Quiz

0. Which of the following statements are correct?
   A. Σ_s is a union of subspaces of dimension s
   B. Σ_s is a union of subspaces of different dimensions
   C. Σ_s is a subspace of dimension s
1. Show that (1) $\Leftrightarrow$ (2)
2. Which of the following statements might be correct? What is your intuition? And why? (we will establish the proof of one of these statements)
   if $\exists \epsilon > 0$ s.t.
   A. M is (ϵ, s)-RIP, then M is a **good** sensing matrix (i.e. allows reconstruction).
   B. M is (ϵ, s)-RIP, then M is a **bad** sensing matrix.
   C. M is $(\epsilon, 2s)$-RIP, then M is a **good** sensing matrix.
   D. M is $(\epsilon, 2s)$-RIP, then M is a **bad** sensing matrix.
3. Prove Lemma RIP and operator norm. To do so,
   A. **(easy)** first show that if M is (ϵ, s)-RIP, then

$$\sup_{x_S \neq 0} \frac{||M_S x_S||_2^2 - ||x_S||_2^2}{||x_S||_2^2} \leq \epsilon$$

   B. **(advanced)** then show that

$$\sup_{x_S \neq 0} \frac{||M_S x_S||_2^2 - ||x_S||_2^2}{||x_S||_2^2} = \max_{x_S \neq 0} \frac{||(M_S^T M_S - I)x_S||_2}{||x_S||_2}.$$

   C. **(easy)** conclude with

$$\max_{x_S \neq 0} \frac{||(M_S^T M_S - I)x_S||_2}{||x_S||_2} = ||M_S^T M_S - I||_{op}.$$

4. Which of the following statements are correct? If M is (ϵ, s)-RIP, then
   A. $\forall S, M_S^T M_S \approx I$ when applied to any x_S
   B. $M^T M \approx I$ when applied to any s-sparse vector
   C. $M^T M \approx I$ when applied to any n- length vector
5. Which of the following statements are correct?
   A. The theorem is a positive result: RIP guarantees the success of IHT.
   B. The update rule of IHT ($x^l \to x^{l+1}$) is a contraction mapping
   C. $3s$ is a typo. Should be s.
   D. $3s$ is a typo. Should be $2s$.
6. Prove Theorem RIP is good for IHT. To do so, let us denote

$$u^l = x^l + M^T(y - Mx^l) = x^l + M^T M(x - x^l) \quad (1)$$
$$x^{l+1} = H_s(u^l)$$

   - First show that $\forall$ s-sparse vector x,

$$||u^l - x^{l+1}||^2 \leq ||u^l - x||^2 \quad (2)$$

   - Explain all equalities and inequalities below

$$||(u^l - x) - (x^{l+1} - x)||^2 \stackrel{(a)}{=} ||u^l - x||^2 + ||x^{l+1} - x||^2 - 2\langle (u^l - x), (x^{l+1} - x)\rangle$$
$$\stackrel{(b)}{\leq} ||u^l - x||^2$$
$$||x^{l+1} - x||^2 \stackrel{(c)}{\leq} 2\langle (u^l - x), (x^{l+1} - x)\rangle \stackrel{(d)}{=} 2\langle (I - M^T M)(x^l - x), (x^{l+1} - x)\rangle \quad (3)$$

   - We now want to show that

$$\langle (I - M^T M)(x^l - x), (x^{l+1} - x)\rangle \leq \epsilon \; ||x^l - x|| \; ||x^{l+1} - x|| \quad (4)$$

To do so, let us denote

$$
\begin{aligned}
u &= x^l - x \\
v &= x^{l+1} - x \\
T &= supp(u) \cup supp(v) \\
&\subset supp(x^l) \cup supp(x) \cup supp(x^{l+1}) \\
|T| &\leq 3s
\end{aligned}
$$

Explain all equalities and inequalities below

$$
\begin{aligned}
\langle (I - M^T M)u, v\rangle &\overset{(d)}{=} u_T^T \ (I - M_T^T M_T) \ v_T \\
&\overset{(e)}{\leq} ||(I - M_T^T M_T)u_T||_2 \ ||v_T||_2 \\
&\overset{(f)}{\leq} ||I - M_T^T M_T||_{op} \ ||u_T||_2 \ ||v_T||_2 \\
&\overset{(g)}{\leq} \epsilon \ ||u_T||_2 \ ||v_T||_2
\end{aligned}
$$

which shows (4).

- Now, from (4) and (3), we have

$$
\begin{aligned}
||x^{l+1} - x||^2 &\leq \epsilon \ ||x^l - x|| \ ||x^{l+1} - x|| \\
||x^{l+1} - x|| &\leq 2\epsilon \ ||x^l - x|| \qquad (5)
\end{aligned}
$$

Conclude, by showing that if $2\epsilon < 1$, then $x^l \xrightarrow[l \to +\infty]{} x$.

$$
\begin{aligned}
x^0 &= 0 \\
x^{l+1} &= H_s\left(x^l + M^T(y - Mx^l)\right) \\
\hat{x} &= x^l
\end{aligned}
$$

7. The goal of this quiz is to explain why (5) is called a concentration inequality.
   Let $y = Mx$. Compute the distribution of y_i and of $||y||^2$.
   Explain now why this is called concentration inequality.
8. Spot the differences between the two statements.
9. Proof of the Johnson Lindenstrauss lemma. Fill in when there is ??

$$
\mathbb{P}_M\left(\sup_{x\in\mathcal{Q}}\left|\frac{||Mx||_2^2}{||x||_2^2} - 1\right| \leq t\right) = \mathbb{P}_M\left(??\forall \ or \exists ?? \ x \in \mathcal{Q}, \left|\frac{||Mx||_2^2}{||x||_2^2} - 1\right| \leq t\right) \qquad (6)
$$

$$
= 1 - \mathbb{P}_M\left(??\right) =: 1 - p \qquad (7)
$$

$$
p \leq \sum_{x\in\mathcal{Q}} \mathbb{P}_M\left(??\right) =: \sum_{x\in\mathcal{Q}} p_x \qquad (8)
$$

What is p_x?
Show that

$$
\mathbb{P}_M\left(\sup_{x\in\mathcal{Q}}\left|\frac{||Mx||_2^2}{||x||_2^2} - 1\right| > t\right) \leq \sum_{x\in\mathcal{Q}} p_x \qquad (9)
$$

$$
\leq |\mathcal{Q}| 2 e^{-\frac{m\epsilon^2}{6}} \leq \delta \qquad (10)
$$

Therefore, if $m \geq ??$ then ??.

Solution:

$$
\mathbb{P}_M\left(\sup_{x\in\mathcal{Q}}\left|\frac{||Mx||_2^2}{||x||_2^2} - 1\right| \leq t\right) = \mathbb{P}_M\left(\forall \ x \in \mathcal{Q}, \left|\frac{||Mx||_2^2}{||x||_2^2} - 1\right| \leq t\right) \qquad (11)
$$

$$
= 1 - \mathbb{P}_M\left(\exists \ x \in \mathcal{Q}, \left|\frac{||Mx||_2^2}{||x||_2^2} - 1\right| \rangle t\right) =: 1 - p \qquad (12)
$$

$$
p \leq \sum_{x\in\mathcal{Q}} \mathbb{P}_M\left(\left|\frac{||Mx||_2^2}{||x||_2^2} - 1\right| \rangle t\right) =: \sum_{x\in\mathcal{Q}} p_x \qquad (13)
$$

What is p_x? a concentration inequality
From the derivation above, and from the fact that the distribution of $||Mx||_2^2$ is sub-exponential, we have

$$\mathbb{P}_M\left(\sup_{x\in\mathcal{Q}}\left|\frac{||Mx||_2^2}{||x||_2^2}-1\right|>t\right)\leq\sum_{x\in\mathcal{Q}}p_x \tag{14}$$

$$\leq|\mathcal{Q}|2e^{-\frac{m\epsilon^2}{6}}\leq\delta \tag{15}$$

Therefore, if $m\geq\frac{6}{t^2}\log\frac{2|\mathcal{Q}|}{\delta}$, then

$$\mathbb{P}_M\left(\sup_{x\in\mathcal{Q}}\left|\frac{||Mx||_2^2}{||x||_2^2}-1\right|\leq t\right)\geq 1-\delta. \tag{16}$$

10. Covering argument.
Let $\rho\geq 0$. Consider that $\mathcal{Q}$ allows to cover $\mathcal{S}_1(\mathbb{R}^s)$ (unit ball in $\mathbb{R}^s$) i.e.

$$\sup_{x:||x||_2=1}\min_{q\in\mathcal{Q}}||x-q||_2\leq\rho$$

We look for the smallest set $\mathcal{Q}$. Which of the following statements are correct?
$\exists\mathcal{Q}\subset\mathcal{S}_1(\mathbb{R}^s)$ s.t.
 A. $\mathcal{Q}$ is finite
 B. $\mathcal{Q}$ grows exponentially with s