A preconditioning framework for coupled poromechanical problems

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Key words: Multi-physics, Preconditioning, Iterative methods

Introduction  This work discusses a general approach for preconditioning the block Jacobian matrix arising from the discretization and linearization of coupled poro-mechanical problems. The basic idea relies on approximately computing an algebraic operator able to decouple the flow and mechanical processes, which can be then solved independently. The decoupling operator is computed by extending the theory of block sparse approximate inverses in a multilevel framework. The proposed approach is implemented and numerically tested in real-world examples that are used to analyze and discuss the preconditioner properties.

Preconditioning strategy  Using the formulation introduced in [1], the global Jacobian matrix $A$ arising from the stabilized Mixed Finite Element discretization of coupled poromechanical equations has the following $3 \times 3$ block structure:

$$
A = \begin{bmatrix}
K & 0 & -Q \\
0 & A & -B \\
Q^T & \gamma B^T & P
\end{bmatrix},
$$

where $K$ is the SPD elastic stiffness matrix, $A$ is the SPD mass matrix for Darcy’s velocity in a mixed form, $P$ is the diagonal capacity matrix for the fluid flow, $Q$ and $B$ are the blocks coupling displacements and Darcy’s velocities to the pressure unknowns, and $\gamma$ is $\vartheta \Delta t$, with $\vartheta \in [0.5, 1]$ a parameter arising from the time-marching scheme and $\Delta t$ the time integration step. We want to compute two decoupling factors, $G$ and $F$, in the form:

$$
G = \begin{bmatrix}
I & 0 & 0 \\
0 & I & 0 \\
G_3^{(1)} & G_3^{(2)} & I
\end{bmatrix}, \quad F = \begin{bmatrix}
I & 0 & F_3^{(1)} \\
0 & I & F_3^{(2)} \\
0 & 0 & I
\end{bmatrix},
$$

such that $GAF = S$, with $S$ a block diagonal $3 \times 3$ matrix containing the Schur complements of $A$:

$$
S_1 = K, \quad S_2 = A, \quad S_3 = P + Q^T K^{-1} Q + \gamma B^T A^{-1} B.
$$

Hence, the exact decoupling blocks are:

$$
F_3^{(1)} = K^{-1} Q, \quad F_3^{(2)} = A^{-1} B, \quad G_3^{(1)} = -F_3^{(1),T}, \quad G_3^{(2)} = -\gamma F_3^{(2),T}.
$$

The blocks $F_3^{(1)}$ and $F_3^{(2)}$ are computed in an approximate way following the theory of block sparse approximate inverses developed in [2] and [3]. Denoting by $J_{3,k}^{(i)}$ the non-zero pattern of the $k$-th column of $F_3^{(i)}$, $i = 1, 2$, we solve the following dense systems of equations:

$$
\begin{align*}
&K[J_{3,k}^{(1)}, J_{3,k}^{(1)}] F_3^{(1)}[J_{3,k}^{(1)}, k] = Q[J_{3,k}^{(1)}, k], \\
&A[J_{3,k}^{(2)}, J_{3,k}^{(2)}] F_3^{(2)}[J_{3,k}^{(2)}, k] = B[J_{3,k}^{(2)}, k],
\end{align*}
$$

where $n_3$ denotes the number of pressure unknowns. Since $K$ and $A$ are SPD, the patterns $J_{3,k}^{(i)}$ can be selected dynamically using the algorithm developed in [3]. Notice that the small and dense linear systems in [3] are independent one of the other, hence can be efficiently solved in a massively parallel context. Using $F_3^{(1)}$ and $F_3^{(2)}$ it is possible to obtain an approximation of $S_3$ in equation [3]:

$$
S_3 \simeq \tilde{S}_3 = P + F_3^{(1),T} K F_3^{(1)} + \gamma F_3^{(2),T} A F_3^{(2)},
$$

which preserves the positive definiteness of $S_3$ by construction. Finally, a local preconditioner for $S_1$, $S_2$ and $S_3$ is computed. The overall preconditioning strategy is therefore applied in two stages: first, reducing the strength of coupling between the flow and deformation processes by the factors $G$ and $F$; second, applying appropriate local preconditioners to the block diagonal matrix of the Schur complements.

A possible difficulty in the computation of the decoupling blocks may arise for matrices with an increasingly large size. In this case, the dynamic detection of $J_{3,k}^{(i)}$ and the solution to the set of systems [3] may become numerically unstable, because most of the non-zero positions are associated to scores differing one from the other for values on the order of the machine precision. This inconvenience is addressed in a twofold way:
developing a hybrid search strategy that combines a static with a dynamic pattern selection for the decoupling blocks. The set of potential positions for the non-zero entries computed for each column of $F^3_i$ is statically restricted by keeping the connections with neighbouring nodes up to a user-specified level. Then, the dynamic procedure can be efficiently used in a more stable way only for the selected subset of potential non-zero entries;

- reducing the size of the diagonal blocks of $A$ by performing a preliminary inner decoupling, thus giving rise to a multilevel approach. An inner decoupling operator can be applied to each diagonal block of $A$ in order to approximately reduce each single-physics process to the sum of independent smaller problems. Then, the outer decoupling operator can be applied to each inner sub-block. This idea can be naturally extended in a multi-level framework where more than two levels are considered.

**Numerical results** The performance and effectiveness of the proposed approach is tested in real-world coupled poromechanical applications. For instance, the consolidation of a compacting reservoir subject to single-phase flow injection and production is simulated. The characterization in terms of permeability is based on the SPE10 dataset [4]. The computational grid is shown in Figure 1a and consists of $60 \times 220 \times 16$ hexahedral elements in $x$, $y$- and $z$-direction, respectively. The fluid is injected through a single vertical injection well located at the bottom left corner of the domain and is produced at the opposite corner (see Figure 1b), with the wells being modeled according to Peaceman [5].

The model totals 1,550,011 unknowns with 72,632,161 non-zeroes stored in $A$. The numerical experiments are performed on a machine equipped with Intel(R) Xeon(R) E5-2680 v2 processors at 2.80 GHz and 256 Gbyte of RAM. For these tests, a shared-memory implementation, based on OpenMP directives, is used, employing 8 cores. Preliminary results on this test problem are given for different values of $\gamma$ in Table 1 which shows: (i) the preconditioner density $\mu$, computed as the ratio between the non-zeroes stored for the preconditioner and for the overall matrix $A$; (ii) the number of iterations $n_{it}$ for the Bi-CGStab to reduce the 2-norm of the initial residual vector by a factor $10^{-8}$; (iii) the time in seconds $T_p^{(1)}$ for the first stage of the preconditioner setup, which can be performed just once at the beginning of a transient simulation and can be regarded as a preprocessing cost easily amortized in few time steps; (iv) the time in seconds $T_p^{(2)}$ for the part of the preconditioner setup that has to be performed at every step of a transient procedure; and (v) the time in seconds $T_s$ to carry out $n_{it}$ Bi-CGStab iterations.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\mu$</th>
<th>$n_{it}$</th>
<th>$T_p^{(1)}$ [s]</th>
<th>$T_p^{(2)}$ [s]</th>
<th>$T_s$ [s]</th>
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</thead>
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<tr>
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<td>299</td>
<td>16.805</td>
<td>5.068</td>
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<td>16.582</td>
<td>4.926</td>
<td>19.708</td>
</tr>
</tbody>
</table>

Table 1: Preliminary numerical results for the test case of Figure 1.