On Decision Problems for Probabilistic Büchi Automata

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Cachan – 05 février 2008

Probabilistic Büchi Automata [Baier, Größer 05]

PBA = NBA with probabilities instead of non-determinism

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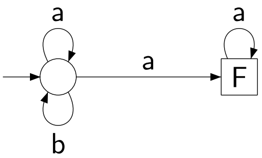

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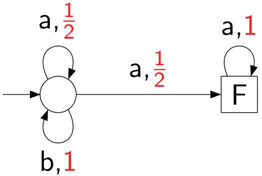

$$\mathcal{L}(\mathcal{A}) = \{w \in \Sigma^\omega \mid \mathbb{P}_{\mathcal{A}}(\{\rho \in \mathsf{Runs}(w) \mid \rho \models \Box\Diamond F\}) > 0\}$$

$$\mathcal{L}_{\mathsf{NBA}}(\mathcal{A}) = (a+b)^* a^\omega = \mathcal{L}_{\mathsf{PBA}}(\mathcal{A})$$

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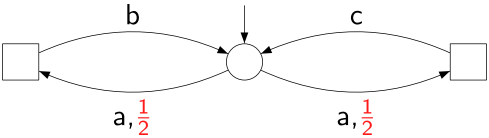

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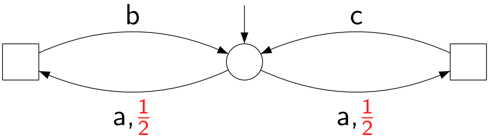

$$\mathcal{L}_{\mathsf{NBA}}(\mathcal{A}) = (ab + ac)^\omega \text{ and } \mathcal{L}_{\mathsf{PBA}}(\mathcal{A}) = \emptyset$$

Expressiveness: PBA vs DBA

PBA are strictly more expressive than DBA

- any DBA $\mathcal{A}$ can be turned into a PBA $\mathcal{P}$

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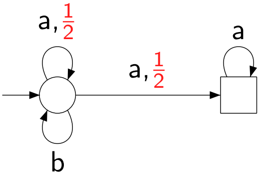
- there is a PBA whose language can't be recognized by a DBA



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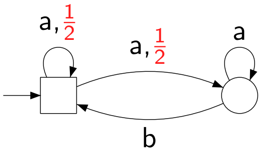

$$\mathcal{L}(\mathcal{A}) = \{a^{k_1} b a^{k_2} b \cdots \mid \prod_i (1 - \frac{1}{2}^{k_i}) > 0)\}$$

Outline

Complementation

Theorem

For each PBA $\mathcal{P}$ there exists a PBA $\mathcal{P}'$ with $|\mathcal{P}'| = \mathcal{O}(\exp(|\mathcal{P}|)$ such that $\mathcal{L}(\mathcal{P}') = \Sigma^\omega \setminus \mathcal{L}(\mathcal{P})$.
Moreover, $\mathcal{P}'$ can be effectively constructed from $\mathcal{P}$.

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Proof Scheme

$$\mathcal{P} \text{ PBA} \quad \longrightarrow \quad \mathcal{P}_R \text{ 0/1-PRA with } \mathcal{L}(\mathcal{P}_R) = \mathcal{L}(\mathcal{P})$$

0/1-PRA: Probabilistic Rabin Automaton s.t. all words have acceptance probability in $\{0, 1\}$

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Difficult step: PBA $\longrightarrow$ equivalent 0/1-PRA

Complementation: First step in details

- From $\mathcal{P}$ build an equivalent 0/1-PRA.

Construction idea: Organize the infinite computation tree into a finite-state automaton by merging runs meeting at some point.

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States: tuples $\langle p_1, \xi_1, \cdots, p_k, \xi_k, R \rangle$
$p_i \in Q$ pairwise distinct, $\xi_i \in \{0, 1\}$ and $R \subseteq Q$.

- R-component: usual powerset construction
- p_i state witnessing sample runs
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Rabin condition: for some index j, the j-th run visits F infinitely often and from some point on the attached bit is 0.

Complementation: First step in details (2)

Possible a-successors of $\bar{p} = \langle p_1, \xi_1, \cdots, p_k, \xi_k, R \rangle$:

$$\bar{q} = \langle q_1, \zeta_1, \cdots, q_k, \zeta_k, q_{k+1}, \zeta_{k+1} \cdots q_m, \zeta_m, S \rangle$$

1. $q_i \in \delta(p_i, a)$ for $1 \leq i \leq k$
2. $\{q_{k+1}, \cdots, q_m\} = (\delta(R, a) \cap F) \setminus \{q_1, \cdots, q_k\}$
3. $\zeta_1 = \cdots = \zeta_k = 0$ and $\zeta_{k+1} = \cdots = \zeta_m = 1$
4. $S = \delta(R, a)$

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$$\mathbb{P}_{\mathsf{PRA}}(\rho) > 0 \quad \Leftrightarrow \quad \mathbb{P}_{\mathsf{PBA}}(\rho) > 0 \quad \Leftrightarrow \quad \mathbb{P}_{\mathsf{PRA}}(\rho) = 1$$

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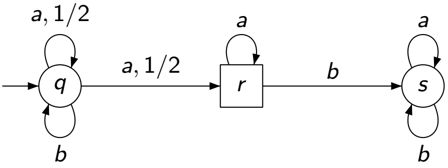
Example



$\mathcal{L}(\mathcal{P}) = (a + b)^* a^\omega$

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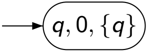

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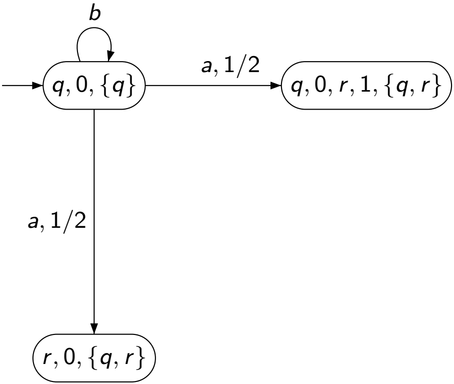

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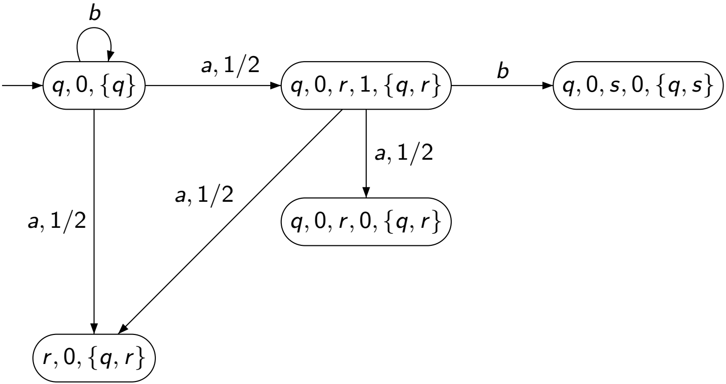

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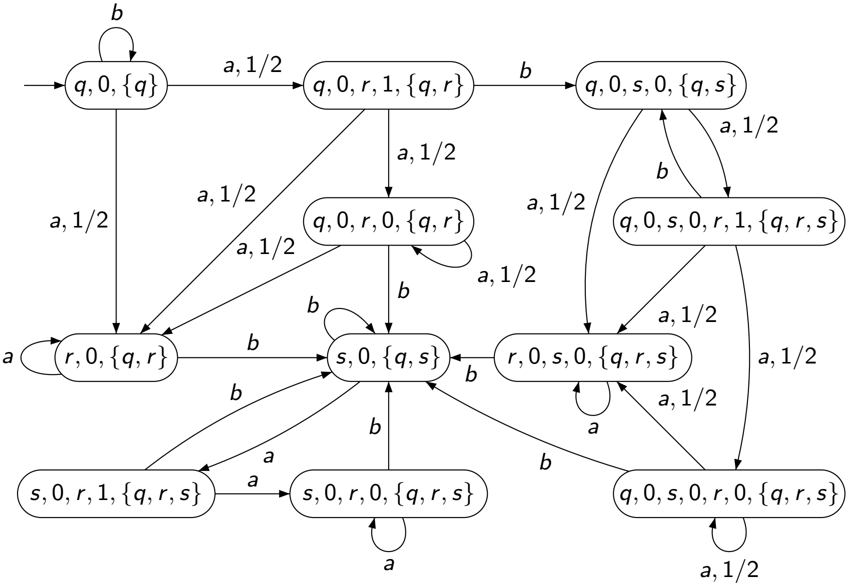

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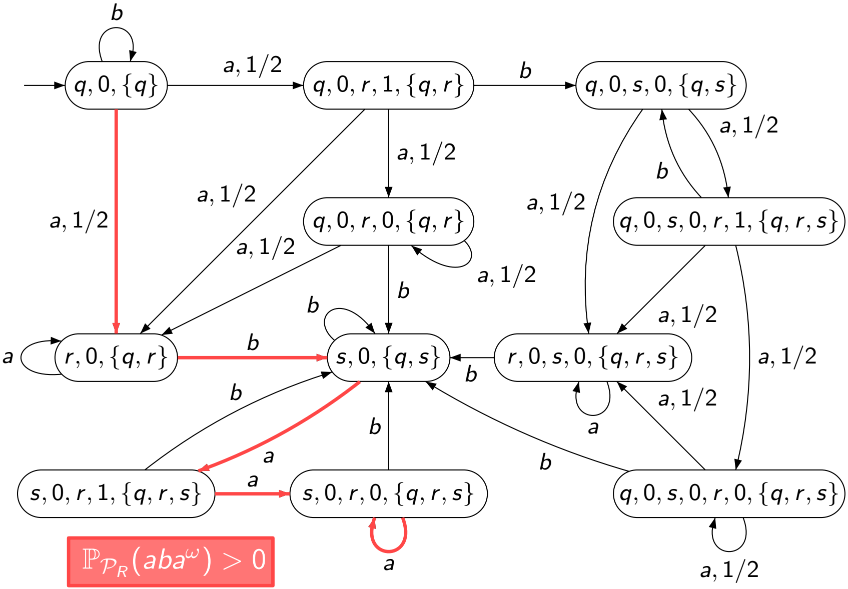

$\mathbb{P}_{\mathcal{P}_R}(aba^\omega) > 0$

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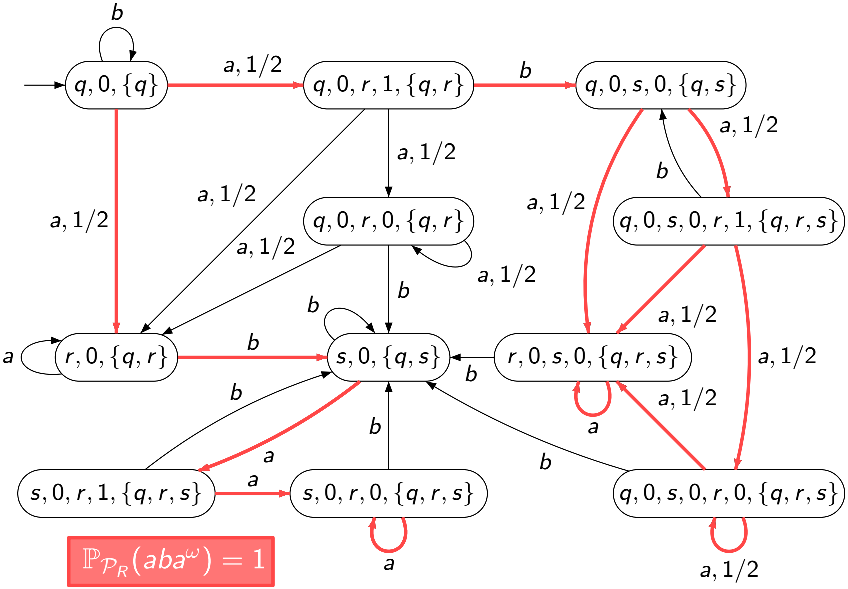

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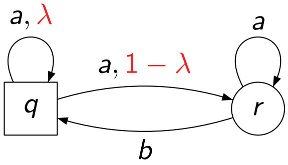
Outline

A weird example

$\mathcal{P}_\lambda$:



$$\mathcal{L}(\mathcal{P}_\lambda) = \{a^{k_1} b a^{k_2} b \cdots \mid \prod_i (1 - \lambda^{k_i}) > 0)\}$$

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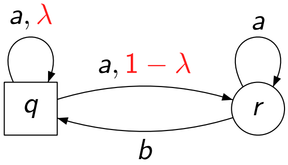

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Lemma

For $0 < \lambda < \frac{1}{2} < \mu < 1$, $\mathcal{L}(\mathcal{P}_\lambda) \neq \mathcal{L}(\mathcal{P}_\mu)$.

Hint $w = a^{k_1} b a^{k_2} b \cdots$ with for all m, 2^m elements of (k_i) set to m.

$\longrightarrow w \in \mathcal{L}(\mathcal{P}_\lambda) \setminus \mathcal{L}(\mathcal{P}_\mu)$

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Undecidability result for PFA [MHC03]

The following problem is undecidable:
Given $0 < \varepsilon < 1$ and $\mathcal{P}$ a PFA such that

- either $\exists w \; \mathbb{P}_{\mathcal{P}}(w) > 1 - \varepsilon$
- or $\forall w \; \mathbb{P}_{\mathcal{P}}(w) \leq \varepsilon$

tell which is the case.

Emptiness problem for PBA

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Proof Sketch

Reduction of the modified emptiness problem for PFA

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Reduction of the modified emptiness problem for PFA

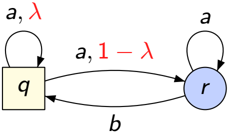
$$\mathcal{R} \text{ PFA with } \begin{cases} \forall w \ \mathbb{P}_{\mathcal{R}}(w) \leq \varepsilon & \text{or} \\ \exists w \ \mathbb{P}_{\mathcal{R}}(w) > 1 - \varepsilon \end{cases}$$

$$\downarrow$$

$\mathcal{P}_1$ and $\mathcal{P}_2$ PBA s.t.

$$\mathcal{L}^{>\varepsilon}(\mathcal{R}) = \emptyset \quad \Leftrightarrow \quad \mathcal{L}(\mathcal{P}_1) \cap \mathcal{L}(\mathcal{P}_2) = \emptyset$$

Proof in more details: $\mathcal{P}_1$



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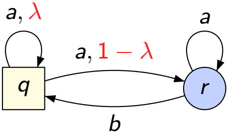

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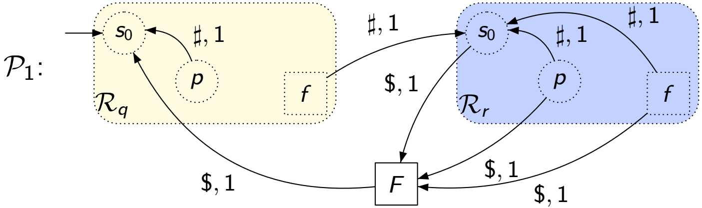

From s_0 in $\mathcal{R}_q$, reading $w\#$ leads to $\mathcal{R}_r$ with probability $\mathbb{P}_{\mathcal{R}}(w)$.

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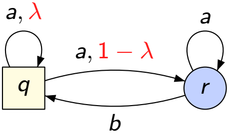

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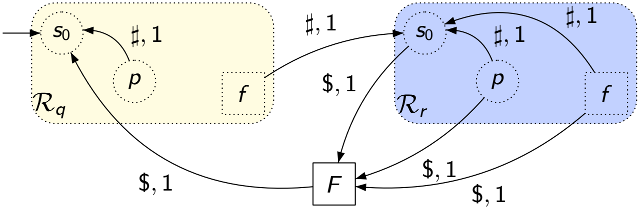

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$$\mathcal{L}(\mathcal{P}_1) = \{w_1^1 \# \cdots w_{k_1}^1 \$\$ w_1^2 \# \cdots w_{k_2}^2 \$\$ \cdots \mid$$
$$\prod_j \Big(1 - (\prod_{i=1}^{k_j - 1} (1 - \mathbb{P}_\mathcal{R}(w_i^j)))\Big) > 0\}$$

Proof in more details: $\mathcal{P}_2$

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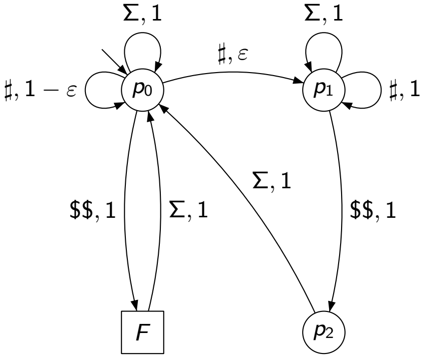

From p_0, reading $v \in (\Sigma \cup \{\#\})^*$ leads to p_1 with probability $1 - (1 - \varepsilon)^{|v_i|_\#}$

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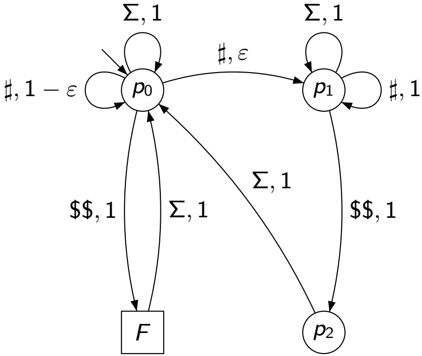

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$$\mathcal{L}(\mathcal{P}_2) = \{v_1\$\$v_2\$\$\cdots \mid v_i \in (\Sigma \cup \#)^* \text{ and } \prod_i (1 - (1-\varepsilon)^{|v_i|_\#}) = 0\}$$

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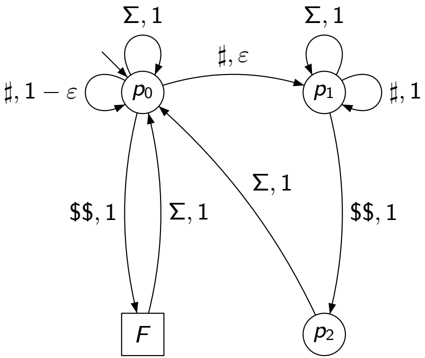

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- $\mathcal{L}(\mathcal{P}_1) = \{w_1^1 \# \cdots w_{k_1}^1 \$\$ w_1^2 \# \cdots w_{k_2}^2 \$\$ \cdots \mid \prod_j \left(1 - \left(\prod_{i=1}^{k_j-1}(1 - \mathbb{P}_{\mathcal{R}}(w_i^j))\right)\right) > 0\}$
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If $\forall w,\ \mathbb{P}_{\mathcal{R}}(w) \leq \varepsilon$

$$\forall \tilde{w},\ \ \tilde{w} \in \mathcal{L}(\mathcal{P}_2) \Rightarrow \tilde{w} \notin \mathcal{L}(\mathcal{P}_1)$$

$$\Longrightarrow \mathcal{L}(\mathcal{P}_1) \cap \mathcal{L}(\mathcal{P}_2) = \emptyset$$

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If $\exists w,\ \mathbb{P}_{\mathcal{R}}(w) > 1 - \varepsilon$

Let $\tilde{w} = (w\#)^{k_1} w\$\$(w\#)^{k_2} w\$\$ \cdots$

$\mathbb{P}_{\mathcal{P}_1}(\tilde{w}) > \prod_j (1 - \varepsilon^{k_j-1})$ and $\mathbb{P}_{\mathcal{P}_2}(\tilde{w}) = \prod_i (1 - (1 - \varepsilon)^{k_i-1})$

$$\Longrightarrow \mathcal{L}(\mathcal{P}_1) \cap \mathcal{L}(\mathcal{P}_2) \neq \emptyset$$

Proof conclusion

- $\mathcal{L}(\mathcal{P}_1) = \{w_1^1\# \cdots w_{k_1}^1\$\$w_1^2\# \cdots w_{k_2}^2\$\$ \cdots \mid \prod_j \left(1 - \left(\prod_{i=1}^{k_j-1}(1 - \mathbb{P}_{\mathcal{R}}(w_i^j))\right)\right) > 0\}$
- $\mathcal{L}(\mathcal{P}_2) = \{w_1^1\# \cdots w_{k_1}^1\$\$w_1^2\# \cdots w_{k_2}^2\$\$ \cdots \mid \prod_i (1 - (1-\varepsilon)^{k_i-1}) = 0\}$

If $\forall w,\ \mathbb{P}_{\mathcal{R}}(w) \leq \varepsilon$

$$\forall \tilde{w},\ \ \tilde{w} \in \mathcal{L}(\mathcal{P}_2) \Rightarrow \tilde{w} \notin \mathcal{L}(\mathcal{P}_1)$$

$$\Longrightarrow \mathcal{L}(\mathcal{P}_1) \cap \mathcal{L}(\mathcal{P}_2) = \emptyset$$

If $\exists w,\ \mathbb{P}_{\mathcal{R}}(w) > 1 - \varepsilon$

Let $\tilde{w} = (w\#)^{k_1} w\$\$(w\#)^{k_2} w\$\$ \cdots$

$\mathbb{P}_{\mathcal{P}_1}(\tilde{w}) > \prod_j (1 - \varepsilon^{k_j-1})$ and $\mathbb{P}_{\mathcal{P}_2}(\tilde{w}) = \prod_i (1 - (1-\varepsilon)^{k_i-1})$

$$\Longrightarrow \mathcal{L}(\mathcal{P}_1) \cap \mathcal{L}(\mathcal{P}_2) \neq \emptyset$$

$$\mathcal{L}^{>\varepsilon}(\mathcal{R}) = \emptyset \quad \Leftrightarrow \quad \mathcal{L}(\mathcal{P}_1) \cap \mathcal{L}(\mathcal{P}_2) = \emptyset$$

PBA-related Consequences

Immediate consequences of the undecidability result.

Corollary

The following problems are undecidable. Given $\mathcal{P}_1$ and $\mathcal{P}_2$ PBA

- $\mathcal{L}(\mathcal{P}_1) = \Sigma^\omega$?
- $\mathcal{L}(\mathcal{P}_1) = \mathcal{L}(\mathcal{P}_2)$?
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Verification against PBA specifications

The following problems are undecidable. Given a transition system T and a PBA $\mathcal{P}$

- is there a path in T whose trace is in $\mathcal{L}(\mathcal{P})$?
- do the traces of all paths in T belong to $\mathcal{L}(\mathcal{P})$?

Consequences for POMDP

Partially Observable MDP

A POMDP $(\mathcal{M}, \sim)$ consists of an MDP $\mathcal{M}$ equipped with an equivalence relation $\sim$ over states of $\mathcal{M}$.

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Undecidability results

The following problems are undecidable

- Given $(\mathcal{M}, \sim)$ and F set of states of $\mathcal{M}$, is there an observation-based $\mathcal{U}$ such that $\mathbb{P}_{\mathcal{U}}(\Box\Diamond F) > 0$.
- Given $(\mathcal{M}, \sim)$ and F set of states of $\mathcal{M}$, is there an observation-based $\mathcal{U}$ such that $\mathbb{P}_{\mathcal{U}}(\Diamond\Box F) = 1$.

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First undecidability results in qualitative verification of POMDP.

Outline

Almost-sure semantics for PBA

Alternative semantics

$$L(\mathcal{A}) = \{w \in \Sigma^\omega \mid \mathbb{P}_{\mathcal{A}}(\{\rho \in \mathsf{Runs}(w) \mid \rho \models \Box\Diamond F\}) = 1\}$$

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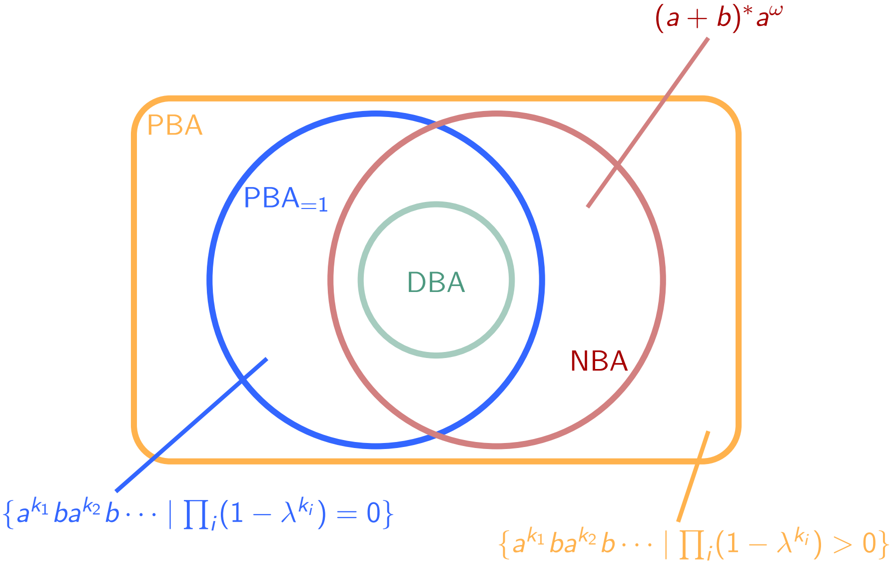
Expressivity

- almost-sure PBA are strictly less expressive than PBA
- almost-sure PBA and ω-regular languages are incomparable
- almost-sure PBA are not closed under complementation

Recap: expressivity



Emptiness problem and related results

Decidability result for POMDP

Almost-sure reachability in POMDP is decidable (EXPTIME).

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Decidability result for POMDP

Almost-sure reachability in POMDP is decidable (EXPTIME).

Corollary

The emptiness problem is decidable for almost-sure PBA.

Proof Sketch

- for PBA almost-sure reachability and almost-sure repeated reachability are interreducible
- PBA are a special instance of POMDP

Outline

Conclusion

Results concerning PBA

- complementation operator
- emptiness (and related problems) undecidable for PBA
- expressivity of almost-sure PBA
- emptiness decidable for almost-sure PBA

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Results concerning PBA

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Results concerning POMDP

- positive repeated reachability undecidable for POMDP
- almost-sure reachability decidable for POMDP

Conclusion

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Results concerning POMDP

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Open questions

- emptiness problem for PBA with small alphabet
- efficient transformation from LTL to PBA

Thank you for your attention!

Questions?