0478</td>
<td>-0.0436</td>
<td>0.0454</td>
<td>0.0507</td>
<td>0.0529</td>
<td>0</td>
<td>1.61</td>
<td>920.74</td>
<td>1862</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0214)</td>
<td>(0.0219)</td>
<td>(0.0220)</td>
<td>(0.0217)</td>
<td>(0.0217)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Volatility estimation with particle filter

3. The IBEX 35 case (ix)

3.2. Threshold stochastic volatility model (ii)

Our approach (ii):

B) Fit to the residuals obtained from the SETAR model, a threshold stochastic volatility model for the variance. The parameter estimation will be made by means the SIRJ procedure

\[
\begin{align*}
\delta_i = \begin{cases} 0 & \text{if } r_{t-1} < 0 \\ 1 & \text{if } r_{t-1} \geq 0 \end{cases}
\end{align*}
\]

\[
y_t = \sigma \epsilon_t, \quad \epsilon_t \sim N(0,1)
\]

Now, \(y_t\) are the residuals from the SETAR model

\[
\log \sigma^2_t = \alpha_x + \beta_x \log \sigma^2_{t-1} + \sigma_x \nu_t, \quad \nu_t \sim N(0,1)
\]

Estimated parameters

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\hat{\theta}_0$</th>
<th>$\hat{\theta}_1$</th>
<th>$\sigma_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.962 (0.002)</td>
<td>0.079 (0.009)</td>
<td>-0.037 (0.009)</td>
<td>0.172 (0.014)</td>
</tr>
</tbody>
</table>

Volatility estimation with particle filter
3. The IBEX 35 case (x)

3.2. Threshold stochastic volatility model (iii)

Model checking

The residual analysis for this model is satisfactory:

\[ \hat{y}_t / \hat{\sigma}_t \]

<table>
<thead>
<tr>
<th>Mean</th>
<th>S. Dev.</th>
<th>Skew.</th>
<th>Kurtosi</th>
<th>Q(20)</th>
<th>Q(20)</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.007</td>
<td>0.955</td>
<td>-0.112</td>
<td>2.634</td>
<td>15.705</td>
<td>25.459</td>
<td>3509</td>
</tr>
</tbody>
</table>

- The model captures the asymmetric behaviour in the IBEX returns and the excess-kurtosis observed
- The residuals and the squared residuals are not autocorrelated
- The residuals follow a Gaussian distribution

3.3. Summary

First generation models

<table>
<thead>
<tr>
<th>Returns equation</th>
<th>Volatility equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_t = \alpha + \varepsilon_t ) ( \varepsilon_t \sim N(0,1) )</td>
<td>( \ln(\sigma^2_t) = 0.01 + 0.97 \ln(\sigma^2_{t-1}) + 0.192 \varepsilon_{t-1} ) ( \varepsilon_t \sim N(0,1) )</td>
</tr>
<tr>
<td>SETAR</td>
<td>- ( -0.05 \alpha + 0.05 \varepsilon_t + 0.1 \alpha \varepsilon_{t-1} + y_t, \ \alpha_t &lt; 0 ) |</td>
</tr>
<tr>
<td>( +0.5 \varepsilon_t + 0.04 \varepsilon_{t-1} + 0.05 \varepsilon_{t-2} + 0.05 \varepsilon_{t-3} + y_t, \ \alpha_t \geq 0 )</td>
<td></td>
</tr>
<tr>
<td>SECOND GENERATION MODEL</td>
<td></td>
</tr>
<tr>
<td>SETAR-THSV</td>
<td>- ( -0.05 \varepsilon_t + 0.06 \varepsilon_{t-1} + 0.1 \varepsilon_{t-2} + y_t, \ \alpha_t &lt; 0 )</td>
</tr>
<tr>
<td>( +0.5 \varepsilon_t + 0.04 \varepsilon_{t-1} + 0.05 \varepsilon_{t-2} + 0.05 \varepsilon_{t-3} + y_t, \ \alpha_t \geq 0 )</td>
<td></td>
</tr>
<tr>
<td>( \ln(\sigma^2_t) = 0.07 + 0.96 \ln(\sigma^2_{t-1}) + 0.172 \varepsilon_{t-1} ) ( \varepsilon_t \sim N(0,1) )</td>
<td></td>
</tr>
<tr>
<td>( -0.037 + 0.96 \ln(\sigma^2_{t-1}) + 0.172 \varepsilon_{t-1} ) ( \varepsilon_t \sim N(0,1) )</td>
<td></td>
</tr>
</tbody>
</table>
4. Conclusions

- SIRJ works well with both models and it is easy to implement.
- SETAR-THSV model:
  - can be implemented using bayesian methods and gives good results.
  - captures the dynamic behaviour of the returns and volatility series.
  - captures the asymmetries and reduces the excess of kurtosis in the volatility series

References (i)

References (ii)


