Probabilistic ω-Automata

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Probabilistic ω-automata are variants of nondeterministic automata over infinite words where all choices are resolved by probabilistic distributions. Acceptance of a run for an infinite input word can be defined using traditional acceptance criteria for ω-automata, such as Büchi, Rabin or Streett conditions. The accepted language of a probabilistic ω-automata is then defined by imposing a constraint on the probability measure of the accepting runs. In this paper, we study a series of fundamental properties of probabilistic ω-automata with three different language-semantics: (1) the probable semantics that requires positive acceptance probability, (2) the almost-sure semantics that requires acceptance with probability 1, and (3) the threshold semantics that relies on an additional parameter $\lambda \in]0, 1[$ that specifies a lower probability bound for the acceptance probability. We provide a comparison of probabilistic ω-automata under these three semantics and nondeterministic ω-automata concerning expressiveness and efficiency. Furthermore, we address closure properties under the Boolean operators union, intersection and complementation and algorithmic aspects, such as checking emptiness or language containment.

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1. INTRODUCTION

Automata as acceptors for infinite words play a crucial role in logic, for verification purposes and other areas, see for example, Vardi [1994], Thomas [1997], and Grädel et al. [2002]. Several types of automata for languages over infinite words have been studied in the literature. They can be classified via their branching structure (deterministic, nondeterministic, universal, alternating) and acceptance condition (Büchi, Müller, Rabin, Streett, etc.).

The purpose of this article is to study probabilistic variants of ω-automata that serve as acceptors for languages over infinite words. The essential idea is to equip nondeterministic ω-automata with probabilistic distributions that resolve the nondeterministic

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choices and to define the recognition of an infinite input word by a requirement on the measure of the set of accepting runs. While probabilistic finite automata (PFA) have been introduced by Rabin [1963] for almost 50 years and studied extensively in the literature, see for example, Paz [1966], Freivalds [1981], Condon [2001], Dwork and Stockmeyer [1990], and Blondel and Canterini [2003], probabilistic automata as acceptors for infinite words have been addressed only recently. The first approach to use probabilistic Büchi automata (PBA) for scanning infinite words has been presented by Reisz [1999a; 1999b]. In this approach, an infinite word is accepted if it has an infinite accepting run of positive probability. For finite-state automata, this notion of acceptance requires that for some infinite suffix of the input string there are no proper probabilistic branches, that is, the behavior is deterministic from some moment on. Thus, the concept of PBA under the semantics of Reisz is rather close to nondeterministic Büchi automata that are deterministic in limit [Vardi and Wolper 1986; Courcoubetis and Yannakakis 1995]. To exert the characterstic features of randomization for the recognition of infinite words, it is more natural to deal with acceptance conditions where the probability measure of the set of accepting runs is taken into account. PBA interpreted as a randomized language acceptor that accepts an infinite input string if the probability for an accepting run is positive (probable semantics), equals 1 (almost-sure semantics) or larger than a given cutpoint $\lambda \in]0,1[$ (threshold semantics) have been studied first in Baier and Grösser [2005] and Baier et al. [2008][1] and also addressed in Chadha et al. [2009b] and Chatterjee et al. [2009]. Probabilistic finite-state monitors [Chadha et al. 2009a] can be seen as a special instance of PBA with a single final trap state.

The main focus of this article are basic properties of probabilistic ω-automata concerning their expressiveness and efficiency to represent ω-regular languages. We consider several acceptance conditions for the runs (Büchi, Streett, Rabin) and three semantics for the language: (1) the probable semantics for which the language consists of the set of infinite words for which the set of accepting runs has positive probability, (2) the almost-sure semantics where the set of accepting runs must have probability 1 for a word to be accepted, and (3) the threshold semantics, where given a threshold $\lambda \in (0,1)$ the probability of accepted runs must be greater than λ.

The first surprising result is that probabilistic Büchi automata under the probable semantics (denoted $\mathrm{PBA}^{>0}$) are more powerful than nondeterministic ω-automata. This stands in contrast to the two well-known facts: first, deterministic Büchi automata do not have the full power of ω-regular languages contrary to their nondeterministic counterpart and second, PFA with the acceptance criteria "the accepting runs have a positive probability measure" can be viewed as nondeterministic finite automata, and hence, have exactly the power of regular languages.

Regarding the efficiency, $\mathrm{PBA}^{>0}$ are not comparable with nondeterministic ω-automata. On the one hand, there exists a family of ω-regular languages that can be recognized by $\mathrm{PBA}^{>0}$ of linear size, while even the smallest nondeterministic Streett automaton for them has at least $\Omega(2^n/n)$ states.

Boolean operations on languages defined by probabilistic Büchi automata under the probable semantics can be performed. The interesting case is the complementation of $\mathrm{PBA}^{>0}$, for which we propose a technique that relies on the switch to an equivalent probabilistic Rabin automaton that accepts all words either with probability 0 or 1 and whose size is exponential in the size of the original PBA. To do this we develop an advanced powerset construction that shares its basic ideas with Safra's determinization procedure [Safra 1988]. Using the duality of Rabin and Streett acceptance and a polynomial transformation from probabilistic Streett automata to PBA this yields a

[1]This article is based on the material of Baier and Grösser [2005] and Baier et al. [2008].

method for the complementation of PBA$^{>0}$ with a possible exponential blow-up. The low complexity of the transformation from PSA to PBA might be surprising, as in the nondeterministic case the switch from Streett to Büchi acceptance can cause an exponential blow-up [Safra and Vardi 1989].

The emptiness problem for PBA$^{>0}$ turns out to be undecidable, while it is decidable for PBA$^{=1}$. We found this result surprising, since for both PBA$^{>0}$ and PBA$^{=1}$ the accepted language does not only depend on its topological structure, but also on the precise transition probabilities. The negative result stating the undecidability of the emptiness problem for PBA$^{>0}$ has many important consequences, including the undecidability of various qualitative verification problems for probabilistic multi-agent systems. (See the explanations in this article.) Vice-versa, the decididability of the emptiness problem for PBA$^{=1}$ is a consequence of a more general result, the decidability of partially observable Markov decision processes under almost-sure Büchi objective.

PBA under the threshold semantics can be strictly more expressive than PBA$^{>0}$, under some conditions on the threshold value, contrary to PBA under the almost-sure semantics (denoted PBA$^{=1}$) which are strictly less expressive than ω-regular languages. However, for the Rabin and Streett acceptance criterion the class of recognizable languages under the almost-sure semantics agrees with the class of PBA-recognizable languages under the probable semantics.

There is a wide range of publications where probabilistic automata are used as operational model for randomized systems and basis for verification purposes, see for example, Vardi and Wolper [1986], van Glabbeek et al. [1990], Pnueli and Zuck [1993], Bianco and de Alfaro [1995], Courcoubetis and Yannakakis [1995], and Segala [1995]. These approachs are opposed to our setting as we use probabilistic automata as a formalism for ordinary non-probabilistic languages over infinite words rather than an operational model for systems with probabilistic behaviors. The classical task for verifying a probabilistic automaton against a temporal logic specification relies on a worst-case analysis where all possible decision functions that select the next action to be performed (often called schedulers, policies or adversaries) are taken into account. If a probabilistic automaton is used for modeling the operational behavior of a multi-agent system then the worst-case analysis ranging over all schedulers can be too pessimistic. Besides fairness assumptions on the schedulers that rule out certain unrealistic interleavings of the agents' activities, it is also often desirable to deal with separate decision functions for the agents. These do not have perfect information on the history, but have to make their choices on the basis of what the corresponding agent has observed from the history (e.g., his own local states and actions). Such notions of randomized multi-agent systems have been studied in the context of distributed scheduling [Cheung et al. 2006; Chatzikokolakis and Palamidessi 2010] or stochastic games with partial information [Gripon and Serre 2009; Bertrand et al. 2009; Chatterjee et al. 2010; Chatterjee and Henzinger 2010] and have concrete applications in security, for example, for information hiding [Andrés et al. 2010]. Randomized multi-agent systems can be seen as multi-player variants of partially observable Markov decision processes (POMDP), that in turn generalize PBA. POMDP have been extensively studied, see, for example, Sondik [1971], Monahan [1982], Papadimitriou and Tsitsiklis [1987], Kaelbling et al. [1995], Lovejoy [1991], and Burago et al. [1996], and their verification against finite-horizon properties was applied to various areas, such as machine maintenance, autonomous robots, or moving target search. POMDP with "long-run" objectives (infinite-horizon properties) have more recently attracted attention [de Alfaro 1999; Chatterjee et al. 2007; Giro and D'Argenio 2007].

Organization of the Article. Section 2 briefly recalls the basics on (non)deterministic ω-automata and Markov decision processes. Probabilistic ω-automata are introduced in Section 3. Section 4 explores expressiveness and efficiency questions for probabilistic Büchi automata. In Section 5, the expressiveness of probabilistic ω-automata with other acceptance conditions is studied. Section 6 is concerned with composition operators (union, intersection and complementation) on PBA. The emptiness problem for PBA under the probable semantics is investigated in Section 7. In Section 8, we consider qualitative questions on partially observable MDP. The article ends with a brief conclusion in Section 9.

2. PRELIMINARIES

2.1. Ordinary ω-Automata

Throughout this article, we assume some familiarity with formal languages, finite automata and ω-automata. We briefly recall the basic concepts and explain our notations concerning nondeterministic ω-automata with the Büchi, Rabin and Streett acceptance criteria. For further details, see, for example, Thomas [1997], Grädel et al. [2002], and Perrin and Pin [2004].

Σ will denote a nonempty finite alphabet. Letters $a, b, c, \ldots$ will be used to denote the elements of Σ. Σ^ω denotes the set of infinite words over Σ, while Σ^* stands for the set of finite words over Σ, and Σ^+ is the set of non-empty finite words over Σ.

Definition 2.1 (*Nondeterministic ω-automata*). A nondeterministic ω-automaton is a tuple $\mathcal{A} = (Q, \Sigma, \delta, Q_0, \mathsf{Acc})$, where Q is a finite nonempty set of states, Σ is a finite nonempty input alphabet, $\delta : Q \times \Sigma \to 2^Q$ is a transition function, $Q_0 \subseteq Q$ is a nonempty set of initial states and Acc is an *acceptance condition*.

The type of the acceptance condition depends on the type of ω-automata. Within this article, we consider the following acceptance conditions.

—Büchi acceptance condition: $\mathsf{Acc} \subseteq Q$ (we then write F instead of Acc)
—Rabin or Streett acceptance condition: $\mathsf{Acc} = \{(H_1, K_1), \ldots, (H_n, K_n)\}$,
$H_i, K_i \subseteq Q, 1 \leq i \leq n$

Let $T \subseteq Q$ be a subset of states. Given a Büchi acceptance condition F, the set T is *accepting*, if $T \cap F \neq \emptyset$. Given a Rabin acceptance condition, T is *accepting*, if there exists $1 \leq i \leq n$ such that $T \cap H_i = \emptyset$ and $T \cap K_i \neq \emptyset$. Given a Streett acceptance condition, T is *accepting*, if for all $1 \leq i \leq n$ it holds that $T \cap H_i \neq \emptyset$ or $T \cap K_i = \emptyset$. Thus, Rabin and Streett acceptance are complementary to each other.

The automaton $\mathcal{A} = (Q, \Sigma, \delta, Q_0, \mathsf{Acc})$ is *deterministic* if $|Q_0| = 1$ and $|\delta(p, a)| \leq 1$ for all $p \in Q$ and $a \in \Sigma$. It is *total* if $|\delta(p, a)| \geq 1$ for all $p \in Q$ and $a \in \Sigma$. We write NBA, NRA, NSA, DBA, DRA, and DSA to denote the nondeterministic and deterministic versions of Büchi, Rabin, or Streett automata, respectively. We write NBA, NRA, NSA, DBA, DRA, and DSA for the class of the respective automata.

Remark 2.2. Note that any given Büchi acceptance condition F can be expressed by the equivalent Rabin condition $\{(\emptyset, F)\}$ and also by the equivalent Streett condition $\{(F, Q)\}$.

ω-automata serve as language acceptors for languages of infinite words over the input alphabet as follows. A *run* for an infinite word $w = a_1 a_2 \ldots$ is an infinite state sequence $\pi = p_0, p_1, \ldots$ such that $p_0 \in Q_0$ and $p_i \in \delta(p_{i-1}, a_i), i \in \mathbb{N}_{\geq 1}$. We write $\mathsf{inf}(\pi)$ to denote the set of states that occur infinitely often in π. An infinite run π is called *accepting*, if $\mathsf{inf}(\pi)$ is accepting with respect to the acceptance condition. We will sometimes refer to finite runs, that is, finite state sequences $p_0, p_1, \ldots p_n$ such that $p_0 \in Q_0$, $p_i \in \delta(p_{i-1}, a_i), 1 \leq i \leq n$ and $\delta(p_n, a_{n+1}) = \emptyset$. That is, the automaton cannot consume the input letter a_{n+1} in state p_n and *rejects*. The *accepted language* of a nondeterministic ω-automaton $\mathcal{A}$ is defined as

$$\mathcal{L}(\mathcal{A}) = \{w \in \Sigma^\omega \mid \exists \text{ accepting run for } w \text{ in } \mathcal{A}\}.$$

Given an automata class, for example, NBA, we denote by *e.g.* $\mathbb{L}$(NBA) the class of languages definable by this type of automata. It is well known (see, for example, Thomas [1990] and Grädel et al. [2002]) that

$$\mathbb{L}(\mathsf{DBA}) \subsetneq \mathbb{L}(\mathsf{NBA}) = \mathbb{L}(\mathsf{DRA}) = \mathbb{L}(\mathsf{NRA}) = \mathbb{L}(\mathsf{DSA}) = \mathbb{L}(\mathsf{NSA}) = \omega\text{-}\mathsf{reg},$$

where ω-reg denotes the class of ω-regular languages. We will often identify an ω-regular language $L \subseteq \Sigma^\omega$ with some ω-regular expression that describes L. For example, $(a+b)^*a^\omega$ is identified with the set of infinite words over $\Sigma = \{a, b\}$ that contain only finitely many b's.

For acceptors of languages over *finite* words, we use the abbreviation PFA for probabilistic finite automata, while NFA and DFA stand for nondeterministic and deterministic finite automata, respectively.

Notation 2.3. Throughout this article, we will sometimes use the LTL notations $\Diamond$ for "eventually" and $\Box$ for "always". Thus, the combination $\Box\Diamond$ denotes "infinitely often" and $\Diamond\Box$ denotes "continuously from some moment on". Given a set of states $F \subseteq Q$, a run $\pi = p_0, p_1, \ldots$ is said to satisfy $\Diamond F$, denoted $\pi \models \Diamond F$, if there exists an index i such that $p_i \in F$. Similar definitions apply to the other operators, for example, π satisfies a Büchi acceptance condition F ($\pi \models \Box\Diamond F$) or it satisfies a Streett acceptance condition $\{(H_1, K_1), \ldots, (H_n, K_n)\}$ ($\pi \models \wedge_{i=1}^{n}(\Box\Diamond H_i \vee \Diamond\Box\neg K_i)$).

2.2. Markov Decision Processes

To define probabilistic ω-automata, we will need the concept of Markov decision processes (MDP). In an MDP, any state s might have several enabled actions. Each of the actions that are enabled in state s is associated with a probability distribution that yields the probabilities for the successor states.

A distribution on a countable set S is a function $\mu : S \to [0, 1]$ such that $\sum_{s\in S} \mu(s) = 1$. If $\mu(s) = 1$ for some $s \in S$, then μ is called a Dirac distribution.

Definition 2.4 (*Markov Decision Process (MDP)*). A Markov decision process is a tuple $\mathcal{M} = (S, \mathsf{Act}, \delta, \mu)$, where S is a finite nonempty set of states, Act is a finite nonempty set of actions, $\delta : S \times \mathsf{Act} \times S \to [0, 1]$ is a transition probability function such that for all $s \in S$ and $\alpha \in \mathsf{Act}$, either $\delta(s, \alpha, .)$ is a probability distribution on S or $\delta(s, \alpha, .)$ is the null-function (i.e., $\delta(s, \alpha, t) = 0$ for all $t \in S$), and μ is a probability distribution on S (called the *initial* distribution). $\mathsf{Act}(s) = \{\alpha \in \mathsf{Act} \mid \exists t \in S : \delta(s, \alpha, t) > 0\}$ denotes the set of actions that are *enabled* in state s. We require for each state $s \in S$, that $\mathsf{Act}(s)$ is nonempty. If $\mathsf{Act}(s) = \mathsf{Act}$ for all states $s \in S$, we call the MDP *total*.

The intuitive operational behavior of an MDP is as follows. If s is the current state, then first one of the actions $\alpha \in \mathsf{Act}(s)$ is chosen nondeterministically. Afterwards, action α is executed leading to state t with probability $\delta(s, \alpha, t)$. By $\delta(s, \alpha) = \{t \mid \delta(s, \alpha, t) > 0\}$, we denote the set of α-successors of s. Given a state set $S' \subseteq S$, then $\delta(S', \alpha) = \cup_{s\in S'} \delta(s, \alpha)$ denotes the set of α-successors of S'. Moreover, given an action sequence $\alpha_1 \cdots \alpha_{i+1}$, we define inductively $\delta(s, \alpha_1 \cdots \alpha_i \alpha_{i+1}) = \delta(\delta(s, \alpha_1 \cdots \alpha_i), \alpha_{i+1})$.

Definition 2.5 (*Path and Corresponding Notation*). An infinite path of an MDP is an infinite sequence $\pi = s_0, \alpha_1, s_1, \alpha_2, \ldots \in (S \times \mathsf{Act})^\omega$ such that $\alpha_i \in \mathsf{Act}(s_{i-1})$ for $i \in \mathbb{N}_{\geq 1}$. We write paths in the form

$$\pi \;=\; s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} s_2 \xrightarrow{\alpha_3} \ldots$$

$\mathsf{first}(\pi) = s_0$ denotes the starting state of π and $\pi\!\uparrow^i = s_0 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_i} s_i$ its i-th prefix and $\pi\!\uparrow_i = s_i \xrightarrow{\alpha_{i+1}} s_{i+1} \xrightarrow{\alpha_{i+2}} \ldots$ its i-th suffix.

Finite paths are finite prefixes of infinite paths that end in a state. We use the notations $\mathsf{first}(\pi)$ (respectively, $\mathsf{last}(\pi)$) for the first (respectively, last) state of a finite path π and $|\pi|$ for the length (number of actions). $\pi_i = s_i$ denotes the $(i+1)$st state of π and $\mathsf{Act}_i(\pi)$ denotes the ith action on π. $\mathsf{Path}_{\mathsf{fin}}(s)$ (respectively, $\mathsf{Path}_{\mathsf{inf}}(s)$) denotes the set of all finite (respectively, infinite) paths of $\mathcal{M}$ with starting state s. $\mathsf{Path}_{\mathsf{fin}}$ (respectively, $\mathsf{Path}_{\mathsf{inf}}$) stands for the set of all finite (respectively infinite) paths in $\mathcal{M}$.

If $\pi = s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} s_2 \xrightarrow{\alpha_3} \ldots$ is an infinite path, then $\mathsf{Lim}(\pi)$ denotes the pair (T, A) where $T = \mathsf{inf}(\pi)$ is the set of states in π that are visited infinitely often and $A : T \to 2^{\mathsf{Act}}$ is the function that assigns to any state $t \in T$ the set $A(t)$ of actions $\alpha \in \mathsf{Act}$ such that $(s_i = t) \wedge (\alpha_{i+1} = \alpha)$ for infinitely many indices i.

A scheduler denotes an instance that resolves the nondeterminism in the states, and thus, yields a Markov chain and a probability measure on the paths. Intuitively, a scheduler takes as input the “history” of a computation (formalized by a finite path π) and chooses a distribution on the actions available.

Definition 2.6 (*Scheduler*). Given a Markov decision process $\mathcal{M} = (S, \mathsf{Act}, \delta, \mu)$, a history-dependent randomized scheduler is a function

$$\mathcal{U} : \mathsf{Path}_{\mathsf{fin}} \to \mathsf{Distr}(\mathsf{Act}),$$

such that $\mathsf{supp}(\mathcal{U}(\pi)) \subseteq \mathsf{Act}(\mathsf{last}(\pi))$ for all $\pi \in \mathsf{Path}_{\mathsf{fin}}$.

A scheduler $\mathcal{U}$ is called *deterministic*, if $\mathcal{U}(\pi)$ is a Dirac distribution for all $\pi \in \mathsf{Path}_{\mathsf{fin}}$. $\mathcal{U}$ is called memoryless if $\mathcal{U}(\pi) = \mathcal{U}(\mathsf{last}(\pi))$ for all $\pi \in \mathsf{Path}_{\mathsf{fin}}$. $\mathsf{Sched}_{\mathsf{HR}}$ (respectively, $\mathsf{Sched}_{\mathsf{HD}}$) denotes the set of history dependent, randomized (respectively, deterministic) schedulers and $\mathsf{Sched}_{\mathsf{MR}}$ (respectively, $\mathsf{Sched}_{\mathsf{MD}}$) denotes the set of memoryless randomized (respectively, deterministic) schedulers.

Given an MDP $\mathcal{M} = (S, \mathsf{Act}, \delta, \mu)$ and a scheduler $\mathcal{U}$ for $\mathcal{M}$, the behavior of $\mathcal{M}$ under $\mathcal{U}$ can be formalized by an infinite-state Markov chain $\mathcal{M}_{\mathcal{U}} = (\mathsf{Path}^{\mathcal{M}}_{\mathsf{fin}}, \mathsf{p}, \mu)$, where

$$\mathsf{p}(\pi,\ \pi') = \mathcal{U}(\pi)(\alpha) \cdot \delta(\mathsf{last}(\pi), \alpha, \mathsf{last}(\pi')),$$

for $\pi, \pi' \in \mathsf{Path}^{\mathcal{M}}_{\mathsf{fin}}$ with $|\pi'| = |\pi| + 1$, $\pi'\!\uparrow^{|\pi|} = \pi$ and α is the last action on the path π', that is, $\pi' = \pi \xrightarrow{\alpha} \mathsf{last}(\pi')$. As the states of $\mathcal{M}_{\mathcal{U}}$ are finite paths of $\mathcal{M}$, this notation is somewhat inconvenient. Let $\Omega = (\mathsf{Path}^{\mathcal{M}_{\mathcal{U}}}_{\mathsf{inf}}, \Delta^{\mathcal{M}_{\mathcal{U}}})$ and $\Omega' = (\mathsf{Path}^{\mathcal{M}}_{\mathsf{inf}}, \Delta^{\mathcal{M}})$ be the measurable spaces where $\Delta^{\mathcal{M}_{\mathcal{U}}}$ is the σ-algebra generated by the empty set and the set of basic cylinders over $\mathcal{M}_{\mathcal{U}}$ and $\Delta^{\mathcal{M}}$ is the σ-algebra generated by the empty set and the set of basic cylinders over $\mathcal{M}$. As usual, a basic cylinder consists of all infinite paths that have some fixed common prefix. We define

$$\mathsf{f} : \mathsf{Path}^{\mathcal{M}_{\mathcal{U}}}_{\mathsf{inf}} \to \mathsf{Path}^{\mathcal{M}}_{\mathsf{inf}},$$

as $\mathsf{f}(\pi_0 \xrightarrow{\alpha_1} \pi_1 \xrightarrow{\alpha_2} \ldots) = \mathsf{last}(\pi_0) \xrightarrow{\alpha_1} \mathsf{last}(\pi_1) \xrightarrow{\alpha_2} \ldots$ (note that the π_i's are finite paths of $\mathcal{M}$). Then f is a measurable function and we define the following probability measure on $\Delta^{\mathcal{M}}$

$$\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(A') = \mathsf{Pr}^{\mathcal{M}_{\mathcal{U}}}(\mathsf{f}^{-1}(A')),\ \text{ for } A' \in \Delta^{\mathcal{M}},$$

where $\mathsf{Pr}^{\mathcal{M}_{\mathcal{U}}}(.)$ is the unique probability measure on $\Omega = (\mathsf{Path}^{\mathcal{M}_{\mathcal{U}}}_{\mathsf{inf}}, \Delta^{\mathcal{M}_{\mathcal{U}}})$ with

$$\mathsf{Pr}^{\mathcal{M}_{\mathcal{U}}}(\{\pi \mid \pi\!\uparrow^k = \pi'\}) = \mu(\mathsf{first}(\pi')) \cdot \prod_{i=1}^{k} \mathsf{p}(\pi'\!\uparrow^{i-1}, \pi'\!\uparrow^i)$$

for any $\pi' \in \mathsf{Path}_{\mathsf{fin}}^{\mathcal{M}_{\mathcal{U}}}$ with $|\pi'| = k$. Then given a scheduler $\mathcal{U}$ for $\mathcal{M}$, the probability measure $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}$ formalizes the behavior of $\mathcal{M}$ under $\mathcal{U}$, where we have the convenience to talk about measures of sets of infinite paths of $\mathcal{M}$. Given a state $s \in S$, we denote by $\mathsf{Pr}_s^{\mathcal{M},\mathcal{U}}$ the probability measure that is obtained if $\mathcal{M}$ is equipped with the starting distribution μ_s^1, where $\mu_s^1(s) = 1$. For more information on measure theory, see, for example, Feller [1950].

We will also fix the following notation for convenience. Given an MDP $\mathcal{M}$, a scheduler $\mathcal{U}$ and a path property E, then

$$\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(E) = \mathsf{Pr}^{\mathcal{M},\mathcal{U}}\big(\{\pi \in \mathsf{Path}_{\mathsf{inf}}^{\mathcal{M}} \mid \pi \text{ satisfies } E\}\big)$$

denotes the probability that the property E holds in $\mathcal{M}$ under the scheduler $\mathcal{U}$.

Throughout this article, we shall use the concepts of end components [Rosier and Yen 1986; de Alfaro 1997, 1998], which can be seen as the MDP counterpart to terminal strongly connected components in Markov chains. Intuitively, an end component of an MDP is a nonempty strongly connected subMDP, that means an end component consists of a nonempty state set $T \subseteq S$ and a nonempty action set $A(t)$ for each state $t \in T$ such that, once T is entered and only actions in $A(t)$ are chosen, the set T will not be left and any state of T can be reached from any other state in T.

Definition 2.7 (*End Components*). Let $\mathcal{M} = (S, \mathsf{Act}, \delta, \mu)$ be an MDP. An end component of $\mathcal{M}$ is a pair (T, A) where $\emptyset \neq T \subseteq S$ and $A : T \to 2^{\mathsf{Act}}$ is a function such that

—$\emptyset \neq A(s) \subseteq \mathsf{Act}(s)$ for all states $s \in T$,
—$\sum_{t \in T} \delta(s, \alpha, t) = 1$ for all states $s \in T$ and actions $\alpha \in A(s)$,
—the underlying digraph $(T, \to_A)$ of (T, A) is strongly connected.

Here, $\to_A$ denotes the edge-relation induced by A, that is $s \to_A t$ if and only if $\delta(s, \alpha, t) > 0$ for some action $\alpha \in A(s)$.

Given an MDP $\mathcal{M}$ and a scheduler $\mathcal{U}$ it holds that in the process induced by $\mathcal{U}$, almost all path of $\mathcal{M}$ (following $\mathcal{U}$) "end" in an end component, that is their limit $\mathsf{Lim}(.)$ forms an end component. For the following lemma, see de Alfaro [1997, 1998].

LEMMA 2.8 (ALMOST-SURE END COMPONENT). *For any MDP $\mathcal{M}$ and scheduler $\mathcal{U}$,* $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\{\pi \in \mathsf{Path}_{\mathsf{inf}}^{\mathcal{M}} \mid \mathsf{Lim}(\pi)$ *is an end component*$\}) = 1$.

3. PROBABILISTIC ω-AUTOMATA

We introduce probabilistic variants of ω-automata that serve as acceptors for languages over infinite words. The essential idea is to equip nondeterministic ω-automata with probabilistic distributions that resolve the nondeterministic choices and to define the acceptance of an infinite input word by some requirement on the probability measure of the set of accepting runs: this probability should be either positive, or equal to 1 or greater than a given threshold λ, depending on the semantics we consider.

3.1. Definition of Probabilistic ω-Automata

In this section, we introduce probabilistic ω-automata which can be viewed as nondeterministic ω-automata where the nondeterminism is resolved by a probabilistic choice. That is, for any state p and letter $a \in \Sigma$ either p does not have any a-successor or there is a probability distribution for the a-successors of p. We will consider various semantics that pose qualitative (respectively, quantitative) conditions on the probability measure of the accepting runs for an input word.

Note that probabilistic ω-automata have also been defined in Reisz [1999a, 1999b], where an infinite word is accepted if and only if there exists an infinite accepting run for this word such that this run has a positive probability. We use here different syntax and semantics as introduced in Baier and Grösser [2005].

Definition 3.1 (*Probabilistic ω-Automata*). A probabilistic ω-automaton is a tuple $\mathcal{P} = (Q, \Sigma, \delta, \mu_0, \mathsf{Acc})$, where

—Q is a finite nonempty set of states,
—Σ is a finite nonempty input alphabet,
—$\delta : Q \times \Sigma \times Q \to [0, 1]$ is a transition probability function such that for all $p \in Q$ and $a \in \Sigma$, either $\delta(p, a, .)$ is a probability distribution on Q or $\delta(p, a, .)$ is the null-function (i.e., $\delta(p, a, q) = 0$ for all $q \in Q$),
—μ_0 is a probability distribution on Q (called the *initial* distribution) and
—Acc is a Büchi, Rabin, or Streett acceptance condition.

As for nondeterministic automata, we write PBA, PRA, PSA to denote the probabilistic version of Büchi, Rabin, or Streett automata, respectively. We write PBA, PRA and PSA for the class of the respective automata.

Remark 3.2. Note that probability distributions are not restricted to rational coefficients. When considering algorithmic issues however, we will assume that probabilities appearing in the distributions are always rational. Almost all what follows would hold as well if we restricted from now on to rational values only. It however makes a crucial difference when we investigate particularities of probabilistic ω-automata under the threshold semantics (see Section 4.3).

Apparently, a probabilistic ω-automaton $\mathcal{P}$ is an MDP equipped with an acceptance condition. We will use the notation $\mathcal{P}$ also to denote only the underlying MDP of the automaton. We call the automaton *total* if the underlying MDP is total. The operational behavior of $\mathcal{P} = (Q, \Sigma, \delta, \mu_0, \mathsf{Acc})$ for a given input word $w = a_1a_2a_3 \ldots \in \Sigma^\omega$ is as follows. The automaton chooses at random an initial state p_0 according to the initial distribution μ_0. After having consumed the first i input symbols $a_1, \ldots, a_i$, $\mathcal{P}$ in state p_i tries to read the next input symbol $a = a_{i+1}$ in state p_i. If there is no outgoing a-transition from the current state p_i, that is, if $a \notin \mathsf{Act}(p_i)$, then $\mathcal{P}$ rejects. Otherwise, $\mathcal{P}$ moves with probability $\delta(p_i, a, p)$ to the next state $p = p_{i+1}$. As for nondeterministic automata, a resulting infinite state-sequence $p_0, p_1, \ldots$ is called a run for w in $\mathcal{P}$. This behavior interprets the input word as a scheduler.

Given an input word $w = a_1a_2a_3 \ldots \in \Sigma^\omega$ we define the scheduler $\mathcal{U}(w)$ such that $\mathcal{U}(w)(p_0, \ldots, p_{n-1})(a_n) = 1$. That is, in step n, the scheduler chooses with probability 1 the letter a_n as the next action. Then, the operational behavior of $\mathcal{P}$ reading the input word w, is formalized by the Markov chain $\mathcal{P}_{\mathcal{U}(w)}$. In contrast to nondeterministic automata, where an input word w is accepted by the automaton if there exists an accepting run for w, the requirement for probabilistic automata is a constraint on the measure of the set of accepting runs for w. Throughout this article, we will identify an input word w with its associated scheduler $\mathcal{U}(w)$, thus we will write $\mathsf{Pr}^{\mathcal{P},w}(.)$ instead of $\mathsf{Pr}^{\mathcal{P},\mathcal{U}(w)}(.)$. For the sake of convenience, we also fix the following notation for the acceptance probability of a word w and a given probabilistic ω-automaton $\mathcal{P}$:

$$\mathsf{Pr}^{\mathcal{P}}(w) = \mathsf{Pr}^{\mathcal{P},w}\big(\big\{\pi \in \mathsf{Path}^{\mathcal{P}}_{\mathsf{inf}} \;\big|\; \mathsf{inf}(\pi) \text{ is accepting}\big\}\big).$$

By the results of Vardi [1985] and Courcoubetis and Yannakakis [1995] the set of accepting runs for w is measurable when dealing with Büchi, Rabin, or Streett acceptance. We consider three different semantics, and hence three possible definitions for the accepted language:

probable semantics: $\mathcal{L}^{>0}(\mathcal{P}) = \{w \in \Sigma^\omega \mid \mathsf{Pr}^{\mathcal{P}}(w) > 0\}$,
almost-sure semantics: $\mathcal{L}^{=1}(\mathcal{P}) = \{w \in \Sigma^\omega \mid \mathsf{Pr}^{\mathcal{P}}(w) = 1\}$,
threshold semantics: $\mathcal{L}^{>\lambda}(\mathcal{P}) = \{w \in \Sigma^\omega \mid \mathsf{Pr}^{\mathcal{P}}(w) > \lambda\}$, given $\lambda \in]0, 1[$.

Note that the accepted language of a probabilistic ω-automaton under any semantics is included in the language that is accepted by the underlying nondeterministic ω-automaton, that is, the nondeterministic ω-automaton that stems from the given probabilistic ω-automaton by ignoring the probabilities.

Simplifying notations, we will index an automaton type (respectively, automata class) with a certain type of semantics, that is, we write PBA$^{>0}$ to denote a probabilistic Büchi automaton equipped with the probable semantics or $\mathsf{PRA}^{=1}$ for the class of probabilistic Rabin automata equipped with the almost-sure semantics. Given an automata class indexed by a type of semantics, for example, $\mathsf{PBA}^{>0}$, we denote by $\mathbb{L}(\mathsf{PBA}^{>0})$ the class of languages definable by this class of automata under the given semantics.

Let a probabilistic ω-automaton $\mathcal{P}$ and an input word w be given. Given an end component (T, A), then

$$\mathsf{Pr}^{\mathcal{P},w}((T, A)) = \mathsf{Pr}^{\mathcal{P},w}\{\pi \in \mathsf{Path}^{\mathcal{P}}_{\mathsf{inf}} \mid \mathsf{Lim}(\pi) = (T, A)\}$$

denotes the probability of all paths π such that $\mathsf{Lim}(\pi) = (T, A)$. We call the end component (T, A) *accepting* if the state set T is accepting (with respect to the acceptance condition of $\mathcal{P}$). Recall Lemma 2.8 stating that given an MDP and a scheduler, almost all runs form an end-component in their limit. Thus, the acceptance probability $\mathsf{Pr}(w)$ agrees with the probability measure of the set of runs π for w such that $\mathsf{Lim}(\pi)$ is an accepting end component (AEC for short). As $\mathcal{P}$ has only finitely many end components, this yields the following lemma.

LEMMA 3.3 (AEC-LEMMA). *For any probabilistic ω-automaton $\mathcal{P}$ and any input word w, it holds that $w \in \mathcal{L}^{>0}(\mathcal{P})$ if and only if $\mathsf{Pr}^{\mathcal{P},w}((T, A)) > 0$ for some accepting end component (T, A).*

3.2. Examples of Probabilistic Büchi Automata

We now provide a few examples of probabilistic ω-automata to illustrate the definition. For simplicity, we only give examples of probabilistic ω-automata with a Büchi acceptance condition.

Given a PBA $\mathcal{P}$ with the acceptance condition F, intuitively, $\mathsf{Pr}^{\mathcal{P}}(w)$ denotes the probability for the event "infinitely often F" under the scheduling policy induced by w. Recalling our LTL notations, this means that $\mathsf{Pr}^{\mathcal{P}}(w) = \mathsf{Pr}^{\mathcal{P},w}(\Box\Diamond F)$.

In the pictures of PBA, we use boxes to denote the accepting states and circles for the non-accepting states. We might simply write a as label for a transition from p to q if $\delta(p, a, q) = 1$. Label a, x with $x \in]0, 1[$ for a transition from p to q denotes that $\delta(p, a, q) = x$. An initial state will be indicated through an incoming edge, labeled with the initial probability of the state. Again, if the probability is 1, we might omit it.

Example 3.4. Figure 1 shows a PBA $\mathcal{P}$ over the alphabet $\Sigma = \{a, b\}$ such that

$$\mathcal{L}^{>0}(\mathcal{P}) = (a + b)^* a^\omega, \quad \mathcal{L}^{=1}(\mathcal{P}) = b^* a^\omega, \quad \mathcal{L}^{>\frac{1}{5}}(\mathcal{P}) = b^* a b^* a b^* a^\omega$$

To see this, we first notice that only the words in $(a + b)^* a^\omega$ have an accepting run, because the a-labeled self-loop in the accepting state p_1 is the only outgoing transition of state p_1. On the other hand, $\mathsf{Pr}(a^\omega) = 1$ (as the nonaccepting run $p_0, p_0, p_0, \ldots$ has probability 0 while all other runs for a^ω are accepting). Given a word $\sigma = c_1 c_2 \ldots c_\ell b a^\omega$

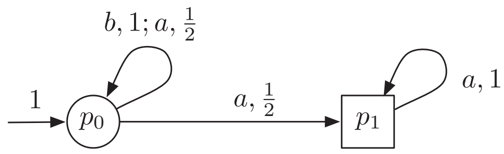


Fig. 1. PBA $\mathcal{P}$ with $\mathcal{L}^{>0}(\mathcal{P}) = (a+b)^*a^\omega$.

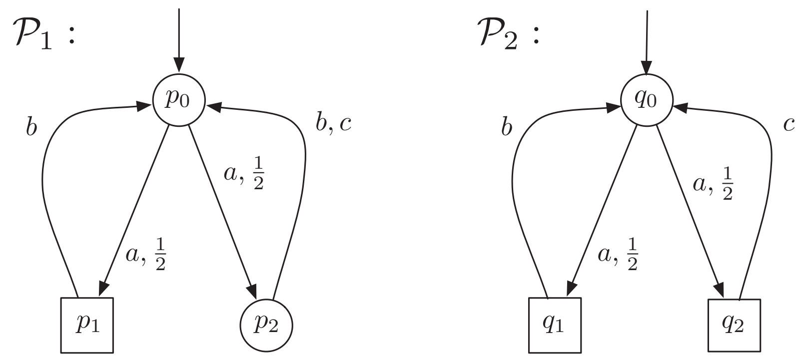


Fig. 2. PBA$^{>0}$ for $(ab+ac)^*(ab)^\omega$ and for $\emptyset$.

with $c_i \in \{a, b\}$, the precise acceptance probability is therefore $\Pr^{\mathcal{P}}(\sigma) = \left(\frac{1}{2}\right)^k$ where $k = |\{i \in \{1, \ldots, \ell\} : c_i = a\}|$. Together, this yields $\mathcal{L}^{>0}(\mathcal{P}) = (a+b)^*a^\omega$, $\mathcal{L}^{=1}(\mathcal{P}) = b^*a^\omega$ and $\mathcal{L}^{>\frac{1}{5}}(\mathcal{P}) = b^*ab^*ab^*a^\omega$.

Clearly, any DBA can be viewed as a PBA under any semantics, with $\delta_{\mathsf{PBA}}(p, a, q) = 1$ if $\delta_{\mathsf{DBA}}(p, a) = \{q\}$ and $\mu_0^{\mathsf{PBA}}(q_0) = 1$. On the other hand, it is well known that the language $(a+b)^*a^\omega$ cannot be recognized by a DBA, thus this example shows that PBA$^{>0}$ are strictly more expressive than DBA.

It is worth mentioning that the qualitative criteria “accepting runs have positive probability” is different from the acceptance criteria “there is an accepting run” in the context of languages of *infinite* words, while they agree for probabilistic automata viewed as acceptors for *finite* words. In fact, the naïve transformation from PBA to NBA which relies on ignoring the probabilities, in general fails to yield an equivalent NBA.

Example 3.5. Consider, for example, the automaton $\mathcal{P}_1$ on the left of Figure 2. Its underlying NBA (that we obtain by ignoring the probabilities) accepts the language $((ac)^*ab)^\omega$ whereas $\mathcal{P}_1$, under the probable semantics, accepts the language $(ab+ac)^*(ab)^\omega$. The intuitive argument why any word w in $(ab+ac)^\omega$ with infinitely many c's is rejected relies on the observation that almost all runs[2] for w are finite and end in state p_1 (where the next input symbol is c and cannot be consumed in state p_1). Under the almost-sure semantics, the language accepted by $\mathcal{P}_1$ is $(ab)^\omega$.

Another example is the PBA $\mathcal{P}_2$ on the right of Figure 2. It accepts (under any semantics) the empty language as any infinite word in $(ab+ac)^\omega$ has exactly one accepting run in $\mathcal{P}_2$, but its probability is 0. However, the underlying NBA accepts the language $(ab+ac)^\omega$.

[2]The formulation “almost all runs have property X” means that the probability measure of the runs where property X does not hold is 0.

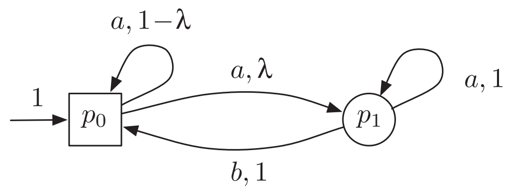


Fig. 3. PBA$^{>0}$ $\mathcal{P}_\lambda$ accepts a non-ω-regular language (where $0 < \lambda < 1$).

4. A CLOSER LOOK ON PBA

In the last section, we introduced general probabilistic ω-automata, and presented a few simple examples of PBA. We now examine PBA a little closer.

4.1. PBA under the Probable Semantics

In this section, we report on results on the expressiveness and efficiency of PBA under the probable semantics. We also show that the precise transition probabilities matter for the accepted language of a PBA$^{>0}$ and present a pumping lemma.

4.1.1. Expressiveness. First, we establish that the class of languages that can be accepted by a PBA$^{>0}$ strictly contains the class of ω-regular languages.

THEOREM 4.1 (PBA$^{>0}$ ARE STRICTLY MORE EXPRESSIVE THAN NBA).

$$\omega\text{-reg} \subsetneq \mathbb{L}(\mathrm{PBA}^{>0})$$

The proof of Theorem 4.1 is split into two parts. In Lemma 4.2, we show that for any NBA $\mathcal{A}$ there exists a PBA $\mathcal{P}$ such that $\mathcal{L}^{>0}(\mathcal{P}) = \mathcal{L}(\mathcal{A})$. Then, in Lemma 4.3 we provide an example of a PBA$^{>0}$ for which the accepted language is not ω-regular.

LEMMA 4.2 (FROM NBA TO PBA$^{>0}$). *For any NBA $\mathcal{A}$, there is a PBA $\mathcal{P}$ such that $\mathcal{L}^{>0}(\mathcal{P}) = \mathcal{L}(\mathcal{A})$ and $|\mathcal{P}| = \mathcal{O}(\exp(|\mathcal{A}|))$.*

PROOF. Following Courcoubetis and Yannakakis [1995], we call an NBA $\mathcal{A}$ *deterministic in limit* if $|\delta(p, a)| \leq 1$ for any state p that is reachable from an accepting state $q \in F$ and any symbol $a \in \Sigma$. If we regard an NBA $\mathcal{A}$ that is deterministic in limit as a PBA $\mathcal{P}$ (with arbitrary probability distributions to resolve the nondeterministic choices) then $\mathcal{L}(\mathcal{A}) = \mathcal{L}^{>0}(\mathcal{P})$. Courcoubetis and Yannakakis [1995] provided a transformation from a given NBA $\mathcal{A}$ into an equivalent NBA that is deterministic in limit and whose size is (single) exponential in $|\mathcal{A}|$. This yields the proof of the lemma. □

It remains to provide an example of a PBA$^{>0}$ that recognizes a language that is not ω-regular.

LEMMA 4.3 (A PBA$^{>0}$ THAT ACCEPTS A NON-ω-REGULAR LANGUAGE). *The PBA$^{>0}$ depicted in Figure 3 accepts a language that is not ω-regular.*

PROOF. For each real number $\lambda \in]0, 1[$, the PBA$^{>0}$ $\mathcal{P}_\lambda$ depicted in Figure 3 recognizes the language:

$$L_\lambda = \left\{ a^{k_1} b a^{k_2} b a^{k_3} b \ldots \mid k_1, k_2, k_3 \ldots \in \mathbb{N}_{\geq 1} \text{ s.t. } \prod_{i=1}^{\infty} (1 - (1-\lambda)^{k_i}) > 0 \right\}.$$

Note that $\mathcal{L}^{>0}(\mathcal{P}_\lambda) \subseteq (a^+b)^\omega$ as every accepting run for an infinite word w that has only finitely many b's has to stay in state p_0 from some point on. But such runs occur with probability 0. Let $w = a^{k_1} b a^{k_2} \ldots \in (a^+b)^\omega$. Starting in p_0 and reading the first k_1 letters a, the automaton reaches state p_0 with probability $(1-\lambda)^{k_1}$ and thus state p_1 with probability $1-(1-\lambda)^{k_1}$. Reading the first b the automaton thus rejects with probability $(1-\lambda)^{k_1}$ and carries on to read the input word with probability $1-(1-\lambda)^{k_1}$. This shows that the probability not to reject while reading the word w is

$$\prod_{i=0}^{\infty}(1-(1-\lambda)^{k_i})$$

and moreover this agrees with the probability to visit infinitely often the final state p_0. Therefore, $\mathcal{L}^{>0}(\mathcal{P}_\lambda) = L_\lambda$.

The following argument shows that $\mathcal{L}^{>0}(\mathcal{P}_\lambda)$ is not ω-regular. It is easily seen that $\mathcal{L}^{>0}(\mathcal{P}_\lambda)$ is nonempty and obviously it does not contain any lasso-shaped word, that is, a word of the form $xy^\omega, x \in \Sigma^*$ and $y \in \Sigma^+$. As any nonempty ω-regular language contains a lasso-shaped word (since it can be described by an NBA with an accepting cycle), it follows that $\mathcal{L}^{>0}(\mathcal{P}_\lambda)$ is not ω-regular. □

Remark 4.4 (*Nonregular Convergence*). The nonregular convergence condition for the words accepted by the PBA$^{>0}$ $\mathcal{P}_\lambda$ in Figure 3 can be explained by the observation that there are finite input words that $\mathcal{P}_\lambda$ rejects with arbitrary small probability. More precisely, when $\mathcal{P}_\lambda$ tries to read a finite word $a^k b$ in state p_0 then $\mathcal{P}_\lambda$ fails to consume the last letter b (i.e., rejects) with probability $(1-\lambda)^k$. If k tends to infinity, the rejecting probability $(1-\lambda)^k$ tends to 0.

Similarly, there are PBA and infinite input words that have accepting runs in the underlying nondeterministic automaton, while the probabilities for the run fragments connecting two accepting states tend to zero. Such an example is described in Section 4.2.

We showed that PBA$^{>0}$ are strictly more expressive than ω-regular languages. Note that Baier and Grösser [2005] identified a subclass of PBA$^{>0}$ that corresponds to the class of ω-regular languages. For this purpose, Baier and Grösser [2005] introduced so-called uniform PBA$^{>0}$. The uniformity condition is semantic in nature and is motivated by the observation made in Remark 4.4 and serves to rule out PBA$^{>0}$ with "non-regular converging behaviors", as it is the case for the PBA$^{>0}$ $\mathcal{P}_\lambda$ of Figure 3. Just recently, Chadha et al. [2009b] introduced syntactic restrictions on PBA that capture regularity, both for the probable as well as the almost-sure semantics. The restriction considered imposes a hierarchical structure on the PBA.

4.1.2. Efficiency. We saw in Lemma 4.2 that, for each NBA, there exists an equivalent PBA$^{>0}$ of size exponential in the size of the NBA. We now study the efficiency of PBA$^{>0}$ in more detail and show that for some languages, PBA$^{>0}$ can be exponentially better than nondeterministic ω-automata.

LEMMA 4.5 (PBA$^{>0}$ CAN BE EXPONENTIALLY SMALLER THAN NSA). *There exists a family $(L_n)_{n\in\mathbb{N}}$ of ω-regular languages in $\{a,b\}^\omega$ such that for every n, L_n is accepted by a PBA under the probable semantics with $2n$ states, while any NSA for L_n has at least $\frac{2^n}{n}$ states.*

PROOF. The language $L_n = \{xy^\omega : x \in \{a,b\}^*, y \in \{a,b\}^n\}$ is accepted by the PBA$^{>0}$ $\mathcal{P}$ shown in Figure 4. Note that all states of $\mathcal{P}$ are accepting and that all states except n_a have a b-transition to the state 1_b and all states except n_b have an a-transition to the state 1_a. We assume uniform distributions: All edges except the a-edge in state n_a and the b-edge in state n_b are taken with probability $\frac{1}{2}$.

Let $w = a_1a_2\ldots \notin L_n$. Then there are infinitely many indices i such that $a_i = a \wedge a_{i+n} = b$. Since every state except the state n_b has an a-transition to state 1_a, the stochastic process induced by $\mathcal{P}$ and the input word w will almost surely be infinitely often in state 1_a with the letter b coming up in n steps. But each time (with probability

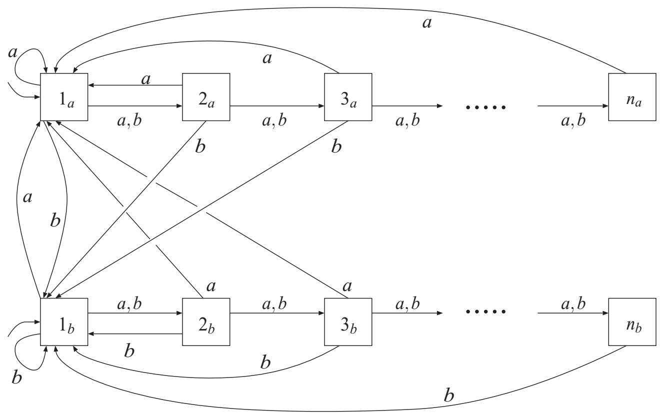


Fig. 4. PBA$^{>0}$ for L_n as in the proof of Lemma 4.5.

$\frac{1}{2^{n-1}}$) the process will have moved to the state n_a while reading the upcoming $n-1$ letters, thus rejecting upon reading the b. Thus, almost surely the process will reject infinitely often with probability $\frac{1}{2^{n-1}}$ which shows that almost all runs are rejecting. Thus, $\mathsf{Pr}^{\mathcal{P}}(w) = 0$ and $w \notin \mathcal{L}^{>0}(\mathcal{P})$. Therefore, $\mathcal{L}^{>0}(\mathcal{P}) \subseteq L_n$.

On the other hand, given a word $w \in L_n$, we can write w as xy^ω with $x \in \{a, b\}^*$ and $y \in \{a, b\}^n$. Then, $\pi = 1_{c_1}, \ldots, 1_{c_k}, 1_{d_1}$ is a run for xd_1, where $x = c_1c_2\ldots c_k$ and $y = d_1d_2\ldots d_n$. (The c's and d's are symbols in $\{a, b\}$.) The probability for this run is strictly greater than zero. Since from that state 1_{d_1} on, the process will never reject while reading the remaining suffix of w and since every infinite run is accepting, this shows that w will be accepted with a probability greater than zero. This yields $L_n \subseteq \mathcal{L}^{>0}(\mathcal{P})$.

It remains to show that any NSA for L_n has at least $\frac{2^n}{n}$ states. Let $\mathcal{A}$ be an NSA with $\mathcal{L}(\mathcal{A}) = L_n$. Let $x = c_1 \ldots c_n$, $y = d_1 \ldots d_n \in \{a, b\}^n$ such that

$$c_1 \ldots c_n \neq d_i \ldots d_n d_1 \ldots d_{i-1} \text{ for all } i = 1, \ldots, n. \tag{+}$$

Then any two accepting cycles for $(c_1 \ldots c_n)^\omega$ and $(d_1 \ldots d_n)^\omega$ are disjoint, or else $\mathcal{A}$ would accept some word of the form $(a+b)^*(c_1 \ldots c_j d_i \ldots d_n d_1 \ldots d_{i-1} c_{j+1} \ldots c_n)^\omega$. But such a word is not in L_n because of (+). Thus, $\mathcal{A}$ has at least $\frac{2^n}{n}$ disjoint accepting cycles, which proves the claim. □

Another example illustrates the efficiency of PBA$^{>0}$ compared to NBA:

LEMMA 4.6 (PBA CAN BE EXPONENTIALLY SMALLER THAN NBA). *Let L_n be the language consisting of all infinite words $w = a_1a_2a_3 \ldots \in \{a, b, c\}^\omega$ such that for all $0 \le i < n$:*

$$\overset{\infty}{\exists}\, k \text{ such that } a_{kn+i} = a \text{ if and only if } \overset{\infty}{\exists}\, k \text{ such that } a_{kn+i} = b. \tag{++}$$

Then, L_n is accepted by a PBA$^{>0}$ with $\mathcal{O}(n^2)$ states, while any NBA that accepts L_n has $\Omega(2^n)$ states.

PROOF. Safra and Vardi [1989] proved that any NBA that accepts L_n has $\Omega(2^n)$ states. But there exists an NSA that accepts L_n and consists of $\mathcal{O}(n)$ states.

It remains to show that there is a PBA$^{>0}$ of quadratic size that accepts L_n. For any word $w = a_1a_2 \ldots \in L_n$, we refer to the suffix $a_{rn+1}a_{rn+2}\ldots$ such that

(1) for all $0 \le i < n$ either $a_{kn+i} = c$ for all $k \ge r$ or there are infinitely many k, ℓ with $a_{kn+i} = a$ and $a_{\ell n+i} = b$ and
(2) r is minimal with respect to (1)

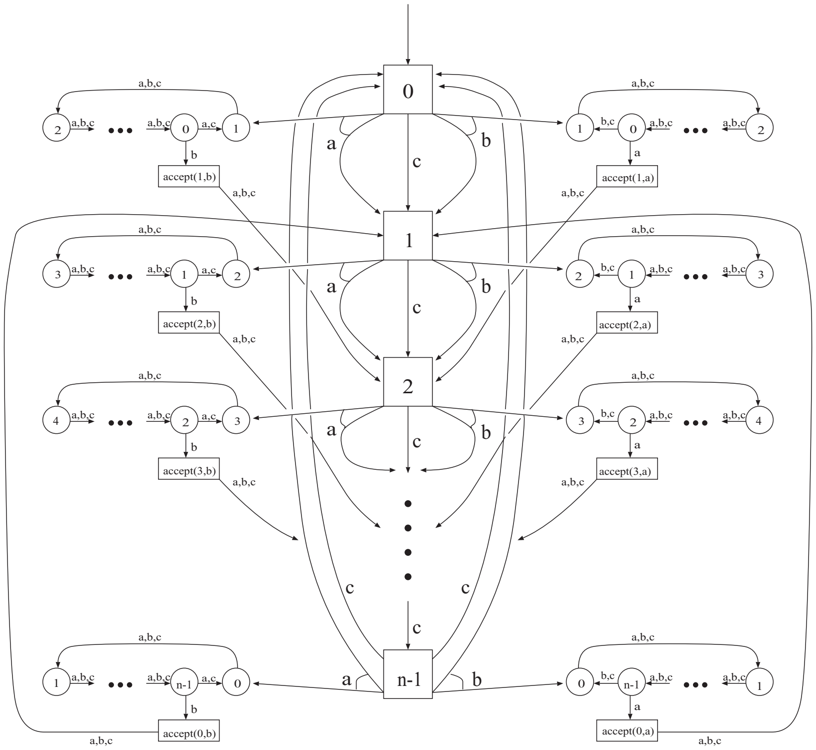


Fig. 5. PBA$^{>0}$ with $\mathcal{O}(n^2)$ states, while any equivalent NBA has $\Omega(2^n)$ states.

as the *legal suffix* of w. A PBA$^{>0}$ $\mathcal{P}$ with $\mathcal{O}(n^2)$ states that accepts L_n is depicted in Figure 5 (we assume uniform distributions). All following calculations with indices $i, j \in \{0, 1, \dots, n-1\}$ are modulo n, that is, we simply write $i+1$ instead of $(i+1)$ mod n.

The automaton $\mathcal{P}$ consists of:

—a subautomaton $\mathcal{P}_{init}$ that serves to wait until the legal suffix of w starts. It consists of a cycle of accepting states $0, 1, \dots, n-1$ that are passed in this order.

—subautomata $\mathcal{P}_{(i,a)}$ and $\mathcal{P}_{(i,b)}$ that are entered from $\mathcal{P}_{init}$ when reading the letter a (respectively, b) in state i. The automata $\mathcal{P}_{(i,a)}$ and $\mathcal{P}_{(i,b)}$ consist of a cycle of nonaccepting states $0, 1, \dots, n-1$ that are passed in this order. They are entered in state $i+1$ (coming from state i of $\mathcal{P}_{init}$ upon reading the letter a (respectively b)) and can be left via an accepting state only when reading the letter b (respectively, a) in state i.

Note that for all words $w \in \{a, b, c\}^\omega$ all runs are infinite and almost all runs leave the subautomaton $\mathcal{P}_{init}$ if w contains infinitely many a's or b's. The automaton rejects if it enters $\mathcal{P}_{(i,a)}$ or $\mathcal{P}_{(i,b)}$, but there is no following position $kn+i$ with $a_{kn+i} = b$ or $a_{kn+i} = a$, respectively.

Let $w = a_1 a_2 \dots \notin L_n$. Without loss of generality, there is some $r \geq 0$ and some $i \in \{0, 1, \dots, n-1\}$ such that $a_{kn+i} = a$ for infinitely many k, but $a_{kn+i} \neq b$ for all $k \geq r$. Assume for simplicity that for all $j \neq i$, condition (++) is fulfilled. We now consider the stochastic process induced by $\mathcal{P}$ and w. As there are infinitely many such k's, the process will almost surely enter $\mathcal{P}_{(i,a)}$ but never leave it. Hence, almost all runs for w are rejecting which yields $w \notin \mathcal{L}^{>0}(\mathcal{P})$. If $(++)$ is violated for several indices $i \in \{0, \ldots, n-1\}$, then the process will almost surely end up in several $\mathcal{P}_{(i,a)}$ and $\mathcal{P}_{(i,b)}$ and never leave those. Hence, almost all runs for w are rejecting which yields $w \notin \mathcal{L}^{>0}(\mathcal{P})$.

Vice-versa, let $w = a_1 a_2 \ldots \in L_n$ and $a_{rn+1}, a_{rn+2}, \ldots$ be the legal suffix of w. Then, all runs for w that stay in $\mathcal{P}_{init}$ for the first rn input symbols (the prefix $a_1 \ldots a_{rn}$ of w) will infinitely often be in $\mathcal{P}_{init}$ and are therefore accepting. Hence, $\mathsf{Pr}^{\mathcal{P}}(w) > 0$ and $w \in \mathcal{L}^{>0}(\mathcal{P})$. □

These examples illustrate that PBA$^{>0}$ can yield much more compact representations of ω-regular languages than nondeterministic automata. Vice-versa, Baier and Grösser [2005] provides an example for a family of ω-regular languages with nondeterministic Büchi automata of linear size, while all uniform PBA$^{>0}$ have at most $\Omega(2^n)$ states. It is open whether the uniformity condition in the proof of Baier and Grösser [2005] can be dropped.

4.1.3. The Precise Probabilities Matter under the Probable Semantics. We will show that the precise transition probabilities of a probabilistic automaton play a role for the language that is accepted under the probable semantics. The PBA $\mathcal{P}_\lambda$, represented in Figure 3, page 11, illustrates this phenomenon.

THEOREM 4.7 (THE PRECISE PROBABILITIES MATTER FOR PBA$^{>0}$). *For any* $0 < \lambda < \mu < 1$, $\mathcal{L}^{>0}(\mathcal{P}_\lambda) \neq \mathcal{L}^{>0}(\mathcal{P}_\mu)$.

PROOF. Recall that $\mathcal{L}^{>0}(\mathcal{P}_\lambda) = \{a^{k_1} b a^{k_2} b \ldots \mid \prod_{i\geq 1}(1-(1-\lambda)^{k_i}) > 0\}$. Assuming $\lambda < \mu$, it is easy to see that $\mathcal{L}^{>0}(\mathcal{P}_\lambda) \subseteq \mathcal{L}^{>0}(\mathcal{P}_\mu)$ since $1 - (1-\mu^k) > 1 - (1-\lambda^k)$ for any $k \in \mathbb{N}$. To prove that the inclusion is strict, let us explain how to build a sequence $(k_i)_{i\in\mathbb{N}}$ such that

$$\prod_{i\geq 1}(1-(1-\lambda)^{k_i}) = 0 \text{and} \prod_{i\geq 1}(1-(1-\mu)^{k_i}) > 0.$$

This suffices to prove that $\mathcal{L}^{>0}(\mathcal{P}_\lambda) \neq \mathcal{L}^{>0}(\mathcal{P}_\mu)$, since the infinite word $a^{k_1} b a^{k_2} b \ldots$ will then be in $\mathcal{L}^{>0}(\mathcal{P}_\mu) \setminus \mathcal{L}^{>0}(\mathcal{P}_\lambda)$.

In order to build such a sequence, let $n, m \in \mathbb{N}$ such that $1 - \mu < \frac{n}{m} < 1 - \lambda$. We define the sequence $(k_i)_{i\in\mathbb{N}}$ as a nondecreasing sequence of natural numbers with $\lfloor (\frac{m}{n})^j \rfloor$ elements equal to j, where given a positive real number x, $\lfloor x \rfloor$ denotes the integer part of x. Observe first that

$$\prod_{i\geq 1}(1-x^{k_i}) \text{ is positive if and only if the series } \sum_{i\geq 1}\log(1-x^{k_i}) \text{ converges.}$$

Since $\log(1-\varepsilon) \sim_{\varepsilon\mapsto 0} -\varepsilon$, the latter series behaves as $\sum_{i\geq 1} x^{k_j}$ (i.e., either both converge, or both diverge). For our chosen sequence $(k_i)_{i\in\mathbb{N}}$, it holds that

$$\sum_{i\geq 1} x^{k_i} \;=\; \sum_{j\geq 1} \left\lfloor \left(\frac{m}{n}\right)^j \right\rfloor \cdot x^j,$$

which converges if and only if $\sum_{j\geq 1}\lfloor (\frac{m}{n})^j \rfloor \cdot x^j + \sum_{j\geq 1}((\frac{m}{n})^j - \lfloor (\frac{m}{n})^j \rfloor)\cdot x^j$ converges. (Note that the second summand is less than $\frac{1}{1-x}$). Altogether,

$$\prod_{i\geq 1}(1-x^{k_i}) \text{ is positive if and only if } \sum_{j\geq 1}\left(\frac{m}{n}\right)^j \cdot x^j \text{ converges,}$$

which is equivalent to $\frac{m}{n} \cdot x < 1$, that is, $x < \frac{n}{m}$. This shows the claim. □

Thus, given two PBA with the same underlying NBA, their accepted languages might differ under the probable semantics.

4.1.4. A Pumping Lemma

LEMMA 4.8 (PUMPING-LEMMA FOR PBA$^{>0}$). *Let $\mathcal{P} = (Q, \Sigma, \delta, \mu, F)$ be a PBA. Then, for each word $\rho \in \mathcal{L}^{>0}(\mathcal{P})$ and for all $i \in \mathbb{N}$ there exist finite words $u, v, w \in \Sigma^*$ and an infinite word $x \in \Sigma^\omega$ such that*

(1) $\rho = uvwx$,
(2) $|u| = i$, $|vw| \leq |Q|$ *and* $|w| \geq 1$, *and*
(3) $uvw^\ell x \in \mathcal{L}^{>0}(\mathcal{P})$ *for all* $\ell \in \mathbb{N}$.

PROOF. Let $n = |Q|$ be the number of states in $\mathcal{P}$. Given $\rho = a_1 a_2 a_3 \ldots \in \mathcal{L}^{>0}(\mathcal{P})$ and $i \in \mathbb{N}$ we regard the finite prefix $y = a_1 a_2 \ldots a_i \ldots a_{i+n}$ of ρ consisting of the first $m = i + n$ letters of ρ. As $\Pr^{\mathcal{P}}(\rho)$ is positive, there is a run

$$\sigma \;=\; q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \ldots \xrightarrow{a_i} q_i \xrightarrow{a_{i+1}} \ldots \xrightarrow{a_m} q_m$$

for y in $\mathcal{P}$ such that the probability of all accepting runs for ρ in $\mathcal{P}$ that start with σ is positive. In particular, the probability for the accepting runs for the infinite suffix $a_{m+1} a_{m+2} a_{m+3} \ldots$ of ρ that start in state q_m is positive. We now pick word positions j and k such that $i \leq j < k \leq m$ and $q_j = q_k$. Then,

$$\begin{array}{llll}
q_0 \xrightarrow{a_1} \ldots \xrightarrow{a_i} q_i & \text{is a run for} & u = & a_1 \ldots a_i \\
q_i \xrightarrow{a_{i+1}} \ldots \xrightarrow{a_j} q_j & \text{is a run for} & v = & a_{i+1} \ldots a_j \\
q_j \xrightarrow{a_{j+1}} \ldots \xrightarrow{a_k} q_k & \text{is a run for} & w = & a_{j+1} \ldots a_k \\
q_k \xrightarrow{a_{k+1}} \ldots \xrightarrow{a_m} q_m & \text{is a run for} & z = & a_{k+1} \ldots a_m
\end{array}$$

Furthermore, let $x = za_{m+1} a_{m+2} a_{m+3} \ldots = a_{k+1} a_{k+2} a_{k+3} \ldots$. Obviously, the first two conditions hold as we have (1) $\rho = uvwx$ and (2) $|u| = i$ and $|vw| = k - i \leq n$. It remains to check the pumping condition (3). Let $\ell \in \mathbb{N}$. As $q_j = q_k$ the above runs for u, v, w and z can be composed to obtain a run for the finite word $uvw^\ell z$ that starts in q_0 and ends in state q_m. Since the probability for the accepting runs for the infinite suffix $a_{m+1}a_{m+2}a_{m+3} \ldots$ of ρ that start in state q_m is positive, we get $\Pr^{\mathcal{P}}(uvw^\ell x) > 0$, and hence $uvw^\ell x \in \mathcal{L}^{>0}(\mathcal{P})$. □

As for other types of automata, the pumping lemma can be useful to establish that a certain language is not PBA$^{>0}$-recognizable. We can show the following, where ω-CF denotes the class of ω-context free languages.

COROLLARY 4.9. ω-CF $\not\subseteq \mathbb{L}(\mathsf{PBA}^{>0})$.

PROOF. For instance, there is no PBA$^{>0}$ that accepts the ω-context free language $L = \{a^n b^n (a+b)^\omega | n \geq 1\}$. Suppose by contradiction that such a PBA$^{>0}$ $\mathcal{P}$ exists. Let n be the number of states in $\mathcal{P}$. We now apply the pumping lemma to the word $\rho = a^n b^n a^\omega \in \mathcal{L}^{>0}(\mathcal{P})$ and the word position $i = 0$, which yields the existence of finite words v, w and an infinite word x such that $\rho = vwx$, w is nonempty and $|vw| \leq n$ and $vw^\ell x \in \mathcal{L}^{>0}(\mathcal{P})$ for all $\ell \geq 0$. Since $|vw| \leq n$, both words v and w contain just a's, say $v = a^j$, $w = a^k$ where $k \geq 1$ (as w is nonempty). Then $x = a^{n-j-k} b^n a^\omega$ and $vw^0 x = vx = a^{n-k} b^n a^\omega \notin L$, which is a contradiction. □

4.2. PBA under the Almost-Sure Semantics

In this section, we investigate the almost-sure semantics for PBA, that is, a PBA accepts a word if the set of accepting runs for this word has measure 1. We start our discussion

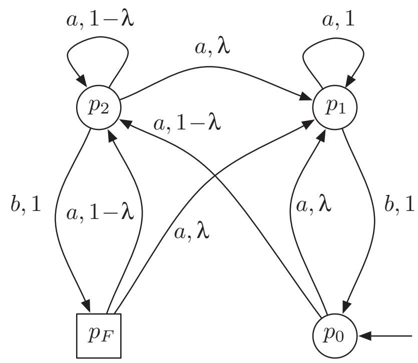


Fig. 6. PBA $\widetilde{\mathcal{P}}_\lambda$ $(0 < \lambda < 1)$ with $\mathcal{L}^{=1}(\widetilde{\mathcal{P}}_\lambda)$ non-ω-regular.

with an example that has the interesting property that each word is either accepted with probability 0 or 1.

4.2.1. Example. Regard the PBA $\widetilde{\mathcal{P}}_\lambda$ shown in Figure 6. Under the almost-sure semantics it accepts the following language

$$\widetilde{L}_\lambda = \left\{ a^{k_1} b a^{k_2} b a^{k_3} b \ldots \mid k_1, k_2, k_3 \ldots \in \mathbb{N}_{\geq 1} \text{ such that} \prod_{i=1}^{\infty} (1 - (1-\lambda)^{k_i}) = 0 \right\}.$$

We now check that $\mathcal{L}^{=1}(\widetilde{\mathcal{P}}_\lambda)$ is indeed $\widetilde{L}_\lambda$. Starting in p_0 (or p_F), $(1-\lambda)^{k_i}$ is the probability to be in p_2 after reading the word a^{k_i}. Hence, $1-(1-\lambda)^{k_i}$ represents the probability to be in state p_1 after the input word a^{k_i}. As a consequence $\prod_i \big(1-(1-\lambda)^{k_i}\big)$ is the probability to avoid forever the final state p_F. The probability to visit p_F after reading the word $a^{k_1} b a^{k_2} b \ldots a^{k_{N-1}} b$ and to avoid p_F from then on is therefore

$$(1-\lambda)^{k_{N-1}} \cdot \prod_{i \geq N} (1 - (1-\lambda)^{k_i})$$

with the convention $k_0 = 0$. Hence, given an input word $w = a^{k_1} b a^{k_2} b a^{k_3} b \ldots$, the probability to avoid q_F from some point on is

$$\sum_{N \geq 1} \left((1-\lambda)^{k_{N-1}} \prod_{i \geq N} \big(1 - (1-\lambda)^{k_i}\big) \right).$$

Thus, $\mathsf{Pr}^{\widetilde{\mathcal{P}}_\lambda}(w) = 1 - \sum_{N \geq 1} ((1-\lambda)^{k_{N-1}} \prod_{i \geq N} (1 - (1-\lambda)^{k_i}))$. To prove that $\mathcal{L}^{=1}(\widetilde{\mathcal{P}}_\lambda) = \widetilde{L}_\lambda$, we need to show that

$$\mathsf{Pr}^{\widetilde{\mathcal{P}}_\lambda}(w) = 1 - \sum_{N \geq 1} \left((1-\lambda)^{k_{N-1}} \prod_{i \geq N} (1 - (1-\lambda)^{k_i}) \right) = 1 \iff \prod_{i \geq 1} (1 - (1-\lambda)^{k_i}) = 0.$$

$\Leftarrow$: Suppose that $\prod_{i \geq 1} (1 - (1-\lambda)^{k_i}) = 0$. Then, $\prod_{i \geq N} (1 - (1-\lambda)^{k_i}) = 0$ for all $N \in \mathbb{N}$. Hence,

$$\sum_{N \geq 1} \left((1-\lambda)^{k_{N-1}} \prod_{i \geq N} (1 - (1-\lambda)^{k_i}) \right) = 0.$$

This yields $\mathsf{Pr}^{\widetilde{\mathcal{P}}_\lambda}(w) = 1$.

$\Rightarrow$: Assume now that $\prod_{i\geq 1}(1-(1-\lambda)^{k_i}) > 0$. Then

$$\sum_{N\geq 1}\left((1-\lambda)^{k_{N-1}}\prod_{i\geq N}(1-(1-\lambda)^{k_i})\right) > (1-\lambda)^{k_0}\prod_{i\geq 1}(1-(1-\lambda)^{k_i}) > 0$$

and therefore $\mathsf{Pr}^{\widetilde{\mathcal{P}}_\lambda}(w) < 1$.

Remark 4.10. We are even able to prove that each word $w \in \{a,b\}^\omega$ is either accepted by $\widetilde{\mathcal{P}}_\lambda$ with probability 0 or with probability 1, that is, $\mathsf{Pr}^{\widetilde{\mathcal{P}}_\lambda}(w) \in \{0,1\}$ for all $w \in \{a,b\}^\omega$. Thus, the automaton $\widetilde{\mathcal{P}}_\lambda$ is quite remarkable as it has the property that $\mathcal{L}^{=1}(\widetilde{\mathcal{P}}_\lambda) = \mathcal{L}^{>0}(\widetilde{\mathcal{P}}_\lambda)$. We call such an automaton a 0/1-automaton.

The following calculation shows the claim. Assume that $\prod_{i\geq 1}(1-(1-\lambda)^{k_i}) > 0$. The goal is to show that

$$\sum_{N\geq 1}\left((1-\lambda)^{k_{N-1}}\prod_{i\geq N}(1-(1-\lambda)^{k_i})\right) = 1$$
.

With $\theta_i = 1-(1-\lambda)^{k_i}$, we obtain:

$$\begin{aligned}\sum_{N\geq 1}\left((1-\lambda)^{k_{N-1}}\prod_{i\geq N}(1-(1-\lambda)^{k_i})\right) &= \sum_{N}\left((1-\theta_{N-1})\prod_{i\geq N}\theta_i\right)\\ &= \sum_{N}\left(\prod_{i\geq N}\theta_i - \theta_{N-1}\prod_{i\geq N}\theta_i\right)\\ &= \sum_{N}\left(\prod_{i\geq N}\theta_i - \prod_{i\geq N-1}\theta_i\right)\\ &= \lim_{N\to\infty}\prod_{i\geq N}\theta_i \qquad \text{since } \theta_0 = 0.\end{aligned}$$

To conclude, we have to show that $\lim_{N\to\infty}\prod_{i\geq N}\theta_i = 1$, using the assumption $c := \prod_{i=1}^{\infty}\theta_i > 0$. But this is obvious since

$$0 \neq c = \prod_{i=1}^{N}\theta_i \cdot \prod_{i=N+1}^{\infty}\theta_i.$$

As the left factor converges to c if N tends to infinity, the right factor has to converge to 1 which shows the claim.

4.2.2. The Precise Probabilities Matter. The previous example shows that alike to the setting of PBA$^{>0}$, the precise transition probabilities also matter for PBA under the almost-sure semantics.

THEOREM 4.11 (THE PRECISE PROBABILITIES MATTER FOR PBA$^{=1}$). *For any* $0 < \lambda < \mu < 1$, $\mathcal{L}^{=1}(\widetilde{\mathcal{P}}_\lambda) \neq \mathcal{L}^{=1}(\widetilde{\mathcal{P}}_\mu)$.

PROOF. The theorem follows immediately from Theorem 4.7 since $\mathcal{L}^{=1}(\widetilde{\mathcal{P}}_\lambda) = (a^+b)^\omega \setminus \mathcal{L}^{>0}(\mathcal{P}_\lambda)$. $\square$

4.2.3. Expressiveness of PBA under the Almost-Sure Semantics. We first give examples to show that the class of probabilistic Büchi automata under the almost-sure semantics is not closed under complementation. We then observe that for probabilistic Büchi automata, the switch from the probable semantics to the almost-sure semantics leads to a loss of expressiveness. At last we show that the class of languages definable by PBA$^{=1}$ (respectively, its complement) is not comparable to the class of ω-regular languages.

LEMMA 4.12. $\mathbb{L}(\mathsf{PBA}^{=1})$ *is not closed under complementation.*

PROOF. It is evident that each DBA $\mathcal{P}$ can be viewed as a PBA and that $\mathcal{L}_{\mathsf{DBA}}(\mathcal{P}) = \mathcal{L}^{=1}(\mathcal{P})$. The language $(a^*b)^\omega$ can be recognized by a DBA and hence by a PBA$^{=1}$. However, its complement $(a+b)^*a^\omega$ cannot be recognized by a PBA$^{=1}$. Suppose by contradiction that there is a PBA $\mathcal{P} = (Q, \{a, b\}, \delta, \mu_0, F)$ such that $\mathcal{L}^{=1}(\mathcal{P}) = (a+b)^*a^\omega$. Without loss of generality, we may assume that all states $p \in Q$ are reachable from some initial state. Let $\rho_p \in \{a, b\}^*$ be a finite word such that $p \in \delta(q_{\mathsf{init}}, \rho_p)$ where q_{init} is an initial state, that is, $\mu_0(q_{\mathsf{init}}) > 0$. Since the word $\rho_p a^\omega$ belongs to $(a+b)^*a^\omega$, it holds that $\mathsf{Pr}^{\mathcal{P}}(\rho_p a^\omega) = 1$. But then the set of accepting runs for a^ω starting in p must have probability measure 1, that is, $\mathsf{Pr}^{\mathcal{P}}_p(a^\omega) = 1$. Thus, there exists $n_p \in \mathbb{N}_{\geq 1}$ such that $\delta(p, a^{n_p}) \cap F \neq \emptyset$. Let $n = \max_{p \in Q} n_p$ and $\tilde{w} = (a^n b)^\omega \notin (a+b)^*a^\omega$. For $p \in Q$, let θ_p be the probability to visit at least once an accepting state $q \in F$ when scanning the word a^n from p. Note that $\theta_p > 0$ since $n_p \leq n$ and $\delta(p, a^{n_p}) \cap F \neq \emptyset$. Let $\theta = \min_{p \in Q} \theta_p$. Then, $\theta > 0$ and for each $k \geq 0$ and each state $p \in \delta(Q_{\mathsf{init}}, (a^n b)^k)$, the probability to enter F at least once while reading a^n from p is at least θ. But then almost all runs for $\tilde{w}$ visit F infinitely often. That is, $\mathsf{Pr}^{\mathcal{P}}(\tilde{w}) = 1$, which contradicts the assumption that $\mathcal{L}^{=1}(\mathcal{P}) = (a+b)^*a^\omega$. Hence, the class of languages $\mathbb{L}(\mathsf{PBA}^{=1})$ is not closed under complementation. $\square$

The next theorem shows that both the class of languages that are accepted by PBA under the almost-sure semantics as well as its complement are strictly included in the class of languages that are accepted by PBA under the probable semantics.

THEOREM 4.13. *(a)* $\mathbb{L}(\mathsf{PBA}^{=1}) \subsetneq \mathbb{L}(\mathsf{PBA}^{>0})$ *and (b)* $\overline{\mathbb{L}(\mathsf{PBA}^{=1})} \subsetneq \mathbb{L}(\mathsf{PBA}^{>0})$.

PROOF. (a) Let $\mathcal{P} = (Q, \Sigma, \delta, \mu_0, F)$ be a PBA and $L = \mathcal{L}^{=1}(\mathcal{P})$ the language of $\mathcal{P}$ under the almost-sure semantics. We will transform $\mathcal{P}$ into an equivalent 0/1-PBA $\mathcal{P}'$, that is a PBA that accepts each word either with probability 0 or 1. The idea to define $\mathcal{P}'$ is to pick at random some word position i where $\mathcal{P}$ could be in a state $p \in Q \setminus F$ and to check whether from this position i on, the probability in $\mathcal{P}$ for the event $\Box\neg F$ is positive. If so, then the input word is rejected by $\mathcal{P}$ with positive probability, and therefore, it does not belong to L. Formally, we define the PBA $\mathcal{P}'$ as $(Q', \Sigma', \delta', \mu'_0, F')$ where

$$Q' = 2^Q \cup Q \times 2^Q, \quad F' = 2^Q$$

and $\mu'_0(Q_{\mathsf{init}}) = 1$ where $Q_{\mathsf{init}} = \{q \in Q : \mu_0(q) > 0\}$. The transition probabilities in $\mathcal{P}'$ are defined as follows. If $R \subseteq Q$ and $a \in \Sigma$ then $\delta'(R, a, S) = 1$ if $S = \delta(R, a) \subseteq F$. For $S = \delta(R, a)$ and $S \setminus F \neq \emptyset$ we define

$$\delta'(R, a, S) = \tfrac{1}{2}, \quad \delta'(R, a, (p, S)) = \tfrac{1}{2 \cdot |S \setminus F|} \text{ for all } p \in S \setminus F.$$

For $p \in R \setminus F$, $q \in Q$ and $S = \delta(R, a)$ we set $\delta'((p, R), a, (q, S)) = \delta(p, a, q)$. For $p \in R \cap F$ and $S = \delta(R, a)$ we set $\delta'((p, R), a, S) = 1$. In all remaining cases, we set $\delta'(\cdot) = 0$.

Now assume $w \in \mathcal{L}^{=1}(\mathcal{P})$, thus $\mathsf{Pr}^{\mathcal{P},w}(\Box\Diamond F) = 1$. This implies $\mathsf{Pr}_t^{\mathcal{P},w\uparrow_i}(\Box\Diamond F) = 1$ for all $i \in \mathbb{N}_{\geq 1}$ and all states $t \in \delta(s, w\uparrow^{i-1})$, where $\mu_0(s) > 0$. Thus, whenever $\mathcal{P}'$ enters its $Q \times 2^Q$ part while reading w, it will afterwards reach a state (p, R) where p is an accepting state of $\mathcal{P}$ with probability one. As from such states $\delta'(.)$ leads to an accepting state of $\mathcal{P}'$ with probability 1 (in one step), this shows that $\mathsf{Pr}^{\mathcal{P}',w}(\Box\Diamond F') = 1$, so $w \in \mathcal{L}^{>0}(\mathcal{P}')$.

Assume $w \notin \mathcal{L}^{=1}(\mathcal{P})$, thus $\mathsf{Pr}^{\mathcal{P},w}(\Diamond\Box\neg F) > 0$. Then, there exists an $i \in \mathbb{N}_{\geq 1}$ such that $\mathsf{Pr}^{\mathcal{P},w}(\Diamond^{=i}\Box\neg F) > 0$, where $\Diamond^{=i}\Box\neg F$ denotes the event that after the $(i-1)$st step only states of $\neg F$ will be visited. Obviously

(i) $\mathsf{Pr}^{\mathcal{P},w}(\Diamond^{=j}\Box\neg F) \geq \mathsf{Pr}^{\mathcal{P},w}(\Diamond^{=i}\Box\neg F) > 0$ for all $j > i$.

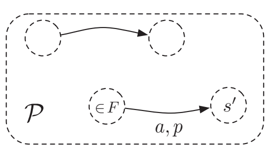
Let $\theta := \mathsf{Pr}^{\mathcal{P},w}(\Diamond^{=i}\Box\neg F)$. As $\theta > 0$, it holds that

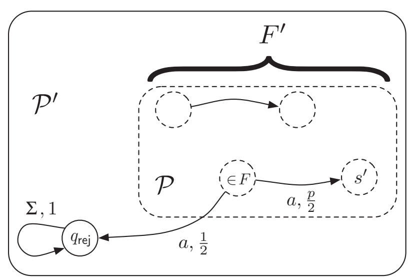
(ii) for all $j > i$ and all runs $q_0', q_1', q_2', \ldots$ of w in $\mathcal{P}'$: $q_j'|_{2^Q} \cap \neg F \neq \emptyset$,

where $q_j'|_{2^Q} = R_j$ if $q_j' = R_j$ is a state in the 2^Q part of $\mathcal{P}'$ and $q_j'|_{2^Q} = R_j$ if $q_j' = (r_j, R_j)$ is a state in the $Q \times 2^Q$ part of $\mathcal{P}'$. Note that for the second statement (ii) the existence of a single run of w in $\mathcal{P}$ satisfying $\Diamond^{=i}\Box\neg F$ suffices as the automaton $\mathcal{P}'$ performs a standard powerset construction on the 2^Q component in all its states. Examining the construction of $\mathcal{P}'$ shows that after reading the first i letters of w, whenever $\mathcal{P}'$ is in its accepting 2^Q part, it will move with probability $\frac{1}{2}$ to its nonaccepting $Q \times 2^Q$ part (because of (ii)) where it will stay forever with probability at least θ (because of (i) and the fact that the nonaccepting $Q \times 2^Q$ part can only be left via a state (r, R) where $r \in F$). But this means that after the ith step (after reading $w\uparrow^i$), the automaton $\mathcal{P}'$ will almost surely reach its nonaccepting part and stay there forever which shows $\mathsf{Pr}^{\mathcal{P}',w}(\Box\Diamond F') = 0$, thus $w \notin \mathcal{L}^{>0}(\mathcal{P}')$. This shows that $\mathcal{L}^{>0}(\mathcal{P}') = \mathcal{L}^{=1}(\mathcal{P})$.

The strictness of the inclusion $\mathbb{L}(\mathsf{PBA}^{=1}) \subsetneq \mathbb{L}(\mathsf{PBA}^{>0})$ immediately follows from ω-reg $\subseteq \mathbb{L}(\mathsf{PBA}^{>0})$ and the fact that $(a+b)^*a^\omega \notin \mathbb{L}(\mathsf{PBA}^{=1})$ (see proof of Lemma 4.12).

(b) Given a PBA $\mathcal{P} = (Q, \Sigma, \delta, \mu_0, F)$ we can trivially construct a PBA $\mathcal{P}' = (Q', \Sigma, \delta', \mu_0', F')$ of the same size such that $\mathcal{L}^{>0}(\mathcal{P}') = \Sigma^\omega \setminus \mathcal{L}^{=1}(\mathcal{P})$. We define $\mathcal{P}'$ as indicated in the following picture (see Größer [2008]).





$Q' = Q \cup \{q_{\mathsf{rej}}\}$. For $q_2 \in Q$, we set $\delta'(q_1, a, q_2) = \delta(q_1, a, q_2)$ if $q_1 \in Q \setminus F$ and $\delta'(q_1, a, q_2) = \frac{1}{2} \cdot \delta(q_1, a, q_2)$ if $q_1 \in F$. For $q_1 \in F$ we set $\delta'(q_1, a, q_{\mathsf{rej}}) = \frac{1}{2}$. Moreover $\delta'(q_{\mathsf{rej}}, a, q_{\mathsf{rej}}) = 1$ for all $a \in \Sigma$, $\mu_0'(q) = \mu_0(q)$ for $q \in Q$ (thus $\mu_0'(q_{\mathsf{rej}}) = 0$) and $F' = Q$.

Now assume $w \in \mathcal{L}^{=1}(\mathcal{P})$, hence $\mathsf{Pr}^{\mathcal{P},w}(\Box\Diamond F) = 1$. Thus, reading w, the automaton $\mathcal{P}'$ will almost surely reach the non-accepting state q_{rej} and will then loop in q_{rej} forever. So $\mathsf{Pr}^{\mathcal{P}',w}(\Box\Diamond F') = 0$ and $w \notin \mathcal{L}^{>0}(\mathcal{P}')$.

Assume $w \notin \mathcal{L}^{=1}(\mathcal{P})$, thus $\mathsf{Pr}^{\mathcal{P},w}(\Diamond\Box\neg F) > 0$. Then there exists an $i \in \mathbb{N}_{\geq 1}$ such that $\mathsf{Pr}^{\mathcal{P},w}(\Diamond^{=i}\Box\neg F) > 0$, where $\Diamond^{=i}\Box\neg F$ denotes the event that after the $(i-1)$st step only states of $\neg F$ will be visited. But $\mathsf{Pr}^{\mathcal{P}',w}(\Box\Diamond F') = \mathsf{Pr}^{\mathcal{P}',w}(\Box F') \geq \mathsf{Pr}^{\mathcal{P}',w}(\Diamond^{=i}\Box F') \geq (\frac{1}{2})^i \cdot \mathsf{Pr}^{\mathcal{P},w}(\Diamond^{=i}\Box\neg F) > 0$, so $w \in \mathcal{L}^{>0}(\mathcal{P}')$. This shows that $\mathcal{L}^{>0}(\mathcal{P}') = \Sigma^\omega \setminus \mathcal{L}^{=1}(\mathcal{P})$.

The strictness of the inclusion $\overline{\mathbb{L}(\mathsf{PBA}^{=1})} \subsetneq \mathbb{L}(\mathsf{PBA}^{>0})$ immediately follows from ω-reg $\subseteq \mathbb{L}(\mathsf{PBA}^{>0})$ and the fact that $(a^*b)^\omega \notin \overline{\mathbb{L}(\mathsf{PBA}^{=1})}$ as $\Sigma^\omega \setminus (a^*b)^\omega = (a+b)^*a^\omega \notin \mathbb{L}(\mathsf{PBA}^{=1})$ (see proof of Lemma 4.12).

Note that the inclusion (b) also follows from (a) and the fact that PBA$^{>0}$ are closed under complementation (as proven later in Section 6). However, the construction in the proof of (a) as well as the complementation yield each an exponential blow-up. □

It remains to show that the class of PBA$^{=1}$-recognizable languages and its complement are not comparable to the class of ω-regular languages.

THEOREM 4.14.

(a) ω-reg $\not\subseteq \mathbb{L}(\mathsf{PBA}^{=1})$,
(b) $\mathbb{L}(\mathsf{PBA}^{=1}) \not\subseteq \omega$-reg,
(c) ω-reg $\not\subseteq \overline{\mathbb{L}(\mathsf{PBA}^{=1})}$ *and*
(d) $\overline{\mathbb{L}(\mathsf{PBA}^{=1})} \not\subseteq \omega$-reg.

PROOF.

(a) The claim immediately follows from the fact that $(a+b)^*a^\omega \notin \mathbb{L}(\mathsf{PBA}^{=1})$ (see proof of Lemma 4.12).
(b) Consider the example PBA$^{=1}$ $\widetilde{\mathcal{P}}_\lambda$ from Figure 6, page 17. It holds that $\mathcal{L}^{=1}(\widetilde{\mathcal{P}}_\lambda) = (a^+b)^\omega \setminus \mathcal{L}^{>0}(\mathcal{P}_\lambda)$, where $\mathcal{P}_\lambda$ is represented in Figure 3, page 11. Hence, $\mathcal{L}^{=1}(\widetilde{\mathcal{P}}_\lambda) \cup (a+b)^*a^\omega \cup b(a+b)^\omega \cup (a+b)^*bb(a+b)^\omega$ is the complement of $\mathcal{L}^{>0}(\mathcal{P}_\lambda)$. As $\mathcal{L}^{>0}(\mathcal{P}_\lambda)$ is not ω-regular, this immediately yields that $\mathcal{L}^{=1}(\widetilde{\mathcal{P}}_\lambda)$ is a non-ω-regular language.
(c)-(d) The claims follow from (b) (respectively, (a)) and the fact that ω-regular languages are closed under complementation. □

4.3. PBA under the Threshold Semantics

In this section, we focus on the expressiveness of PBA under the threshold semantics. First, we will show that the exact threshold is of no importance for the class of accepted languages.

4.3.1. Comparison for Different Threshold Values

LEMMA 4.15. *For all* $\lambda, \mu \in]0,1[$, $\mathbb{L}(\mathsf{PBA}^{>\lambda}) = \mathbb{L}(\mathsf{PBA}^{>\mu})$.

PROOF. Let $\lambda \neq \mu$ be two real numbers in $]0,1[$. From a given PBA $\mathcal{P}$, we construct another PBA $\mathcal{P}'$ with $\mathcal{L}^{>\lambda}(\mathcal{P}) = \mathcal{L}^{>\mu}(\mathcal{P}')$. The construction depends on whether $\lambda < \mu$ or $\lambda > \mu$.

Assume first $\lambda > \mu$. $\mathcal{P}'$ consists of a copy of $\mathcal{P}$ equipped with an extra rejecting sink state q_{rej}. In $\mathcal{P}'$, accepting states are exactly those of $\mathcal{P}$, and the initial distribution μ_0' is defined by $\mu_0'(p) = \frac{\mu}{\lambda}\mu_0(p)$ for states from $\mathcal{P}$ and $\mu_0'(q_{\mathsf{rej}}) = 1 - \frac{\mu}{\lambda}$. Given a word w, it holds that $\mathsf{Pr}^{\mathcal{P}'}(w) = \frac{\mu}{\lambda}\mathsf{Pr}^{\mathcal{P}}(w)$ and thus $\mathcal{L}^{>\mu}(\mathcal{P}') = \mathcal{L}^{>\lambda}(\mathcal{P})$.

Assume now $\lambda < \mu$. Automaton $\mathcal{P}'$ consists of a copy of $\mathcal{P}$ with an extra accepting sink state q_{acc}. The accepting states in $\mathcal{P}'$ are those of $\mathcal{P}$ together with q_{acc}, and the initial probability distribution is defined by $\mu_0'(p) = \frac{1-\mu}{1-\lambda}\mu_0(p)$ for states in $\mathcal{P}$ and $\mu_0'(q_{\mathsf{acc}}) = \frac{\mu-\lambda}{1-\lambda}$. Given a word w, it holds that $\mathsf{Pr}^{\mathcal{P}'}(w) = \frac{\mu-\lambda}{1-\lambda} + \frac{1-\mu}{1-\lambda}\mathsf{Pr}^{\mathcal{P}}(w)$ and thus $\mathcal{L}^{>\mu}(\mathcal{P}') = \mathcal{L}^{>\lambda}(\mathcal{P})$. □

4.3.2. Comparison with Other Semantics. Let us now compare threshold PBA to PBA under other semantics. The class of languages accepted by PBA under the threshold semantics subsumes the class of languages accepted by PBA under the probable or the almost-sure semantics.

LEMMA 4.16. *For every* $\lambda \in]0, 1[$, $\mathbb{L}(\mathsf{PBA}^{>0}) \subseteq \mathbb{L}(\mathsf{PBA}^{>\lambda})$.

PROOF. Let $\mathcal{P}$ be a PBA, and $\lambda \in]0, 1[$. The automaton $\mathcal{P}$ can be transformed into $\mathcal{P}'$ such that $\mathcal{L}^{>0}(\mathcal{P}) = \mathcal{L}^{>\lambda}(\mathcal{P}')$. The new PBA $\mathcal{P}'$ consists of a copy of $\mathcal{P}$ with an extra accepting sink state q_{acc}. Accepting states of $\mathcal{P}'$ are those of $\mathcal{P}$ together with q_{acc}, and the initial distribution μ_0' is defined by: $\mu_0'(p) = (1-\lambda)\mu_0(p)$ for states of $\mathcal{P}$, and $\mu_0'(q_{\mathsf{acc}}) = \lambda$. □

The latter lemma raises the question whether PBA with the probable semantics are as expressive as PBA with the threshold semantics. The following theorem shows that this is not the case.

THEOREM 4.17 (THERE EXIST THRESHOLD-LANGUAGES NOT IN $\mathbb{L}(\mathsf{PBA}^{>0})$). *For all real numbers* $\lambda \in]0, 1[$, *it holds that* $\mathbb{L}(\mathsf{PBA}^{>\lambda}) \not\subseteq \mathbb{L}(\mathsf{PBA}^{>0})$.

PROOF. Note that, because of Lemma 4.15 it is sufficient to show the existence of a PBA $\mathcal{P}$ and a threshold $\lambda \in]0, 1[$ such that $\mathcal{L}^{>\lambda}(\mathcal{P}) \notin \mathbb{L}(\mathsf{PBA}^{>0})$. The proof is based on an adaption of arguments provided by Paz [1971] for probabilistic finite automata (PFA). We identify any real number $\lambda \in]0, 1[$ with the infinite word $a_1a_2a_3\ldots \in \{0,1\}^\omega$ obtained by its binary representation

$$\lambda = \sum_{i=1}^{\infty} a_i 2^{-i} = 0.a_1a_2a_3\ldots$$

(where we assume that $a_i \neq 0$ for infinitely many indices i). We now consider the following languages $K_\lambda \subseteq \{0,1\}^*$:

$$K_\lambda = \left\{ b_1\ldots b_n \mid b_1,\ldots,b_n \in \{0,1\} \text{ such that } \sum_{i=1}^{n} b_i 2^{-i} > \lambda \right\}$$

Paz [1971] has shown that K_λ is regular if and only if λ is rational. Rabin [1963] provided a PFA $\mathcal{R}$ such that for all finite words $\rho \in \{0,1\}^*$, $\mathsf{Pr}^{\mathcal{R}}(\rho) > \lambda$ if and only if $\rho \in K_\lambda$. We modify this PFA $\mathcal{R}$ to a PBA $\mathcal{P}$ over the alphabet $\Sigma = \{0,1,c\}$ that under the threshold semantics accepts the language

$$L_\lambda = K_\lambda c^\omega = \{\rho c^\omega \mid \rho \in K_\lambda\}.$$

when dealing with the threshold λ.

For this, we add a new accepting state q_{acc} with a c-self-loop and no other transitions (i.e., we set $\delta_{\mathcal{P}}(q_{\mathsf{acc}}, c, q_{\mathsf{acc}}) = 1$ and $\delta_{\mathcal{P}}(q_{\mathsf{acc}}, b, \cdot) = 0$ for $b \in \{0,1\}$) and c-transitions from each final state p in $\mathcal{R}$ to q_{acc} (i.e., we set $\delta_{\mathcal{P}}(p, c, q_{\mathsf{acc}}) = 1$ for all final states p of $\mathcal{R}$). The remaining transitions are as in $\mathcal{R}$. Finally the acceptance set of $\mathcal{P}$ is defined as $F = \{q_{\mathsf{acc}}\}$ and the initial distribution of $\mathcal{P}$ is the same as in $\mathcal{R}$. It then holds for all words $\rho \in \{0,1\}^*$ that $\mathsf{Pr}^{\mathcal{P}}(\rho c^\omega) = \mathsf{Pr}^{\mathcal{R}}(\rho)$. Furthermore, $\mathsf{Pr}^{\mathcal{P}}(w) = 0$ if w contains infinitely many 0's or 1's. Thus:

$$\mathcal{L}^{>\lambda}(\mathcal{P}) = \{\rho c^\omega \mid \mathsf{Pr}^{\mathcal{R}}(\rho) > \lambda\} = K_\lambda c^\omega = L_\lambda.$$

Now fix an arbitrary irrational number $\lambda \in]0, 1[$. Thus, by Paz [1971], K_λ is nonregular. It remains to show that there is no PBA $\mathcal{P}'$ such that $\mathcal{L}^{>0}(\mathcal{P}') = L_\lambda$. The intuitive argument will be the following. Suppose by contradiction that $\mathcal{P}'$ is a PBA with

$\mathcal{L}^{>0}(\mathcal{P}') = L_\lambda \subseteq \{0,1\}^* c^\omega$. Thus, each word that will be accepted with a positive probability has a suffix consisting only of c's. But then, there is some "kind of underlying (finite word) automaton" in $\mathcal{P}'$, that decides which of the prefixes "to accept". Although this is a probabilistic automaton, as the acceptance threshold in $\mathcal{P}'$ is 0, this "underlying (finite word) automaton" will accept a regular language, which will contradict the assumption that λ is irrational.

More formally, we first observe that whenever (T, A) is an accepting end component of $\mathcal{P}'$ with $\mathsf{Pr}^{\mathcal{P}',w}((T, A)) > 0$ for some word $w \in \Sigma^\omega$ then $A(p) = \{c\}$ for all states $p \in T$. (Otherwise, $\mathcal{P}'$ would accept some words that do not have a suffix consisting of c's.) Let T_0 be the set of states p in $\mathcal{P}'$ such that $p \in T$ for some accepting end component (T, A) of $\mathcal{P}'$ with $\mathsf{Pr}^{\mathcal{P}',w}((T, A)) > 0$ for some word $w \in \Sigma^\omega$. Furthermore, let T_0^+ be the set of all states q in $\mathcal{P}'$ such that $p \in \delta_{\mathcal{P}'}(q, c^n)$ for some $n \geq 0$ and $p \in T_0$. That is T_0^+ consists of all states from which a relevant accepting end component can be reached via a finite sequence of c's. Whenever $\rho \in \{0,1\}^*$ such that $\mathsf{Pr}^{\mathcal{P}'}(q_0 \xrightarrow{\rho} q) > 0$ for some initial state q_0 and some state $q \in T_0^+$, then

$$\mathsf{Pr}^{\mathcal{P}'}(\rho c^\omega) \geq \mathsf{Pr}^{\mathcal{P}'}(q_0 \xrightarrow{\rho} q) \cdot \mathsf{Pr}_q^{\mathcal{P}'}(q \xrightarrow{c^n} p) > 0$$

for some $n \in \mathbb{N}_{\geq 0}$, some state $q \in T_0^+$ and some state $p \in \delta_{\mathcal{P}'}(q, c^n)$, where p is in a relevant accepting end component. This yields

$$\rho c^\omega \in \mathcal{L}^{>0}(\mathcal{P}') = L_\lambda \ = K_\lambda c^\omega,$$

and therefore $\rho \in K_\lambda$. Vice-versa, if $\rho \in K_\lambda$, then $\rho c^\omega \in L_\lambda = \mathcal{L}^{>0}(\mathcal{P}')$. Hence, there exists a state $q \in T_0^+$ such that $q \in \delta_{\mathcal{P}'}(q_0, \rho)$ for some initial state q_0.

This shows that K_λ agrees with the set of finite words $\rho \in \{0,1\}^*$ such that $\delta_{\mathcal{P}'}(q_0, \rho) \cap T_0^+ \neq \emptyset$. But then, K_λ agrees with the language of the NFA resulting from $\mathcal{P}'$ by discarding all c-transitions, interpreting the probabilistic branches by nondeterministic choices and declaring the states in T_0^+ to be final. Thus, K_λ is regular. This contradicts the assumption that λ is irrational in which case K_λ is not regular, as shown by Paz [1971]. $\square$

Remark 4.18 (*Irrational Coefficients*). Note that the PFA $\mathcal{R}$ used in the proof of Theorem 4.17 only contains rational transition probabilities as well as rational initial probabilities. So the constructed PBA $\mathcal{P}$ also only contains rational transition probabilities as well as rational initial probabilities. It is only the irrational threshold λ that makes the language $\mathcal{L}^{>\lambda}(\mathcal{P})$ not PBA$^{>0}$-recognizable.

In the transformations provided in this section (i.e., from a PBA$^{>0}$ to a PBA$^{>\lambda}$ or from a PBA$^{>\lambda}$ to a PBA$^{>\mu}$), it is important to notice that the transition probabilities are not changed. Only the initial probabilities are changed and as soon as λ is irrational, the resulting automaton might contain irrational initial probabilities (even if the original automaton did not).

Also we can detail the relationships between $\mathbb{L}(\mathsf{PBA}^{>0})$ and languages defined by PBA with rational or irrational thresholds. First, on the one hand there are only countably many languages defined by PBA under the threshold semantics and with rational thresholds, whereas as a consequence of Theorem 4.17 there are uncountably many languages defined by PBA under the threshold semantics with irrational thresholds. As a consequence, irrational thresholds do enhance the expressive power of PBA under the threshold semantics. Second, the class $\mathbb{L}(\mathsf{PBA}^{>0})$ is closed under complementation, as we shall prove in Section 6.2, whereas it was shown in Chadha et al. [2011] that languages defined by PBA under the threshold semantics with rational thresholds are not. Hence, $\mathbb{L}(\mathsf{PBA}^{>0}) \subsetneq \mathbb{L}(\mathsf{PBA}^{>\lambda})$ for irrational values of λ.

5. PRA AND PSA

After the investigation of properties of PBA in the latter section, we now study other acceptance criteria than the Büchi condition for probabilistic automata as acceptors for infinite words. We concentrate here on Streett and Rabin acceptance since they will play a central role for our complementation operator.

For each semantics, any PBA can be trivially seen as a PSA or PRA by replacing the Büchi acceptance set F with the singleton Streett (respectively, Rabin) acceptance set $\mathsf{Acc} = \{(F, Q)\}$ (respectively, $\{(\emptyset, F)\}$). We explore here transformations from $\text{PRA}^{>0}$ and $\text{PSA}^{>0}$ to $\text{PBA}^{>0}$, and compare the expressiveness of probable and almost-sure semantics for PRA and PSA. In the following, in order to stress the acceptance condition of a given probabilistic ω-automaton $\mathcal{P}$, we often index Pr and $\mathcal{L}$ with Rabin, Streett, or Büchi, and write, for example, $\mathsf{Pr}^{\mathcal{P}}_{\mathit{Streett}}$ or $\mathcal{L}^{>0}_{\mathit{Rabin}}(\mathcal{P})$.

5.1. From $\text{GPBA}^{>0}$, $\text{PSA}^{>0}$ and $\text{PRA}^{>0}$ to $\text{PBA}^{>0}$

We show that the transformations known for nondeterministic automata can easily be adapted to the probabilistic setting. Moreover, even more efficient transformations can be provided as we will see in the case of Streett acceptance.

THEOREM 5.1 (FROM $\text{GPBA}^{>0}$, $\text{PRA}^{>0}$ AND $\text{PSA}^{>0}$ TO $\text{PBA}^{>0}$).

(a) *For any generalized probabilistic Büchi automaton $\mathcal{P}$, there exists a PBA $\mathcal{P}'$ with $\mathcal{L}^{>0}(\mathcal{P}) = \mathcal{L}^{>0}(\mathcal{P}')$, $\mathcal{L}^{=1}(\mathcal{P}) = \mathcal{L}^{=1}(\mathcal{P}')$, and $|\mathcal{P}'| = \mathcal{O}(m|\mathcal{P}|)$ where m is the number of acceptance sets in $\mathcal{P}$.*

(b) *For any PRA $\mathcal{P}_R$, there exists a PBA $\mathcal{P}$ with $\mathcal{L}^{>0}_{\mathit{Rabin}}(\mathcal{P}_R) = \mathcal{L}^{>0}(\mathcal{P})$ and $|\mathcal{P}| = \mathcal{O}(m|\mathcal{P}_R|)$.*

(c) *For any PSA $\mathcal{P}_S$, there exists a PBA $\mathcal{P}$ with $\mathcal{L}^{>0}_{\mathit{Streett}}(\mathcal{P}_S) = \mathcal{L}^{>0}(\mathcal{P})$ and $|\mathcal{P}| = \mathcal{O}(m2^m|\mathcal{P}_S|)$.*

PROOF. (a) In generalized Büchi automata, the acceptance condition consists of a set $F = \{F_1, \dots, F_n\}$ where the F_i's are subsets of the state space Q. Acceptance of a run requires that each of the F_i's is visited infinitely often. In the nondeterministic case, it is well known that the Büchi and generalized Büchi acceptance conditions have equal power, in the sense that any generalized NBA $\mathcal{G}$ can be transformed into an equivalent NBA $\mathcal{A}$. The idea behind this transformation is to work with n copies $\mathcal{G}_1, \dots, \mathcal{G}_n$ of $\mathcal{G}$ where the outgoing transitions from an F_i-state in the ith copy are redirected to the $(i+1)$th copy (modulo n), while outgoing transitions from other states stay in the same copy. The Büchi acceptance condition in the so-built NBA $\mathcal{A}$ consists of the F_1-states in $\mathcal{G}_1$. The exact same technique yields a polynomial transformation from $\text{GPBA}^{>0}$ to $\text{PBA}^{>0}$, and from $\text{GPBA}^{=1}$ to $\text{PBA}^{=1}$.

(b) Let $\mathcal{P}_R = (Q_R, \delta_R, \mu_R, \{(H_1, K_1), \dots, (H_n, K_n)\})$ be a PRA. Then, we define the PBA $\mathcal{P}$ by extending $\mathcal{P}_R$ by copies of the subautomata for the states not in H_i. From any $(K_i \setminus H_i)$-state in the main automaton, we have a probabilistic choice to stay in the main automaton or to move to the $(Q_R \setminus H_i)$-copy in which we try to visit infinitely often a K_i-state and which can only be left to a rejecting sink state. Formally, $\mathcal{P} = (Q, \delta, \mu, F)$, where $Q = \{q_{\mathsf{rej}}\} \cup Q_R \cup \bigcup_{1 \le i \le n}((Q_R \setminus H_i) \times \{i\})$, $\mu = \mu_R$ and $F = \cup_{1 \le i \le n}(K_i \times \{i\})$. The transition relation δ of $\mathcal{P}$ is chosen in such a way that for $a \in \Sigma$:

$$\begin{array}{lll}
\delta(p, a, q) > 0 & \text{iff} & \delta_R(p, a, q) > 0 \\
\delta(p, a, \langle q, i\rangle) > 0 & \text{iff} & \delta_R(p, a, q) > 0 \text{ and } p \in K_i \setminus H_i \\
\delta(\langle p, i\rangle, a, \langle q, i\rangle) & = & \delta_R(p, a, q) \\
\delta(\langle p, i\rangle, a, q_{\mathsf{rej}}) & = & \Sigma_{q \in H_i} \delta_R(p, a, q) \\
\delta(q_{\mathsf{rej}}, a, q_{\mathsf{rej}}) & = & 1.
\end{array}$$

It is straightforward to show that $\mathcal{L}^{>0}(\mathcal{P}) = \mathcal{L}^{>0}_{\mathit{Rabin}}(\mathcal{P}_R)$.

(c) Let $\mathcal{P}_S = (Q_S, \delta_S, \mu_S, \{(H_1, K_1), \ldots, (H_n, K_n)\})$ be a PSA. We first observe that the Streett acceptance condition $\bigwedge_{i=1}^{n}(\Box\Diamond H_i \vee \Diamond\Box\neg K_i)$ is equivalent to $\bigvee_{I\subseteq\{1,\ldots,n\}}\big(\bigwedge_{i\in I}\Box\Diamond H_i \wedge \bigwedge_{i\notin I}\Diamond\Box\neg K_i\big)$. We define the PBA $\mathcal{P}$ by extending $\mathcal{P}_S$ by subautomata $\mathcal{P}_I$ for any subset I of $\{1,\ldots,n\}$. In each state of the main automaton we decide randomly to stay in the main automaton or to move to one of the subautomata $\mathcal{P}_I$. Once $\mathcal{P}_I$ has been entered, it can only be left to a rejecting sink state. Intuitively, $\mathcal{P}_I$ realizes the condition $\bigwedge_{i\in I}\Box\Diamond H_i \wedge \bigwedge_{i\notin I}\Diamond\Box\neg K_i \equiv \bigwedge_{i\in I}\Box\Diamond H_i \wedge \Diamond\Box\neg(\cup_{i\notin I}K_i)$. Note that such a condition is similar to a single Rabin condition except that it enforces a generalized Büchi condition on the H_i's, where $i \in I$. Thus the construction of the $\mathcal{P}_I$'s is a combination of the constructions used in (a) and (b). $\mathcal{P}_I$ consists of copies of $\mathcal{P}_S$ without the states in $\bigcup_{i\notin I} K_i$. Formally, the PBA $\mathcal{P} = (Q, \delta, \mu, F)$ is defined as follows. The state space is $Q = \{q_{\mathsf{rej}}\} \cup Q_S \cup \bigcup_{I\subseteq\{1,\ldots,n\}} Q_I$ where $Q_\emptyset = \{\langle q, \emptyset\rangle : q \in Q_S \setminus (\cup_{i=1}^{n} K_i)\}$ and $Q_I = \big\{\langle q, I, i\rangle : i \in I, q \in Q_S \setminus (\cup_{i\notin I}K_i)\big\}$ for $\emptyset \neq I \subseteq \{1,\ldots,n\}$. The initial distribution is $\mu = \mu_S$. For states $p, q \in Q_S$ with $\delta_S(p, a, q) > 0$, we let:

—$\delta(p, a, q) > 0$,
—$\delta(p, a, \langle q, \emptyset\rangle) > 0$ and
—$\delta(p, a, \langle q, I, \mathsf{first}(I)\rangle) > 0$ for any nonempty subset I of $\{1,\ldots,n\}$, assuming a fixed enumeration $i_0, \ldots, i_{k-1}$ of the elements in I and putting $\mathsf{first}(I) = i_0$.

The transitions inside Q_I are defined in a similar way as in the transformation from GPBA to PBA:

$$\begin{array}{ll}
\delta(\langle p, I, i_\ell\rangle, a, \langle q, I, i_r\rangle) & = \delta_S(p, a, q) \text{ if either } (r = \ell) \wedge (p \notin H_\ell) \text{ or} \\
& \ \delta_S(p, a, q)\ (r = (\ell+1) \mod |I|) \wedge (p \in H_\ell) \\
\delta(\langle p, I, i_\ell\rangle, a, q_{\mathsf{rej}}) & = \Sigma_{q\in\cup_{i\notin I}K_i}\delta_S(p, a, q) \\
\delta(\langle p, \emptyset\rangle, a, \langle q, \emptyset\rangle) & = \delta_S(p, a, q) \\
\delta(\langle p, \emptyset\rangle, a, q_{\mathsf{rej}}) & = \Sigma_{q\in\cup_{i=1}^{n}K_i}\delta_S(p, a, q) \\
\delta(q_{\mathsf{rej}}, a, q_{\mathsf{rej}}) & = 1.
\end{array}$$

The Büchi acceptance set in $\mathcal{P}$ is $F = Q_\emptyset \cup \bigcup_I\big\{\langle p, I, \mathsf{first}(I)\rangle \in Q_I : p \in H_{\mathsf{first}(I)}\big\}$ where I ranges over all nonempty subsets of $\{1,\ldots,n\}$. This construction ensures $\mathcal{L}^{>0}(\mathcal{P}) = \mathcal{L}^{>0}_{\mathit{Streett}}(\mathcal{P}_S)$. $\Box$

As for nondeterministic automata, the transformation of a PSA$^{>0}$ into an equivalent PBA$^{>0}$ is more complicated than for Rabin automata and can cause an exponential blowup. However, we show here that a polynomial transformation can be provided. This stands in contrast to the nondeterministic case where it is known that there exist NSA of polynomial size, for which any equivalent NBA has exponentially many states [Safra and Vardi 1989].

THEOREM 5.2 (POLYNOMIAL TRANSFORMATION FROM PSA$^{>0}$ TO PBA$^{>0}$). *For any PSA $\mathcal{P}_S$, there exists a PBA $\mathcal{P}$ with $\mathcal{L}^{>0}_{\mathit{Streett}}(\mathcal{P}_S) = \mathcal{L}^{>0}(\mathcal{P})$ and $|\mathcal{P}| = \mathcal{O}(n^2|\mathcal{P}_S|)$, where n is the number of acceptance pairs in $\mathcal{P}_S$.*

PROOF. Let $\mathcal{P}_S = (Q_S, \Sigma, \delta_S, \mu_S^0, \{(H_1, K_1), \ldots, (H_n, K_n)\})$ be a PSA. For simplicity, we may assume that $H_i \cap K_i = \emptyset$ as otherwise K_i could be replaced with $K_i \setminus H_i$. Intuitively, the PBA $\mathcal{P}$ arises from $\mathcal{P}_S$ by making several copies of $\mathcal{P}_S$: a subautomaton $\mathcal{P}_{init}$ in which the process starts, a subautomaton $\mathcal{P}_{accept}$ which has to be visited infinitely often and which is reachable with positive probability via any outgoing transition from the states in $\mathcal{P}_{init}$, and subautomata $\mathcal{P}_i$ and $\mathcal{P}_{i,j}$ for $i, j \in \{1,\ldots,n\}$, $i \neq j$, that are reached from $\mathcal{P}_{accept}$ whenever a state in K_i is visited in $\mathcal{P}_{accept}$. Subautomaton $\mathcal{P}_i$ can only be left via transitions from a H_i-state in $\mathcal{P}_i$ from which we move back to $\mathcal{P}_{accept}$. Subautomaton $\mathcal{P}_{i,j}$ behaves as $\mathcal{P}_i$, but in addition it takes into account the Streett-acceptance pair

$$
\begin{array}{lcll}
\delta(\langle q, init\rangle, a, \langle p, init\rangle) & = & \frac{1}{2}\cdot\delta_S(q,p) & \\
\delta(\langle q, init\rangle, a, \langle p, accept\rangle) & = & \frac{1}{2}\cdot\delta_S(q,p) & \\
 & & & \\
\delta(\langle q, accept\rangle, a, \langle p, accept\rangle) & = & \delta_S(q,p) & \text{if } q \notin K_1 \cup \ldots \cup K_n \\
\delta(\langle q, accept\rangle, a, \langle p, i\rangle) & = & \frac{1}{|\{\ell | q\in K_\ell\}|\cdot n}\cdot\delta_S(q,p) & \text{if } q \in K_i \\
\delta(\langle q, accept\rangle, a, \langle p, i, j\rangle) & = & \frac{1}{|\{\ell | q\in K_\ell\}|\cdot n}\cdot\delta_S(q,p) & \text{if } q \in K_i \\
 & & & \\
\delta(\langle q, i\rangle, a, \langle p, accept\rangle) & = & \delta_S(q,p) & \text{if } q \in H_i \\
\delta(\langle q, i\rangle, a, \langle p, i\rangle) & = & \delta_S(q,p) & \text{if } q \notin H_i \\
 & & & \\
\delta(\langle q, i, j\rangle, a, \langle p, accept\rangle) & = & \delta_S(q,p)\cdot \begin{cases} 0, q \notin H_i \cup K_j \\ 0, q \in K_j \setminus H_i \\ 1, q \in H_i \setminus K_j \\ \frac{1}{n+2}, q \in H_i \cap K_j \end{cases} & \\
 & & & \\
\delta(\langle q, i, j\rangle, a, \langle p, i, j\rangle) & = & \delta_S(q,p)\cdot \begin{cases} 1, q \notin H_i \cup K_j \\ \frac{1}{n+1}, q \in K_j \setminus H_i \\ 0, q \in H_i \setminus K_j \\ \frac{1}{n+2}, q \in H_i \cap K_j \end{cases} & \\
 & & & \\
\delta(\langle q, i, j\rangle, a, \langle p, j\rangle) & = & \delta_S(q,p)\cdot \begin{cases} 0, q \notin H_i \cup K_j \\ \frac{1}{n+1}, q \in K_j \setminus H_i \\ 0, q \in H_i \setminus K_j \\ \frac{1}{n+2}, q \in H_i \cap K_j \end{cases} & \\
 & & & \\
\delta(\langle q, i, j\rangle, a, \langle p, j, k\rangle) & = & \delta_S(q,p)\cdot \begin{cases} 0, q \notin H_i \cup K_j \\ \frac{1}{n+1}, q \in K_j \setminus H_i \\ 0, q \in H_i \setminus K_j \\ \frac{1}{n+2}, q \in H_i \cap K_j \end{cases} &
\end{array}
$$

Fig. 7. Transition probabilities of the PBA constructed in the proof of Theorem 5.2 (where $i, j, k \in \{1, \ldots, n\}$ s.th. $i \neq j$ and $j \neq k$ (but possibly $i = k$)).

(H_j, K_j). When a K_j-state in $\mathcal{P}_{i,j}$ is reached, we randomly choose to stay in $\mathcal{P}_{i,j}$ or to move to $\mathcal{P}_j$ or one of the subautomata $\mathcal{P}_{j,k}$. Formally, the PBA $\mathcal{P} = (Q, \Sigma, \delta, \mu_0, F)$ is defined as follows. The state space is

$$Q = Q_{init} \cup Q_{accept} \cup \bigcup_{1\leq i\leq n} Q_i \cup \bigcup_{\substack{1\leq i,j\leq n \\ i\neq j}} Q_{i,j}$$

where $Q_* = \{\langle q, *\rangle : q \in Q_S\}$. The set of accepting states is $F = Q_{accept}$. The initial distribution is given by $\mu_0(\langle q, init\rangle) = \mu_S^0(q)$ and $\mu_0(\langle q, *\rangle) = 0$ for all other states $\langle q, *\rangle \in Q$. The transition probabilities in $\mathcal{P}$ are shown in Figure 7 where $q, p \in Q_S$. Here, i, j, k range over all indices in $\{1, \ldots, m\}$ with $i \neq j$ and $j \neq k$ (but possibly $i = k$). In the sequel, we refer to the fragment of the Q_*-states as the $\mathcal{P}_*$-subautomaton.

To explain the role of subautomata $\mathcal{P}_{i,j}$ and $\mathcal{P}_i$, let us assume for simplicity that the acceptance condition consists of two pairs $\{(H_1, K_1), (H_2, K_2)\}$ such that $K_1 \setminus K_2 \neq \emptyset$. $\mathcal{P}_{i,j}$ subautomata avoid to accept paths visiting infinitely many K_2 states in $\mathcal{P}_1$, without taking care of visiting infinitely many H_2 states as well. Let $k_1 \in K_1 \setminus K_2$, $k_2 \in K_2$, $h_1 \in H_1$. Without loss of generality, assume $k_2 \notin H_1$. Then, the nonaccepting (possible) path

$$k_1, k_2, h_1, k_1, k_2, h_1, \ldots$$

in the Streett automaton would be lifted with positive probability to accepting paths of the form

$$\ldots, \langle k_1, accept\rangle, \langle k_2, 1\rangle, \langle h_1, 1\rangle, \langle k_1, accept\rangle, \langle k_2, 1\rangle, \langle h_1, 1\rangle, \ldots$$

in the Büchi automaton. This kind of behavior is avoided thanks to the $\mathcal{P}_{i,j}$ subautomata. However, $\mathcal{P}_i$ subautomata need to be added as well. Indeed, without the $\mathcal{P}_i$

subautomata, the Büchi automata could visit infinitely many K_1 states while being in $\mathcal{P}_{accept}$ moving to $\mathcal{P}_{1,2}$. If the automaton also visits infinitely many K_2 states but no H_1 states, the Büchi automaton will almost surely leave $\mathcal{P}_{1,2}$ and move to $\mathcal{P}_{2,1}$ which it can leave to $\mathcal{P}_{accept}$ if it visits infinitely many H_2 states. Thus, it could accept although it might not satisfy the Streett condition $\{(H_1, K_1)\}$. This is shown in the following example. Assume the previous construction without the $\mathcal{P}_i$ subautomata. Assume that $K_1 \setminus K_2 \neq \emptyset$, $K_2 \setminus H_1 \neq \emptyset$ and $H_2 \setminus K_1 \neq \emptyset$ and let $k_1 \in K_1 \setminus K_2$, $k_2 \in K_2 \setminus H_1$, $h_2 \in H_2 \setminus K_1$. Then, the nonaccepting (possible) path

$$k_1, k_2, h_2, k_1, k_2, h_2, \ldots$$

in the Streett automaton would be lifted with positive probability to accepting paths of the form

$$\ldots, \langle k_1, accept\rangle, \langle k_2, 1, 2\rangle, \langle h_2, 2, 1\rangle, \langle k_1, accept\rangle, \langle k_2, 1, 2\rangle, \langle h_2, 2, 1\rangle, \ldots$$

in the Büchi automaton. To avoid this, we need the $\mathcal{P}_i$ subautomata.

In the following, we will denote by $\mathsf{Acc}^{\mathcal{P}_S}_{Streett} = \bigwedge_{1\le j\le n}(\Box\Diamond K_j \Rightarrow \Box\Diamond H_j)$ the Streett acceptance condition of $\mathcal{P}_S$ and by $\mathsf{Acc}^{\mathcal{P}_S}_{Rabin} = \bigvee_{1\le j\le n}(\Diamond\Box\neg H_j \wedge \Box\Diamond K_j)$ the acceptance condition obtained from the acceptance pairs in $\mathcal{P}_S$ by interpreting them as a Rabin acceptance condition. We now show that $\mathcal{L}^{>0}_{Büchi}(\mathcal{P}) = \mathcal{L}^{>0}_{Streett}(\mathcal{P}_S)$.

$\subseteq$: Let $w \notin \mathcal{L}^{>0}_{Streett}(\mathcal{P}_S)$, thus $\mathsf{Pr}^{\mathcal{P}_S,w}(\mathsf{Acc}^{\mathcal{P}_S}_{Streett}) = 0$ and hence $\mathsf{Pr}^{\mathcal{P}_S,w}(\mathsf{Acc}^{\mathcal{P}_S}_{Rabin}) = 1$. Consider a run π of $\mathcal{P}_S$ that satisfies the Rabin condition $\bigvee_{1\le j\le n}(\Diamond\Box\neg H_j \wedge \Box\Diamond K_j)$, thus there exists an index j such that $\pi \models \Diamond\Box\neg H_j \wedge \Box\Diamond K_j$. Consider the liftings of π in the constructed Büchi automaton $\mathcal{P}$. (By a lifting of π, we mean any run in $\mathcal{P}$ for w whose projection to the Q_S-components agrees with π.) As the above construction ensures that whenever a K_j-state is visited in $\mathcal{P}_{accept}$ or $\mathcal{P}_{i,j}$ for some $i \neq j$, then with equal positive probability one of the subautomaton $\mathcal{P}_j$ or $\mathcal{P}_{j,k}$ is entered. Hence, if infinitely often a K_j-state is visited and the process does not stay forever in one of the subautomata $\mathcal{P}_i$ (for some $i \neq j$) or $\mathcal{P}_{k,\ell}$ (where it cannot accept), then almost surely $\mathcal{P}_j$ is entered. But $\mathcal{P}_j$ can only be left via a H_j-state. As $\pi \models \Diamond\Box\neg H_j$ this ensures that almost all liftings of π will eventually stay in one of the subautomata $\mathcal{P}_i$ or $\mathcal{P}_{i,k}$, hence they will almost surely not be accepting. This shows that $\mathsf{Pr}^{\mathcal{P}_S,w}(\mathsf{Acc}^{\mathcal{P}_S}_{Rabin}) = 1 \Rightarrow \mathsf{Pr}^{\mathcal{P}}_{Büchi}(w) = 0$ and $w \notin \mathcal{L}^{>0}_{Streett}(\mathcal{P}_S) \Rightarrow w \notin \mathcal{L}^{>0}_{Büchi}(\mathcal{P})$.

$\supseteq$: Let $w \in \mathcal{L}^{>0}_{Streett}(\mathcal{P}_S)$, thus $\mathsf{Pr}^{\mathcal{P}_S,w}(\bigwedge_{1\le j\le n}(\Box\Diamond K_j \Rightarrow \Box\Diamond H_j)) > 0$. As $\{\pi \mid \pi \models \bigwedge_{1\le i\le n}(\Box\Diamond K_i \Rightarrow \Box\Diamond H_i)\} = \bigsqcup_{J\subseteq\{1,\ldots,n\}}\{\pi \mid \pi \models \mathsf{Acc}^{\mathcal{P}_S}_{Streett} \wedge \bigwedge_{j\in J}\Box\Diamond K_j \wedge \bigwedge_{j\notin J}\Diamond\Box\neg K_j\}$, there exists $J \subseteq \{1,\ldots,n\}$ such that

$$\mathsf{Pr}^{\mathcal{P}_S,w}\left(\mathsf{Acc}^{\mathcal{P}_s}_{Streett} \wedge \bigwedge_{j\in J}\Box\Diamond K_j \wedge \bigwedge_{j\notin J}\Diamond\Box\neg K_j\right) > 0.$$

Since $\Diamond\Box\neg K_j$ is the disjoint union of $(\Diamond^{=\ell-1}K_j \wedge \Box^{\ge\ell}\neg K_j)$, for $\ell \in \mathbb{N}$, and since $\{j \mid j \notin J\}$ is finite, there exists $r \in \mathbb{N}_{\ge 0}$ such that

$$\mathsf{Pr}^{\mathcal{P}_S,w}\left(\mathsf{Acc}^{\mathcal{P}_s}_{Streett} \wedge \bigwedge_{j\in J}\Box\Diamond K_j \wedge \bigwedge_{j\notin J}\Box^{>r}\neg K_j\right) > 0.$$

Let π_S be a run in $\mathcal{P}_S$ with $\pi_S \models \mathsf{Acc}^{\mathcal{P}_s}_{Streett} \wedge \bigwedge_{j\in J}\Box\Diamond K_j \wedge \bigwedge_{j\notin J}\Box^{>r}\neg K_j$. Then almost all liftings of π_S to runs for w in $\mathcal{P}$ that stay in $\mathcal{P}_{init}$ for the first r input symbols and eventually enter $\mathcal{P}_{accept}$ are accepting. This yields

$$\mathsf{Pr}^{\mathcal{P}}_{Büchi}(w) \geq \frac{1}{2^{r+1}} \cdot \mathsf{Pr}^{\mathcal{P}_S,w}\left(\mathsf{Acc}^{\mathcal{P}_s}_{Streett} \wedge \bigwedge_{j\in J} \Box\Diamond K_j \wedge \bigwedge_{j\notin J} \Box^{>r}\neg K_j\right) > 0$$

and $w \in \mathcal{L}^{>0}_{Büchi}(\mathcal{P})$. $\Box$

5.2. From PBA$^{>0}$ to 0/1-PRA

In this section, we will describe how a given PBA$^{>0}$ $\mathcal{P}$ can be transformed into an equivalent 0/1-PRA $\mathcal{P}_R$, that is a PRA such that every infinite word is either accepted with probability 0 or 1. This result will play a crucial role in Section 5.3 and for the complementation of PBA under the probable semantics (see Section 6.2). Before we present the theorem let us briefly comment on this transformation which has some similarities with Safra's determinization algorithm for NBA and also relies on some kind of powerset construction. However, we argue that the probabilistic setting is slightly simpler. Instead of organizing the potential accepting runs in Safra trees, we may deal with up to n independent *sample runs* (where n is the number of states in $\mathcal{P}$) that are representative for all potential accepting runs. The idea is to represent the current states of the sample runs by tuples $\langle p_1, \ldots, p_k\rangle$ of pairwise distinct states in $\mathcal{P}$. Whenever two sample runs meet at some point, say the next states p'_1 and p'_2 in the first two sample runs agree, then they are merged, which requires a shift operation for the other sample runs and yields a tuple of the form $\langle p'_1, p'_3, \ldots, p'_k, \ldots, q, \ldots\rangle$ where p'_i is a successor of p_i in the ith sample run. Additionally, new sample runs are generated in case the original PBA $\mathcal{P}$ can be in an accepting state $q \notin \{p'_1, \ldots, p'_k\}$. The Rabin condition serves to express that at least one of the sample runs enters the set F of accepting states in $\mathcal{P}$ infinitely often and is a proper run in $\mathcal{P}$ (i.e., is affected by the shift operations only finitely many times). Intuitively, the automaton $\mathcal{P}_R$ "simulates" $\mathcal{P}$ and moreover each time $\mathcal{P}$ could be in an accepting state, $\mathcal{P}_R$ starts a new sample run (if necessary). Let $w \in \mathcal{L}(\mathcal{P})$, thus $\mathsf{Pr}^{\mathcal{P}}(w) > 0$ and with positive probability $\mathcal{P}$ can be in an accepting state infinitely often. But then $\mathcal{P}_R$ almost surely either already is in a corresponding sample run or starts a new sample run infinitely often and from there on accepts the remaining suffix with positive probability $> c$ for some $c > 0$ (as it "simulates" $\mathcal{P}$ and $\mathsf{Pr}^{\mathcal{P}}(w) > 0$). This yields that the automaton $\mathcal{P}_R$ accepts w with probability 1. This idea is formalized in the proof of the following theorem.

THEOREM 5.3 (FROM PBA$^{>0}$ TO 0/1-PRA). *For any PBA $\mathcal{P}$, there exists a PRA $\mathcal{P}_R$ with $\mathcal{L}^{>0}(\mathcal{P}) = \mathcal{L}^{>0}_{Rabin}(\mathcal{P}_R)$ and such that for every infinite word w, $\mathsf{Pr}^{\mathcal{P}_R}(w) \in \{0, 1\}$.*

PROOF. Let $\mathcal{P} = (Q, \delta, \mu_0, F)$ be the given PBA$^{>0}$. Without loss of generality, we may suppose that $\mathcal{P}$ is total. The idea for the definition of $\mathcal{P}_R$ is to deal with states of the form $\langle p_1, \ldots, p_k, R\rangle$ where $p_1, \ldots, p_k$ are pairwise distinct states that represent the current states of "independent" runs for the given input word. The acceptance condition of $\mathcal{P}_R$ will then require that at least one of these runs in $\mathcal{P}$ is accepting. The last component R is a subset of Q, representing the set of all potential states in which the original automaton $\mathcal{P}$ could be. It will be obtained by the standard powerset construction for finite automata.

To organize the independent runs in a finite-state automaton (rather than an infinite tree), we abstract away from multiple occurrences of some states and merge runs that meet at some point. This causes some technical difficulties because $\mathcal{P}_R$ has to recover fictitious sample runs that enter F infinitely often by combining fragments of infinitely many runs. For this reason, we attach a bit $\xi_j \in \{0, 1\}$ for each of the states p_j which indicates whether the last step results from a proper transition in $\mathcal{P}$ (in which case

$\xi_j = 0$) or p_j is the first state of a newly generated run (in which case $\xi_j = 1$). These bits will be used in the definition of the Rabin acceptance condition of $\mathcal{P}_R$ which requires that for some j, the jth run visits F infinitely often and in some infinite suffix, the attached bits are 0.

We will structure our states in $\mathcal{P}_R$ in such a way that we first list the states that result from a proper transition in $\mathcal{P}$ (having the attached bit 0) and then we list the states that are newly generated (because the automaton $\mathcal{P}$ could be in an accepting state). The latter have attached bit 1. Thus, for each state $\langle p_1, \xi_1, \dots, p_n, \xi_n, R\rangle$,

$$\xi_i = 1 \Rightarrow \forall i < j \le n,\ \xi_j = 1 \tag{$\dagger$}$$

Since several sample runs could be in the same next state (with the attached bit 0), we may need to merge them. Therefore, we define a normalization operator ν that takes as input k states $p_1, \dots, p_k$ in $\mathcal{P}$ augmented with bits $\xi_1, \dots, \xi_k$, possibly with multiple occurrences of some states, and returns a normalized tuple where each state in $\{p_1, \dots, p_k\}$ appears exactly once, with an appropriate bit. Formally, given a $2k$-tuple $\langle p_1, \xi_1, \dots, p_k, \xi_k\rangle \in (Q \times \{0,1\})^k$ where $k \ge 1$ and the ξ_i's satisfy ($\dagger$) we now define $\nu(\langle p_1, \xi_1, \dots, p_k, \xi_k\rangle)$ to be the unique tuple $\langle p_{i_1}, \xi'_{i_1}, \dots, p_{i_\ell}, \xi'_{i_\ell}\rangle$ where $i_1, \dots, i_\ell \in \{1, \dots, k\}$ are indices such that

—$i_1 < i_2 < \cdots < i_\ell$ and $\{p_1, \dots, p_k\} = \{p_{i_1}, \dots, p_{i_\ell}\}$,
—$p_{i_1}, \dots, p_{i_\ell}$ are pairwise distinct and $p_{i_h} \notin \{p_1, \dots, p_{i_h - 1}\}$ for $1 \le h \le \ell$,
—$\xi'_{i_h} = 1$ if $h < i_h$ and $\xi'_{i_h} = \xi_{i_h}$ if $h = i_h$.

Note that $\xi'_1, \dots, \xi'_\ell$ satisfy ($\dagger$). For example, $\nu(\langle p, 0, q, 1, p, 1\rangle) = \nu(\langle p, 0, p, 0, q, 0\rangle) = \langle p, 0, q, 1\rangle$. The idea is to identify all tuples $\langle p_1, \xi_1, \dots, p_k, \xi_k\rangle$ and $\langle q_1, \zeta_1, \dots, q_j, \zeta_j\rangle$ such that $\nu(\langle p_1, \xi_1, \dots, p_k, \xi_k\rangle) = \nu(\langle q_1, \zeta_1, \dots, q_j, \zeta_j\rangle)$. The reason why the normalization operator ν requires $\xi_{i_h} = 1$ if $h < i_h$ is that the bit 1 serves as a separation symbol in the state sequence induced by the $(2h-1)$-st component of the states in a run in $\mathcal{P}_R$. Given a run $\overline{\pi}$ in $\mathcal{P}_R$ such that for infinitely many states in $\overline{\pi}$ the bit in the $2h$-th component is 1, then the state sequence obtained by the $(2h-1)$-st components of the states in $\overline{\pi}$ results from the concatenation of fragments of infinitely many runs in $\mathcal{P}$. Hence, it does not necessarily represent a run in $\mathcal{P}$. This will be important for the acceptance condition in $\mathcal{P}_R$.

We now present the precise definition of the PRA $\mathcal{P}_R$. The state space of the PRA $\mathcal{P}_R$ is

$$\overline{Q} = \bigcup_{1 \le k \le n} \overline{Q}_k$$

where $n = |Q|$ and $\overline{Q}_k$ is the set of all tuples $\langle p_1, \xi_1, \dots, p_k, \xi_k, R\rangle \in (Q \times \{0,1\})^k \times 2^Q$ such that $p_i \ne p_j$ for $1 \le i < j \le k$ and that $\xi_1, \dots, \xi_k$ satisfy ($\dagger$). Let us fix the notation $\overline{Q}_{\ge j} = \bigcup_{j \le k \le n} \overline{Q}_k$ to denote the set of states of $\mathcal{P}_R$ that represent at least j sample runs. Similarly, $\overline{Q}_{<j} = \bigcup_{1 \le k < j} \overline{Q}_k$ denotes the set of states of $\mathcal{P}_R$ that represent less than j sample runs. Intuitively, when reading letter a in state $\overline{q} = \langle q_1, \xi_1, \dots, q_k, \xi_k, R\rangle$ in $\mathcal{P}_R$, then the possible successors are the tuples

$$\overline{p} = \langle p_1, \zeta_1, \dots, p_k, \zeta_k, p_{k+1}, \zeta_{k+1}, \dots, p_m, \zeta_m, S\rangle$$

where

(i) $p_i \in \delta(q_i, a)$ for $1 \le i \le k$,
(ii) $p_{k+1}, \dots, p_m$ are pairwise distinct states in $\mathcal{P}$ such that
$$\{p_{k+1}, \dots, p_m\} = (\delta(R, a) \cap F) \setminus \{p_1, \dots, p_k\},$$
(iii) $\zeta_1 = \cdots = \zeta_k = 0$ and $\zeta_{k+1} = \cdots = \zeta_m = 1$,
(iv) $S = \delta(R, a)$.

These tuples $\overline{p}$ might be not contained in $\overline{Q}$, but they will be turned into states of $\mathcal{P}_R$ by applying the ν-operator. The intuitive meaning of condition (i) is the independence of the transitions $q_i \xrightarrow{a} p_i$, $i = 1, \ldots, k$, that serve to mimick $\mathcal{P}$'s behavior by sample runs. Condition (ii) can be understood as the creation of new sample runs that are potential accepting runs in $\mathcal{P}$. We attach the bit 0 to the first k components to denote that the last step of the sample runs $1, \ldots, k$ was a proper transition in $\mathcal{P}$, while the attached bit 1 for runs $k+1, \ldots, m$ indicate that new runs have been generated (condition (iii)). The last condition (iv) states that the last component is obtained with the standard powerset construction. The probability to obtain the tuple $\overline{p}$ (note that $\overline{p} \notin \overline{Q}$ is possible as there might be multiple occurrences of states with the attached bit 0) from state $\overline{q} \in \overline{Q}$ by reading letter a is given by

$$\Delta(\overline{q}, a, \overline{p}) \;=\; \prod_{1 \le i \le k} \delta(q_i, a, p_i),$$

provided that these conditions (i), (ii), (iii), and (iv) hold. For all other tuples, we set $\Delta(\overline{q}, a, \overline{p}) = 0$.

For given states $\overline{q} \in \overline{Q}$ and $\overline{q}' \in \overline{Q}$ in $\mathcal{P}_R$, the transition probability $\delta_{\mathcal{P}_R}(\overline{q}, a, \overline{q}')$ in $\mathcal{P}_R$ is obtained by summing up the values $\Delta(\overline{q}, a, \overline{p})$ where $\overline{p}$ ranges over all tuples that are represented by state $\overline{q}'$ in $\mathcal{P}_R$ and satisfy conditions (i), (ii), (iii), and (iv). Formally, given a state

$$\overline{q}' \;=\; \langle q_1', \xi_1', \ldots, q_\ell', \xi_\ell', R' \rangle \in \overline{Q},$$

let $[\![\overline{q}']\!]$ be the set of all tuples $\overline{p} = \langle p_1, \zeta_1, \ldots, p_m, \zeta_m, S \rangle$ such that

$$\nu(\langle p_1, \zeta_1, \ldots, p_m, \zeta_m \rangle) \;=\; \langle q_1', \xi_1', \ldots, q_\ell', \xi_\ell' \rangle \quad \text{and} \quad R' = S.$$

The transition probabilities in $\mathcal{P}_R$ are defined by:

$$\delta_{\mathcal{P}_R}(\overline{q}, a, \overline{q}') \;=\; \sum_{\overline{p} \in [\![\overline{q}']\!]} \Delta(\overline{q}, a, \overline{p}).$$

The acceptance condition of the probabilistic Rabin Automaton $\mathcal{P}_R$ consists of n acceptance pairs $(H_1, K_1), \ldots, (H_n, K_n)$. Intuitively, the jth pair (H_j, K_j) formalizes the condition stating that the state sequence obtained by the $(2j-1)$-st components of a given run $\overline{\pi}$ in $\mathcal{P}_R$ stands for an accepting run in $\mathcal{P}$. This requires that F is visited infinitely often and that from some moment on the attached bit at position $2j$ is 0. Intuitively, these conditions assert that the state sequence in Q obtained by the $(2j-1)$-st components of the states in $\overline{\pi}$ contains an infinite suffix which is the suffix of an accepting run in $\mathcal{P}$. Formally, the set $K_j \subseteq \overline{Q}$ consists of all states

$$\langle p_1, \xi_1, \ldots, p_j, \xi_j, \ldots, p_k, \xi_k, R \rangle \in \overline{Q}_{\ge j} \;\text{ such that }\; p_j \in F.$$

The set $H_j \subseteq \overline{Q}$ consists of all states

$$\langle p_1, \xi_1, \ldots, p_j, \xi_j, \ldots, p_k, \xi_k, R \rangle \in \overline{Q}_{\ge j} \;\text{ such that }\; \xi_j = 1.$$

The initial distribution in $\mathcal{P}_R$ is given by

$$\overline{\mu}_0(\langle p, 0, Q_{\mathsf{init}} \rangle) \;=\; \mu_0(p)$$

where Q_{init} is the set of initial states in $\mathcal{P}$, that is, $Q_{\mathsf{init}} = \{q \in Q : \mu_0(q) > 0\}$.

Given an infinite word $w = a_1 a_2 a_3 \ldots \in \Sigma^\omega$, we show the equivalence of the following three statements:

$$(1)\ \Pr^{\mathcal{P}_R}_{\mathit{Rabin}}(w) > 0 \quad (2)\ \Pr^{\mathcal{P}}_{\mathit{Büchi}}(w) > 0 \quad (3)\ \Pr^{\mathcal{P}_R}_{\mathit{Rabin}}(w) = 1.$$

This equivalence yields

$$\mathcal{L}^{>0}_{Rabin}(\mathcal{P}_R) = \mathcal{L}^{>0}_{Büchi}(\mathcal{P}) \text{ and } \mathsf{Pr}^{\mathcal{P}_R}_{Rabin}(w) \in \{0, 1\} \text{ for all } w \in \Sigma^\omega.$$

(3) $\Rightarrow$ (1) is obvious.

(1) $\Rightarrow$ (2): Suppose that $\mathsf{Pr}^{\mathcal{P}_R}_{Rabin}(w) > 0$. Then, there is some $j \in \{1, \dots, n\}$ such that

$$\mathsf{Pr}^{\mathcal{P}_R, w}(\Diamond\Box\neg H_j \wedge \Box\Diamond K_j) > 0.$$

As the set of runs that satisfy $\Diamond\Box\neg H_j$ is the disjoint union of the sets of runs satisfying $\Diamond^{=k-1} H_j \wedge \Box^{\geq k}\neg H_j$, $k = 0, 1, 2, \dots$, there exists $m \in \mathbb{N}_{\geq 0}$ such that

$$\mathsf{Pr}^{\mathcal{P}_R, w}(\Box^{\geq m}\neg H_j \wedge \Box\Diamond K_j) > 0.$$

Here, for every run π, $\pi \models \Box^{\geq k}\neg H_j$ if and only if $\pi^\ell \notin H_j$, for $\ell \geq k$. As the set $\overline{Q} \setminus H_j$ is finite, there exists a state $\overline{r} \notin H_j$, such that

$$\mathsf{Pr}^{\mathcal{P}_R, w}\big\{\pi \mid \pi \models \Diamond^{=m}\overline{r} \wedge \Box^{\geq m}\neg H_j \wedge \Box\Diamond K_j\big\} > 0,$$

where $\pi \models \Diamond^{=m}\overline{r}$ if and only if $\pi^m = \overline{r}$.

It follows from the transition relation of $\mathcal{P}_R$ that whenever there is a transition from a state $\overline{q} \in \overline{Q}_i$ to a state $\overline{p} \in Q_j$, where $i < j$, then the jth bit in $\overline{p}$ is set to 1. Thus, $\overline{Q}_{\geq j}$ can only be entered from $\overline{Q}_{<j}$ via a state in H_j and therefore a run that satisfies $\Box^{\geq m}\neg H_j \wedge \Box\Diamond\overline{Q}_{\geq j}$ satisfies $\Box^{\geq m-1}\overline{Q}_{\geq j}$.

As $K_j \subseteq \overline{Q}_{\geq j}$, the condition $\Diamond^{=m}\overline{r} \wedge \Box^{\geq m}\neg H_j \wedge \Box\Diamond K_j$ can only hold for runs

$$\overline{\pi} = \overline{q}_0, \overline{q}_1, \overline{q}_2, \dots$$

in $\mathcal{P}_R$ that have an infinite suffix $\overline{q}_m, \overline{q}_{m+1}, \overline{q}_{m+2}, \dots$ consisting of states $\overline{q}_i = \langle p_{1,i}, \xi_{1,i}, \dots, p_{j,i}, \xi_{j,i}, \dots, R_i\rangle$ in $\overline{Q}_{\geq j}$ where $\xi_{j,i} = 0$ for all $i \geq m$. Moreover $\overline{q}_m = \overline{r}$ and there are infinitely many indices i such that $p_{j,i} \in F$.

But then the projection to the $(2j-1)$-st components in $\overline{q}_m, \overline{q}_{m+1}, \overline{q}_{m+2}, \dots$ yields an infinite suffix $p_{j,m}, p_{j,m+1}, p_{j,m+2}, \dots$ of an accepting run for w in $\mathcal{P}$. Furthermore, state $r_j = p_{j,m}$ is reachable from an initial state $q_0 \in Q_{\mathsf{init}}$ via a run for the prefix $a_1 \dots a_m$ of w, where r_j denotes the $(2j-1)$st component of $\overline{r}$. Thus,

$$\mathsf{Pr}^{\mathcal{P}}(q_0 \xrightarrow{a_1 \dots a_m} r_j) > 0$$

in $\mathcal{P}$. Hence

$$\mathsf{Pr}^{\mathcal{P}}_{Büchi}(w) \geq \mathsf{Pr}^{\mathcal{P}}(q_0 \xrightarrow{a_1 \dots a_m} r_j) \cdot \mathsf{Pr}^{\mathcal{P}_R, w}(\Diamond^{=m}\overline{r} \wedge \Box^{\geq m}\neg H_j \wedge \Box\Diamond K_j).$$

Hence, $\mathsf{Pr}^{\mathcal{P}}_{Büchi}(w) > 0$, and therefore, $w \in \mathcal{L}(\mathcal{P})$.

(2) $\Rightarrow$ (3): Let us suppose that $\theta = \mathsf{Pr}^{\mathcal{P}}_{Büchi}(w) > 0$ and show that $\mathsf{Pr}^{\mathcal{P}_R}_{Rabin}(w) = 1$. We pick some state $p \in F$ such that $\mathsf{Pr}^{\mathcal{P}, w}(\Box\Diamond p) > 0$. Let $R_i = \delta(Q_{\mathsf{init}}, a_1 \dots a_i)$ for $i \geq 0$. Then, $p \in R_i \cap F$ for infinitely many $i \in \mathbb{N}_{\geq 1}$. For each such index i, let $\theta_i = \mathsf{Pr}^{\mathcal{P}}\{\pi : \pi \text{ is a run for } a_{i+1}a_{i+2}a_{i+3}\dots \text{ starting in } p \text{ such that } \pi \models \Box\Diamond p\}$. Note that θ_i can be written as a sum

$$\theta_i = \sum_{j=i+1}^{\infty} \varsigma[i, j] \cdot \theta_j$$

where $\varsigma[i, j]$ denotes the probability of the set of runs $q_i, q_{i+1}, \dots, q_j$ for the finite subword $a_{i+1} \dots a_j$ of w with $q_i = q_j = p$ and $p \notin \{q_{i+1}, \dots, q_{j-1}\}$. As

$$0 \leq \varsigma[i, j] \leq 1 \quad \text{and} \quad \sum_{j>i} \varsigma[i, j] \leq 1,$$

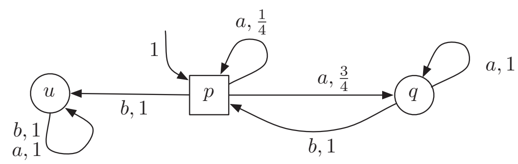


Fig. 8. PBA $\mathcal{P}$: Example for the transformation into 0/1-PRA.

for each $i \in \mathbb{N}_{\geq 1}$ there exists some $j > i$ with $\theta_i \leq \theta_j$. Hence, there exists an infinite sequence $i_1 < i_2 < i_3 < \ldots$ of natural numbers such that $p \in R_{i_h} \cap F$ for all $h \geq 1$ and

$$0 < \theta = \theta_{i_1} \leq \theta_{i_2} \leq \theta_{i_3} \leq \ldots$$

We now regard the stochastic process induced by $\mathcal{P}_R$ and the input word w. Let $I = \{i_1, i_2, i_3, \ldots\}$. For each index $i \in I$, the process enters a state

$$\overline{p}_i \ = \ \langle p_{1,i}, \xi_{1,i}, \ldots, p_{k,i}, \xi_{k,i}, R_i \rangle \quad \text{where } p \in R_i \cap F \subseteq \{p_{1,i}, \ldots, p_{k,i}\}.$$

Say $p = p_{j,i}$. With probability θ_i, the state sequence obtained by scanning the suffix $a_{i+1}a_{i+2}a_{i+3}\ldots$ of w from $p = p_{j,i}$ is a run $p_i, p_{i+1}, p_{i+2}, \ldots$ in $\mathcal{P}$ that visits p infinitely often. Thus, with probability at least θ_i, the stochastic process induced by $\mathcal{P}_R$ and w will generate from position i on a run $\overline{p}_i, \overline{p}_{i+1}, \overline{p}_{i+2}, \ldots$ where after at most $j-1$ shifts via the ν-operator an infinite suffix $p_i, p_{i+1}, p_{i+2}, \ldots$ (with $p_i = p$) of an accepting run in $\mathcal{P}$ will be generated in the $(2\ell-1)$-st component for some $\ell \leq j$. This holds for each index $i \in I$. Hence, the probability for $\mathcal{P}_R$ to generate an accepting run for w is at least

$$\begin{aligned}
\sum_{h=1}^{\infty} \left(\theta_{i_h} \cdot \prod_{1 \leq k < h} (1 - \theta_{i_k}) \right) \ &= \ \lim_{N \to \infty} \sum_{h=1}^{N} \left(\theta_{i_h} \cdot \prod_{1 \leq k < h} (1 - \theta_{i_k}) \right) \\
&= \ \lim_{N \to \infty} \left(1 - \prod_{1 \leq k \leq N} (1 - \theta_{i_k}) \right) \\
&\geq \ \lim_{N \to \infty} \left(1 - (1-\theta)^N \right) \ = \ 1.
\end{aligned}$$

This yields $\mathsf{Pr}^{\mathcal{P}_R}_{Rabin}(w) = 1$ and shows the theorem. $\square$

Remark 5.4. We will see in Section 6 that PBA$^{>0}$ are closed under complementation. Together with the previous transformation from PBA$^{>0}$ to 0/1-PRA, this provides a transformation from PBA$^{>0}$ to 0/1-PSA due to the duality of Rabin and Streett acceptance conditions.

To conclude this section, we illustrate the construction of an equivalent 0/1-PRA on a simple example of a PBA$^{>0}$. Recall the automaton $\mathcal{P}_\lambda$ from Figure 3, page 11 that has two remarkable properties. Namely, it accepts a non-ω-regular language and its accepted language depends on the precise transition probabilities. In Figure 8 we depict the automaton $\mathcal{P}$ which resembles $\mathcal{P}_{\frac{3}{4}}$, but is moreover total.

Given a word $w = a^{k_1}ba^{k_2}ba^{k_3}b\ldots \in \mathcal{L}(\mathcal{P})$, $\mathsf{Pr}^{\mathcal{P}}(w) = \prod_{i=1}^{\infty}(1-(\frac{1}{4})^{k_i}) > 0$. Applying the transformation described in the proof of Theorem 5.3 yields the 0/1-PRA $\mathcal{P}_R$ depicted in Figure 9. States in $\mathcal{P}_R$ consist of a sequence of states of $\mathcal{P}$, each with an associated bit, and a subset of the states of $\mathcal{P}$. For sake of readability the associated bit is subscripted in Figure 9, for example, $\langle u_0, p_1, \{p, u\}\rangle$ stands for $\langle u, 0, p, 1, \{p, u\}\rangle$. The crucial parts of the automaton $\mathcal{P}_R$ are two very similar subautomata that we denote by $\mathcal{P}_{\mathsf{left}}$ and $\mathcal{P}_{\mathsf{right}}$. $\mathcal{P}_{\mathsf{left}}$ consists of the states in the left dashed rectangular box and the left dotted parallelogram and $\mathcal{P}_{\mathsf{right}}$ consists of the states in the right dashed rectangular box

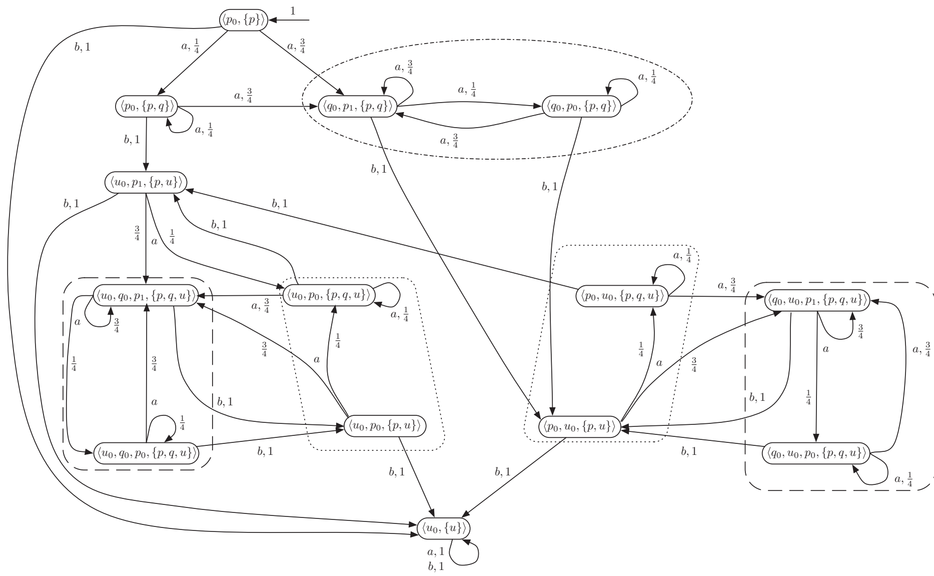


Fig. 9. The resulting 0/1-PRA $\mathcal{P}_R$.

and the right dotted parallelogram. Both $\mathcal{P}_{\mathsf{left}}$ and $\mathcal{P}_{\mathsf{right}}$ simulate $\mathcal{P}$, the two states in the corresponding dashed box simulate the state q of $\mathcal{P}$ and the two states in the corresponding dotted parallelogram simulate the state p. Note that the automaton $\mathcal{P}$ basically rejects if it reads the letter b in state p. This is simulated in $\mathcal{P}_R$ as follows. If two consecutive b's are read then $\mathcal{P}_R$ moves with the first b from the dashed box to the lower state of the dotted parallelogram. With the second b, it moves to state $\langle u_0, \{u\}\rangle$ from which it can never accept. If a word $a^k b$ is read, then $\mathcal{P}$'s behavior is simulated, but instead of rejecting with probability $(\frac{1}{4})^k$, $\mathcal{P}_R$ moves to the state $\langle u_0, p_1, \{p, u\}\rangle$ from where the process of $\mathcal{P}$ is simulated in the subautomaton $\mathcal{P}_{\mathsf{left}}$ for the remaining suffix of the input word. Note that for an input word $a^{k_j} b a^{k_{j+1}} b \ldots$, the probability that the b's are not read in the parallelogram but rather in the box is $\prod_{i=j}^{\infty}(1 - (\frac{1}{4})^{k_i})$. This quantity is greater than $\prod_{i=1}^{\infty}(1 - (\frac{1}{4})^{k_i})$ and thus positive if $a^{k_1} b a^{k_2} b \ldots \in \mathcal{L}^{>0}(\mathcal{P})$. Hence, with positive probability (bounded from below), the process stays in one of the subautomata $\mathcal{P}_{\mathsf{left}}$ or $\mathcal{P}_{\mathsf{right}}$ where it accepts (in the second component for the automaton $\mathcal{P}_{\mathsf{left}}$ and in the first component for the automaton $\mathcal{P}_{\mathsf{right}}$). This ensures that it accepts the words in $\mathcal{L}^{>0}(\mathcal{P})$ with probability 1.

Note that a word with only finitely many b's will not be accepted as almost all runs are not accepting. If the automaton enters a dashed box (after reading the last b), it will almost surely reject, as it will visit both states in the box almost surely. But such a run does not satisfy the Rabin acceptance condition as there is only an accepting state of $\mathcal{P}$ (namely p) in the third component, but one of the states in the box has the p in the third component associated with the bit 1. If the input word contains no b, the same reasoning applies to the two states in the oval.

5.3. Probable and Almost-Sure Semantics

While the threshold semantics is more powerful for PBA than the probable semantics, the almost-sure acceptance criterion is too strong for PBA and even fails to cover the full class of ω-regular languages. In constrast to this constellation for PBA, we prove in the remainder of this subsection that for Streett and Rabin acceptance conditions, probable and almost-sure semantics are equally expressive.

PROPOSITION 5.5. $\mathbb{L}(\mathsf{PSA}^{>0}) = \mathbb{L}(\mathsf{PSA}^{=1}) = \mathbb{L}(\mathsf{PRA}^{=1}) = \mathbb{L}(\mathsf{PRA}^{>0})$.

PROOF. The proposition follows from the duality of the Streett and Rabin acceptance conditions, the results presented in Sections 5.1 and 5.2 and the fact that PBA$^{>0}$ are closed under complementation (see Section 6.2). More precisely, we show a ring of inclusions

$$\mathbb{L}(\mathsf{PSA}^{>0}) \overset{(i)}{\subseteq} \mathbb{L}(\mathsf{PSA}^{=1}) \overset{(ii)}{\subseteq} \mathbb{L}(\mathsf{PRA}^{=1}) \overset{(iii)}{\subseteq} \mathbb{L}(\mathsf{PRA}^{>0}) \overset{(iv)}{\subseteq} \mathbb{L}(\mathsf{PSA}^{>0}).$$

For the sake of clarity we may index Pr or $\mathcal{L}$ by Rabin, Streett, or Büchi to stress that the automaton is of such a kind.

(i) Let $\mathcal{P}_S$ be a PSA. By Theorem 5.2, there exists a PBA $\mathcal{P}_B$ such that $\mathcal{L}^{>0}(\mathcal{P}_S) = \mathcal{L}^{>0}(\mathcal{P}_B)$. Theorem 6.3 from next section implies the existence of a PBA $\overline{\mathcal{P}}_B$ such that $\mathcal{L}^{>0}(\overline{\mathcal{P}}_B) = \Sigma^\omega \setminus \mathcal{L}^{>0}(\mathcal{P}_B)$. This PBA can be transformed into an equivalent PRA $\overline{\mathcal{P}}_R$ (see Remark 2.2, page 4). Thus,

$$\mathcal{L}^{>0}_{\mathit{Streett}}(\mathcal{P}_S) = \mathcal{L}^{>0}_{\mathit{Büchi}}(\mathcal{P}_B) = \Sigma^\omega \setminus \mathcal{L}^{>0}_{\mathit{Büchi}}(\overline{\mathcal{P}}_B) = \Sigma^\omega \setminus \mathcal{L}^{>0}_{\mathit{Rabin}}(\overline{\mathcal{P}}_R) = \mathcal{L}^{=1}_{\mathit{Streett}}(\overline{\mathcal{P}}_R).$$

Note that $\Sigma^\omega \setminus \mathcal{L}^{>0}_{\mathit{Rabin}}(\mathcal{P}) = \mathcal{L}^{=1}_{\mathit{Streett}}(\mathcal{P})$ holds for any probabilistic ω-automaton $\mathcal{P}$ since for every $w \in \Sigma^\omega$, $\mathsf{Pr}^{\mathcal{P}}_{\mathit{Rabin}}(w) = 1 - \mathsf{Pr}^{\mathcal{P}}_{\mathit{Streett}}(w)$.

(ii) Let $\mathcal{P}_S$ be a PSA. Then, $\mathcal{L}^{=1}_{\mathit{Streett}}(\mathcal{P}_S) = \Sigma^\omega \setminus \mathcal{L}^{>0}_{\mathit{Rabin}}(\mathcal{P}_S)$. Interpreting $\mathcal{P}_S$ as a Rabin automaton, thanks to the second item in Theorem 5.1, page 24, there exists an equivalent PBA$^{>0}$ $\mathcal{P}_B$. $\mathcal{P}_B$ can then be complemented into $\overline{\mathcal{P}}_B$ (see Theorem 6.3). Theorem 5.3 yields then an equivalent 0/1-PRA $\mathcal{P}_R$. Hence:

$$\begin{aligned}\mathcal{L}^{=1}_{\mathit{Streett}}(\mathcal{P}_S) = \Sigma^\omega \setminus \mathcal{L}^{>0}_{\mathit{Rabin}}(\mathcal{P}_S) &= \Sigma^\omega \setminus \mathcal{L}^{>0}_{\mathit{Büchi}}(\mathcal{P}_B)\\ &= \mathcal{L}^{>0}_{\mathit{Büchi}}(\overline{\mathcal{P}}_B) = \mathcal{L}^{>0}_{\mathit{Rabin}}(\mathcal{P}_R) = \mathcal{L}^{=1}_{\mathit{Rabin}}(\mathcal{P}_R).\end{aligned}$$

Note that the last equality holds, because $\mathcal{P}_R$ is a 0/1-automaton, that is, each input word is either accepted with probability 0 or 1.

(iii) Let $\mathcal{P}_R$ be a PRA. Then $\mathcal{L}^{=1}_{\mathit{Rabin}}(\mathcal{P}_R) = \Sigma^\omega \setminus \mathcal{L}^{>0}_{\mathit{Streett}}(\mathcal{P}_R)$. Interpreting $\mathcal{P}_R$ as a Streett automaton, thanks to Theorem 5.2 (or third item in Theorem 5.1) there exists an equivalent Büchi automaton $\mathcal{P}_B$ which can be complemented into $\overline{\mathcal{P}}_B$ (see Theorem 6.3 in next section). PBA $\overline{\mathcal{P}}_B$ can be seen as a PRA$^{>0}$ $\mathcal{P}'_R$ (see Remark 2.2) yielding

$$\mathcal{L}^{=1}_{\mathit{Rabin}}(\mathcal{P}_R) = \Sigma^\omega \setminus \mathcal{L}^{>0}_{\mathit{Streett}}(\mathcal{P}_R) = \Sigma^\omega \setminus \mathcal{L}^{>0}_{\mathit{Büchi}}(\mathcal{P}_B) = \mathcal{L}^{>0}_{\mathit{Büchi}}(\overline{\mathcal{P}}_B) = \mathcal{L}^{>0}_{\mathit{Rabin}}(\mathcal{P}'_R).$$

(iv) Let $\mathcal{P}_R$ be a PRA$^{>0}$. Using the second item in Theorem 5.1 $\mathcal{P}_R$ can be transformed into an equivalent Büchi automaton $\mathcal{P}_B$ under probable semantics that itself can be seen as a PSA$^{>0}$ $\mathcal{P}_S$ (see Remark 2.2). This yields

$$\mathcal{L}^{>0}_{\mathit{Rabin}}(\mathcal{P}_R) = \mathcal{L}^{>0}_{\mathit{Büchi}}(\mathcal{P}_B) = \mathcal{L}^{>0}_{\mathit{Streett}}(\mathcal{P}_S). \quad \square$$

Remark 5.6. Note how the following series of equalities/containments:

$$\mathbb{L}(\mathsf{PSA}^{>0}) = \mathbb{L}(\mathsf{PSA}^{=1}) = \mathbb{L}(\mathsf{PRA}^{=1}) = \mathbb{L}(\mathsf{PRA}^{>0}) = \mathbb{L}(\mathsf{PBA}^{>0}) \supsetneq \mathbb{L}(\mathsf{PBA}^{=1}).$$

compares to the nonprobabilistic setting of deterministic and nondeterministic ω-automata where

$$\mathbb{L}(\mathsf{NSA}) = \mathbb{L}(\mathsf{DSA}) = \mathbb{L}(\mathsf{DRA}) = \mathbb{L}(\mathsf{NRA}) = \mathbb{L}(\mathsf{NBA}) \supsetneq \mathbb{L}(\mathsf{DBA}).$$

This correspondence between nondeterminism and probable semantics as well as determinism and almost-sure semantics is further reflected in the results of Section 6

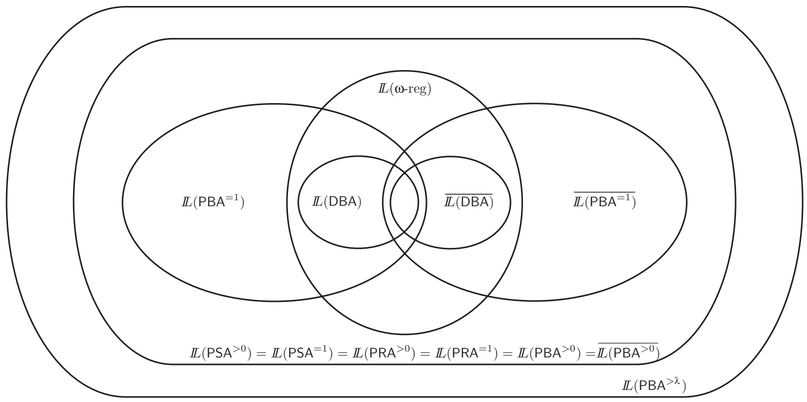


Fig. 10. Overview of expressiveness of variants of probabilistic ω-automata.

and Lemma 4.12, where it is shown that $\mathbb{L}(\mathsf{PBA}^{>0})$ is closed under union, intersection and complementation (just as $\mathbb{L}(\mathsf{NBA})$) whereas $\mathbb{L}(\mathsf{PBA}^{=1})$ is closed under union and intersection but not under complementation (just as $\mathbb{L}(\mathsf{DBA})$). It is also strengthened by several results of Chadha et al. [2009b], where it is shown that every language in $\mathbb{L}(\mathsf{PBA}^{>0})$ is a Boolean combination of languages in $\mathbb{L}(\mathsf{PBA}^{=1})$ (like every language in $\mathbb{L}(\mathsf{NBA})$ is a Boolean combination of languages in $\mathbb{L}(\mathsf{DBA})$). Moreover Chadha et al. [2009b] shows that the ω-regular part of $\mathbb{L}(\mathsf{PBA}^{=1})$ is exactly the class of languages that can be accepted by a DBA, that is,

$$\mathbb{L}(\mathsf{PBA}^{>0}) \cap \omega\text{-}\mathsf{reg} = \mathbb{L}(\mathsf{NBA}) \quad \text{and} \quad \mathbb{L}(\mathsf{PBA}^{=1}) \cap \omega\text{-}\mathsf{reg} = \mathbb{L}(\mathsf{DBA}).$$

An overview of the expressiveness results of variants of probabilistic ω-automata is depicted in Figure 10.

6. COMPOSITION OPERATORS

After the general discussion on PBA in Section 4, we will now present composition operators for PBA under the probable as well as the almost-sure semantics. As we saw in Lemma 4.12, the class $\mathbb{L}(\mathsf{PBA}^{=1})$ is not closed under complementation. Thus, we first present operators for union and intersection for PBA under the probable and the almost-sure semantics. We then show that the class $\mathbb{L}(\mathsf{PBA}^{>0})$ is also closed under complementation.

6.1. Union and Intersection of PBA under the Probable as Well as the Almost-Sure Semantics

LEMMA 6.1. *$\mathbb{L}(\mathsf{PBA}^{>0})$ and $\mathbb{L}(\mathsf{PBA}^{=1})$ are closed under union and intersection.*

PROOF. Let $\mathcal{P}_1 = (Q_1, \Sigma, \delta_1, \mu_0^1, F_1)$ and $\mathcal{P}_2 = (Q_2, \Sigma, \delta_2, \mu_0^2, F_2)$ be two PBA over the same alphabet.

Union. The union of $\mathcal{P}_1$ and $\mathcal{P}_2$ is realized (similar to the case of DBA) through a parallel composition $\mathcal{P}_1 \bowtie \mathcal{P}_2$. Formally, the state space of $\mathcal{P}_1 \bowtie \mathcal{P}_2$ is the cartesian product $Q_1 \times Q_2$. The transition probabilities in the product are given by $\delta((p_1, p_2), a, (q_1, q_2)) = \delta_1(p_1, a, q_1) \cdot \delta_2(p_2, a, q_2)$. The initial distribution in $\mathcal{P}_1 \bowtie \mathcal{P}_2$ is defined by $\mu_0((p_1, p_2)) = \mu_0^1(p_1) \cdot \mu_0^2(p_2)$. The set of accepting states is $F_1 \times Q_2 \cup Q_1 \times F_2$.

This parallel composition yields a PBA satisfying for every $w \in \Sigma^\omega$:

$$\begin{aligned}\mathsf{Pr}^{\mathcal{P}_1 \bowtie \mathcal{P}_2}(w) &= \mathsf{Pr}^{\mathcal{P}_1 \bowtie \mathcal{P}_2, w}(\Box\Diamond F_1 \times Q_2 \vee \Box\Diamond Q_1 \times F_2)\\ &= \mathsf{Pr}^{\mathcal{P}_1}(w) + (1 - \mathsf{Pr}^{\mathcal{P}_1}(w)) \cdot \mathsf{Pr}^{\mathcal{P}_2}(w)\\ &= \mathsf{Pr}^{\mathcal{P}_1}(w) + \mathsf{Pr}^{\mathcal{P}_2}(w) - \mathsf{Pr}^{\mathcal{P}_1}(w) \cdot \mathsf{Pr}^{\mathcal{P}_2}(w).\end{aligned}$$

As a consequence, $\mathcal{L}^{>0}(\mathcal{P}_1 \bowtie \mathcal{P}_2) = \mathcal{L}^{>0}(\mathcal{P}_1) \cup \mathcal{L}^{>0}(\mathcal{P}_2)$ and $\mathcal{L}^{=1}(\mathcal{P}_1 \bowtie \mathcal{P}_2) = \mathcal{L}^{=1}(\mathcal{P}_1) \cup \mathcal{L}^{=1}(\mathcal{P}_2)$.

Intersection. For the intersection operator, we use the same trick as for NBA (respectively, DBA) and construct a generalized PBA (GPBA) $\mathcal{P}_1 \bowtie_{\mathsf{G}} \mathcal{P}_2$. Recall that a generalized Büchi automaton is equipped with several acceptance sets and runs have to visit each of the acceptance sets infinitely often in order to be accepting. The GPBA $\mathcal{P}_1 \bowtie_{\mathsf{G}} \mathcal{P}_2$ can then be turned into an equivalent PBA (see Theorem 5.1(a)). Formally, the state space of $\mathcal{P}_1 \bowtie_{\mathsf{G}} \mathcal{P}_2$ is the cartesian product $Q_1 \times Q_2$. The transition probabilities in the product are given by $\delta((p_1, p_2), a, (q_1, q_2)) = \delta_1(p_1, a, q_1) \cdot \delta_2(p_2, a, q_2)$. The initial distribution in $\mathcal{P}_1 \bowtie_{\mathsf{G}} \mathcal{P}_2$ is defined by $\mu_0((p_1, p_2)) = \mu_0^1(p_1) \cdot \mu_0^2(p_2)$. $\mathcal{P}_1 \bowtie_{\mathsf{G}} \mathcal{P}_2$ has two acceptance sets, namely $F_1 \times Q_2$ and $Q_1 \times F_2$. Then, for every word $w \in \Sigma^\omega$,

$$\begin{aligned}\mathsf{Pr}^{\mathcal{P}_1 \bowtie_{\mathsf{G}} \mathcal{P}_2}(w) &= \mathsf{Pr}^{\mathcal{P}_1 \bowtie_{\mathsf{G}} \mathcal{P}_2, w}(\Box\Diamond F_1 \times Q_2 \wedge \Box\Diamond Q_1 \times F_2)\\ &= \mathsf{Pr}^{\mathcal{P}_1}(w) \cdot \mathsf{Pr}^{\mathcal{P}_2}(w).\end{aligned}$$

As a consequence, $\mathcal{L}^{>0}_{\mathsf{GPBA}}(\mathcal{P}_1 \bowtie_{\mathsf{G}} \mathcal{P}_2) = \mathcal{L}^{>0}(\mathcal{P}_1) \cap \mathcal{L}^{>0}(\mathcal{P}_2)$ and $\mathcal{L}^{=1}_{\mathsf{GPBA}}(\mathcal{P}_1 \bowtie_{\mathsf{G}} \mathcal{P}_2) = \mathcal{L}^{=1}(\mathcal{P}_1) \cap \mathcal{L}^{=1}(\mathcal{P}_2)$. Thanks to Theorem 5.1(a), the GPBA $\mathcal{P}_1 \bowtie_{\mathsf{G}} \mathcal{P}_2$ can be transformed into an equivalent PBA (under both semantics). $\Box$

Note that closure under union and intersection of $\mathbb{L}(\mathsf{PBA}^{=1})$ has also been showed in Chadha et al. [2009b] using a "partial" complementation via finite probabilistic monitors.

Remark 6.2. The union of two probable PBA as well as the intersection of two almost-sure PBA can also be obtained by a very simple construction similar to the one used for the union of NBA. Given two PBA $\mathcal{P}_1 = (Q_1, \Sigma, \delta_1, \mu_0^1, F_1)$ and $\mathcal{P}_2 = (Q_2, \Sigma, \delta_2, \mu_0^2, F_2)$ over the same alphabet, we define the PBA $\mathcal{P}$ as $\frac{1}{2}\mathcal{P}_1 + \frac{1}{2}\mathcal{P}_2$. Formally, $\mathcal{P}$ has the state space $Q_1 \mathbin{\dot\cup} Q_2$, equipped with the same transitions than $\mathcal{P}_1$ and $\mathcal{P}_2$, with the set of accepting states $F_1 \cup F_2$ and with the initial distribution μ_0 defined by $\mu_0(q) = \frac{1}{2}\mu_0^i(q)$ for every state $q \in Q_i$. Intuitively, $\mathcal{P}$ simulates both $\mathcal{P}_1$ and $\mathcal{P}_2$ with probability $\frac{1}{2}$. Clearly $\mathsf{Pr}^{\mathcal{P}}(w) = \frac{1}{2}\mathsf{Pr}^{\mathcal{P}_1}(w) + \frac{1}{2}\mathsf{Pr}^{\mathcal{P}_2}(w)$, and thus $\mathcal{L}^{>0}(\mathcal{P}) = \mathcal{L}^{>0}(\mathcal{P}_1) \cup \mathcal{L}^{>0}(\mathcal{P}_2)$ and $\mathcal{L}^{=1}(\mathcal{P}) = \mathcal{L}^{=1}(\mathcal{P}_1) \cap \mathcal{L}^{=1}(\mathcal{P}_2)$.

6.2. Complementation of PBA under the Probable Semantics

The question of whether the class of languages recognizable by PBA$^{>0}$ is closed under complementation was left open in Baier and Grösser [2005]. In Baier et al. [2008], we showed that, for each PBA$^{>0}$ $\mathcal{P}$, there exists a PBA$^{>0}$ that accepts the complement of $\mathcal{L}^{>0}(\mathcal{P})$. The construction relies on the transformation of a given PBA$^{>0}$ $\mathcal{P}$ into an equivalent PRA $\mathcal{P}_R$ that accepts each word with probability 0 or 1 (thanks to Theorem 5.3). This PRA can easily be turned into a PSA for the complement language, which will at last be transformed into an equivalent PBA$^{>0}$ (due to Theorem 5.2).

THEOREM 6.3 ($\mathbb{L}(\mathsf{PBA}^{>0})$ IS CLOSED UNDER COMPLEMENTATION). *For each PBA $\mathcal{P}$, there exists a PBA $\mathcal{P}'$ of size $\mathcal{O}(\exp(|\mathcal{P}|))$ such that $\mathcal{L}^{>0}(\mathcal{P}') = \Sigma^\omega \setminus \mathcal{L}^{>0}(\mathcal{P})$. Moreover, $\mathcal{P}'$ can be effectively constructed from $\mathcal{P}$.*

PROOF. The idea for the complementation of a given PBA$^{>0}$ $\mathcal{P}$ is to provide the following series of transformations:

$$\begin{aligned}\text{PBA } \mathcal{P} &\stackrel{(1)}{\Longrightarrow} \text{0/1-PRA } \mathcal{P}_R \text{ with } \mathcal{L}^{>0}(\mathcal{P}_R) = \mathcal{L}^{=1}(\mathcal{P}_R) = \mathcal{L}^{>0}(\mathcal{P})\\ &\stackrel{(2)}{\Longrightarrow} \text{0/1-PSA } \mathcal{P}_S \text{ with } \mathcal{L}^{>0}(\mathcal{P}_S) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{P}_R)\\ &\stackrel{(3)}{\Longrightarrow} \text{PBA } \mathcal{P}' \text{ with } \mathcal{L}^{>0}(\mathcal{P}') = \mathcal{L}^{>0}(\mathcal{P}_S),\end{aligned}$$

where 0/1-PRA denotes a PRA with $\mathsf{Pr}^{\mathcal{P}_R}(w) \in \{0,1\}$ for each word $w \in \Sigma^\omega$. The transformation of step (1) uses Theorem 5.3, we will explain step (2) right now and refer to Theorem 5.2 for step (3).

Let $\mathcal{P} = (Q, \Sigma, \delta, \mu_0, F)$ be a PBA$^{>0}$. Applying Theorem 5.3, we construct an equivalent 0/1 PRA $\mathcal{P}_R$ such that

$$\mathcal{L}^{>0}(\mathcal{P}) = \mathcal{L}^{>0}(\mathcal{P}_R) = \mathcal{L}^{=1}(\mathcal{P}_R).$$

For the sake of clarity we may index Pr or $\mathcal{L}$ by Rabin, Streett, or Büchi to stress that the automaton is of such a kind. Using the duality between the Rabin acceptance condition $\bigvee_{1\le j\le n}(\Diamond\Box\neg H_j \wedge \Box\Diamond K_j)$ and the Streett acceptance condition $\bigwedge_{1\le j\le n}(\Box\Diamond K_j \rightarrow \Box\Diamond H_j)$, we note that

$$\Sigma^\omega \setminus \mathcal{L}^{>0}_{\mathit{Rabin}}(\mathcal{P}_R) = \mathcal{L}^{=1}_{\mathit{Streett}}(\mathcal{P}_R) = \mathcal{L}^{>0}_{\mathit{Streett}}(\mathcal{P}_R)$$

since $\mathsf{Pr}^{\mathcal{P}_R}_{\mathit{Rabin}}(w) = 1 - \mathsf{Pr}^{\mathcal{P}_R}_{\mathit{Streett}}(w)$ and $\mathcal{P}_R$ is a 0/1 automaton. Thus, we view the automata $\mathcal{P}_R$ as a PSA $\mathcal{P}_S$ and apply Theorem 5.2 to transform the PSA $\mathcal{P}_S$ into an equivalent PBA$^{>0}$ $\mathcal{P}'$, which yields

$$\mathcal{L}^{>0}_{\mathit{Büchi}}(\mathcal{P}') = \mathcal{L}^{>0}_{\mathit{Streett}}(\mathcal{P}_S) = \Sigma^\omega \setminus \mathcal{L}^{>0}_{\mathit{Rabin}}(\mathcal{P}_R) = \Sigma^\omega \setminus \mathcal{L}^{>0}_{\mathit{Büchi}}(\mathcal{P}).$$

Let n be the number of states in the original PBA $\mathcal{P}$. The construction presented in the proof of Theorem 5.3 implies that the number of states in $\mathcal{P}_R = \mathcal{P}_S$ is bounded by $2^{\mathcal{O}(n\log n)}$, while the number of acceptance pairs in $\mathcal{P}_R = \mathcal{P}_S$ is n. Therefore, the size of the PBA$^{>0}$ $\mathcal{P}'$ generated from $\mathcal{P}_S$ by Theorem 5.2 is bounded by $n^2 \cdot 2^{\mathcal{O}(n\log n)}$. $\mathcal{P}'$ is thus at most exponentially larger than $\mathcal{P}$. □

7. EMPTINESS CHECKING FOR PBA$^{>0}$ AND RELATED PROBLEMS

For finite probabilistic automata (PFA), it has been shown that the emptiness problem is undecidable. Recall that a PFA $\mathcal{P}_{\mathsf{fin}}$ is equipped with a threshold $0 < \lambda < 1$ and that the accepted language $\mathcal{L}^{>\lambda}(\mathcal{P}_{\mathsf{fin}})$ consists of all input words for which the set of runs that end in an accepting state has a probability greater than λ. From this, it easily follows that the emptiness problem for PBA under the threshold semantics (see Section 4.3) is undecidable. Indeed, any given PFA $\mathcal{P}_{\mathsf{fin}}$ over the alphabet Σ can be transformed into a PBA $\mathcal{P}$ such that, for each finite word $\rho \in \Sigma^*$, it holds that $\mathsf{Pr}^{\mathcal{P}_{\mathsf{fin}}}(\rho) = \mathsf{Pr}^{\mathcal{P}}(\rho c^\omega)$, where c is an additional letter which is not in Σ. Moreover, the automaton $\mathcal{P}$ can only produce (Büchi) accepting runs for input words that are of the form ρc^ω, where ρ is in Σ^* (for each accepting state of the PFA $\mathcal{P}_{\mathsf{fin}}$, add a c-transition with probability one to a new state q_{acc} which then is the only accepting state of the PBA $\mathcal{P}$ and has a c-loop attached to it with probability one).

Any probabilistic finite automata with threshold $\lambda = 0$ accepts the same language as its underlying nondeterministic finite automaton, and therefore, the emptiness problem for this restricted class is decidable. In contrast, we will show that the emptiness problem for PBA$^{>0}$ is undecidable. We also discuss immediate consequences of this result.

7.1. Emptiness Problem for PBA$^{>0}$

The proof for the undecidability of the emptiness problem for PBA under the probable semantics relies on a reduction from a variant of the emptiness problem for PFA, using the fact that modifying the transition probabilities can affect the accepted language of a PBA$^{>0}$ (Theorem 4.7). The emptiness problem for PFA is known to be undecidable [Paz 1971]. Here we use the following variant of this result, due to Madani et al. [2003].

THEOREM 7.1 (UNDECIDABILITY RESULT FOR PFA [MADANI ET AL. 2003]). *The following problem is undecidable: Given a constant* $0 < \varepsilon < 1$ *and a PFA that either accepts some string with probability at least* $1 - \varepsilon$ *or accepts all strings with probability at most* ε*, decide which is the case.*

THEOREM 7.2 (UNDECIDABILITY OF THE EMPTINESS PROBLEM FOR PBA$^{>0}$). *Checking emptiness is undecidable for PBA under the probable semantics.*

PROOF. To provide an undecidability proof of the emptiness problem for PBA$^{>0}$, we reduce the variant of the emptiness problem for PFA recalled in Theorem 7.1 to the *intersection problem* for PBA$^{>0}$, which takes as input two PBA $\mathcal{P}_1$ and $\mathcal{P}_2$ and asks whether $\mathcal{L}^{>0}(\mathcal{P}_1) \cap \mathcal{L}^{>0}(\mathcal{P}_2)$ is empty. As PBA are closed under intersection (see Section 6.1), this will complete the proof for Theorem 7.2.

Let $\mathcal{R}$ be a PFA over some alphabet Σ and $0 < \varepsilon < \frac{1}{2}$ as in Theorem 7.1, that is, such that either there exists some word ρ accepted by $\mathcal{R}$ with probability strictly greater than $1 - \varepsilon$, or all words are accepted with probability less than ε. For $\rho \in \Sigma^*$, let $\mathsf{Pr}^{\mathcal{R}}(\rho)$ denote the probability that the word ρ is accepted by $\mathcal{R}$. From the PFA $\mathcal{R}$ and the constant ε we construct two PBA $\mathcal{P}_1$ and $\mathcal{P}_2$ such that

$$\mathcal{L}^{>\varepsilon}(\mathcal{R}) = \emptyset \text{ if and only if } \mathcal{L}^{>0}(\mathcal{P}_1) \cap \mathcal{L}^{>0}(\mathcal{P}_2) = \emptyset,$$

where $\mathcal{L}^{>\varepsilon}(\mathcal{R}) = \{\rho \in \Sigma^* \mid \mathsf{Pr}^{\mathcal{R}}(\rho) > \varepsilon\}$. The alphabet for both $\mathcal{P}_1$ and $\mathcal{P}_2$ arise from the alphabet Σ of $\mathcal{R}$ by adding two new symbols $\sharp$ and \$, that is, $\mathcal{P}_1$ and $\mathcal{P}_2$ are PBA over the alphabet $\Sigma' = \Sigma \cup \{\sharp, \$\}$. The rough idea is to use the somehow complementary acceptance behavior of the automata $\mathcal{P}_\lambda$ and $\widetilde{\mathcal{P}}_\lambda$ (see Figure 3 on page 11, and Figure 6 on page 17). The automata $\mathcal{P}_1$ and $\mathcal{P}_2$ are designed to read words of the form $\rho_1^1\sharp\rho_2^1\sharp\ldots\rho_{k_1}^1\$\$\rho_1^2\sharp\rho_2^2\sharp\ldots\rho_{k_2}^2\$\$\ldots$ where $\rho_i^j \in \Sigma^*$ and $k_i \geq 2$. Roughly speaking, $\mathcal{P}_1$ will mimick the automaton $\mathcal{P}_\lambda$ and $\mathcal{P}_2$ will mimick $\widetilde{\mathcal{P}}_\lambda$, where reading a word $\rho_i^j\sharp$ in $\mathcal{P}_1$ (respectively, $\mathcal{P}_2$) corresponds to reading a single letter a in $\mathcal{P}_\lambda$ (respectively, $\widetilde{\mathcal{P}}_\lambda$). Recall that $\mathcal{P}_\lambda$ and $\widetilde{\mathcal{P}}_\lambda$ accept infinite words of the form $a^{k_1}ba^{k_2}b\ldots$ (depending on the k_i). The two consecutive \$-symbols serve as a separator for $\mathcal{P}_1$ and $\mathcal{P}_2$, just like the letter b does for $\mathcal{P}_\lambda$ and $\widetilde{\mathcal{P}}_\lambda$. Thus, the number of $\sharp$-symbols between the $(j-1)$st and the jth occurence of \$\$ (and therefore the number of words ρ_i^j) corresponds to the value of k_j. Automaton $\mathcal{P}_1$ evolves from the automaton $\mathcal{P}_\lambda$ by replacing each of its two states p_0, p_1 by a copy of the PFA $\mathcal{R}$. The transitions for the $\sharp$-symbol will be defined, such that after reading a word $\rho_i^j\sharp$ in the copy of $\mathcal{R}$ that corresponds to the state p_0 (recall that this corresponds to reading a single letter a in state p_0 of in $\mathcal{P}_\lambda$) the automaton $\mathcal{P}_1$ is still in this copy of $\mathcal{R}$ with probability $1 - \mathsf{Pr}^{\mathcal{R}}(\rho_i^j)$ and has moved to the other copy with probability $\mathsf{Pr}^{\mathcal{R}}(\rho_i^j)$, similar to the behavior of automaton $\mathcal{P}_\lambda$ upon reading the letter a in state p_0 (it stays in p_0 with probability $1 - \lambda$ and moves to p_1 with probability λ). The structure of $\mathcal{P}_1$ and $\mathcal{P}_2$ is shown in Figures 11 and 12, respectively.

The PBA $\mathcal{P}_1$ is composed of two copies of the PFA $\mathcal{R}$ (represented in dashed lines) augmented with new edges using the additional symbols $\sharp$ and \$. The initial states of $\mathcal{P}_1$ are the initial states of the first copy of $\mathcal{R}$ according to the initial distribution of $\mathcal{R}$. Reading the symbol $\sharp$ in any final state of the first copy of $\mathcal{R}$, the PBA $\mathcal{P}_1$ proceeds to the

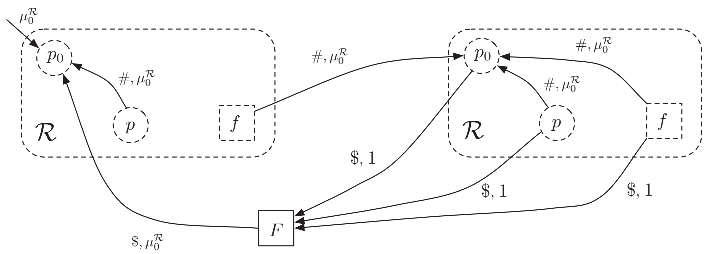


Fig. 11. PBA $\mathcal{P}_1$.

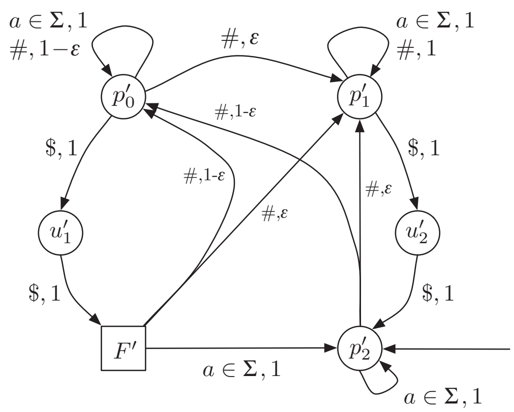


Fig. 12. PBA $\mathcal{P}_2$.

initial state of $\mathcal{R}$ in the second copy according to the initial distribution of $\mathcal{R}$. Reading the symbol $\sharp$ in any nonfinal state of the first copy of $\mathcal{R}$, the PBA $\mathcal{P}_1$ proceeds to the initial state of $\mathcal{R}$ in the first copy according to the initial distribution of $\mathcal{R}$. Consuming the symbol \$ in some (final or nonfinal) state of the second copy of $\mathcal{R}$, $\mathcal{P}_1$ moves with probability 1 to the special state F, which is the unique accepting state of $\mathcal{P}_1$. Reading the second \$ symbol, $\mathcal{P}_1$ proceeds on to an initial state according to the initial distribution of $\mathcal{R}$. As justified at the end of this proof, the accepted language of this PBA$^{>0}$ is:

$$L_1 = \mathcal{L}^{>0}(\mathcal{P}_1) = \Bigg\{ \rho_1^1 \sharp \rho_2^1 \sharp \dots \rho_{k_1}^1 \$\$ \, \rho_1^2 \sharp \rho_2^2 \sharp \dots \rho_{k_2}^2 \$\$ \, \dots \mid \rho_i^j \in \Sigma^*, k_i \geq 2 \\ \text{and} \prod_{j \geq 1} \left(1 - \left(\prod_{i=1}^{k_j - 1} \left(1 - \Pr^{\mathcal{R}}(\rho_i^j)\right) \right) \right) > 0 \Bigg\}.$$

The PBA $\mathcal{P}_2$ does not depend on the structure of the given PFA $\mathcal{R}$, but only on ε and the alphabet Σ. As the automaton $\mathcal{P}_2$ is only a slight variant of the automaton $\widetilde{\mathcal{P}}_\lambda$ from Example 4.2.1, we derive from the expression for $\mathcal{L}^{>0}(\widetilde{\mathcal{P}}_\lambda)$ the language accepted by $\mathcal{P}_2$

under the probable semantics:

$$L_2 = \mathcal{L}^{>0}(\mathcal{P}_2) = \left\{ v_1\$\$\, v_2\$\$ \ldots \mid v_i \in (\Sigma \cup \{\sharp\})^*, |v_i|_\sharp \geq 1 \text{ and } \prod_{i\geq 1}(1-(1-\varepsilon)^{|v_i|_\sharp}) = 0 \right\},$$

where $|v|_\sharp$ is the number of $\sharp$ symbols in the word $v \in (\Sigma \cup \{\sharp\})^*$.

We now show that the language $\mathcal{L}^{>\varepsilon}(\mathcal{R})$ accepted by $\mathcal{R}$ for the threshold ε is empty if and only if $L_1 \cap L_2 = \emptyset$.

"$\Longrightarrow$": Assume that $\mathcal{L}^{>\varepsilon}(\mathcal{R})$ is empty, that is, for every finite word $\rho \in \Sigma^*$ it holds that $\mathsf{Pr}^{\mathcal{R}}(\rho) \leq \varepsilon$. Let $w \in L_2$. The goal is to prove that $w \notin L_1$. Since $w \in L_2$, w can be written as $w = v_1\$\$\, v_2\$\$ \ldots$ with $v_i \in (\Sigma \cup \{\sharp\})^*$, $|v_i|_\sharp \geq 1$ and $\prod_i(1-(1-\varepsilon)^{|v_i|_\sharp}) = 0$. The subwords v_i can be decomposed according to the occurrences of the symbol $\sharp$. That is,

$$w = \rho_1^1\sharp\rho_2^1\sharp \ldots \rho_{k_1}^1\$\$\, \rho_1^2\sharp\rho_2^2\sharp \ldots \rho_{k_2}^2\$\$ \ldots \text{ with } \rho_j^i \in \Sigma^* \text{ and } k_i = |v_i|_\sharp + 1.$$

Hence $w \in L_2$ implies $\prod_i(1-(1-\varepsilon)^{k_i-1}) = 0$. However:

$$\begin{aligned}
\prod_{j\geq 1}\left(1 - \prod_{i=1}^{k_j-1}\left(1-\mathsf{Pr}^{\mathcal{R}}(\rho_i^j)\right)\right) &\leq \prod_{j\geq 1}\left(1-\prod_{i=1}^{k_j-1}(1-\varepsilon)\right) \quad \text{since } \mathcal{L}^{>\varepsilon}(\mathcal{R}) = \emptyset \\
&= \prod_{j\geq 1}(1-(1-\varepsilon)^{k_j-1}) \\
&= 0 \qquad\qquad \text{since } w \in L_2.
\end{aligned}$$

Hence, $w \notin L_1$. Since this holds for every $w \in L_2$, we conclude that $L_1 \cap L_2 = \emptyset$.

"$\Longleftarrow$": Assume now that $\mathcal{L}^{>\varepsilon}(\mathcal{R}) \neq \emptyset$. By assumption on the PFA $\mathcal{R}$, this means that there exists a finite word $\rho \in \Sigma^*$ such that $\mathsf{Pr}^{\mathcal{R}}(\rho) \geq 1-\varepsilon$. For every sequence $(k_j)_{j\in\mathbb{N}}$ of natural numbers, we define:

$$w_{k_1,k_2,\ldots} = (\rho\sharp)^{k_1}\rho\$\$\,(\rho\sharp)^{k_2}\rho\$\$ \ldots,$$

and prove that there exists a sequence (k_j), such that $w_{k_1,k_2,\ldots} \in L_1 \cap L_2$. The acceptance probability of $w_{k_1,k_2,\ldots}$ in $\mathcal{P}_1$ is

$$\begin{aligned}
\prod_{j\geq 1}\left(1-\prod_{i=1}^{k_j}(1-\mathsf{Pr}^{\mathcal{R}}(\rho))\right) &= \prod_{j\geq 1}(1-(1-\mathsf{Pr}^{\mathcal{R}}(\rho))^{k_j}) \\
&\geq \prod_{j\geq 1}(1-(1-(1-\varepsilon))^{k_j}) = \prod_{j\geq 1}(1-\varepsilon^{k_j}).
\end{aligned}$$

On the other hand, the word $w_{k_1,k_2,\ldots}$ can be written as $v_1\$\$v_2\$\$\ldots$ with $v_i \in (\Sigma \cup \{\sharp\})^*$ and $|v_i|_\sharp = k_i$. Hence, the acceptance probability of $w_{k_1,k_2,\ldots}$ in $\mathcal{P}_2$ is:

$$\prod_{i\geq 1}(1-(1-\varepsilon)^{|v_i|_\sharp}) = \prod_{i\geq 1}(1-(1-\varepsilon)^{k_i}).$$

We finally apply Theorem 4.7 which yields the existence of a sequence $(k_i')_{i\geq 1}$ that will ensure at the same time $\prod_{j\geq 1}(1-\varepsilon^{k_j'}) > 0$ and $\prod_{i\geq 1}(1-(1-\varepsilon)^{k_i'}) = 0$. Hence, $w_{k_1',k_2',\ldots} \in L_1 \cap L_2$ and $\mathcal{L}(\mathcal{P}_1) \cap \mathcal{L}(\mathcal{P}_2) \neq \emptyset$.

Let us now come back to the justification for the language accepted by $\mathcal{P}_1$ under the probable semantics.We claimed that

$$\mathcal{L}^{>0}(\mathcal{P}_1) = \Big\{ \rho_1^1 \sharp \rho_2^1 \sharp \dots \rho_{k_1}^1 \$\$ \, \rho_1^2 \sharp \rho_2^2 \sharp \dots \rho_{k_2}^2 \$\$ \dots \mid \rho_i^j \in \Sigma^*, k_i \geq 2 \\ \text{and} \prod_{j \geq 1} \left(1 - \left(\prod_{i=1}^{k_j - 1} \left(1 - \mathsf{Pr}^{\mathcal{R}}(\rho_i^j)\right) \right) \right) > 0 \Big\}.$$

Indeed, starting in the first copy of $\mathcal{R}$, $1 - \mathsf{Pr}^{\mathcal{R}}(\rho)$ is the probability for reading the word ρ and ending in some non-final state p. Hence, $\prod_{i=1}^{k_j-1}(1 - \mathsf{Pr}^{P}(\rho_i^j))$ represents the probability to stay in the first copy of $\mathcal{R}$ after having read the finite word $\rho_1^j \sharp \rho_2^j \sharp \dots \rho_{k_j-1}^j \sharp$. The complement of this probability is then exactly the probability to jump to the second copy at some point before reading $\rho_{k_j}^j$. This corresponds to the probability to be able to read the symbol \$ after the prefix $\rho_1^j \sharp \rho_2^j \sharp \dots \rho_{k_j-1}^j \sharp \rho_{k_j}^j$. Thus, the infinite product

$$\prod_{j} \left(1 - \left(\prod_{i=1}^{k_j - 1} \left(1 - \mathsf{Pr}^{\mathcal{R}}(\rho_i^j)\right) \right) \right)$$

is the probability to be able to read the two consecutive \$-symbols each time they appear in the input word. This agrees with the probability to visit infinitely often the final state F. This shows that the given expression for $\mathcal{L}^{>0}(\mathcal{P}_1)$ is correct and completes the proof of Theorem 7.2. □

Remark 7.3. An alternative proof for Theorem 7.2 is based on a recent result by Gimbert and Oualhadj: the undecidability of the value 1 problem for probabilistic finite automata [Gimbert and Oualhadj 2010]. The latter problem asks, given a probabilistic finite automaton (PFA), whether there are words accepted with probability arbitrarily close to 1. Their proof is short, elegant, and only relies on the undecidability of emptiness for PFA with threshold $0 < \lambda < 1$. Let us know show to derive Theorem 7.2 from their result. Take a PFA $\mathcal{P}$ and turn it into a PBA$^{>0}$ $\mathcal{P}'$ by adding a new transition from every final state to every initial state labelled with a new symbol $\sharp$ and with probability 1. This simple construction ensures that $\mathcal{P}$ has value 1 if and only if $\mathcal{L}^{>0}(\mathcal{P}') \neq \emptyset$. Indeed, if $\mathcal{P}$ has value 1, a sequence $(w_n)_{n \in \mathbb{N}}$ of words such that w_n is accepted with probability say greater than $1 - 1/n^2$ can be used to derived an infinite word, namely $w_1 \sharp w_2 \sharp \dots$, accepted with positive probability in the PBA$^{>0}$ $\mathcal{P}'$. Assuming now that the value of $\mathcal{P}$ is strictly less than 1, there exists $\varepsilon > 0$ such that for all finite words the acceptance probability in $\mathcal{P}$ is bounded by $1 - \varepsilon$. Hence, the concatenation of any sequence of finite words, alternated with $\sharp$ symbols can never yield an infinite word accepted with positive probability in $\mathcal{P}'$.

7.2. Immediate Consequences of the Undecidability of the Emptiness Problem

Since complementation is effective for PBA$^{>0}$, the undecidability of the emptiness problem yields immediately that many other interesting algorithmic problems for PBA$^{>0}$ are undecidable too.

COROLLARY 7.4 (OTHER UNDECIDABILITY RESULTS FOR PBA). *Given two PBA $\mathcal{P}_1$ and $\mathcal{P}_2$, the following problems are undecidable.*

$$\begin{array}{rl} \textit{universality:} & \mathcal{L}^{>0}(\mathcal{P}_1) \;=\; \Sigma^\omega? \\ \textit{equivalence:} & \mathcal{L}^{>0}(\mathcal{P}_1) \;=\; \mathcal{L}^{>0}(\mathcal{P}_2)? \\ \textit{language containment:} & \mathcal{L}^{>0}(\mathcal{P}_1) \;\subseteq\; \mathcal{L}^{>0}(\mathcal{P}_2)? \end{array}$$

Remark 7.5. Chadha et al. [2009b] shows that the emptiness, universality (respectively, language containment) problem for PBA$^{>0}$ with rational coefficients is Σ_2^0-complete, where Σ_2^0 is on the second level of the arithmetical hierarchy.

Another immediate consequence of Theorem 7.2 is that the verification problem for finite nondeterministic transition systems $\mathcal{T}$ and PBA-specifications is undecidable. Here we assume that the states in $\mathcal{T}$ are labeled with sets of atomic propositions of some finite set AP and consider the traces of the paths in $\mathcal{T}$ that arise by the projection to the labels of the states. Furthermore, we assume that the given PBA has the alphabet 2^{AP}.

COROLLARY 7.6 (VERIFICATION AGAINST PBA-SPECIFICATIONS (I)). *The following problems are undecidable.*

(a) *Given a finite transition system $\mathcal{T}$ and a PBA $\mathcal{P}$, is there a path in $\mathcal{T}$ whose trace is in $\mathcal{L}^{>0}(\mathcal{P})$?*
(b) *Given a finite transition system $\mathcal{T}$ and a PBA $\mathcal{P}$, do the traces of all paths in $\mathcal{T}$ belong to $\mathcal{L}^{>0}(\mathcal{P})$?*

PROOF. Consider a transition system $\mathcal{T}$ such that each infinite word over the alphabet of $\mathcal{P}$ is a trace of $\mathcal{T}$ and vice-versa. Then, the emptiness problem for PBA reduces to (a) and the universality problem for PBA reduces to (b). We define $\mathcal{T}$ as follows. Given a PBA with the alphabet $\Sigma = 2^{\mathsf{AP}} = \{a_1, \ldots, a_n\}$, we define the state set of $\mathcal{T}$ to be $S = \{s_1, \ldots, s_n\}$ and the set of actions to be $\mathsf{Act} = \{\alpha_1, \ldots, \alpha_n\}$. Each state s_i is labeled with the set of atomic propositions a_i. There is a transition from each s_i to each s_j, $1 \le i, j \le n$ via action α_j and every state is an initial state of $\mathcal{T}$. Thus $\Sigma^\omega = \{\mathit{trace}(\pi) \mid \pi \text{ is an infinite path in } \mathcal{T}\}$. □

As transition systems are special instances of state-labeled Markov decision processes, the following four cases of the qualitative verification problem for finite state-labeled Markov decision processes $\mathcal{M}$ and PBA-specifications $\mathcal{P}$ are undecidable too.

COROLLARY 7.7 (VERIFICATION AGAINST PBA-SPECIFICATIONS (II)). *Given a finite state-labeled Markov decision process $\mathcal{M}$ and a PBA-specification $\mathcal{P}$, the problems whether there is a scheduler $\mathcal{U}$ for $\mathcal{M}$ such that:*

$$\begin{array}{llll} (i) & \mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\mathcal{L}^{>0}(\mathcal{P})) > 0? & (ii) & \mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\mathcal{L}^{>0}(\mathcal{P})) = 1? \\ (iii) & \mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\mathcal{L}^{>0}(\mathcal{P})) < 1? & (iv) & \mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\mathcal{L}^{>0}(\mathcal{P})) = 0? \end{array}$$

are undecidable.

PROOF. Indeed, problem (a) of Corollary 7.6 reduces to (i) and problem (b) reduces to (iii) when $\mathcal{T}$ is viewed as an MDP $\mathcal{M}_{\mathcal{T}}$, where we assume the initial distribution to be uniform over the initial states of $\mathcal{T}$. Similarly, problem (a) of Corollary 7.6 reduces to (ii) and problem (b) to (iv). □

Remark 7.8. Note that owing to the effective constructions given in Section 5 and owing to Proposition 5.5 all the undecidability results from this section also hold for probabilistic Rabin and Streett automata under the probable and the almost-sure semantics. However, in the next section, we will show that the emptiness problem is decidable for PBA$^{=1}$.

8. POMDP

Partially obserable Markov decision processes (POMDP) have been deeply investigated in the literature. Most of the publications on POMDP focus on finite-horizon properties, and provide algorithms to solve them. More recently, infinite-horizon objectives have been studied in the verification community. On the one hand probabilistic Büchi automata form a very specific instance of POMDP, but on the other hand, they seem to contain the core problems for partial observation. As a consequence, combining PBA specificities with standard MDP techniques can yield new results for POMDP.

Definition 8.1 (*Partially Observable MDP*). A *partially observable* MDP (POMDP) is a pair $(\mathcal{M}, \sim)$ consisting of an MDP and an equivalence relation $\sim \subseteq S \times S$ over the states of $\mathcal{M}$ such that for all states $s, t \in S$, if $s \sim t$ then the sets of actions enabled in s and t are equal. Given a POMDP $(\mathcal{M}, \sim)$, an *observation-based scheduler* $\mathcal{U}$ is a scheduler for the underlying MDP $\mathcal{M}$ that is consistent with $\sim$, that is, which satisfies $\mathcal{U}(s_0 s_1 \ldots s_n) = \mathcal{U}(t_0 t_1 \ldots t_m)$ if $n = m$ and $s_i \sim t_i$ for $0 \le i \le m$. The set of observation-based schedulers is denoted by $\mathsf{Sched}^{(\mathcal{M},\sim)}$.

Note that probabilistic ω-automata can be seen as particular instances of POMDP. Indeed given a total PBA (respectively, PRA, PSA) $\mathcal{P}$ and the trivial equivalence relation over states $\sim = Q \times Q$, the pair $(\mathcal{P}, \sim)$ forms a POMDP, where an observation-based deterministic scheduler represents an input word for $\mathcal{P}$.

8.1. Undecidability Results

Since PBA are a special case of partially observable Markov decision processes, our negative results from Section 7 immediately imply undecidability results for POMDP and qualitative properties. In the literature, some undecidability results for POMDP (or similar models) and quantitative properties (e.g., expected rewards, approximation of the maximal reachability problem) can be found [Madani et al. 2003; Giro and D'Argenio 2007]. However, as far as we know, the undecidability of qualitative ω-regular properties for POMDP is a new result. As POMDP are $1\frac{1}{2}$-player games, the following results also apply to the setting of stochastic multi-player games with incomplete information.

COROLLARY 8.2 (UNDECIDABILITY RESULTS FOR POMDP). *The following problems are undecidable:*

(a) *Given $(\mathcal{M}, \sim)$ a finite POMDP and F a set of states in $\mathcal{M}$, is there a deterministic observation-based scheduler $\mathcal{U}$ for $(\mathcal{M}, \sim)$ such that* $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Box\Diamond F) > 0$*?*
(b) *Given $(\mathcal{M}, \sim)$ a finite POMDP and F a set of states in $\mathcal{M}$, is there a deterministic observation-based scheduler $\mathcal{U}$ for $(\mathcal{M}, \sim)$ such that* $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Diamond\Box F) = 1$*?*

PROOF. Given a total PBA $\mathcal{P}$ (i.e., a PBA that has transitions for each pair of a state and input letter) we define the equivalence relation $\sim = Q \times Q$. Note that each PBA can be trivially transformed into an equivalent total PBA. The pair $(\mathcal{P}, \sim)$ forms a POMDP with the action set Σ where a deterministic observation-based scheduler $\mathcal{U}$ represents an input word $w_{\mathcal{U}}$ for the PBA $\mathcal{P}$ (and vice-versa). Consider F to be the set of accepting states of $\mathcal{P}$.

The undecidability of (a) is an immediate consequence of the undecidability of the emptiness problem for PBA$^{>0}$ as $\mathsf{Pr}^{\mathcal{P},\mathcal{U}}(\Box\Diamond F) = \mathsf{Pr}^{\mathcal{P}}(w_{\mathcal{U}})$.

The undecidability of (b) follows from the undecidability of the universality problem for PBA$^{>0}$. Indeed letting $\mathcal{P} = \mathcal{M}$ with the set of accepting states $Q \setminus F$, the answer to

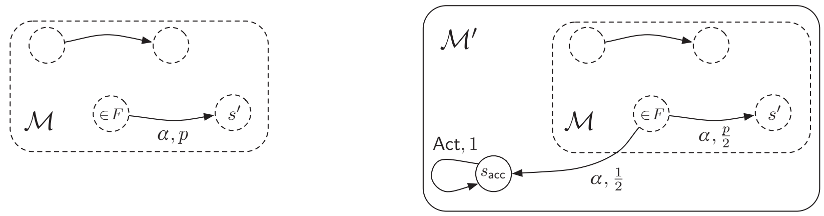


Fig. 13. Transformation from $\mathcal{M}$ to $\mathcal{M}'$.

(b) is "yes" is and only if $\mathcal{L}^{>0}(\mathcal{P}) \neq \Sigma^\omega$ since:

$$\mathsf{Pr}^{\mathcal{P}}(w_{\mathcal{U}}) \;=\; \mathsf{Pr}^{\mathcal{P},\mathcal{U}}(\Box\Diamond(Q \setminus F)) \;=\; 1 - \mathsf{Pr}^{\mathcal{P},\mathcal{U}}(\Diamond\Box F).$$

As the universality problem for PBA$^{>0}$ is undecidable, this shows the claim. $\Box$

8.2. Decidability Results

Before we show any decidability results for POMDP, we first prove that almost-sure repeated reachability and almost-sure reachability are interreducible for POMDP.

LEMMA 8.3. *The two following problems are reducible to each other:*

(1) *Given a POMDP* $(\mathcal{M}, \sim)$ *and a set of states* F*, is there an observation-based scheduler* $\mathcal{U}$ *with* $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Box\Diamond F) = 1$*?*
(2) *Given a POMPD* $(\mathcal{M}, \sim)$ *and a set of states* F*, is there an observation-based scheduler* $\mathcal{U}$ *with* $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Diamond F) = 1$*?*

PROOF.

"$\Leftarrow$" Problem (2) reduces to (1) in a straightforward manner: Given an instance for (2) we transform it into an instance for (1) by making all F-states absorbing, that is, by removing all outgoing edges from states in F, and adding self loops for all actions with probability one (to the states of F).

"$\Rightarrow$" We now show that problem (1) is reducible to problem (2). Let $(\mathcal{M}, \sim)$, F be an instance for (1). We define $\mathcal{M}'$ as follows: $\mathcal{M}'$ consists of a copy of $\mathcal{M}$ and some additional state s_{acc}. All transitions (s, α, s') in $\mathcal{M}$ with $s \notin F$ are left unchanged. The transitions (s, α, s') in $\mathcal{M}$ with $s \in F$ are kept, but their probabilities are divided by 2 in $\mathcal{M}'$. Moreover, for all $s \in F$ and $\alpha \in \mathsf{Act}$, we add a new transition $(s, \alpha, s_{\mathsf{acc}})$ with probability $\frac{1}{2}$. Finally, we add a self-loop with probability 1 to state s_{acc} for all action $\alpha \in \mathsf{Act}$. The transformation is depicted in Figure 13. The equivalence relation $\sim'$ on $S \,\dot\cup\, \{s_{\mathsf{acc}}\}$ agrees with $\sim$ on S and $\{s_{\mathsf{acc}}\}$ forms its own equivalence class, *i.e.*, $[s]_{\sim'} = [s]_\sim$ for $s \in S$ and $[s_{\mathsf{acc}}]_{\sim'} = \{s_{\mathsf{acc}}\}$. With $F' = \{s_{\mathsf{acc}}\}$, $(\mathcal{M}', \sim')$, F' is an instance for problem (2) satisfying the equivalence:

$$\exists \mathcal{U} \in \mathsf{Sched}^{(\mathcal{M},\sim)}.\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Box\Diamond F) = 1 \Leftrightarrow \exists \mathcal{U}' \in \mathsf{Sched}^{(\mathcal{M}',\sim')}.\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Diamond F') = 1.$$

Indeed, if F is visited almost surely infinitely often in $\mathcal{M}$ under the scheduler $\mathcal{U}$, F' will be almost surely visited in $\mathcal{M}'$ under the scheduler $\mathcal{U}'$ that mimics $\mathcal{U}$. That is, $\mathcal{U}'(\pi') = \mathcal{U}(\pi')$, if π' is not only a finite path in $\mathcal{M}'$ but also in $\mathcal{M}$ and $\mathcal{U}(\pi') = \alpha$ if $\mathsf{last}(\pi') = s_{\mathsf{acc}}$ (where $\alpha \in \mathsf{Act}$ is arbitrary). Note that all other cases (π' does not end in s_{acc} and is not a path in $\mathcal{M}$) are irrelevant.
Conversely, given $\mathcal{U}' \in \mathsf{Sched}^{(\mathcal{M}',\sim')}$ with $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Diamond F') = 1$, we define $\mathcal{U} \in \mathsf{Sched}^{(\mathcal{M},\sim)}$ to be the restriction of $\mathcal{U}'$ on the set of path of $\mathcal{M}$, that is, $\mathcal{U}(\pi) = \mathcal{U}'(\pi)$ for all $\pi \in \mathsf{Path}_{\mathsf{fin}}^{\mathcal{M}}$. Then, $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Box\Diamond F) = 1$, since $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Diamond\Box\neg F) > 0$ implies $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Box\neg F') > 0$. The last

claim is easy to see. We denote by $[(F)_{=j}(\neg F)_{>j}]$ the set of infinite paths π such that $\pi^j \in F$ and $\pi^k \notin F, k > j$. But then it holds that

$$\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Box\neg F') \geq \tfrac{1}{2} \cdot \mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}([(F)_{=j}(\neg F)_{>j}]) \geq \tfrac{1}{2^{j+1}} \cdot \mathsf{Pr}^{\mathcal{M},\mathcal{U}}([(F)_{=j}(\neg F)_{>j}]).$$

As $\{\pi \mid \pi \models \Diamond\Box\neg F\} = \dot\bigcup_{j\geq -1} [(F)_{=j}(\neg F)_{>j}]$, assuming $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Diamond\Box\neg F) > 0$ yields the existence of an index k, such that $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}([(F)_{=k}(\neg F)_{>k}]) > 0$ which, together with the above chain of inequalities (for j = k), shows the claim.
Note, that $\mathcal{U}$ and $\mathcal{U}'$ are of the same type, that is, $\mathcal{U}$ is deterministic (respectively, memoryless) if and only if $\mathcal{U}'$ is. $\Box$

Remark 8.4. Note that the construction in Figure 13 also ensures that

$$\forall\mathcal{U} \in \mathsf{Sched}^{(\mathcal{M},\sim)}.\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Box\Diamond F) = 1 \Leftrightarrow \forall\mathcal{U}' \in \mathsf{Sched}^{(\mathcal{M}',\sim')}.\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Diamond F') = 1.$$

Indeed, let us assume that there exists an observation-based scheduler $\mathcal{U}$ of $\mathcal{M}$ such that $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Box\Diamond F) < 1$. By Lemma 2.8, it follows that there exists an end component (T, A) of $\mathcal{M}$ with $T \cap F = \emptyset$ such that $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\{\pi \in \mathsf{Path}^{\mathcal{M}}_{\mathsf{inf}} \mid \mathsf{Lim}(\pi) = (T, A)\}) > 0$. This immediately shows that $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\{\pi \in \mathsf{Path}^{\mathcal{M}'}_{\mathsf{inf}} \mid \mathsf{Lim}(\pi) = (T, A)\}) > 0$ and therefore $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Diamond F') < 1$. Here, $\mathcal{U}'$ is the scheduler of $\mathcal{M}'$ that mimics $\mathcal{U}$ (as in the proof of Lemma 8.3). On the other hand, assume that there exists an observation-based scheduler $\mathcal{U}'$ of $\mathcal{M}'$ such that $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Diamond F') < 1$. Note that for each end component (T, A) of $\mathcal{M}'$, either $T = F'$ or $T \cap F = \emptyset$. By Lemma 2.8, it follows that there exists an end component (T, A) of $\mathcal{M}'$ with $F' \neq T$ and $T \cap F = \emptyset$ such that $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\{\pi \in \mathsf{Path}^{\mathcal{M}'}_{\mathsf{inf}} \mid \mathsf{Lim}(\pi) = (T, A)\}) > 0$. Thus, for the restriction $\mathcal{U}$ of $\mathcal{U}'$ to $\mathcal{M}$, we derive that $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\{\pi \in \mathsf{Path}^{\mathcal{M}}_{\mathsf{inf}} \mid \mathsf{Lim}(\pi) = (T, A)\}) > \mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\{\pi \in \mathsf{Path}^{\mathcal{M}'}_{\mathsf{inf}} \mid \mathsf{Lim}(\pi) = (T, A)\}) > 0$ and therefore $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Box\Diamond F) < 1$, which was to show.

By Lemma 8.3, we can reduce the almost-sure repeated reachability problem for POMDP to the almost-sure reachability problem for POMDP for which we now show decidability (see Alur et al. [1995] and Littman [1996] for related results).

THEOREM 8.5 (ALMOST-SURE REACHABILITY PROBLEM FOR POMDP). *Let a POMDP $(\mathcal{M}, \sim)$ and a state set $F \subseteq S$ be given. It is decidable, whether there exists an observation-based scheduler $\mathcal{U} \in \mathsf{Sched}^{(\mathcal{M},\sim)}$ such that $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Diamond F) = 1$.*

PROOF. We reduce the almost-sure reachability problem for POMDP to the almost-sure reachability problem for (fully observable) MDPs, which is known to be solvable by means of graph-algorithms. Let $\mathcal{M} = ((S, \mathsf{Act}, \delta, \mu), \sim)$ be a (without loss of generality, total) POMDP and $F \subseteq S$. Without loss of generality, we assume that the states in F are absorbing, that is, for all states $q \in F$, $\delta(q, \alpha, q) = 1$ for all $\alpha \in \mathsf{Act}$. We define an MDP $\mathcal{M}' = (S', \mathsf{Act}, \delta', \mu')$ as follows. The set of states S' of $\mathcal{M}$ consists of pairs (r, R) with $r \in R \subseteq [r]_\sim$ and an extra state q_F that has a self-loop with probability one for all $\alpha \in \mathsf{Act}$. Given $\alpha \in \mathsf{Act}$ and $R \subseteq S$, let $R' = \delta(R\backslash F, \alpha)$.
If $\delta(r, \alpha) \cap F = \emptyset$, then $\delta'((r, R), \alpha, (r', R' \cap [r']_\sim) = \delta(r, \alpha, r')$ for each $r' \in S$.
If $\delta(r, \alpha) \cap F \neq \emptyset$, then $\delta'((r, R), \alpha, (r', R' \cap [r']_\sim)) = \frac{1}{2\cdot|R'\backslash F|}$ for all $r' \in R' \backslash F$ and $\delta'((r, R), \alpha, q_F) = \frac{1}{2}$ (in case $R'\backslash F = \emptyset$, $\delta'((r, R), \alpha, q_F) = 1$).
Moreover $\mu'(q, [q]_\sim) = \mu(q)$ for all $q \notin F$ and $\mu'(q_F) = \Sigma_{r\in F}\mu(r)$. We set $F' = \{q_F\}$.
Before we show that this construction ensures that there exists an observation-based scheduler $\mathcal{U}$ of $\mathcal{M}$ with $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Diamond F) = 1$ if and only if there exists a scheduler $\mathcal{U}'$ of $\mathcal{M}'$ such that $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Diamond F') = 1$, we fix some notation. For each action α we define the set of pre-final states of $\mathcal{M}'$ as $F'_{\mathsf{pre}}(\alpha) = \{(r, R) \mid \delta(r, \alpha) \cap F \neq \emptyset\}$. So $F'_{\mathsf{pre}}(\alpha)$ is the set of states

($\neq q_F$) from which $\mathcal{M}'$ reaches its accepting state via the action α. Given a position in some path π we denote by $\mathsf{Next}_{\mathsf{Act}}$ the action that occurs after this position in π. So $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Box\Diamond(\vee_\alpha(F'_{\mathsf{pre}}(\alpha) \wedge \mathsf{Next}_{\mathsf{Act}} = \alpha)))$ denotes the probability under the scheduler $\mathcal{U}'$ of the set of paths in which infinitely often a pre-final state for some action α appears and is followed by the action α, that is, it denotes the value

$$\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}\big(\{\pi' \mid \overset{\infty}{\exists} i : \vee_\alpha(\pi'_i \in F'_{\mathsf{pre}}(\alpha) \wedge \mathsf{Act}_{i+1}(\pi') = \alpha)\}\big).$$

Similarly, $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Diamond\Box(\wedge_\alpha(\neg F'_{\mathsf{pre}}(\alpha) \vee \mathsf{Next}_{\mathsf{Act}} \neq \alpha)))$ denotes the probability under the scheduler $\mathcal{U}'$ of the set of paths for which from some point it holds that, whenever a pre-final state for some action α appears, then the following action is not α.

Now assume that there exists an observation-based scheduler $\mathcal{U} \in \mathsf{Sched}^{(\mathcal{M},\sim)}$ such that $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Diamond F) = 1$. We define $\mathcal{U}' \in \mathsf{Sched}^{\mathcal{M}'}$ as follows:

$$\mathcal{U}'((r_0, R_0) \xrightarrow{\alpha_1} (r_1, R_1) \ldots \xrightarrow{\alpha_n} (r_n, R_n)) \;=\; \mathcal{U}([r_0]_\sim \xrightarrow{\alpha_1} [r_1]_\sim \ldots \xrightarrow{\alpha_n} [r_n]_\sim)$$

We claim that $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Box\Diamond(\vee_\alpha(F'_{\mathsf{pre}}(\alpha) \wedge \mathsf{Next}_{\mathsf{Act}} = \alpha)) \;\vee\; \Diamond\Box q_F) = 1$. Assume the contrary. So $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Diamond\Box(\wedge_\alpha(\neg F'_{\mathsf{pre}}(\alpha) \vee \mathsf{Next}_{\mathsf{Act}} \neq \alpha)) \;\wedge\; \Box\Diamond\neg q_F) > 0$. As q_F is absorbing, this implies $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Diamond\Box(\wedge_\alpha(\neg F'_{\mathsf{pre}}(\alpha) \vee \mathsf{Next}_{\mathsf{Act}} \neq \alpha)) \;\wedge\; \Box\neg q_F) > 0$. Since $\mathcal{M}'$ is a finite state system, there exists a finite path $\tilde{\pi}' = (r_0, R_0), (r_1, R_1), \ldots, (r_n, R_n)$ of $\mathcal{M}'$ such that

$$\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\{\pi' \mid \pi'{\uparrow}^n = \tilde{\pi}' \;\wedge \pi' \models \Diamond^{=n}\Box(\wedge_\alpha(\neg F'_{\mathsf{pre}}(\alpha) \vee \mathsf{Next}_{\mathsf{Act}} \neq \alpha)) \;\wedge\; \Box\neg q_F\}) > 0.$$

Then,

$$\mathsf{Pr}_{(r_n,R_n)}^{\mathcal{M}',\mathcal{U}'_{\tilde{\pi}'}}(\Box(\wedge_\alpha(\neg F'_{\mathsf{pre}}(\alpha) \vee \mathsf{Next}_{\mathsf{Act}} \neq \alpha)) \;\wedge\; \Box\neg q_F) > 0,$$

where $\mathcal{U}'_{\tilde{\pi}'}(\hat{\pi}') = \mathcal{U}'(\tilde{\pi}'\hat{\pi}')$ for all finite paths $\hat{\pi}'$ with $\mathsf{first}(\hat{\pi}') = \mathsf{last}(\tilde{\pi}')$. For all other paths $\hat{\pi}'$ with $\mathsf{first}(\hat{\pi}') \neq \mathsf{last}(\tilde{\pi}')$, let $\mathcal{U}'_{\tilde{\pi}'}(\hat{\pi}')$ be defined arbitrarily. Note that

$$\mathsf{Pr}_{r_n}^{\mathcal{M},\mathcal{U}_{\tilde{\pi}}}(\Box\neg F) \geq \mathsf{Pr}_{(r_n,R_n)}^{\mathcal{M}',\mathcal{U}'_{\tilde{\pi}'}}(\Box(\wedge_\alpha(\neg F'_{\mathsf{pre}}(\alpha) \vee \mathsf{Next}_{\mathsf{Act}} \neq \alpha)) \;\wedge\; \Box\neg q_F) > 0 \qquad (+),$$

where $\tilde{\pi}$ is the state-wise projection of $\tilde{\pi}'$ to its first component, that is, $\tilde{\pi} = r_0, r_1, \ldots, r_n$. This implies

$$\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Box\neg F) \;\geq\; \underbrace{\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\{\pi \mid \pi{\uparrow}^n = \tilde{\pi}\})}_{>0} \cdot \underbrace{\mathsf{Pr}_{r_n}^{\mathcal{M},\mathcal{U}_{\tilde{\pi}}}(\Box\neg F)}_{>0} > 0,$$

which is a contradiction as we assumed $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Diamond F) = 1$. This shows our claim that $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Box\Diamond(\vee_\alpha(F'_{\mathsf{pre}}(\alpha) \wedge \mathsf{Next}_{\mathsf{Act}} = \alpha)) \;\vee\; \Diamond\Box q_F) = 1$. Inspecting the construction of $\mathcal{M}'$ it easily follows that $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Diamond\Box q_F) = 1$, so $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Diamond q_F) = 1$, which we wanted to show. It remains to show (+), that is

$$\mathsf{Pr}_{r_n}^{\mathcal{M},\mathcal{U}_{\tilde{\pi}}}(\Box\neg F) \geq \mathsf{Pr}_{(r_n,R_n)}^{\mathcal{M}',\mathcal{U}'_{\tilde{\pi}'}}(\Box(\wedge_\alpha(\neg F'_{\mathsf{pre}}(\alpha) \vee \mathsf{Next}_{\mathsf{Act}} \neq \alpha)) \;\wedge\; \Box\neg q_F).$$

Indeed, consider the infinite Markov chains $\mathcal{M}'_{\mathcal{U}'_{\tilde{\pi}'}}$ and $\mathcal{M}_{\mathcal{U}_{\tilde{\pi}}}$ that evolve when applying the scheduler $\mathcal{U}'_{\tilde{\pi}'}$ to $\mathcal{M}'$ and the scheduler $\mathcal{U}_{\tilde{\pi}}$ to $\mathcal{M}$. Then, the state-wise projection on the first component of each path π' of $\mathcal{M}'_{\mathcal{U}'_{\tilde{\pi}'}}$ is also a path of $\mathcal{M}_{\mathcal{U}_{\tilde{\pi}}}$. Moreover, the construction of $\mathcal{M}'$ ensures that if π' satisfies $\Box(\wedge_\alpha(\neg F'_{\mathsf{pre}}(\alpha) \vee \mathsf{Next}_{\mathsf{Act}} \neq \alpha)) \;\wedge\; \Box\neg q_F$[3],

[3]Note that the states of $\mathcal{M}'_{\mathcal{U}'_{\tilde{\pi}'}}$ are finite paths of $\mathcal{M}'$. A state $x_1, x_2, \ldots, x_n$ of $\mathcal{M}'_{\mathcal{U}'_{\tilde{\pi}'}}$ is said to satisfy a property, if the last $\mathcal{M}'$-state of its sequence, namely x_n, satisfies the property.

then the transition probabilities of π' in $\mathcal{M}'_{\mathcal{U}'_{\tilde{\pi}'}}$ agree with the transition probabilities of its projection in $\mathcal{M}_{\mathcal{U}_{\tilde{\pi}}}$. As the projection of each such path satisfies $\Box\neg F$, this shows (+).

We now show the other direction. So we assume that there exists a scheduler $\mathcal{U}'$ of $\mathcal{M}'$ such that $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}'}(\Diamond F') = 1$. We have to construct an observation-based scheduler $\mathcal{U} \in \mathsf{Sched}^{(\mathcal{M},\sim)}$ such that $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Diamond F) = 1$. Note that given a standard MDP $\tilde{\mathcal{M}}$ and a state set $\tilde{F}$, the existence of a scheduler under which $\tilde{\mathcal{M}}$ reaches $\tilde{F}$ almost surely also ensures the existence of a memoryless deterministic scheduler under which $\tilde{\mathcal{M}}$ reaches $\tilde{F}$ almost surely Hart et al. [1983] and Bianco and de Alfaro [1995]. So, without loss of generality, we assume that $\mathcal{U}'$ is memoryless and deterministic.

Let $S = S_1 \dot\cup \ldots \dot\cup S_n$ be the partition of the state set of $\mathcal{M}$ with respect to $\sim$, that is, for all $p \in S_i$ it holds that $[p]_\sim = S_i$. For each equivalence class S_i and each set $R \subseteq S_i$, we define a representative $p_i^R \in S_i$ such that the state (p_i^R, R) is reachable in $\mathcal{M}'$ (if possible). If no such state exists, the representative is undefined (R is then of no importance with respect to, to the equivalence class S_i). First, we define a new scheduler $\mathcal{U}''$ of $\mathcal{M}'$ that makes the same decision for states of $\mathcal{M}'$ that have a state of the same equivalence class in their first component and have the same second component. That is,

$$\mathcal{U}''((p, R)) \;:=\; \mathcal{U}'((p_i^R, R)),$$

where the index i is such that $p \in S_i$. Note that $R \subseteq [p]_\sim = [p_i^R]_\sim$ and that the scheduler $\mathcal{U}''$ is memoryless and deterministic. The construction of $\mathcal{M}'$ ensures that $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}''}(\Diamond F') = 1$. Now we define a scheduler $\mathcal{U}$ for $\mathcal{M}$ for all finite paths $p_0 \xrightarrow{\alpha_1} p_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_n} p_n$ of M with $p_1, \ldots, p_n \notin F$ (recall that the states in F are absorbing). For such a path, there is a unique corresponding run

$$(p_0, [p_0]_\sim) \xrightarrow{\alpha_1} (p_1, R_1) \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_n} (p_n, R_n)$$

in $\mathcal{M}'$. We define the scheduler $\mathcal{U}$ of $\mathcal{M}$ as

$$\mathcal{U}(p_0 \xrightarrow{\alpha_1} p_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_n} p_n) \;:=\; \mathcal{U}''((p_n, R_n)).$$

Note that $\mathcal{U}$ is not only an observation-based scheduler, but also $\mathsf{Pr}^{\mathcal{M},\mathcal{U}}(\Diamond F) = 1$. This can be seen as follows. Any infinite path of $\mathcal{M}$ that never visits the set F has a corresponding path in $\mathcal{M}'$. As $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}''}(\Diamond F') = 1$, such a path almost surely satisfies the condition that under the scheduler $\mathcal{U}''$ at infinitely many indices, the next action had the state q_F as a successor (since $\mathsf{Pr}^{\mathcal{M}',\mathcal{U}''}\big(\Box\Diamond\big(\vee_\alpha(F'_{\mathsf{pre}}(\alpha) \wedge \mathsf{Next}_{\mathsf{Act}} = \alpha)\big) \;\vee\; \Diamond\Box q_F\big) = 1$). But this means that the original path in $\mathcal{M}$ (which never visits F) almost surely satisfies the condition that under the scheduler $\mathcal{U}$ at infinitely many indices the next action had a successor in F. Since $\mathcal{M}$ is finite, all the transition probabilities are bounded by some $\varepsilon > 0$. This then ensures that the set of infinite paths never visiting F has measure zero under the scheduler $\mathcal{U}$. $\Box$

Our algorithm uses a powerset construction and hence runs in time exponential in the size of the given POMDP. However, given the EXPTIME-hardness results established by Reif [1984] and by Chatterjee et al. [2006] for 2-player games with incomplete information and by de Alfaro [1999] for POMDP, we do not expect more efficient algorithms.

Remark 8.6. Inspecting the proof of Theorem 8.5, we see that given a POMDP $(\mathcal{M}', \sim')$ and a state set F', the existence of a scheduler under which $\mathcal{M}'$ reaches F' almost surely also ensures the existence of a finite-memory deterministic scheduler under which $\mathcal{M}'$ reaches F' almost surely. But then the construction used in the proof of Lemma 8.3 ensures that given a POMDP $(\mathcal{M}'', \sim'')$ and a state set F'', the existence of a scheduler under which $\mathcal{M}''$ repeatedly reaches F'' almost surely also ensures the existence of a finite-memory deterministic scheduler under which $\mathcal{M}''$ repeatedly reaches F'' almost surely.

THEOREM 8.7 (DECIDABILITY RESULTS FOR POMDP). *Let a POMDP $(\mathcal{M}, \sim)$ and a state set $F \subseteq S$ be given. It is decidable,*

(a) *whether there exists an observation-based scheduler $\mathcal{U}$ for $(\mathcal{M}, \sim)$ such that $\mathrm{Pr}^{\mathcal{M},\mathcal{U}}(\Box\Diamond F) = 1$.*
(b) *whether there exists an observation-based scheduler $\mathcal{U}$ for $(\mathcal{M}, \sim)$ such that $\mathrm{Pr}^{\mathcal{M},\mathcal{U}}(\Diamond\Box F) > 0$.*

PROOF.

(a) The claim follows immediately from Lemma 8.3 together with Theorem 8.5.
(b) It holds that

$$\begin{array}{lll} \exists \mathcal{U} \in \mathsf{Sched}^{(\mathcal{M},\sim)} \text{ such that } & \mathrm{Pr}^{\mathcal{M},\mathcal{U}}(\Diamond\Box F) > 0 & \Leftrightarrow \\ \neg\big(\forall \mathcal{U} \in \mathsf{Sched}^{(\mathcal{M},\sim)}. & \mathrm{Pr}^{\mathcal{M},\mathcal{U}}(\Diamond\Box F) = 0\big) & \Leftrightarrow \\ \neg\big(\forall \mathcal{U} \in \mathsf{Sched}^{(\mathcal{M},\sim)}. & \mathrm{Pr}^{\mathcal{M},\mathcal{U}}(\Box\Diamond\neg F) = 1\big) & \overset{\text{(see Remark 8.4)}}{\Leftrightarrow} \\ \neg\big(\forall \mathcal{U}' \in \mathsf{Sched}^{(\mathcal{M}',\sim')}. & \mathrm{Pr}^{\mathcal{M}',\mathcal{U}'}(\Diamond F') = 1\big) & \Leftrightarrow \\ \exists \mathcal{U}' \in \mathsf{Sched}^{(\mathcal{M}',\sim')} \text{ such that } & \mathrm{Pr}^{\mathcal{M}',\mathcal{U}'}(\Diamond F') < 1 & \Leftrightarrow \\ \exists \mathcal{U}' \in \mathsf{Sched}^{(\mathcal{M}',\sim')} \text{ such that } & \mathrm{Pr}^{\mathcal{M}',\mathcal{U}'}(\Box\neg F') > 0. & \end{array}$$

The latter problem (confinement with positive probability: $\mathrm{Pr}^{\mathcal{U}}(\Box F) > 0$) has been proven to be EXPTIME-complete by de Alfaro [1999]. $\Box$

As PBA are a special case of POMDP, we can now show

THEOREM 8.8 (DECIDABILITY OF THE EMPTINESS PROBLEM FOR $\mathsf{PBA}^{=1}$). *Checking emptiness is decidable for PBA under the almost-sure semantics.*

PROOF. As PBA are a special case of POMDP, the claim is an immediate consequence of Theorem 8.7 and Remark 8.6 (since each deterministic scheduler can be seen as an input word). $\Box$

Remark 8.9. The decidability of the emptiness problem for PBA under the almost-sure semantics might be surprising at a first glance, since we proved earlier (see Theorem 4.11, page 18) that the language of a PBA (under the almost-sure semantics) depends on the exact probability distributions. However, whether the language is empty or not does not depend on the precise values of probability distributions. Indeed solving the almost-sure reachability problem for standard MDPs is done by means of graph algorithms [Hart et al. 1983; Vardi 1985; Courcoubetis and Yannakakis 1995] that do not take into account the precise transition probabilities (just whether they are $\neq 0$). It shows that the almost-sure repeated reachability problem for POMDP and therefore the emptiness problem for PBA under the almost-sure semantics do not depend on the precise transition probabilities. For each PBA $\mathcal{P}'$ that evolves from $\mathcal{P}$ by altering the transition probabilities in a legal way, that is, $\delta(s, \alpha, t) > 0$ if and only if $\delta'(s, \alpha, t) > 0$, if $\mathcal{L}^{=1}(\mathcal{P}) \neq \emptyset$, then there exists a word in $\mathcal{L}^{=1}(\mathcal{P})$ that is contained in $\mathcal{L}^{=1}(\mathcal{P}')$ for all such PBA $\mathcal{P}'$, that is,

$$\text{if } \mathcal{L}^{=1}(\mathcal{P}) \neq \emptyset, \text{ then } \bigcap_{\mathcal{P}'} \mathcal{L}^{=1}(\mathcal{P}') \neq \emptyset,$$

where $\mathcal{P}'$ ranges over all PBA evolving from $\mathcal{P}$ by legally altering the transition probabilities. This follows immediately from Remark 8.6, as $\mathcal{L}^{=1}(\mathcal{P}) \neq \emptyset$ ensures the existence of a finite-memory word w, such that $\Pr^{\mathcal{P}}(w) = 1$. The behavior of $\mathcal{P}$ under this finite-memory word can be described by a finite Markov chain $\mathcal{P}^w$ and the almost-sure acceptance of the word in $\mathcal{P}$ is equivalent to the almost-sure repeated reachability of a set F' in $\mathcal{P}^w$. As the latter does not depend on the exact transition probabilities of $\mathcal{P}^w$, but only on its underlying graph [Hart et al. 1983], this shows the claim.

Remark 8.10. Note that Chadha et al. [2009b] shows that the emptiness (respectively, universality) problem for $\mathrm{PBA}^{=1}$ with rational coefficients is PSPACE-complete. However it is also shown that the language containment problem for $\mathrm{PBA}^{=1}$ with rational coefficients is Σ^0_2-complete. It is moreover shown that given a PBA $\mathcal{P}$ with $\mathcal{L}^{=1}(\mathcal{P}) \neq \emptyset$ (respectively, $\mathcal{L}^{=1}(\mathcal{P}) \neq \Sigma^\omega$), then $\mathcal{L}^{=1}(\mathcal{P})$ (respectively, $\Sigma^\omega \setminus \mathcal{L}^{=1}(\mathcal{P})$) contains a lasso-shaped word (called ultimately periodic in Chadha et al. [2009b]), that is, a word of the type uv^ω with $u, v \in \Sigma^*$, $|v| \geq 1$.

9. CONCLUSION

We introduced and studied probabilistic ω-automata with Büchi, Rabin, and Streett acceptance conditions under several acceptance semantics, namely positive acceptance, almost-sure acceptance and threshold acceptance. An overview of the expressiveness of the different classes is depicted in Figure 10, page 35. In this context we established a couple of remarkable results. First, under the probable semantics PBA are more expressive than NBA. The analogue result is known for PFA and NFA, but PFA are equipped with a threshold $\lambda \in]0, 1[$ for the acceptance probability. Second, PBA can be exponentially more efficient than nondeterministic Streett automata. We are not aware of such a result for finite automata. As far as we know, the best known result to illustrate the efficiency of PFA in contrast to (non-probabilistic) finite automata has been established by Ambainis [1996] who proved the existence of a PFA with $\mathcal{O}(n)$ states while any equivalent DFA has $\Omega\big(2^{n/\log n}\big)$ states. Surprisingly, there is a polynomial transformation from probabilistic Streett to probabilistic Büchi automata, both under the probable semantics, which is impossible in the nondeterministic case [Safra and Vardi 1989]. As for nondeterministic ω-automata, complementation of PBA is difficult. We proposed a complementation operator that has some similarities with Safra's algorithm for the generation of an equivalent DRA for a given NBA [Safra 1988], but is simpler since the probabilistic setting permits to argue by means of sample runs that avoid the organization of the potential accepting runs in Safra-trees. However, the asymptotic bound is the same as in the nondeterministic case. The undecidability of the emptiness problem for $\mathrm{PBA}^{>0}$ has many important consequences, including undecidability results for stochastic games under incomplete information and ω-regular winning objectives and the undecidability of the model-checking problem for transition systems and (possiby non-ω-regular) linear time properties specified by PBA. In the introduction we mentioned several potential application areas, for example, security in multi-agent systems, where the model of PBA appears to be natural and could play a central role.

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