Local Geoid Determination and an Inverse Problem

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Outlines

I. Geoid determination.

II. “Equivalent” discrete system of point mass.

I. Geoid determination.

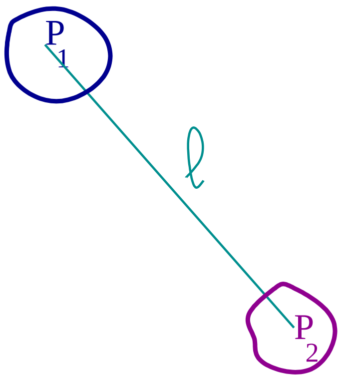
Gravitational Attraction

Newton attraction law (1687) :

$$F = G\frac{m_1 m_2}{\ell^2}$$

with $G = 6.6742 \,.\, 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{s}^{-2}$

P_1 the potentiant point and by supposing that $m_1 = m$ et $m_2 = 1$



$$\mathbf{F} = G\,\frac{m}{\ell^2}\,\mathbf{u}$$

$$\ell \;=\; \sqrt{(x_{P_2} - x_{P_1})^2 + (y_{P_2} - y_{P_1})^2 + (z_{P_2} - z_{P_1})^2}$$

$$\Rightarrow \mathbf{F} = \nabla V$$

$$V = -\frac{Gm}{\ell}$$

Earth potential

Mass density is given by $\rho = \dfrac{dm}{dv}$

Gravitational potential at a given point is

$$V = -G \iiint_{\mathcal{V}} \frac{dm}{\ell} = -G \iiint_{\mathcal{V}} \frac{\rho}{\ell}\, dv$$

Centrifugal potential : The centrifugal force

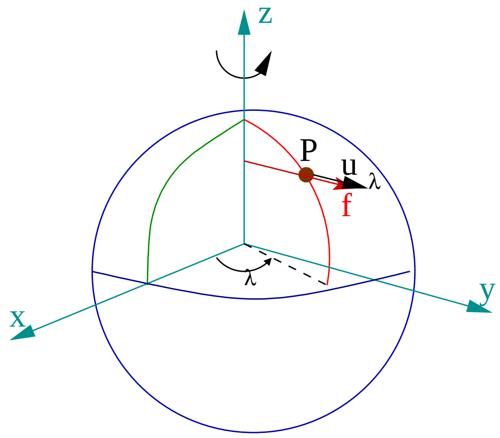
$$\mathbf{f} = m_P\, \omega^2 \sqrt{x^2 + y^2}\, \mathbf{u}_\lambda$$

derives from a centrifugal potential Φ :

$$\mathbf{f} = \nabla\Phi \;\; \text{where } \Phi = \frac{1}{2}\omega^2 (x^2 + y^2)$$

Total earth gravity field :

$$W = V + \Phi$$



$\Delta W = -4\pi G\rho + 2\omega^2.$

Out of the earth :

V is harmonic and $V(\infty) = 0$.

What is the geoid and what is it built for ?

Any water surface on the earth, corresponds to an equipotential of the earth gravity field W.

The mean level of the seas on the earth is called : geoid

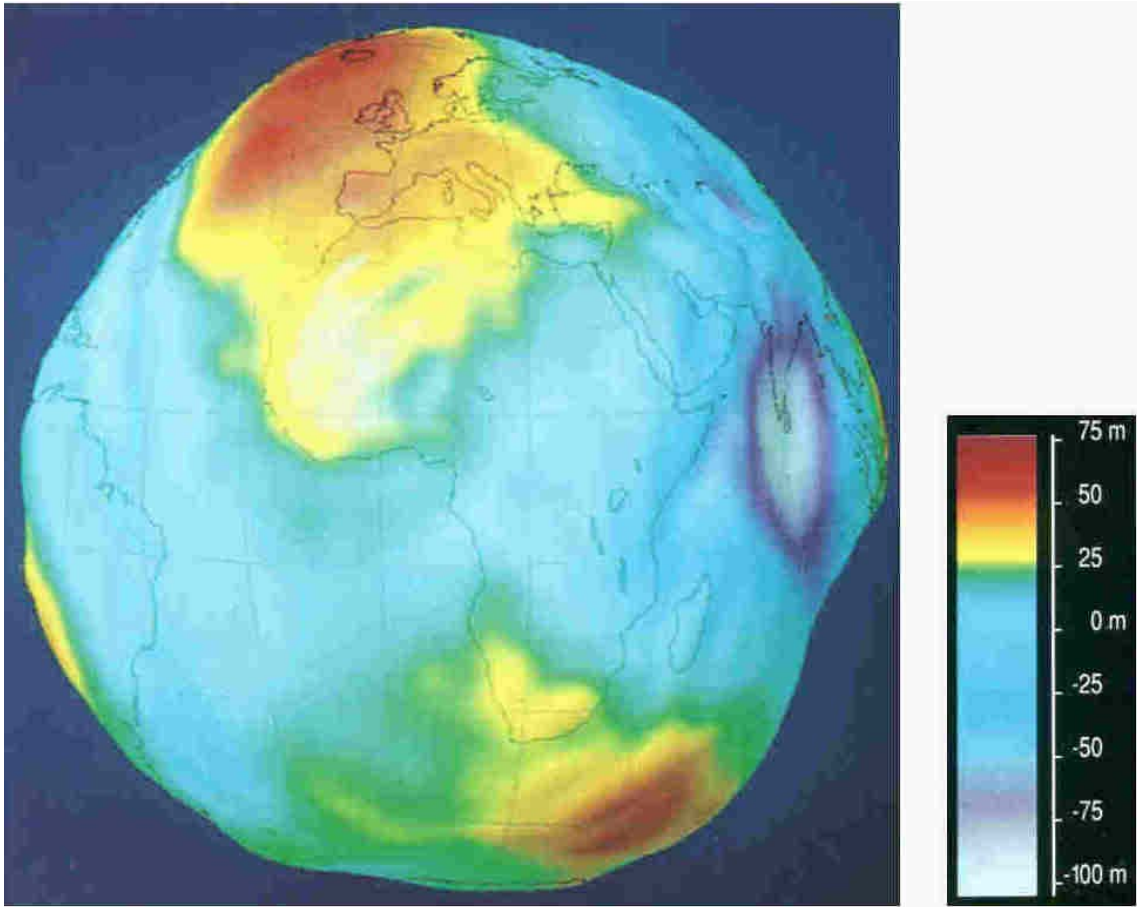
It can be considered on a region in order to :

- study the topography of the zone,
- study control air traffic,
- develop some civil engineering applications.

and at a wider extent, to

- provide a new precise and useful reference surface for topography : combined with GPS to obtain precise altitudes or the sea level.
- determine a global model of the international geoid.
- eliminate the problems in connexion with the distinct reference systems.
- find a new interpretation and understanding for the internal physics of the earth.

Earth geoid



From CNES

Geoid determination

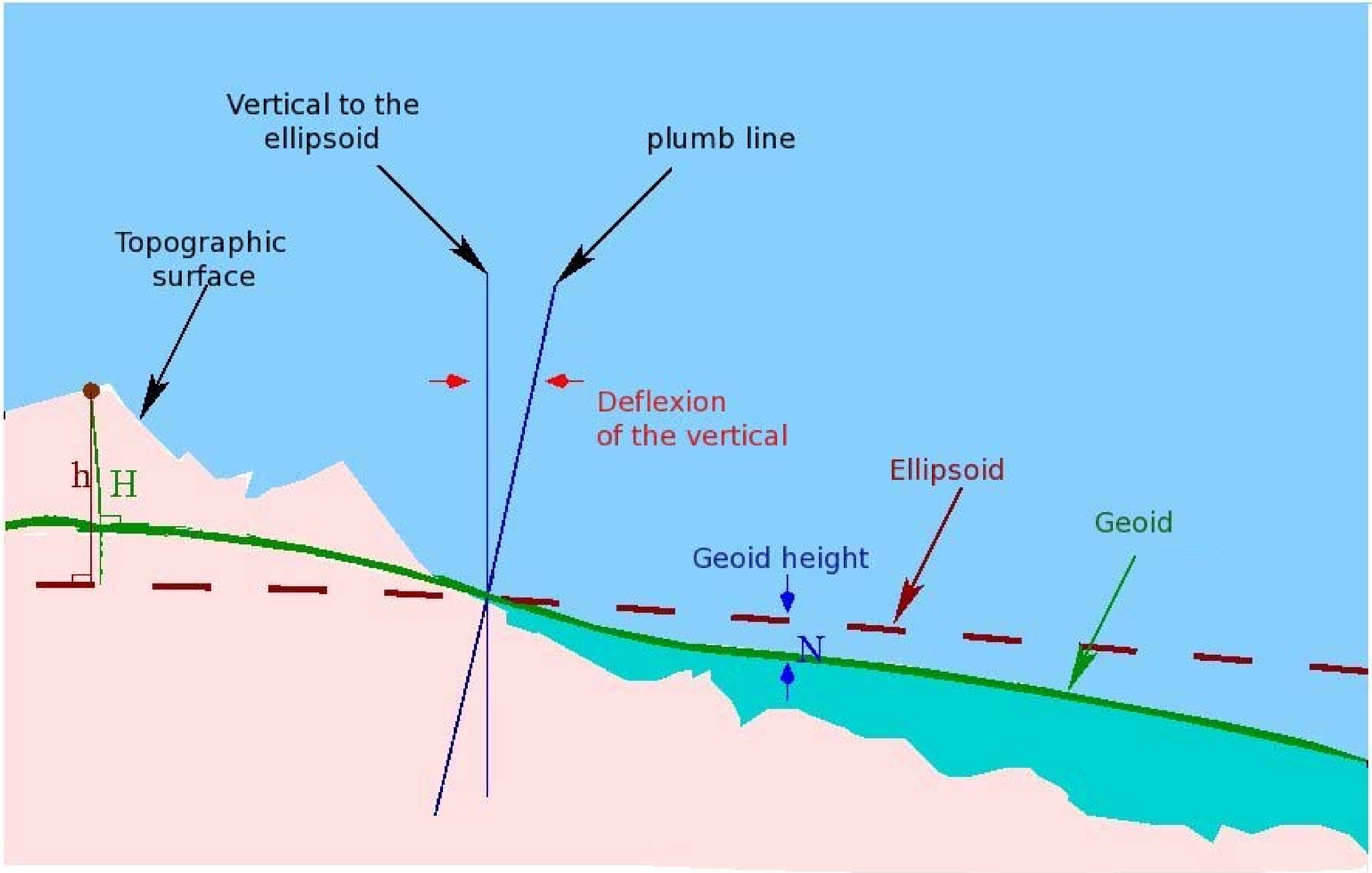
Because of the irregular distribution of mass inside the earth, the geoid cannot be directly computed. It is obtained by adjusting a model to given measurements.

An approximation of the earth is defined by an ellipsoid (homogeneous mass distribution, ellipsoid mass = earth mass + atmosphere mass, rotational symmetry).

From the reference ellipsoid a mathematical expression of the corresponding gravity field U is calculated; it is called normal potential .

A correction $T = W - U$, called anomalous potential , is computed by adjusting given measurements in geodetic fundamental relations.

Fondamental quantities



Existing reference ellipsoids

GRS 1980 :

$$
\begin{aligned}
a &= 6,378,137\ m \\
b &= 6,356,752.3141\ m \\
GM &= 3,986,005 \times 10^8\ m^3 s^{-2}
\end{aligned}
$$

WGS84 :

$$
\begin{aligned}
a &= 6,378,137\ m \\
b &= 6,356,752.3142\ m \\
GM &= 3,986,004.418 \times 10^8\ m^3 s^{-2}
\end{aligned}
$$

Clarke 1880 :

$$
\begin{aligned}
a &= 6,378,249.20\ m \\
b &= 6,356,515\ m
\end{aligned}
$$

Computation of the normal potential

The normal potential is $U = \mathcal{V} + \Phi$ where

- $\mathcal{V}$ ellipsoidal gravitational potential
- Φ centrifugal force potential

$\mathcal{V}$ is a harmonic function outside the ellipsoid. In spherical coordinates :

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \mathcal{V}}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial \mathcal{V}}{\partial\theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 \mathcal{V}}{\partial\lambda^2} = 0$$

This is solved by a classical technique of separation of variables :

$$\mathcal{V}(r, \theta, \lambda) = f(r)Y(\theta, \lambda)$$

where $r =$ radius, $\lambda =$ longitude and $\theta =$ co-latitude

Series expansion of $\mathcal{V}$ (spherical harmonics)

The normal gravitational potential can be writen as a Legendre series expansion

$$\mathcal{V} = \frac{GM}{r}\left[1 - \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{2n} J_{2n}P_{2n}(\cos\theta)\right]$$

where

- $J_{2n} = (-1)^{n+1}\dfrac{3\,e^{2n}}{(2n+1)(2n+3)}\left(1 - n + 5n\dfrac{J_2}{e^2}\right)$
- $J_2 = \dfrac{1}{3}e^2\left(1 - \dfrac{2}{15}\dfrac{me'}{q_0}\right)$
- $m = \dfrac{\omega^2 a^2 b}{GM},\ e = \dfrac{\sqrt{a^2-b^2}}{a},\ e' = \dfrac{\sqrt{a^2-b^2}}{b}$
- $q_0 \ = \ \dfrac{1}{2}\left(1 + \dfrac{3}{e'^2}\right)\text{arctg}\ e' - \dfrac{3}{2e'}$

Collocation Method

This is the most appropriate method when the measured data are a mixture of several types of geodetic measurements (e.g. gravimetric data, GPS measurements, deflection of the vertical, etc.)

It is used to characterize the geoid by computing its height N from the ellipsoid surface.

Brun's formula links the geoidal height N to the anomalous potential $T = W - U$ and the magnitude of the normal gravity vector γ :

$$N = \frac{T}{\gamma}.$$

Some relations

T is related to the geodetic parameters by linear expressions obtained by linearizing fundamental relations:

The Vening Meinesz formulae :

$$\xi = \frac{1}{\gamma r}\frac{\partial T}{\partial \phi} \qquad , \qquad \eta = \frac{1}{\gamma r \sin\phi}\frac{\partial T}{\partial \lambda}$$

ξ and η are the deflection of vertical components (North-South and Est-West)

The so-called fondamental formula of physical geodesy :

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{r}T$$

where $\Delta g(P) = g(P) - \gamma(Q)$ is gravity anomaly and P is a point on the geoid and Q is its projection on the ellipsoid.

Application to the geoid determination

$$\ell = \begin{pmatrix} \Delta g_1 \\ \vdots \\ \Delta g_p \\ \xi_1 \\ \vdots \\ \xi_p \\ \eta_1 \\ \vdots \\ \eta_p \end{pmatrix} \qquad\qquad e = \begin{pmatrix} e_{g_1} \\ \vdots \\ e_{g_p} \\ e_{\xi_1} \\ \vdots \\ e_{\xi_p} \\ e_{\eta_1} \\ \vdots \\ e_{\eta_p} \end{pmatrix}$$

observation vector | standard error vector

- $\ell = L\,t + e$
- the gravity field parameters : $T, N, \Delta g, \xi, \eta, etc.$, are considered as random centered and correlated variables
- hence, the problem of the geoid determination becomes a generalized least squares estimation problem.

$\Longrightarrow$ The collocation method requires the determination of the empirical covariance function and the analytic associated model.

Modeling the covariance function

The empirical covariance function from Δ_g

$$C_{\Delta g \Delta g} = \frac{1}{n_k} \sum_{E_k} \Delta g_i \Delta g_j$$

$E_k = \{(P_i, Pj); \psi_{ij} \in [\psi_k - \Delta\psi, \psi_k + \Delta\psi]\}$
$\psi_k =$ the k$^{\text{th}}$ spherical distance class
$\cos\psi_k = \cos\phi_i \cos\phi_j + \sin\phi_i \sin\phi_j \cos(\lambda_i - \lambda_j)$
$E_k =$ set of the couples distant with $\psi_k \pm \Delta\psi$.

An analytic model (Rapp & Tscherning) is given by an approximation in the exterior of a sphere totally enclosed in the earth (Bjerhammar–sphere) :

$$\begin{aligned} \mathrm{cov}(\Delta g_P, \Delta g_Q) &= \alpha \sum_{n=2}^{N_{\max}} \sigma_n^e \left(\frac{R}{r_P\, r_Q}\right)^{n+2} P_n(\cos\psi_{PQ}) \\ &+ \sum_{n=N_{\max}+1}^{\infty} \sigma_n \left(\frac{R_B}{r_P\, r_Q}\right)^{n+2} P_n(\cos\psi_{PQ}). \end{aligned}$$

Modeling the covariance function (2)

- Where :

  $R_B =$ the Bjerhammer sphere radius,

  $R =$ the mean earth radius,

  $\alpha =$ the scale factor,

  $\psi =$ the spherical distance,

  $P, Q =$ the spatial points,

  $\sigma_n^e =$ the error degree variances related to the geopotential coefficients,

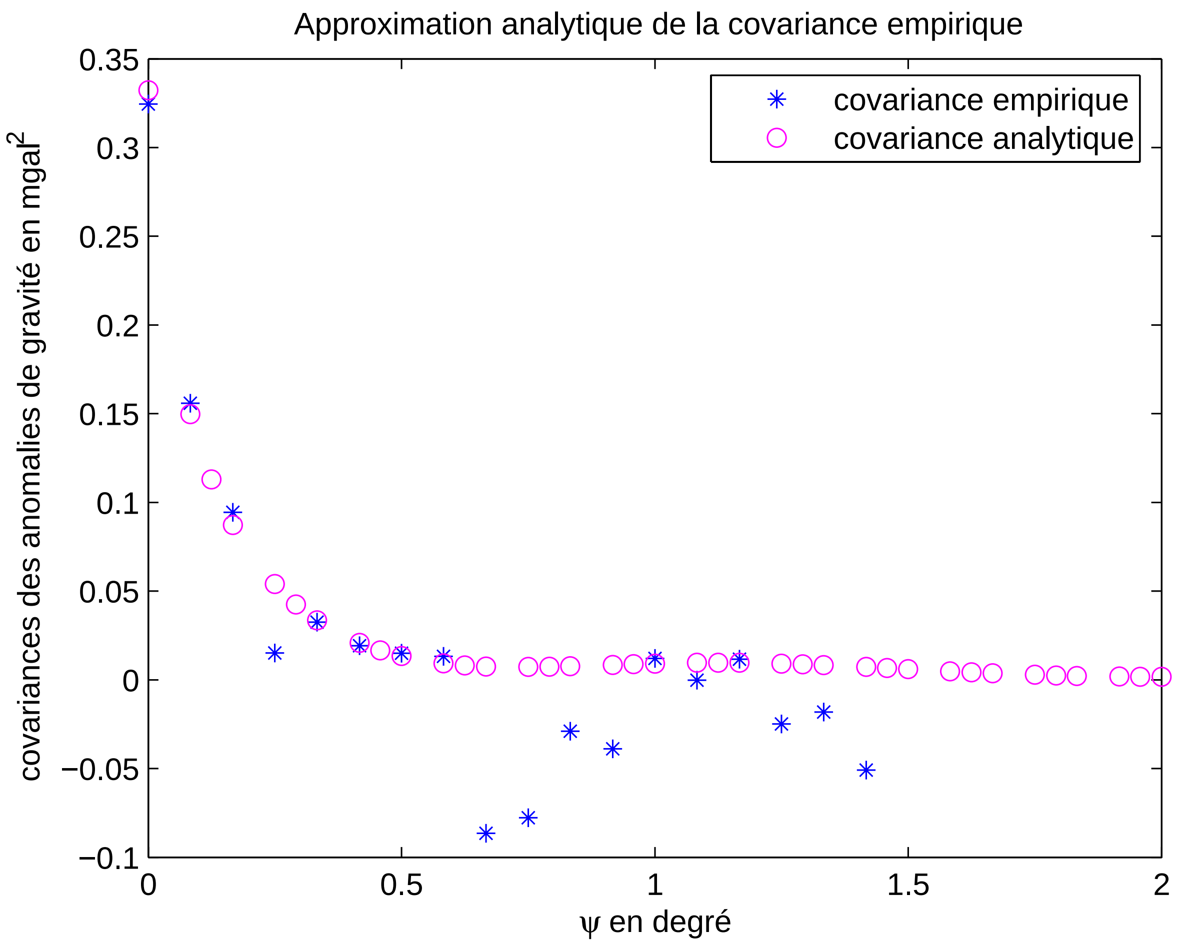
  and $\sigma_n = \dfrac{A(n-1)}{(n-2)(n+B)}$.

- For a choice of B, α, A and R_B must be computed using the data within the specified area

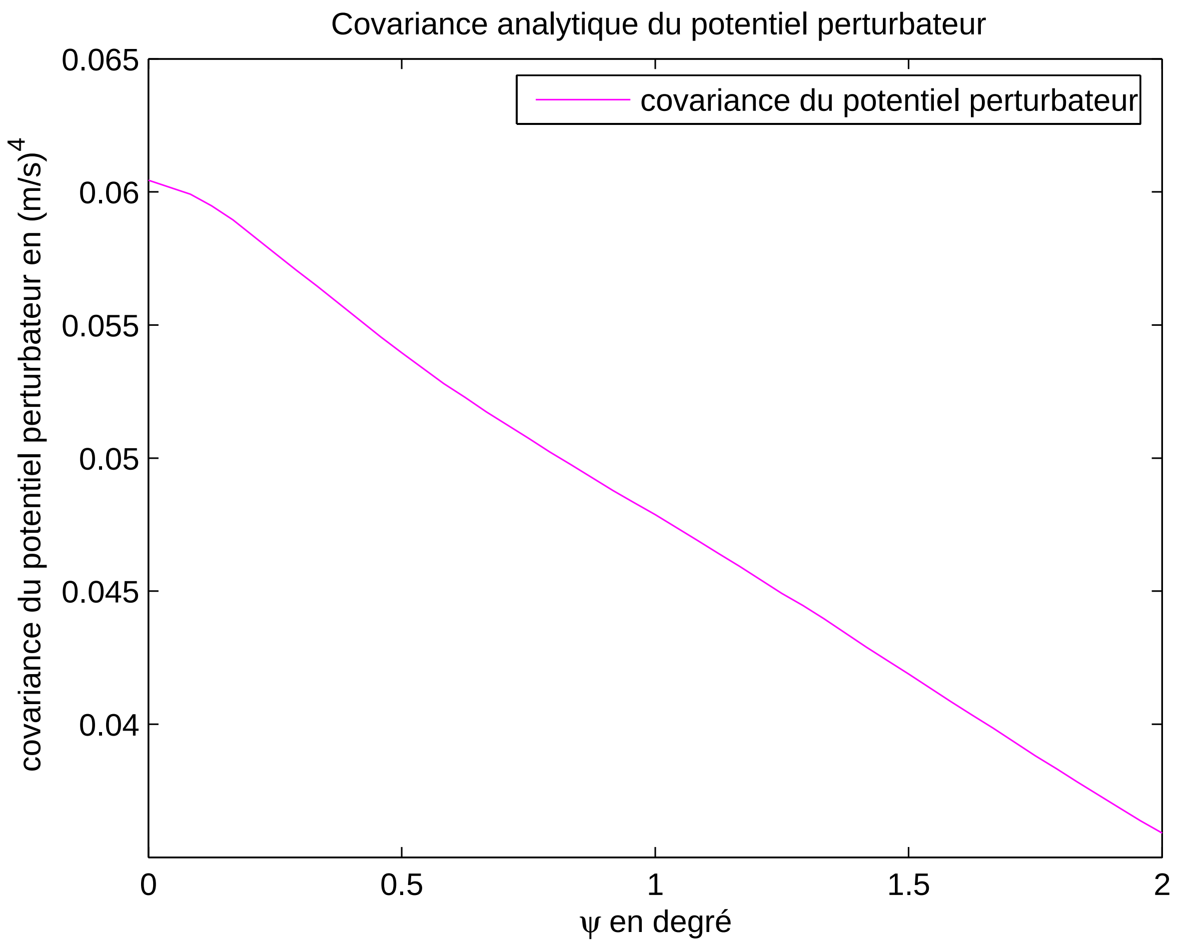
- We use the program COVFIT (Tscherning and Knudsen)

- The other auto-covariances, and cross-covariances are deduced from the linear expressions and propagation laws of covariance

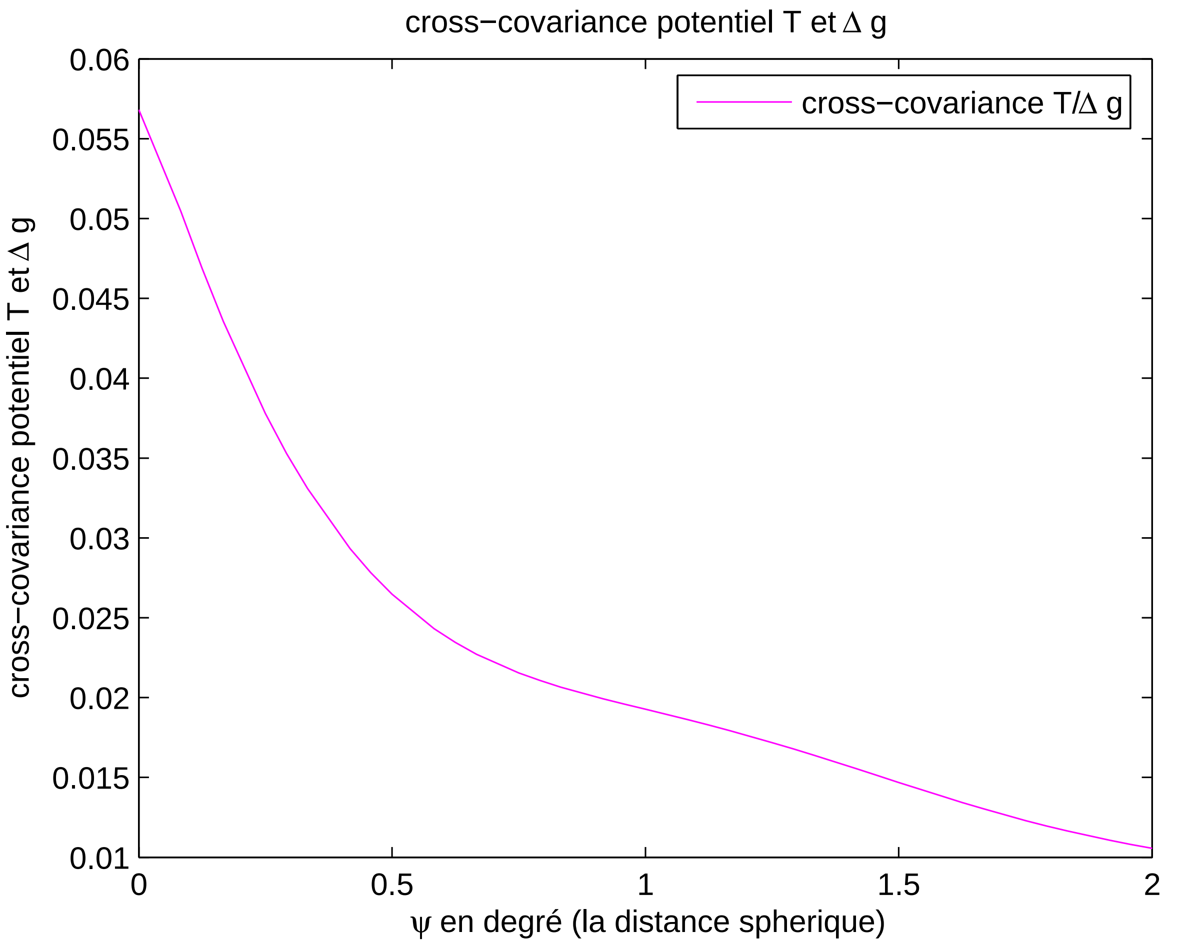
Representation of the covariance function



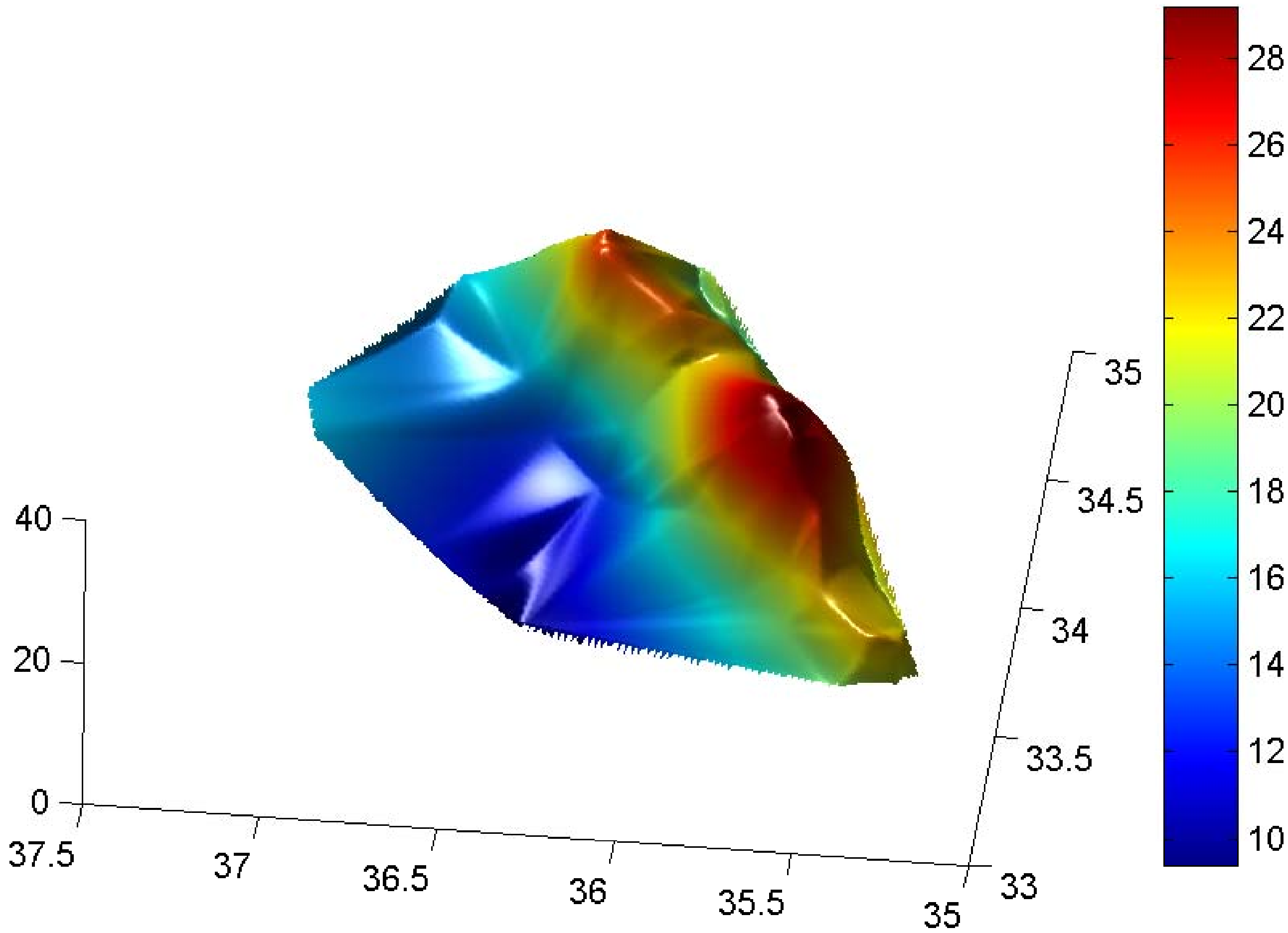
Representation of the covariance function



Representation of the covariance function



Example



40
20
0
37.5
37
36.5
36
35.5
35
33
33.5
34
34.5
35
28
26
24
22
20
18
16
14
12
10

II. Inverse problem.

Basic description of the problem

The goal is to define a point-mass system which generates a field of gravitation as close as possible to the actual field on a given geographical region.

Therefore, from the region a (regular) grid is defined on a burried spherical surface (centered in the center of the earth, radius R).

In every point (R, λ_i, θ_i) $(i = 1, \cdots, M)$ of the grid a point-mass m_i is put.

These masses generates a potential which is linear with respect to the masses.

When considering variable depths for the masses, the potential is non linear with respect to these unknowns.

Least Square problems in a subspace of $\mathcal{L}^2(S^2)$

The normal potential U is defined by a series expansion (spherical harmonics):

$$
\begin{aligned}
U(r,\theta,\phi) &= \frac{GM}{r}\left(1+\sum_{n=2}^{+\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}\left(\bar{c}_{nm}\cos m\phi+\bar{s}_{nm}\sin m\phi\right)\bar{P}_{nm}(\cos\theta)\right)\\
&= \frac{GM}{r}\left(1+\sum_{n=2}^{+\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}\left(\bar{c}_{nm}\bar{\mathcal{R}}_{nm}+\bar{s}_{nm}\bar{\mathcal{S}}_{nm}\right)\right).
\end{aligned}
$$

The anomalous potential T is defined on the geoid. By an integral representation, it is known everywhere in the exterior of the geoid.

Therefore $W = U + T$ can be known on a given sphere S^2 centered at the earth center and containing the geoid in its interior.

Least Square problems in a subspace of $\mathcal{L}^2(S^2)$ (2)

The harmonic functions restricted to S^2 are functions

$$f(\theta,\phi) = \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \left[a_{nm}\, \bar{\mathcal{R}}_{nm}(\theta,\phi) + b_{nm}\, \bar{\mathcal{S}}_{nm}(\theta,\phi) \right]$$

which are expressed in the orthonormal basis

$$(\bar{\mathcal{R}}_{nm}, \quad \bar{\mathcal{S}}_{nm}, \quad 0 \le m \le n).$$

The goal is to find the mass distribution which minimizes the quantity :

$$\frac{1}{2} \iint_{S^2} (W - \widetilde{W})^2 d\sigma$$

where $\widetilde{W}$ is the potential generated by the system of masses.

Expressing the least squares problem

Let $\mathbf{m} \in \mathbb{R}^M$ the vector of masses and $\mathbf{x} \in \mathbb{R}^M$ the vector of the unknown position parameters (i.e. the depths).

The potential W is represented by the vector w in the basis on S^2.

For all i, vector $a_i(x_i)$ represents the potential of a unit mass located at the i-th position of the grid at a depth x_i.

By denoting $A(\mathbf{x}) = [a_1(x_1), \cdots, a_M(x_M)]$, the linear least squares problem to solve is expressed by :

$$\begin{aligned}\rho &= \min_{\mathbf{x}} \min_{\mathbf{m}} \|w - A(\mathbf{x})\mathbf{m}\|_2 \\ &= \min_{\mathbf{x}} \|w - A(\mathbf{x})A(\mathbf{x})^{+}w\|_2.\end{aligned}$$

Ongoing work and perspectives

First tests for the linear problem are done.

A theoretical and practical study must be done to measure the sensitivity of the minimum w.r.t. the depth of the grid.

A gradient algorithm must be defined to ajust the depth of the masses.

Amine is working hard on that...