Petri Algebras

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Abstract. The firing rule of Petri nets relies on a residuation operation for the commutative monoid of natural numbers. We identify a class of residuated commutative monoids, called Petri algebras, for which one can mimic the token game of Petri nets to define the behaviour of generalized Petri net whose flow relation and place contents are valued in such algebraic structures. We show that Petri algebras coincide with the positive cones of lattice-ordered commutative groups and constitute the subvariety of the (duals of) residuated lattices generated by the commutative monoid of natural numbers. We introduce a class of nets, termed lexicographic Petri nets, that are associated with the positive cones of the lexicographic powers of the additive group of real numbers. This class of nets is universal in the sense that any net associated with some Petri algebras can be simulated by a lexicographic Petri net. All the classical decidable properties of Petri nets however are undecidable on the class of lexicographic Petri nets. Finally we turn our attention to bounded nets associated with Petri algebras and show that their dynamics can be reformulated in term of MV-algebras.

1 Introduction

The Petri net model is a graphical and mathematical modeling tool that, since its introduction in the early sixties, have come to play a pre-eminent role in the formal study of concurrent discrete-event dynamic systems. A Petri net $(P, T, Pre, Post)$ consists of a finite set P of places, a finite set T of transitions (disjoint from P), and flow relations $Pre, Post : P \times T \to \mathbb{N}$. Places can contain some tokens representing the resources available in this place for the current configuration. A configuration of a Petri net is given as a vector $M : P \to \mathbb{N}$, called marking, indicating the number of tokens available in each place. Tokens are consumed and produced by the firing of transitions according to the so-called token game

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$$M\,[t\rangle\,M' \Leftrightarrow (\forall p \in P)\ M(p) \sqsupseteq Pre(p,t) \wedge M'(p) = (M(p) - Pre(p,t)) + Post(p,t)$$

The token game of Petri net says that in order for a transition t to fire in marking M it should be the case that each place contains enough resources as it is expressed by the condition $M(p) \sqsupseteq Pre(p,t)$ where $\sqsubseteq$ is the usual order relation on $\mathbb{N}$. Then the firing of transition t proceeds in two stages : a consumption of resources ($Pre(p,t)$ tokens are removed from place p) followed by a production of resources ($Post(p,t)$ tokens are added to place p). The notation $M\,[t\rangle\,M'$ expresses the fact that transition t is allowed to fire in marking M and that firing t in marking M produces the new marking M'. Numerous techniques, supported and automated by software tools, can be used to verify that some required properties are met for systems specified using Petri nets. For instance reachability, coverability, place-boundedness, deadlock and liveness can be decided on the class of Petri nets [13].

Numerous extensions of this basic model of Petri nets have been introduced over the years. Some of them are high level nets that allow for more compact representations but do not increase the expressive power of Petri nets: these high level nets can be unfolded into equivalent, even though in general much larger, Petri nets. Some extensions however change more dramatically the semantics of the original model. For instance timing constraints may be added, as in timed Petri nets or stochastic Petri nets for the purpose of enabling performance analysis. With continuous Petri nets the discrete state transition rule is replaced by a notion of trajectory using a continuum of intermediate states. In Fuzzy Petri nets one has a possibilistic measure of the firing of a transition in the given marking thus enabling to deal with incertainty. Our purpose in this paper is to put forward an axiomatisation of the token game of Petri nets. More precisely we identify a class of commutative residuated monoids, called Petri algebras, for which one can mimic the token game of Petri nets to define the behaviour of generalized Petri nets whose flow relations and place contents are valued in such algebraic structures. The sum and its associated residuation capture respectively how resources within places are produced and consumed through the firing of a transition. The class of usual Petri nets is associated with the commutative monoid of natural numbers. We show that Petri algebras coincide with the positive cones of lattice-ordered commutative groups and constitute the subvariety of the (duals of) residuated lattices generated by the commutative monoid of natural numbers. The basic Petri net model is thus associated with the generator of the variety of Petri algebras which shows that these extended nets share all algebraic properties of Petri nets, in particular they have the same equational and inequational theory. We however exhibit a Petri algebra whose corresponding class of nets is strictly more expressive than the class of Petri nets, i.e. their class of marking graphs is strictly larger. More precisely, we introduce a class of nets, termed lexicographic Petri nets, that are associated with the positive cones of the lexicographic powers of the additive group of real numbers. This class of nets is proved to be universal in the sense that any net associated with some Petri algebra can be simulated by a lexicographic Petri net. All the classical decidable properties of Petri nets however (termination, covering, boundedness, structural boundedness, accessibility, deadlock, liveness ...) are proved to be undecidable on the class of lexicographic Petri nets. Finally we turn our attention to bounded nets associated with Petri algebras and show that their dynamics can be reformulated in term of MV-algebras.

2 An Axiomatisation of the Token Game

In order to obtain an axiomatisation of the token game of Petri nets we represent the marking of a net as a map $M : P \to \bigsqcup_{p \in P} A_p$ that associates with each place $p \in P$ the local value of the current configuration $M(p) \in A_p$ in this place. Content of places are resources that are consumed and produced according to the token game. Thus we assume that each place $p \in P$ is associated with a commutative divisibility monoid $A_p = (A_p, \oplus, 0)$, i.e. a monoid such that

$$\text{the relation } a \sqsupseteq b \;\; \Leftrightarrow \;\; \exists c \cdot a = b \oplus c \;\; \text{is an order relation} \tag{1}$$

The constant 0 represents the absence of resource and the binary operator $\oplus$ the accumulation of resources in places. Immediate consequences of condition (1) are the following:

$$\begin{array}{l} a \oplus b \sqsupseteq a, b \\ 0 \sqsubseteq a \\ a \oplus b = 0 \;\; \Rightarrow \;\; a = b = 0 \end{array}$$

Moreover we need to have a residuation operation $\ominus$ such that $a \ominus b$ represents the residual resource obtained by substracting b from a when $b \sqsubseteq a$. Thus the following should hold true:

$$b \sqsubseteq a \;\; \Rightarrow \;\; a = (a \ominus b) \oplus b \tag{2}$$

Usual Petri nets corresponds to the situation where, for every place p, $A_p = (\mathbb{N}, +, 0)$ is the commutative monoid of natural numbers with the truncated difference $n \ominus m = \max(0; n - m)$ as residuation. This operation is characterized by the universal property that for every natural numbers n, m and p

$$n + m \sqsupseteq p \;\; \Leftrightarrow \;\; n \sqsupseteq p \ominus m$$

Up to the reversal of the order relation, it is a *commutative residuated monoid i.e.* a commutative monoid $(A, \oplus, 0)$ with an order relation $\leq$ and a residuation operation $\ominus$ which is a right adjoint to the addition, in the sense that

$$a \oplus b \leq c \;\; \Leftrightarrow \;\; a \leq c \ominus b \tag{3}$$

It follows immediately from this definition that a commutative monoid is residuated if and only if its addition is order preserving in each argument and the inequation $a \oplus b \leq c$ has a largest solution for a (namely $c \ominus b$). In particular the residual is uniquely determined by the addition and the order relation.

When the monoid is a divisibility monoid the order relation itself is defined in terms of the addition and thus the whole structure is characterized by its monoid reduct.

Proposition 1. *Let $(A, \oplus, 0, \sqsubseteq)$ be a commutative monoid where the neutral element is also the least element for the order relation, we assume that this monoid is co-residuated in the sense that there exists a residuation operation $\ominus$ such that*

$$a \oplus b \sqsupseteq c \;\; \Leftrightarrow \;\; a \sqsupseteq c \ominus b \tag{4}$$

then the following conditions are equivalent

(i) It is a divisibility monoid: $a \sqsupseteq b \;\; \Leftrightarrow \;\; \exists c \cdot a = b \oplus c$
(ii) It is an upper semi-lattice with: $a \sqcup b \; = \; (a \ominus b) \oplus b$
(iii) $b \sqsubseteq a \;\; \Rightarrow \;\; a = (a \ominus b) \oplus b$

Definition 2. *A* Petri pre-structure *is a commutative monoid equipped with a residuation operation* $(M, \oplus, 0, \ominus)$ *satisfying the conditions (1) and (4).*

The firing of a transition proceeds in two stages: a consumption of resources in the input places followed by a production of resources in the output places. More precisely, the transition relation $M\,[t\rangle\,M'$ stating that transition t can fire in marking M and leads, when it is fired, to the new marking M' is given by:

$$M\,[t\rangle\,M' \;\Leftrightarrow\; \forall p \in P \;\; M(p) \sqsupseteq Pre(p,t) \wedge M'(p) = (M(p) \ominus Pre(p,t)) \oplus Post(p,t)$$

A net is called *homogeneous* if all the algebras A_p are identical. We will stick to homogeneous nets until Section 3 where it will be noticed that the "multi-sorted" case adds in fact no extra generality. By the way we also restrict our attention in this paper to commutative algebras. With non commutative monoids it would be possible [1] for example to take fifo nets [11] into account.

For any non empty sequence of transitions $u = a_0 \ldots a_{n-1} \in T^+$we let $M\,[u\rangle\,M'$ state the existence of markings $M = M_0,\, M_1,\, \ldots,\, M_n = M'$ such that $M_i\,[a_i\rangle\,M_{i+1}$ for every $0 \leq i < n$. Moreover we set $M\,[\varepsilon\rangle\,M$ where $\varepsilon \in E^*$ is the empty sequence and M an arbitrary marking. We use $M\,[u\rangle$ (respectively $[u\rangle\,M'$) as a shorthand for $\exists M' \;\; M\,[u\rangle\,M'$ (resp. $\exists M \;\; M\,[u\rangle\,M'$). If $a, b \in T$ are transitions in a (n usual) Petri net we have the following equivalences (using the vectorial notations $\mathcal{P}\,(t) = (\mathcal{P}\,(p,t)\,; p \in P) \in \mathbb{N}^P$ for $\mathcal{P} \in \{Pre,\, Post\}$)

$$\begin{aligned}
M\,[ab\rangle \;\; &\Leftrightarrow \;\; M \sqsupseteq Pre(a) \text{ and } (M - Pre(a)) + Post(a) \sqsupseteq Pre(b) \\
&\Leftrightarrow \;\; M \sqsupseteq \max\,(Pre(a); Pre(a) + (Pre(b) - Post(a))) \\
&\Leftrightarrow \;\; M \sqsupseteq Pre(a) + \max\,(0; Pre(b) - Post(a)) \\
&\Leftrightarrow \;\; M \sqsupseteq Pre(a) \oplus (Pre(b) \ominus Post(a))
\end{aligned}$$

This suggests to let $Pre(uv) = Pre(u) \oplus (Pre(v) \ominus Post(u))$ for any sequences $u, v \in T^*$ and symmetrically $Post(uv) = (Post(u) \ominus Pre(v)) \oplus Post(v)$. For these definitions to make sense however, it remains to show that they do not depend upon the specific chosen decomposition $w = uv$; otherwise stated, the product defined on $A \times A$ by $(x, y) \otimes (x', y') = (x \oplus (x' \ominus y), (y \ominus x') \oplus y')$ should be associative.

Theorem 3. *For any Petri pre-structure, the following conditions are equivalent:*
(i) Operation $\otimes$ is associative,
(ii) the identity $(b \oplus c) \ominus a = (b \ominus (a \ominus c)) \oplus (c \ominus a)$ holds,
(iii) the monoid is cancellable: $a \oplus b = a \oplus c \;\Rightarrow\; b = c$, and
(iv) the identity $(a \oplus b) \ominus b = a$ holds.

Definition 4. *A Petri algebra is a Petri pre-structure with a cancellable monoid reduct.*

Corollary 5. *Petri algebras satisfy the following equivalence*

$$a \sqsupseteq b \oplus c \quad \Leftrightarrow \quad a \sqsupseteq b \text{ and } a \ominus b \sqsupseteq c \tag{5}$$

Identity (ii) of Theorem 3 is an internalization of (5) using the axiomatization of the order relation: $a \sqsubseteq b \Leftrightarrow a \ominus b = 0$.

Let us consider a net over a Petri algebra A, then we can inductively define the applications Pre, $Post : P \times T^* \to A$ by letting $\varphi(p, u) = (Pre(p, u), Post(p, u))$ where $\varphi(p, -) : T^* \to A \times A$ is the unique monoid morphism such that the images $\varphi(p, t) = (Pre(p, t), Post(p, t))$ of the generators $t \in T$ be given by the flow relations of the net. Then the following holds:

$$\begin{array}{l} Pre(p, \varepsilon) = Post(p, \varepsilon) = 0 \\ Pre(p, uv) = Pre(p, u) \oplus (Pre(p, v) \ominus Post(p, u)) \\ Post(p, uv) = (Post(p, u) \ominus Pre(p, v)) \oplus Post(p, v) \end{array}$$

Theorem 6. *The generalized transition relation $M\,[u\rangle\, M'$ stating the existence of a sequence u of transitions leading from M to M' is given by any of the three following equivalent conditions*

1. $\forall p \in P \;\; M(p) \sqsupseteq Pre(p, u)$ *and* $M'(p) = (M(p) \ominus Pre(p, u)) \oplus Post(p, u)$
2. $\forall p \in P \;\; M'(p) \sqsupseteq Post(p, u)$ *and* $M(p) = (M'(p) \ominus Post(p, u)) \oplus Pre(p, u)$
3. $\forall p \in P \;\; M(p) \sqsupseteq Pre(p, u)$; $M'(p) \sqsupseteq Post(p, u)$ *and* $M(p) \ominus Pre(p, u) = M'(p) \ominus Post(p, u)$

We have so far identified the set of conditions that should be fulfilled by Petri algebras so that we can play the token game and the resulting firing rule is associative. To sum up, these structures are duals of commutative residuated lattices whose joins and meets are given by the formulas $a \sqcup b = a \ominus (a \ominus b)$ and $a \sqcap b = b \oplus (a \ominus b)$. Moreover this lattice is integral in the sense that the neutral element for the sum is also the least element of the lattice. Finally the underlying monoid is cancellable and this condition is equivalent to the identity $(a \oplus b) \ominus b = a$.

Using [3, 9] we can conclude that Petri algebras coincide with the (duals of) *integral, cancellative and commutative GMV-algebras.* These algebras form a sub-variety of the variety of residuated lattices and the following result is a direct consequence of [10–Theorem 5.6 and corollaries].

Theorem 7. *Petri algebras coincide with the positive cones of lattice-ordered abelian groups. Moreover lattice-ordered abelian groups constitute the subvariety of lattice-ordered groups generated by the group $\mathbb{Z}$ of integer, and their positive cones (i.e. Petri algebras) is the subvariety of residuated lattices generated by $\mathbb{N}$.*

3 Lexicographic Petri Nets

We define a (generalized) Petri net as a structure $\mathcal{N} = (P, T, Pre, Post, M_0)$ where P is a finite set of places with a Petri algebra A_p associated with each place $p \in P$, T is a finite set of transitions disjoint from P and Pre, $Post : P \times T \to \bigsqcup_{p\in P} A_p$, the flow relations, are such that $\forall p \in P\ \forall t \in T \quad Pre(p,t),\ Post(p,t) \in A_p$. A marking is a map $M : P \to \bigsqcup_{p\in P} A_p$ that associates with each place $p \in P$ the local value of the current configuration $M(p) \in A_p$ in this place. M_0 is some fixed marking, called the *initial marking.* The transition relation $M\,[t\rangle\,M'$ stating that transition t can fire in marking M and leads, when it is fired, to the new marking M' is given by:

$$M\,[t\rangle\,M' \ \Leftrightarrow\ \forall p \in P\ \ M(p) \sqsupseteq Pre(p,t) \wedge M'(p) = (M(p) \ominus Pre(p,t)) \oplus Post(p,t)$$

This relation can be extended inductively to sequences $u \in T^*$ of transitions by letting $M\,[\varepsilon\rangle\,M$ for every marking M and $M\,[t\cdot u\rangle\,M'$ if and only if there exists some marking M'' such that $M\,[t\rangle\,M''$ and $M''\,[u\rangle\,M'$ for every $t \in T$ and $u \in T^*$. The set of *reachable markings* is $Reach(\mathcal{N}) = \{M \mid \exists u \in T^*\ M_0\,[u\rangle\,M\}$, and the *marking graph* of a generalized net $\mathcal{N} = (P, T, Pre, Post, M_0)$ is the labelled graph $\Gamma_{\mathcal{N}} = (V, \Lambda, v_0)$ whose set of vertices is given by the set $V = Reach(\mathcal{N})$ of reachable markings with $v_0 = M_0$ and whose set of arcs $\Lambda \subseteq V \times T \times V$ is the restriction of the transition relation to the set of reachable markings: $\Lambda = \{(M, t, M') \mid M,\ M' \in V\ \wedge\ M\,[t\rangle\,M'\}$. Two generalized Petri nets are termed *equivalent* when they have isomorphic marking graphs.

We immediately see that a place p whose type A_p is a sub-algebra of a product of Petri algebras ($A_p \subseteq A_1 \times \cdots \times A_n$) can be replaced by n places $p_1, \ldots, p_n$ with respective types $A_1, \ldots, A_n$ without changing the marking graph (at least up to isomorphism). A classical result of universal algebra says that any algebra of a variety is a sub-direct product of sub-directly irreducible algebras. Thus we can assume without loss of generality that all algebras A_p are sub-directly irreducible algebras in the variety of Petri algebras. Now any $M(p)$ belongs to the sub-algebra of A_p generated by the set $\{M_0(p)\} \cup \bigcup_{t\in T} \{Pre(p,t),\ Post(p,t)\}$. Thus:

Theorem 8. *Every generalized Petri net is equivalent to a generalized Petri net all of whose types are sub-directly irreducible and finitely generated Petri algebras.*

Let $Irr\,(V)$ denote the set of sub-directly irreducible algebras of a variety V , then if V is a subvariety of W one has $Irr(W) \cap V = Irr(V)$; using the fact that the sub-directly irreducible commutative GMV-algebras are chains (totally ordered sets) we deduce that

Proposition 9. *sub-directly irreducible Petri algebras are chains.*

An algebra is sub-directly irreducible if and only if it admits a least non trivial congruence [4]. Now we know [5, 6] that the congruences of Petri algebras are in bijective correspondance with their convex sub-monoids. On the one hand we can associate each congruence θ of a Petri algebra A with the class of the neutral element which is a convex sub-monoid $M_\theta = [0]_\theta$ of A. Conversely we associate each such monoid M to the congruence $\theta_M = \{(a, b) \in A^2 \mid b \ominus a, a \ominus b \in M\}$. The correspondances $\theta \mapsto M_\theta$ and $M \mapsto \theta_M$ are inverses to each other and they establish an isomorphism between the lattice of congruences of A and the lattice of the convex sub-monoids of A. Moreover for every $a \in A$, the principal congruence generated by the equation $a = 0$ corresponds to the convex sub-monoid generated by a. A Petri algebra is then sub-directly irreducible if and only if it admits a least non trivial convex sub-monoid. Let us assume that A is a totally ordered Petri algebra. Let

$$M(x) = \{y \in A \mid \exists k \in \mathbb{N} \cdot y \sqsubseteq k \cdot x = \underbrace{x \oplus \cdots \oplus x}_{k \text{ times}}\}$$

denote the principal convex sub-monoid generated by $x \in A$. $M(x)$ is non-trivial if and only if $x \neq 0$. Now if x is some element of a convex sub-monoid M of A one necessarily has $M(x) \subseteq M$; thus a minimal convex sub-monoid is principal and is generated by any of its non null elements. Since A is totally ordered and $x \leq y \Rightarrow M(x) \subseteq M(y)$ we deduce that A admits at most one minimal non trivial sub-monoid. $M(x)$ is minimal if and only if $y \ll x \Rightarrow y = 0$ where relation $\ll$ is given by $y \ll x \Leftrightarrow \forall k \in \mathbb{N} \cdot k \cdot y \sqsubset x$. Otherwise stated $y \ll x$ if and only if $y \sqsubset x$ and $M(y)$ is strictly included in $M(x)$. Therefore A has no non trivial minimal sub-monoid if and only if for every $x \in A \setminus \{0\}$ one can find some $y \in A \setminus \{0\}$ such that $y \ll x$. Under that condition one can form an infinite strictly decreasing chain thus proving that the order relation $\ll$ is not well-founded. Conversely if this order is well-founded then any non empty subset of A, and thus in particular $A \setminus \{0\}$ if A is not trivial, admits a least element for this order which shows the existence of a minimal non trivial sub-monoid. We thus have established the following:

Theorem 10. *A Petri algebra is sub-directly irreducible if and only if it is a chain and the order relation* $y \ll x \Leftrightarrow \forall k \in \mathbb{N} \cdot k \cdot y \sqsubset x$ *is well-founded.*

The lexicographic product $\mathbb{G} \circ \mathbb{H}$ of two ordered groups $\mathbb{G}$ and $\mathbb{H}$ is the product group $\mathbb{G} \times \mathbb{H}$ equipped with the lexicographic order relation:

$$(x, y) \leq_{\mathbb{G} \circ \mathbb{H}} (x', y') \quad \Leftrightarrow \quad x <_{\mathbb{G}} x' \text{ or } (x = x' \text{ and } y \leq_{\mathbb{H}} y')$$

If $\mathbb{G}$ and $\mathbb{H}$ are simply ordered abelian groups then the same holds for their lexicographic product. This product is associative and we can define inductively

$L_n(\mathbb{G}) = (\mathbb{G}^n)^+$ for every simply ordered abelian group $\mathbb{G}$ and integer $n \in \mathbb{N}$ by letting $\mathbb{G}^0 = \{0\}$ be the trivial group and $\mathbb{G}^{n+1} = \mathbb{G}^n \circ \mathbb{G}$, and where $\mathbb{G}^+$ denote the positive cone of group $\mathbb{G}$. The group $\mathbb{G}^n$ naturally embedds into $\mathbb{G}^m$ when $n \leq m$; the projective limit of this sequence of embeddings is the group $\mathbb{G}^\omega$ whose elements are the infinite sequences of elements in $\mathbb{G}$, with componentwise composition and the lexicographic order relation defined as follows: $u \leq_{lex} v \;\; \Leftrightarrow \;\; u <_{lex} v$ or $u = v$ where $u <_{lex} v \;\; \Leftrightarrow \;\; \exists n \in \mathbb{N}\, \forall m \leq n \;\; u_m = v_m$ and $u_n <_{\mathbb{G}} v_n$. The inductive limit, or "union" $\bigcup_{n<\omega} \mathbb{G}^n$, is the subgroup of $\mathbb{G}^\omega$ consisting of the sequences u of finite support ($supp(u) = \sup\{k \in \mathbb{N} \mid u_k \neq 0\} < \omega$) with $\mathbb{G}^n$ identified with the subgroup of $u \in \mathbb{G}^\omega$ such that $supp(u) \leq n$.

Definition 11. *The set $Lex(\mathbb{G})$ of lexicographic Petri nets based on a totally ordered abelian group $\mathbb{G}$ is the set of (homogeneous) generalized Petri net of type $(\mathbb{G}^\omega)^+$. $Lex(\mathbb{G}, n) \subseteq Lex(\mathbb{G})$ is the set of n-dimensional lexicographic Petri nets with type $L_n(\mathbb{G}) = (\mathbb{G}^n)^+ \subseteq (\mathbb{G}^\omega)^+$, i.e. all flow arc inscriptions and initial place contents, and hence all place contents in every accessible marking, are elements in $(\mathbb{G}^n)^+$.*

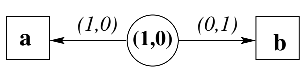
If $\mathcal{K}$ and $\mathcal{L}$ are subclasses of generalized Petri nets we let $\mathcal{K} \lesssim \mathcal{L}$ when every net in $\mathcal{K}$ is equivalent to some net in $\mathcal{L}$. This is a pre-order relation, we let $\approx$ denote its associated equivalence relation and $\lnapprox$ the corresponding strict relation: $\mathcal{K} \lnapprox \mathcal{L}$ when every net in $\mathcal{K}$ is equivalent to some net in $\mathcal{L}$ but there exists some net in $\mathcal{L}$ not equivalent to any net in $\mathcal{K}$. Notice that $Lex(\mathbb{G}, n) \lesssim Lex(\mathbb{H}, m)$ when $\mathbb{G} \subseteq \mathbb{H}$ and $n \leq m$; and that $Lex(\mathbb{Z}, 1)$ is the class of Petri nets.

Lemma 12. *Any finitely generated sub-directly irreducible Petri algebra A is isomorphic to a sub-algebra of the positive cone of some finite power of the additive group of real numbers: $A \subseteq (\mathbb{R}^n)^+$.*

By Theorem 8 we deduce the following result.

Theorem 13. *Every generalized Petri net is equivalent to some lexicographic Petri net, more precisely : $GenPetri \approx Lex(\mathbb{R})$*

We provide an example showing that $Lex(\mathbb{Z}, 1) \lnapprox Lex(\mathbb{Z}, 2)$, i.e. that lexicographic Petri nets based on the group of integers of dimension 2 are already strictly more expressive than the class of Petri nets. Let us consider the net of type $L_2 = (\mathbb{Z} \circ \mathbb{Z})^+ = \{(n, m) \mid (n = 0 \text{ and } m \geq 0) \text{ or } (n > 0 \text{ and } m \in \mathbb{Z})\}$.



From the initial marking $(1, 0)$ transition a can fire once $(1, 0)\,[a\rangle\,(0, 0)$ and transition b can fire an infinite number of time leading to the infinite firing sequence $(1, 0)\,[b\rangle\,(1, -1)\,[b\rangle\,(1, -2) \ldots [b\rangle\,(1, -n) \ldots$ and there are no other transitions in the marking graph of the net. Suppose there exists some Petri net with an isomorphic marking graph. Since transition b can fire an infinite number of time, and

$$M_0\,[b^n\rangle\, M_n \quad\Rightarrow\quad \forall p \in P\ M_n(p) = M_0(p) - n \times (Pre(p,b) - Post(p,b))$$

we deduce that for every place p it is the case that $Post(p,b) \sqsupseteq Pre(p,b)$ and thus $M_n(p) \sqsupseteq M_0(p)$. By monotony of the firing rule, any transition that can fire in the initial marking M_0 can also fire in any of the markings M_n obtained by firing b. Transition a is in contradiction with this property. Thus

Proposition 14. *$Petri = Lex(\mathbb{Z},1) \lnapprox GenPetri$*

It can also be shown that $Lex(\mathbb{Z},n) \lnapprox Lex(\mathbb{Z},m) \lnapprox Lex(\mathbb{Z})$ for $n < m < \omega$, and $Lex(\mathbb{Z},n) \approx Lex(\mathbb{Q},n) \lnapprox Lex(\mathbb{R},n)$.

It appears to be difficult to obtain strict extensions of the class of Petri nets that preserve all of its decidable properties. Many of these extensions, like the class of Petri nets with inhibitor arcs, are indeed Turing-powerful. We recall that an inhibitor arc from a place p to a transition t (one such arc is depicted in Fig. 1) is intended to inhibit the firing of transition t as long as place p is not empty.

Theorem 15. *Lexicographic Petri nets are a strict extension of the class of Petri nets with inhibitor arcs. Thus Reachability, Coverability, Place-boundedness, Boundedness, Deadlock and Liveness are undecidable for the class of lexicographic Petri nets.*

The translation of a Petri net with inhibitor arcs $\mathcal{N}$ into an equivalent lexicographic Petri net $\overline{\mathcal{N}}$, illustrated in Fig. 1, consists in splitting every place p with

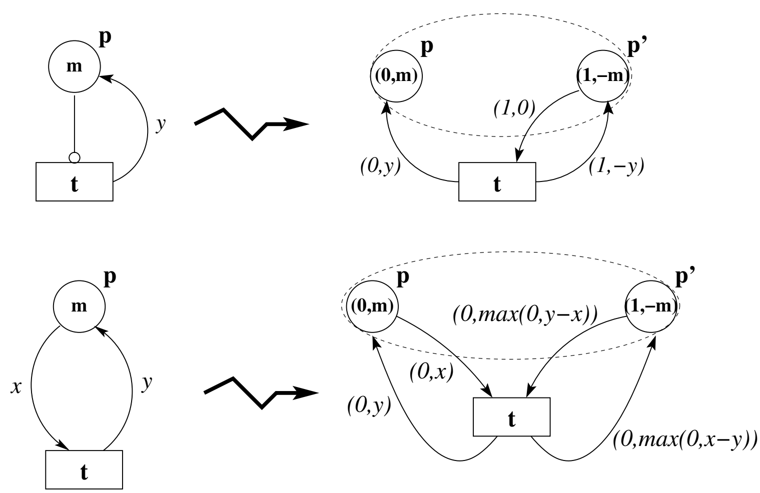


Fig. 1. A translation from Petri nets with inhibitor arcs into lexicographic Petri nets

initial marking $m \in \mathbb{N}$ of the original net into two places denoted p and p' with initial markings $(0, m) \in (\mathbb{Z} \circ \mathbb{Z})^+$ and $(1, -m) \in (\mathbb{Z} \circ \mathbb{Z})^+$respectively.

4 Bounded Nets

A net is bounded if we can find an upper bound on the possible values of places in any accessible marking. Let us start our study on the algebraization of the dynamic of bounded nets by the following observation.

Proposition 16. *Any non trivial commutative Petri algebra is an unbounded lattice.*

However we can enforce boundedness by modifying the rule of the token game. Let us consider first the case of the usual Petri nets: assume that each place $p \in P$ is associated with a capacity $k_p \in \mathbb{N}$ and that we want to ensure that the value of a place p of a Petri net be bounded from above by its capacity k_p. For that purpose we modify the firing rule as follows (where all computations are performed in $\mathbb{Z}$)

$$M\,[t\rangle\,M' \;\Leftrightarrow\; \forall p \in P \;\; \begin{cases} M(p) \sqsupseteq \mathit{Pre}(p,t) \;\wedge\; (M(p) - \mathit{Pre}(p,t)) + \mathit{Post}(p,t) \sqsubseteq k_p \\ M'(p) = (M(p) - \mathit{Pre}(p,t)) + \mathit{Post}(p,t) \end{cases}$$

this rule can be reformulated as:

$$M\,[t\rangle\,M' \;\Leftrightarrow\; \forall p \in P \quad \begin{cases} \mathit{Pre}(p,t) \sqsubseteq M(p) \sqsubseteq (k_p + \mathit{Pre}(p,t)) - \mathit{Post}(p,t) \\ M'(p) = (M(p) - \mathit{Pre}(p,t)) + \mathit{Post}(p,t) \end{cases}$$

Petri algebras are the positive cones G^+ of lattice-ordered abelian groups $G = (G, +, 0, \sqcup, \sqcap)$. It is an algebra with the following operations: the sum (restriction of the group operation $x \oplus y = x + y$), and the truncated difference $(x \ominus y = (x - y) \sqcup 0)$.

Let $k \sqsupseteq 0$ be some element of this positive cone; we suppose that it is a *strong unit* in the sense that $\forall g \in G\; \exists n \in \mathbb{N} \;\cdot\; n \cdot k \sqsupseteq g$. We can by modifying the firing rule ensure that the values of places stay within the interval $I = [0, k] = \{g \in G \mid 0 \sqsubseteq g \sqsubseteq k\}$. This interval with induced order is a bounded lattice that can be equipped with the following operations: the truncated sum ($x \boxplus y = (x + y) \sqcap k$), the truncated difference $(x \ominus y = (x - y) \sqcup 0)$, the product $(x \bullet y = ((x + y) - k) \sqcup 0)$, the implication $(x \rightarrow y = ((y + k) - x) \sqcap k)$, and the negation $(\neg x = (x \rightarrow 0) = k \ominus x)$.

Such a structure is called an MV-algebra. MV-algebras are generalizations of boolean algebras used in the algebraic analysis of Lukasiewicz infinite-valued propositional logic and this class of algebras admits several equivalent definitions [7, 12]. We then state that the firing relation of a so-called bounded net associated with some MV-algebras is given by:

$$M\,[t\rangle\,M' \;\Leftrightarrow\; \forall p \in P \quad \begin{cases} \mathit{Pre}(p,t) \sqsubseteq M(p) \sqsubseteq \mathit{Post}(p,t) \rightarrow \mathit{Pre}(p,t) \\ M'(p) = (M(p) \ominus \mathit{Pre}(p,t)) \boxplus \mathit{Post}(p,t) \end{cases}$$

The boolean algebra $2 = \{0, 1\}$ is an MV-algebra where $x \boxplus y = x \sqcup y$ and $x \bullet y = x \sqcap y$. Let P be the set of places of a net and $B = 2^P = \wp(P)$ the corresponding product structure, the preceding firing rule can be reformulated as:

$$M\,[a\rangle\,M' \;\Leftrightarrow\; M \supseteq Pre(a) \;\wedge\; M \cap Post(a) \subseteq Pre(a) \;\wedge\; M' = (M \setminus Pre(a)) \cup Post(a)$$

which is the usual firing rule of 1-safe nets. If we replace the boolean algebra $2 = \{0, 1\}$ by the interval $[0, 1]$ of the additive group of real numbers, i.e. with $x \boxplus y = min(1, x + y)$ and $x \bullet y = max(0, x + y - 1)$ then we obtain the firing rule of some kind of 1-safe "fuzzy" nets :

$$M\,[a\rangle\,M' \;\Leftrightarrow\; \forall p \in P \;\begin{cases} Pre(a,p) \sqsubseteq M(p) \sqsubseteq Pre(a,p) + 1 - Post(a,p) \\ M'(p) = min(1, M(p) + Post(a,p) - Pre(a,p)) \end{cases}$$

Mundici [12] proved that MV-algebras coincide with $[0, k]$ intervals of abelian lattice-ordered groups where k is a strong unit. More precisely if $G = (G, +, 0)$ is an abelian lattice-ordered group with strong unit k then $\Gamma(G, k) = (A, \boxplus, 0, \bullet, k, \neg)$ where $A = [0, k] = \{x \in G \,|\, 0 \sqsubseteq x \sqsubseteq k\}$, $x \boxplus y = (x + y) \sqcap k$, $x \bullet y = ((x + y) - k) \sqcup 0$, and $\neg x = k - x$ is an MV-algebra such that the restriction of the order relation of the group on the unit interval $[0, k]$ coincides with the order relation of the MV-algebra: $x \sqsubseteq y \Leftrightarrow x \ominus y = 0$ (where $x \ominus y = x \bullet \neg y = (x - y) \sqcup 0$). Moreover Γ extends into an equivalence between the respective categories, i.e. it induces a bijective correspondence between isomorphism classes of abelian lattice-ordered groups with strong unit and isomorphism classes of MV-algebras. Now we have seen that Petri algebras corresponds bijectively, up to isomorphism, to the positive cones of abelian lattice-ordered groups, we thus have a Petri algebra canonically associated with each MV-algebra. The following result shows that, by using complementary places, we can simulate a bounded Petri net by a generalized Petri net defined on the associated Petri algebra.

Theorem 17. *Any bounded net can be simulated by a generalized Petri net.*

The translation of a bounded net $\mathcal{N}$ into an equivalent generalized Petri net $\overline{\mathcal{N}}$, illustrated in Fig. (2), consists in splitting every place p with initial marking $m \in A_p$ of the bounded net into two places denoted p and p' with initial markings $m \in (\mathbb{G}_p)^+$ and $k_p - m \in (\mathbb{G}_p)^+$respectively where $(\mathbb{G}_p, k_p)$ is the lattice-ordered abelian group with strong unit associated with the MV-algebra $A_p \simeq \Gamma(\mathbb{G}_p, k_p)$.

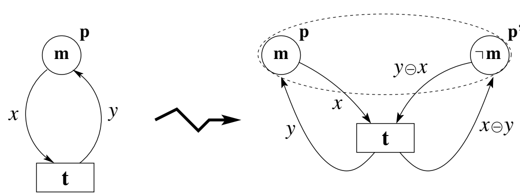


Fig. 2. Translation from bounded nets to generalized Petri nets

5 Conclusion

In this paper we have put forward an axiomatization of the token game of Petri nets by identifying a class of commutative residuated monoids, called Petri algebras, for which one can generalize the rule of token game of Petri nets to define the behaviour of generalized Petri net whose flow relation and place contents are valued in such algebras. In this way we have put the basis for a uniform presentation of various families of Petri nets by recasting them as particular instances of a generic class of Petri nets parametric in algebraic structures representing some concrete notion of resources. We thus have followed the line of research best illustrated in [8], a special issue of *Advances in Petri nets*, dedicated to the development of uniform approaches to Petri nets. However the present approach, centered on the notion of resources, is probably too concrete to be of practical interest in many situations. For instance even though one can describe continuous Petri nets in this framework the obtained semantics is too much extensional. We see two directions that can be used in order to derived more abstract representations for the behaviour of these generic nets. First, one can abstract of the flow arc inscriptions. Such an inscription takes its value in some algebra of abstract properties. A precondition then appears as a guard stating that some property has to be satisfied by the resources contained in the corresponding place and a postcondition is interpreted as adding resources to enforce some property. Second, one can abstract on the firing relation itself by giving the measure in some adequate semiring of the "firability" of a transition in some marking.

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