Localized spectral analysis for an inverse problem in geodesy

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ABSTRACT. A distribution of ponctual masses (caracterized by its intensities and positions) is determinated, in such a way that the associated equivalent potential best approximates a given potential field. For this purpose, a geodetic inverse problem is solved. On the whole unit sphere a potential function is usually expressed in spherical harmonics, basis functions with global support. The identification of the two potentials is done by solving a least-squares problem. When a limited area is studied, the estimation of the point-mass parameters by means of spherical harmonics is prone to error, since they are no longer orthogonal over a partial domain of the sphere. The construction of a local spherical harmonic basis that is orthogonal over the specified limited domain of the sphere, allows us to treat the local point-mass determination problem.

RÉSUMÉ. Une distribution de points masses (caractérisés par leur intensités et leurs positions) est déterminée de telle manière qu'elle génère un potentiel qui approche au mieux un potentiel donné. On définit ainsi un problème inverse (moindres carrés). Sur la sphère unité le problème est résolu en identifiant les développements en harmoniques sphériques des deux potentiels. Lorsque seule une région de la sphére unité est considérée, l'estimation des paramètres des points masses en utilisant la base des harmoniques sphériques est susceptible d'erreur, puisque la propriété d'orthogonalité des éléments de la base n'est plus vérifiée. La construction d'une base locale orthogonale sur la région étudiée permettra de résoudre le problème local des points masses.

KEYWORDS : Inverse problem, point masses, gravitational potential, spherical harmonics, global basis, local basis, limited domain, Slepian functions

MOTS-CLÉS : Problème inverse, masses ponctuelles, potentiel gravitationnel, spheriques harmoniques, base globale, base locale, domaine limité, fonctions de Slepian

1. Introduction

Given a potential gravitational field of the earth, we are interested in constructing a distribution of buried point masses, in such a way that the generated potential best fits the given one. In the case of the gravity potential of the whole earth, the buried masses are determined by solving an inverse problem using the spherical harmonics basis. On the other hand, when only a limited region of the earth is studied, the spherical harmonics basis is no longer orthogonal. Consequently, the spherical harmonic basis for the determination of the local point-mass problem is not appropriate and hence one has to find another basis which is localized in the region of interest. The question of finding a local basis was first studied by Slepian, Pollak and Landau for an interval of the real line and for a rectangular region of the plane [8, 9, 14, 15, 16, 17]. They discovered by serendipity a basis of functions (now called Slepian functions) whose energies are concentrated in the considered region.

The basis functions are orthonormal on the sphere and orthogonal on the specified region. So that any arbitrary concentrated function on this region can be expanded into a local spherical harmonics series. This procedure was very useful in several domains of applied mathematics and physics, notably geophysics [1, 10], cosmology [5] and image processing [3, 11]. In this paper a set of point masses is determined in such a way that the associated potential best approximates the given gravitational potential on a specified region.

2. Spherical harmonics

Let S^2 denote the unit sphere in $\mathbb{R}^3$ and let (θ, ϕ) denote a generic point on the sphere where θ is the colatitude and ϕ is the longitude. On the space of square integrable functions on S^2, we have the inner product $\langle \cdot, \cdot \rangle_{S^2}$ defined by

$$\langle f, g \rangle_{S^2} = \frac{1}{4\pi} \int_{S^2} f(\theta, \phi) g(\theta, \phi)\, d\sigma = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) g(\theta, \phi) \sin\theta\, d\theta d\phi. \quad (1)$$

We recall that any square-integrable function on the unit sphere S^2 can be expanded in terms of spherical harmonics as [4]

$$f(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{mn} Y_{nm}(\theta, \phi), \quad (2)$$

where f_{mn} is the spherical harmonic coefficient associated with $Y_{nm}(\theta, \phi)$, the spherical harmonic of degree n and order m which is given by

$$Y_{nm}(\theta, \phi) = \begin{cases} \overline{P}_{nm}(\cos\theta) \cos m\phi & \text{if m} \geq 0, \\ \overline{P}_{nm}(\cos\theta) \sin |m|\phi & \text{if m} < 0. \end{cases} \quad (3)$$

Here $\overline{P}_{nm}(\cos\theta)$ is a normalized version of $P_{nm}(\cos\theta)$, the Legendre's function of degree n and order m given by

$$\overline{P}_{nm}(\cos\theta) = \sqrt{k(2n+1)\frac{(n-m)!}{(n+m)!}} P_{nm}(\cos\theta), \quad \text{where} \quad \begin{cases} k = 1 & \text{for } m = 0, \\ k = 2 & \text{for } m \neq 0. \end{cases} \quad (4)$$

The spherical harmonics functions Y_{nm} are orthogonal with respect to inner product (1), i.e.,

$$\langle Y_{nm}, Y_{n'm'} \rangle_{S^2} = \frac{1}{4\pi} \int_{S^2} Y_{nm}(\theta, \phi)\ Y_{n'm'}(\theta, \phi)\ d\sigma = \delta_{ll'} \delta_{mm'}.$$

Accordingly, the spherical harmonics coefficients of the function f are given by

$$f_{nm} = \langle f, Y_{nm} \rangle_{S^2} = \frac{1}{4\pi} \int_{S^2} f(\theta, \phi) Y_{nm}(\theta, \phi)\ d\sigma.$$

If we denote by $\boldsymbol{f} = (\ldots, f_{nm}, \ldots)^T$, then we can define the spatial and spectral equivalent norms of the function f, respectively, as follows

$$\|f\|_{S^2}^2 = \int_{S^2} f(\theta, \phi)^2\ d\sigma, \qquad \|\boldsymbol{f}\|_2^2 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{nm}^2. \tag{5}$$

Given two points (θ, ϕ) and (θ', ϕ') on the unit sphere we have the identity

$$\sum_{m=-n}^{n} Y_{nm}(\theta, \phi) Y_{nm}(\theta', \phi') = \frac{2n+1}{4\pi} P_n(\cos \psi), \tag{6}$$

where ψ is the spherical distance between (θ, ϕ) and (θ', ϕ') given by

$$\cos \psi = \cos \theta' \cos \theta + \sin \theta' \sin \theta \cos(\phi' - \phi). \tag{7}$$

3. The gravitational potential in terms of spherical harmonics

The gravitational potential is a harmonic function outside the earth. For a point P with spherical coordinates (r, θ, ϕ) that lies outside the earth we have [7]

$$V(r, \theta, \phi) = \frac{GM}{r} \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} c_{nm} \left(\frac{a}{r}\right)^n Y_{nm}(\theta, \phi). \tag{8}$$

Here G is the universal gravitational constant, M is the total mass of the earth, a is the equatorial radius of the earth and c_{nm} is the geopotential coefficient of degree n and order m. The values of M and c_{nm} depend on the choice of the geopotential model (e.g., EGM96, OSU91, etc.) at the considered zone. Note that the first term in the summation, i.e., for $n = 0$ is GM/r, represents the potential generated by a homogeneous sphere of mass M and radius r. Thanks to satellite missions, such as CHAMP (Challenging Minisatellite Payload) and GRACE (Gravity Recovery and Climate Experiment), the coefficients c_{nm} in this expansion can now be determined up to the order $n = 360$, see e.g., [12, 13].

4. The point-mass generated potential

Let $S = \{M_k \equiv (r_k, \theta_k, \phi_k; m_k)\}_{1,..,N}$ be a distribution of N points located at (r_k, θ_k, ϕ_k) and with masses m_k. Recall that the potential at a spacial point P with coordinates (r, θ, ϕ) generated by a point mass M_k is given by

$$\widetilde{V}_k(r, \theta, \phi) = \frac{Gm_k}{\ell_k}. \tag{9}$$

Here ℓ_k is the Euclidean distance between the two points M_k and P. If we denote by ψ_k the spherical distance between M_k and P which is obtained by taking $(\theta', \phi') = (\theta_k, \phi_k)$ in (7), then $\ell_k = \sqrt{r^2 - 2rr_k \cos\psi_k + r_k^2}$. As the point masses are buried, we have $r_k < r$, $1 \leq k \leq N$, and as a consequence, the inverse of the distance between M_k and P, admits a convergent Legendre series expansion. Indeed, if we set $\alpha_k = r_k/r$ and $u_k = \cos\psi_k$, then $\ell_k = r\sqrt{1 - 2\alpha_k u_k + \alpha_k^2}$, and therefore $r/\ell_k = 1/\sqrt{1 - 2\alpha_k u_k + \alpha_k^2}$ can be expanded into power series with respect to α_k [7]: $r/\ell_k = \sum_{n=0}^{+\infty} \alpha_k^n P_n(u_k)$. Then, it follows from (6) that

$$\frac{1}{\ell_k} = \frac{1}{r}\sum_{n=0}^{+\infty} \alpha_k^n P_n(u_k) = \frac{1}{r}\sum_{n=0}^{+\infty}\sum_{m=-n}^{n} \alpha_k^n \frac{4\pi}{2n+1} Y_{nm}(\theta,\phi) Y_{nm}(\theta_k,\phi_k). \tag{10}$$

The expression (9) of the potential $\widetilde{V}_k$ becomes

$$\widetilde{V}_k(r,\theta,\phi) = \frac{Gm_k}{r}\sum_{n=0}^{+\infty}\sum_{m=-n}^{n} \alpha_k^n \frac{4\pi}{2n+1} Y_{nm}(\theta,\phi) Y_{nm}(\theta_k,\phi_k).$$

The potential $\widetilde{V}$ generated by the N point masses, is the sum of the potentials generated by the considered elementary buried masses

$$\widetilde{V}(r,\theta,\phi) = \frac{G}{r}\sum_{k=1}^{N} m_k \sum_{n=0}^{+\infty}\sum_{m=-n}^{n} \left(\frac{r_k}{r}\right)^n \frac{4\pi}{2n+1} Y_{nm}(\theta,\phi) Y_{nm}(\theta_k,\phi_k). \tag{11}$$

On a selected sphere (i.e., for a fixed value $\tilde{R}$ of r) we seek the masses and positions of the buried points, so that the resulting generated potential best approximates the gravitational potential for the considered region. It consists in solving an inverse problem which we formulate in the following section.

5. Statement of the inverse problem

In the forward problem of point masses, we assume that the masses and their positions are known, and we look for the potential generated by them. Considering the expression of the elementary potential (9), we gave in (11) the potential caused by a distribution of point masses at a spacial point P with spherical coordinates $(\tilde{R}, \theta, \phi)$, as the sum of the elementary potentials generated by all the point masses. The inverse problem, is then to find the masses $\boldsymbol{m} = (m_1, \ldots, m_N)^T$ and positions $\boldsymbol{\xi} = (r_1, \theta_1, \phi_1, \ldots, r_N, \theta_N, \phi_N)^T$ of the point masses for which the difference between the chosen gravitational potential model and the predicted one is minimized. Eventually, some constraints on those unknowns need to be imposed so that uniqueness and stability of the solution are guaranteed. If we assume that the colatitudes and longitudes (θ_k, ϕ_k) of the point masses are distributed on a fixed regular grid on the unit sphere then the unknown parameters are the masses $\boldsymbol{m} = (m_1, \ldots, m_N)^T$ and depths $\boldsymbol{r} = (r_1, \ldots, r_N)^T$. The approach we use here, is based on the identification of the two potentials (i.e., the modeled and the predicted) in the same orthogonal basis. In the following sections, we describe the orthogonal basis to be used which varies with the region of interest: the whole earth or on a limited region of the earth.

5.1. Point-mass determination on the whole earth

To find the distribution of point masses that generates a potential field that best fits (in the sense of least squares) a given potential field we introduce the real-valued function defined on $\mathbb{R}^{3N} \times \mathbb{R}^N$ by

$$F(\boldsymbol{\xi}, \boldsymbol{m}) = \tfrac{1}{2} \int_{S^2} |V(\tilde{R}, \theta, \phi) - \widetilde{V}(\tilde{R}, \theta, \phi)|^2 \, d\sigma,$$

and we consider the following minimization problem

$$\min_{\boldsymbol{\xi} \in \mathbb{R}^{3N}, \boldsymbol{m} \in \mathbb{R}^N} F(\boldsymbol{\xi}, \boldsymbol{m}). \tag{12}$$

That is we are interested in finding the masses $\boldsymbol{m}$ and positions $\boldsymbol{\xi}$ of the points so that the function $F(\boldsymbol{\xi}, \boldsymbol{m})$ attains its minimum.

Using the spherical harmonic expansions (8) and (11) and the orthonormality of the spherical harmonics, this minimization problem can be written as

$$\min_{\boldsymbol{\xi} \in \mathbb{R}^{3N}, \boldsymbol{m} \in \mathbb{R}^N} \tfrac{1}{2} \|\boldsymbol{b} - A(\boldsymbol{\xi}) \boldsymbol{m}\|^2, \tag{13}$$

where $\boldsymbol{b}$ is a vector with infinite numbers of rows and $A(\boldsymbol{\xi})$ is a matrix function of $\boldsymbol{\xi}$ with an infinite number of rows and N columns. The entries of $\boldsymbol{b}$ and $A(\boldsymbol{\xi})$ are

$$b_{I(n,m)} = M c_{nm} \left(\frac{a}{\tilde{R}} \right)^n, \quad I(n,m) = 1, 2, \ldots,$$

$$A_{I(n,m)k}(\boldsymbol{\xi}) = \frac{4\pi}{(2n+1)} \left(\frac{r_k}{\tilde{R}} \right)^n Y_{nm}(\theta_k, \phi_k), \quad I(n,m) = 1, 2, \ldots, \quad k = 1, \ldots, N,$$

where $I(n,m) = n(n+1) + m + 1$. We shall assume that the positions of the point masses are such that the matrix $A(\boldsymbol{\xi})$ has full column rank. Otherwise, we can arrange by adding/removing a point mass so that $A(\boldsymbol{\xi})$ satisfies this hypothesis.

As we shall see, this minimization problem can be formulated as a (nonlinear) minimization problem for $\boldsymbol{\xi}$ only. We first solve the quadratic minimization problem for $\boldsymbol{m}$

$$\min_{\boldsymbol{m} \in \mathbb{R}^N} \tfrac{1}{2} \|\boldsymbol{b} - A(\boldsymbol{\xi}) \boldsymbol{m}\|^2,$$

whose solution is readily given by $\hat{\boldsymbol{m}} = A^+(\boldsymbol{\xi}) \boldsymbol{b}$, where $A^+(\boldsymbol{\xi})$ stands for the pseudo-inverse of $A(\boldsymbol{\xi})$. Then the problem (13) is equivalent to the minimization problem

$$\min_{\boldsymbol{\xi} \in \mathbb{R}^{3N}} \tfrac{1}{2} \|\boldsymbol{b} - A(\boldsymbol{\xi}) A^+(\boldsymbol{\xi}) \boldsymbol{b}\|^2. \tag{14}$$

Let us denote by $\mathcal{P}(\boldsymbol{\xi}) = I - A(\boldsymbol{\xi}) A^+(\boldsymbol{\xi})$, the orthogonal projector onto the complement of the range of $A(\boldsymbol{\xi})$. Then by noting that $\mathcal{P}(\boldsymbol{\xi})^T = \mathcal{P}(\boldsymbol{\xi})$ and $\mathcal{P}(\boldsymbol{\xi})^2 = \mathcal{P}(\boldsymbol{\xi})$ we obtain

$$\min_{\boldsymbol{\xi} \in \mathbb{R}^{3N}} \tfrac{1}{2} \|\mathcal{P}(\boldsymbol{\xi}) \boldsymbol{b}\|^2 = \min_{\boldsymbol{\xi} \in \mathbb{R}^{3N}} \tfrac{1}{2} \boldsymbol{b}^T \mathcal{P}(\boldsymbol{\xi})^T \mathcal{P}(\boldsymbol{\xi}) \boldsymbol{b} = \min_{\boldsymbol{\xi} \in \mathbb{R}^{3N}} \tfrac{1}{2} \boldsymbol{b}^T \mathcal{P}(\boldsymbol{\xi}) \boldsymbol{b}. \tag{15}$$

The first-order optimality condition of this minimization problem is the vanishing of the gradient of the objective function

$$\rho(\boldsymbol{\xi}) = \tfrac{1}{2} \boldsymbol{b}^T \mathcal{P}(\boldsymbol{\xi}) \boldsymbol{b}.$$

Straightforward computation shows that

$$(\nabla\rho(\boldsymbol{\xi}))_i = -\left[\boldsymbol{b}^T\mathcal{P}(\boldsymbol{\xi})\right]\frac{dA(\boldsymbol{\xi})}{d\xi_i}\left[A^+(\xi)\boldsymbol{b}\right], \quad i = 1,\dots,3N. \tag{16}$$

The solution of the minimization problem (14) is given by the solution of the nonlinear equations $(\nabla\rho(\boldsymbol{\xi}))_i = 0, i = 1,\dots,3N$, which we solve iteratively. Thus, the problem of finding the point-mass parameters for the global gravitational potential has been solved.

5.2. Point-mass determination on a limited region of the earth

The fact that, on a limited region of the earth, the spherical harmonics basis is no longer orthogonal, makes the computations of the entries of the matrix $A(\boldsymbol{\xi})$ and vector $\boldsymbol{b}$ more complicated. By using a new basis of functions that are concentrated on the region of interest and orthogonal simplifies the computations.

5.2.1. Slepian basis of localized functions

A stated earlier, a square-integrable function f on S^2 can be represented as

$$f(\theta,\phi) = \sum_{n=0}^{+\infty}\sum_{m=-n}^{n} f_{nm}Y_{nm}(\theta,\phi) \qquad \text{with} \qquad f_{nm} = \int_{S^2} f(\theta,\phi)Y_{nm}(\theta,\phi)\,d\sigma.$$

We say that f is band-limited with band width K if $f_{nm} = 0$ for all $n > K$, i.e., it has the representation

$$f(\theta,\phi) = \sum_{n=0}^{K}\sum_{m=-n}^{n} f_{nm}Y_{nm}(\theta,\phi). \tag{17}$$

To find a basis of functions that are localized on a given region Ω of the unit sphere, we seek functions f such that the ratio

$$\lambda = \frac{\int_\Omega f^2(\theta,\phi)d\sigma}{\int_{S^2} f^2(\theta,\phi)\,d\sigma}. \tag{18}$$

between its energy over Ω and its energy over S^2 is maximized. This maximization problem is equivalent to an eigenvalues and eigenfunctions problem. When the set $\Omega \subset S^2$ has positive measure and is not equal to S^2, we have $0 < \lambda < 1$ for any f. If f is band-limited with band width K then (18) reduces to

$$\lambda = \frac{1}{\|\boldsymbol{f}\|_2^2}\sum_{n=0}^{K}\sum_{m=-n}^{n} f_{nm}\sum_{n'=0}^{K}\sum_{m'=-n'}^{n'} D_{nm,n'm'}f_{n'm'}, \tag{19}$$

where

$$D_{nm,n'm'} = \int_\Omega Y_{nm}(\theta,\phi)Y_{n'm'}(\theta,\phi)\,d\sigma.$$

If we denote by D the $(K+1)^2 \times (K+1)^2$ matrix with coefficients $D_{nm,n'm'}$ and by $\boldsymbol{f} = (f_{00},\dots,f_{nm},\dots,f_{KK})^T$ the f-spherical harmonic coefficients vector, then $\lambda = \dfrac{\boldsymbol{f}^T D\boldsymbol{f}}{\boldsymbol{f}^T\boldsymbol{f}}$, and the maximization problem becomes the algebraic eigenvalue problem

$$D\boldsymbol{f} = \lambda\boldsymbol{f}. \tag{20}$$

The matrix D is real, symmetric and positive definite, so its $(K+1)^2$ eigenvalues λ and associated eigenvectors $\boldsymbol{f}$ are always real. The eigenvalues $\lambda_1, \ldots, \lambda_{(K+1)^2}$ are ordered such that $1 > \lambda_1 \geq \lambda_2 \ldots \geq \lambda_{(K+1)^2} > 0$. The symmetry of D guaranties the orthogonality of the eigenvectors. Consequently,

$$\boldsymbol{f}_p^T \boldsymbol{f}_q = \delta_{pq}, \qquad \boldsymbol{f}_p^T D \boldsymbol{f}_q = \lambda_p \delta_{pq}. \tag{21}$$

Every eigenvector $\boldsymbol{f}_p$, $p = 1, 2, \ldots, (K+1)^2$ defines an associated band-limited spatial eigenfunction $f_p(\theta, \phi)$ of the form (17). We remark that these functions are at the same time orthonormal over the whole sphere S^2 and orthogonal over the region Ω, i.e.,

$$\int_{S^2} f_p(\theta,\phi) f_q(\theta,\phi)\, d\sigma = \delta_{pq}, \qquad \int_{\Omega} f_p(\theta,\phi) f_q(\theta,\phi)\, d\sigma = \lambda_p \delta_{pq}. \tag{22}$$

If now we multiply the eigenvalue equation (20) by $Y_{nm}(\theta, \phi)$ and sum over all $0 \leq n \leq K$ and $-n \leq m \leq n$, we deduce that the eigenfunction f satisfies also the Fredholm integral eigenvalue equation of the second kind in Ω

$$\mathcal{D}(f)(\theta,\phi) := \int_{\Omega} D((\theta,\phi),(\theta',\phi')) f(\theta',\phi') d\sigma = \lambda f(\theta,\phi), \qquad (\theta,\phi) \in \Omega,$$

where $\mathcal{D}$ is the integral operator and D is its associated kernel. The kernel D is symmetric and depends only on the spherical distance, ψ, between (θ, ϕ) and (θ', ϕ')

$$D((\theta,\phi),(\theta',\phi')) = \sum_{n=0}^{K} \frac{2n+1}{4\pi} P_n(\cos\psi).$$

When Ω is the spherical cap $\alpha \leq \cos\theta \leq 1$, Grünbaum *et al.* [6] discovered a second-order differential operator $\mathcal{H}$ that commutes with the integral operator $\mathcal{D}$:

$$\mathcal{H} = \frac{d}{dx}\left[(1-x^2)(\alpha - x)\frac{d}{dx}\right] - K(K+2)x - \frac{m^2(\alpha - x)}{1-x^2}, \quad \forall\, 0 \leq m \leq K,$$

where $x = \cos\theta$. It follows that both $\mathcal{D}$ and $\mathcal{H}$ possess the same eigenfunctions (the Slepian functions), but not necessarily the eigenvalues. They found that the eigenfunctions of the operator $\mathcal{H}$ are much easier to obtain. From a numeric point of view this requires a diagonalization of the tridiagonal matrix of the operator $\mathcal{H}$. Every eigenvector of $\mathcal{H}$ defines a Slepian basis function through its spherical harmonic coefficients.

5.2.2. Point-mass determination using the Slepian basis

In section 5.2.1 we constructed a local basis of the band-limited and spatial-limited concentrated functions on a domain $\Omega \in S^2$. We here briefly discuss the problem of point-mass determination on a limited region of the earth which we solve in an analogous manner to that we used for the global one. Basically, the two main steps are:

i) The point-mass generated potential $\tilde{V}$ and the analytic gravitational potential V over the considered region are expressed in terms of the Slepian basis by exploiting the orthogonality properties (22).

ii) The local point-mass parameters are determined by following exactly the same arguments used in section 5.1 and described in equations (12)$-$(16).

6. Conclusion

The use of localized spectral analysis allowed us to study the inverse problem of point-mass determination and avoid the discrepancies we may have when the spherical harmonics basis is used for the local problem. The numerical solution of the earth and regional point-mass determination problems is the subject of forthcoming paper.

7. References

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