Quiz 2 - (Part3)

1. Show that (1) $\Leftrightarrow$ (2)

2. Which of the following statements might be correct? What is your intuition? And why? (we will establish the proof of one of these statements)
   if $\exists \epsilon > 0$ s.t.

   A. M is (ϵ, s)-RIP, then M is a **good** sensing matrix (i.e. allows reconstruction).

   B. M is (ϵ, s)-RIP, then M is a **bad** sensing matrix.

   C. M is $(\epsilon, 2s)$-RIP, then M is a **good** sensing matrix.

   D. M is $(\epsilon, 2s)$-RIP, then M is a **bad** sensing matrix.

3. Prove Lemma RIP and operator norm. To do so,

   A. **(easy)** first show that if M is (ϵ, s)-RIP, then

   $$\max_{x_S \neq 0} \frac{||M_S x_S||_2^2 - ||x_S||_2^2}{||x_S||_2^2} \leq \epsilon$$

   B. **(advanced)** then show that

   $$\max_{x_S \neq 0} \frac{||M_S x_S||_2^2 - ||x_S||_2^2}{||x_S||_2^2} = \max_{x_S \neq 0} \frac{||(M_S^T M_S - I)x_S||_2}{||x_S||_2}.$$

   C. **(easy)** conclude with

   $$\max_{x_S \neq 0} \frac{||(M_S^T M_S - I)x_S||_2}{||x_S||_2} = ||M_S^T M_S - I||_{op}.$$

4. Which of the following statements are correct? If M is (ϵ, s)-RIP, then

   A. $\forall S, M_S^T M_S \approx I$ when applied to any x_S

   B. $M^T M \approx I$ when applied to any s-sparse vector

   C. $M^T M \approx I$ when applied to any n- length vector

5. Which of the following statements are correct?

   A. The theorem is a positive result: RIP guarantees the success of IHT.

   B. The update rule of IHT ($x^l \to x^{l+1}$) is a contraction mapping

   C. $3s$ is a typo. Should be s.

   D. $3s$ is a typo. Should be $2s$.

6. Prove Theorem RIP is good for IHT. To do so, let us denote

   $$\begin{aligned} u^l &= x^l + M^T(y - Mx^l) = x^l + M^T M(x - x^l) \quad (1) \\ x^{l+1} &= H_s(u^l) \end{aligned}$$

   - First show that $\forall$ s-sparse vector x,

   $$||u^l - x^{l+1}||^2 \leq ||u^l - x||^2 \quad (2)$$

- Explain all equalities and inequalities below

$$
\begin{aligned}
||(u^l - x) - (x^{l+1} - x)||^2 &\overset{(a)}{=} ||u^l - x||^2 + ||x^{l+1} - x||^2 - 2\langle (u^l - x), (x^{l+1} - x)\rangle \\
&\overset{(b)}{\leq} ||u^l - x||^2 \\
||x^{l+1} - x||^2 &\overset{(c)}{\leq} 2\langle (u^l - x), (x^{l+1} - x)\rangle \overset{(d)}{=} 2\langle (I - M^T M)(x^l - x), (x^{l+1} - x)\rangle
\end{aligned} \tag{3}
$$

- We now want to show that

$$
\langle (I - M^T M)(x^l - x), (x^{l+1} - x)\rangle \leq \epsilon \; ||x^l - x|| \; ||x^{l+1} - x|| \tag{4}
$$

To do so, let us denote

$$
\begin{aligned}
u &= x^l - x \\
v &= x^{l+1} - x \\
T &= supp(u) \cup supp(v) \\
&\subset supp(x^l) \cup supp(x) \cup supp(x^{l+1}) \\
|T| &\leq 3s
\end{aligned}
$$

Explain all equalities and inequalities below

$$
\begin{aligned}
\langle (I - M^T M)u, v\rangle &\overset{(d)}{=} u_T^T \; (I - M_T^T M_T) \; v_T \\
&\overset{(e)}{\leq} ||(I - M_T^T M_T)u_T||_2 \; ||v_T||_2 \\
&\overset{(f)}{\leq} ||I - M_T^T M_T||_{op} \; ||u_T||_2 \; ||v_T||_2 \\
&\overset{(g)}{\leq} \epsilon \; ||u_T||_2 \; ||v_T||_2
\end{aligned}
$$

which shows (4).

- Now, from (4) and (3), we have

$$
\begin{aligned}
||x^{l+1} - x||^2 &\leq \epsilon \; ||x^l - x|| \; ||x^{l+1} - x|| \\
||x^{l+1} - x|| &\leq 2\epsilon \; ||x^l - x||
\end{aligned} \tag{5}
$$

Conclude, by showing that if $2\epsilon < 1$, then $x^l \xrightarrow[l \to +\infty]{} x$.

$$
\begin{aligned}
x^0 &= 0 \\
x^{l+1} &= H_s\left(x^l + M^T(y - Mx^l)\right) \\
\hat{x} &= x^l
\end{aligned}
$$

7. The goal of this quiz is to explain why (5) is called a concentration inequality.
   Let $y = Mx$. Compute the distribution of y_i and of $||y||^2$.
   Explain now why this is called concentration inequality.

8. Spot the differences between the two statements.

9. Proof of the Johnson Lindenstrauss lemma. Fill in when there is ??

$$\mathbb{P}_M\left(\sup_{x\in\mathcal{Q}}\left|\frac{||Mx||_2^2}{||x||_2^2}-1\right|\leq t\right)=\mathbb{P}_M\left(??\forall\ or \exists??\ x\in\mathcal{Q},\left|\frac{||Mx||_2^2}{||x||_2^2}-1\right|\leq\epsilon\right) \tag{6}$$

$$=1-\mathbb{P}_M\left(??\right)=:1-p \tag{7}$$

$$p\leq\sum_{x\in\mathcal{Q}}\mathbb{P}_M\left(??\right)=:\sum_{x\in\mathcal{Q}}p_x \tag{8}$$

What is p_x?
Show that

$$\mathbb{P}_M\left(\sup_{x\in\mathcal{Q}}\left|\frac{||Mx||_2^2}{||x||_2^2}-1\right|>t\right)\leq\sum_{x\in\mathcal{Q}}p_x \tag{9}$$

$$\leq|\mathcal{Q}|2e^{-\frac{m\epsilon^2}{6}}\leq\delta \tag{10}$$

Therefore, if $m\geq$?? then ??.

10. Covering argument.
Let $\rho\geq 0$. Consider that $\mathcal{Q}$ allows to cover $\mathcal{S}_1(\mathbb{R}^s)$ (unit ball in $\mathbb{R}^s$) i.e.

$$\sup_{x:||x||_2=1}\min_{q\in\mathcal{Q}}||x-q||_2\leq\rho$$

We look for the smallest set $\mathcal{Q}$. Which of the following statements are correct?
$\exists\mathcal{Q}\subset\mathcal{S}_1(\mathbb{R}^s)$ s.t.

A. $\mathcal{Q}$ is finite

B. $\mathcal{Q}$ grows exponentially with s