CONVERGENCE ANALYSIS OF THE MAP TURBO DETECTOR: NEW REPRESENTATION AND CONVERGENCE TO THE AWGN CASE

Noura Sellami[1*], *Aline Roumy*[2], *and Inbar Fijalkow* [3]

[1] ISECS, Route Menzel Chaker km 0.5, B.P 868, 3018 Sfax, Tunisia
Email: sellami_noura@yahoo.fr, Tel: (216)97059318, Fax: (216)74274088
[2] IRISA-INRIA, Campus de Beaulieu, 35042 Rennes Cedex, France
Email: aline.roumy@irisa.fr, Tel: (33)299847394, Fax: (33)299847171
[3] ETIS, UMR 8051, ENSEA-UCP-CNRS,
6 av. du Ponceau, 95014 Cergy-Pontoise, France
Email: fijalkow@ensea.fr, Tel: (33)130736610, Fax: (33)130736627

ABSTRACT

In this paper, we consider a coded transmission over a frequency selective channel. In [1, 2], we studied analytically the impact of *a priori* information on the MAP equalizer performance and gave the expression of the extrinsic Log Likelihood Ratios (LLRs) at its output. Based on these results, we propose in this paper to study the convergence of the turbo equalizer using a Maximum *a posteriori* (MAP) equalizer and a MAP decoder. We give an analysis of the decoder performance when it is provided with *a priori*. Then, we propose a new representation space for the convergence analysis of the turbo equalizer. This representation is interesting since for the equalizer it is independent of the signal to noise ratio (SNR). We also show that the performance of the turbo equalizer converges at high SNR to the Additive White Gaussian Noise (AWGN) channel performance under a certain condition on the channel and the code that we will derive.

1. INTRODUCTION

An important source of degradation in high data rate communication systems is the presence of intersymbol interference (ISI) between consecutive data symbols which is due to the frequency selectivity of mobile radio channels. In order to improve the quality of the transmission, an error correction code is generally used on top of an equalizer. At the receiver, a solution achieving a good complexity/performance trade-off is to use an iterative receiver constituted of a soft-input soft-output (SISO) equalizer and a SISO decoder [3], following the idea of turbo-codes [4]. The basic idea behind iterative processing is to exchange extrinsic information among the equalizer and the decoder in order to achieve successively refined performance. The optimal SISO algorithm, in the sense of minimum bit error rate (BER), to be used for equalization and decoding is the symbol MAP algorithm [5]. Hence, in this paper, we consider an iterative receiver composed of a MAP equalizer and a MAP decoder (see figure 2). Our aim is to perform the convergence analysis of this iterative receiver. Actually, most analyses are based on extrinsic information transfer (EXIT) charts [6, 7]. These analyses use generally simulations since it is difficult to study analytically the performance of the MAP algorithm having a large number of states. In [1], we analysed the impact of the *a priori* information on the MAP equalizer performance. We assumed that the *a priori* observations at the input of the equalizer are modeled as the outputs of an AWGN channel. Since the equalizer and the decoder are exchanging extrinsic LLRs, we studied in [2] the expression of the distribution of these extrinsic LLRs at the output of the equalizer. In this paper, we perform an analytical analysis of the decoder performance when it is provided with *a priori*. We then propose a new representation space of the convergence analysis of the turbo equalizer. This representation is interesting since for the equalizer it is independent of the SNR. The analysis we propose is analytical at high SNR. At low SNR, the analysis is semi analytical. Indeed, a simulation of the decoder has to be done. Our analysis of the turbo equalizer allows to predict the convergence point which corresponds in most cases to the AWGN performance. We will give analytically the condition on the channel and the code under which we have this convergence to the AWGN performance at high SNR.

The paper is organized as follows. In section 2, we describe the system model. In section 3, we present the iterative receiver. In section 4, we present the analysis of the equalizer. In section 5, we analyse the performance of the MAP decoder with *a priori*. In section 6, we perform the convergence analysis of the turbo equalizer. In section 7, we give simulation results.Throughout this paper scalars are lower case and vectors are underlined lower case. $(.)^T$ denotes the transposition.

2. SYSTEM MODEL

We consider a coded data transmission system over a frequency selective channel depicted in figure 1. The input information bit sequence is first encoded with a convolutional encoder. The output of the encoder is interleaved, mapped to the symbol alphabet $\mathscr{A}$. For simplicity, we will consider only the BPSK modulation ($\mathscr{A} = \{+1, -1\}$). We assume that transmissions are organized into bursts of T symbols. The channel is supposed to be invariant during one burst. The received baseband signal sampled at the symbol rate at time k is

$$x_k = \sum_{l=0}^{L-1} h_l s_{k-l} + n_k \quad (1)$$

where L is the channel memory. In this expression, n_k are modeled as independent random variables of a real white Gaussian noise with normal probability density function (pdf) $\mathscr{N}(0, \sigma^2)$ where $\mathscr{N}(\alpha, \sigma^2)$ denotes a Gaussian distribution with mean α and variance σ^2. The term h_l is the l^{th} tap gain of the channel, which is assumed to be real valued. The optimal receiver for this coded system performs joint equalization and decoding treating the concatenation of the encoder and the ISI channel as one code. However, the complexity of this receiver is in general prohibitive especially when an interleaver is used. A solution achieving a good complexity/performance trade-off is to use an iterative receiver constituted of a soft-input soft-output (SISO) equalizer and a SISO decoder [3]. We present in the following the iterative receiver and propose to analyse its convergence.

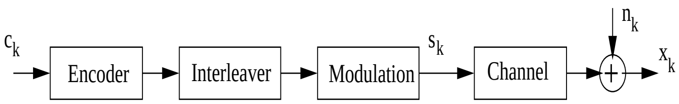

Figure 1: Transmitter structure

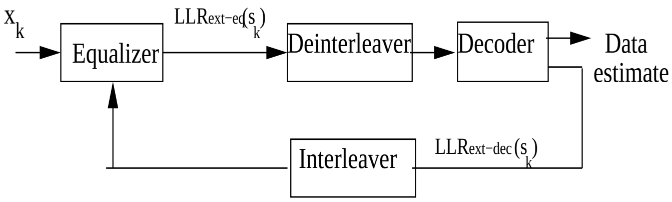

Figure 2: Receiver structure

3. ITERATIVE RECEIVER

We assume that the channel is perfectly known at the receiver. As shown in figure 2, the receiver consists of SISO processors, the equalizer and the decoder. We consider only the MAP approach for both equalization and decoding, using the BCJR algorithm [5]. The MAP equalizer computes the *a posteriori* probabilities (APPs) on the coded bits, $P(s_k = s|\underline{x})$, $s \in \mathscr{A}$, $1-L \leq k \leq T-1$, $\underline{x} = (x_{T-1},...,x_0)^T$, and outputs the extrinsic log-likelihood ratios (LLRs) [7]:

$$\begin{aligned} LLR_{ext\_eq}(s_k) &= LLR_{eq}(s_k) - LLR_{priori\_eq}(s_k) \\ &= \log\frac{P(s_k=+1|\underline{x})}{P(s_k=-1|\underline{x})} - \log\frac{P(s_k=+1)}{P(s_k=-1)} \end{aligned} \quad (2)$$

which are the *a posteriori* LLRs $LLR_{eq}(s_k)$ minus the *a priori* $LLR_{priori\_eq}(s_k)$ provided by the decoder. At the first receiver iteration, $LLR_{priori\_eq}(s_k) = 0$ since no *a priori* information is available. The LLRs $LLR_{ext\_eq}(s_k)$ are then deinterleaved and provided to the decoder as input *a priori* information, $LLR_{priori\_dec}(s_k)$, in order to refine its calculations. The MAP decoder computes the APPs $P(s_k = s|\underline{r})$, $\underline{r} = (LLR_{ext\_eq}(s_{1-L}), \cdots, LLR_{ext\_eq}(s_{T-1}))^T$, and outputs the LLRs

$$\begin{aligned} LLR_{ext\_dec}(s_k) &= LLR_{dec}(s_k) - LLR_{priori\_dec}(s_k) \\ &= \log\frac{P(s_k=+1|\underline{r})}{P(s_k=-1|\underline{r})} - \log\frac{P(s_k=+1)}{P(s_k=-1)}. \end{aligned}$$

These LLRs are then interleaved and provided to the equalizer as *a priori*, $LLR_{priori\_eq}(s_k)$, at the next iteration. After some iterations, hard decisions are taken on the information bits by the decoder.

4. ANALYSIS OF THE EQUALIZER

In order to perform the convergence analysis of the turbo equalizer, we first recall the results of [1, 2]. Then, we test for the validity of these results by simulations. We suppose that the *a priori* observations at the input of the equalizer are modeled as the outputs of an AWGN channel with zero mean and variance $\sigma^2_{eq,in}$. Hence, the *a priori* LLRs are modeled as independent and identically distributed (i.i.d) samples from a random variable with the conditional pdf $\mathscr{N}\left(\frac{2s_k\mu^2_{eq,in}}{\sigma^2}, \frac{4\mu^2_{eq,in}}{\sigma^2}\right)$ where

$$\mu_{eq,in} \triangleq \frac{\sigma}{\sigma_{eq,in}}.$$

This assumption is classically taken in the analysis of iterative receivers [6, 7]. We consider here the case of short channels ($L \leq 6$) since we can derive simple analytical expressions in this case [1].

We distinguish two cases: the case of unreliable *a priori* and the case of reliable *a priori*. Let d_{min} be the channel minimum distance [8]. The limit value $\mu_{eq-\lim}$ of $\mu_{eq,in}$ between both cases is [2]

$$\mu_{eq\_\lim} = \sqrt{1 - \frac{d^2_{\min}}{4}} \quad (3)$$

and is independent of the SNR. When the *a priori* is not reliable ($\mu_{eq,in} < \mu_{eq-\lim}$), the overall probability of error can be approximated at high SNR by [1]:

$$P_e \simeq Q\left(\frac{\sqrt{d^2_{\min} + 8\mu^2_{eq,in}}}{2\sigma}\right). \quad (4)$$

where $Q(\alpha) = \frac{1}{\sqrt{2\pi}}\int_\alpha^\infty \exp(-\frac{y^2}{2})dy$. The distribution of the extrinsic LLRs in this case can be approximated by [2]

$$LLR_{ext\_eq}(s_k) \sim \mathscr{N}\left(s_k\left(\frac{d^2_{\min} + 4\mu^2_{eq,in}}{2\sigma^2}\right), \left(\frac{d^2_{\min} + 4\mu^2_{eq,in}}{\sigma^2}\right)\right) \quad (5)$$

When the *a priori* is reliable ($\mu_{eq,in} > \mu_{eq-\lim}$), the overall probability of error can be approximated at high SNR by [1]:

$$P_e \simeq Q\left(\frac{\sqrt{1 + \mu^2_{eq,in}}}{\sigma}\right) \quad (6)$$

and the distribution of the extrinsic LLRs can be approximated by [2]

$$LLR_{ext\_eq}(s_k) \sim \mathscr{N}\left(s_k\left(\frac{2}{\sigma^2}\right), \frac{4}{\sigma^2}\right). \quad (7)$$

We notice that for good *a priori* information, the extrinsic LLRs are equivalent to the LLRs corresponding to the AWGN channel with zero mean and variance σ^2. Hence, the effect of the ISI is eliminated. In a turbo equalizer, the equalizer provides the decoder with the extrinsic LLRs $LLR_{ext\_eq}(s_k)$, at each iteration. Hence, if the *a priori* information become reliable, the performance of the decoder (which is also the performance of the turbo equalizer) is equivalent to the performance of the coded AWGN channel. In section 6, we will give the condition under which the performance of the turbo equalizer converges to the AWGN performance.

We can define $\mu_{eq,out}$ such as the LLRs $LLR_{ext\_eq}(s_k)$ are modeled as i.i.d samples from a random variable with the conditional pdf $\mathcal{N}(\frac{2s_k\mu^2_{eq,out}}{\sigma^2}, \frac{4\mu^2_{eq,out}}{\sigma^2})$. The quantity $\mu^2_{eq,out}$ is a function of $\mu^2_{eq,in}$ defined as

$$\mu^2_{eq,out} = f_{eq}(\mu^2_{eq,in})$$

with

$$\begin{cases} f_{eq}(\mu^2_{eq,in}) \triangleq \frac{d^2_{\min}+4\mu^2_{eq,in}}{4}, \text{ if } \mu_{eq,in} < \sqrt{1-\frac{d^2_{\min}}{4}} \\ f_{eq}(\mu^2_{eq,in}) \triangleq 1, \text{ elsewhere.} \end{cases} \tag{8}$$

Simulation results

We want to test for the validity of the results given in this section. Hence, we do not consider the channel coding and the turbo equalizer yet. In the simulations, the modulation used is the BPSK and the channel is assumed to be known at the receiver. We consider the channel with the impulse response (0.29;0.50;0.58;0.50;0.29). The minimum distance of the channel is 1.0532 [9] and $\mu_{eq-\lim} = 0.866$.

We provide the equalizer with Gaussian *a priori* LLRs with the conditional pdf $\mathcal{N}(\frac{2s_k\mu^2_{eq,in}}{\sigma^2}, \frac{4\mu^2_{eq,in}}{\sigma^2})$, for a given $\mu_{eq,in} = \frac{\sigma}{\sigma_{eq,in}}$. Figure 3 shows $\mu^2_{eq,out}$ versus $\mu^2_{eq,in}$. The dotted curves are obtained by simulations for different values of the SNR. The solid curve is the theoretical curve obtained using (8). Notice that the theoretical curve does not depend on the SNR, however the curves obtained by simulations vary with the SNR. We can see that the theoretical curve approximates well the curves given by simulations at high SNR. At low SNR, the approximation becomes less accurate. We notice that for $\mu_{eq,in} >> \mu_{eq-\lim}$ and $\mu_{eq,in} << \mu_{eq-\lim}$, the analytical curve approximates well the curves obtained by simulations. Around the limit value $\mu_{eq-\lim}$ the approximation is less accurate. This is due to the fact that in the analyses of the equalizer performance in [1], we assumed that isolated errors dominate the probability of error when the *a priori* is reliable and double errors dominate it when the *a priori* is not reliable. However, when $\mu_{eq,in}$ is close to $\mu_{eq-\lim}$, isolated errors and double errors occur and the error probability becomes then a combination of the probabilities of these two types of error events. Hence, the expressions (4) and (6) do not approximate well the error probability in this case.

5. ANALYSIS OF THE DECODER

In order to perform the whole analytical study of the iterative receiver convergence, we have to find analytically the distribution of the extrinsic outputs of the decoder. The MAP decoder outputs *a posteriori* LLRs on the coded bits and on the information bits. The *a posteriori* LLRs on the coded symbols s_k can be written as the sum of the extrinsic LLRs and the *a priori* LLRs as

$$LLR_{dec}(s_k) = LLR_{ext\_dec}(s_k) + LLR_{priori\_dec}(s_k).$$

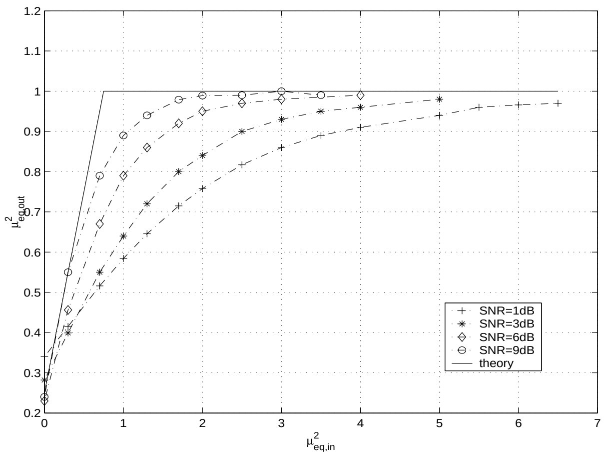

Figure 3: Equalizer analysis: $\mu^2_{eq,out}$ versus $\mu^2_{eq,in}$.

We suppose that the *a priori* observations at the input of the decoder are modeled as the outputs of an AWGN channel with zero mean and variance $\sigma^2_{dec,in}$. Hence, the *a priori* LLRs are modeled as i.i.d samples from a random variable with the conditional pdf $\mathcal{N}(\frac{2s_k\mu^2_{dec,in}}{\sigma^2}, \frac{4\mu^2_{dec,in}}{\sigma^2})$ where

$$\mu_{dec,in} \triangleq \frac{\sigma}{\sigma_{dec,in}}.$$

Let d_{free} be the code minimum distance. At high SNR, the overall probability of error can be approximated by

$$P_e \simeq Q\left(\frac{\sqrt{d_{free}}\mu_{dec,in}}{\sigma}\right). \tag{9}$$

It is equivalent to the performance of an AWGN channel with a noise having a zero mean and variance $\sigma_3^2 \triangleq \frac{\sigma^2}{d_{free}\mu^2_{dec,in}}$. Hence, the *a posteriori* LLRs at the output of the decoder can be modeled as i.i.d samples from a random variable with pdf $\mathcal{N}(\frac{2s_k}{\sigma_3^2}, \frac{4}{\sigma_3^2})$.

Since the *a priori* and extrinsic LLRs are independent by construction, we obtain

$$LLR_{ext\_dec}(s_k) \sim \mathcal{N}\left(s_k\left(\frac{2}{\sigma_3^2} - \frac{2\mu^2_{dec,in}}{\sigma^2}\right), \left(\frac{4}{\sigma_3^2} - \frac{4\mu^2_{dec,in}}{\sigma^2}\right)\right).$$

Hence

$$LLR_{ext\_dec}(s_k) \sim \mathcal{N}\left(s_k\left(\frac{2\mu^2_{dec,in}(d_{free}-1)}{\sigma^2}\right), \frac{4\mu^2_{dec,in}(d_{free}-1)}{\sigma^2}\right). \tag{10}$$

As for the equalizer, we can define $\mu^2_{dec,out}$ such as the LLRs $LLR_{ext\_dec}(s_k)$ are modeled as i.i.d samples from a random variable with the conditional pdf $\mathcal{N}(\frac{2s_k\mu^2_{dec,out}}{\sigma^2}, \frac{4\mu^2_{dec,out}}{\sigma^2})$. The quantity $\mu^2_{dec,out}$ is a function of $\mu^2_{dec,in}$ defined as

$$\mu^2_{dec,out} = f_{dec}(\mu^2_{dec,in}) \triangleq \mu^2_{dec,in}\left(d_{free}-1\right). \tag{11}$$

Simulation results

The information data are encoded using the rate $R_c = 1/2$ convolutional code having 4 states and generator polynomials $(7,5)$, with a minimum distance $d_{free} = 5$. We provide the decoder with Gaussian *a priori* LLRs with the conditional pdf $\mathcal{N}(\frac{2s_k\mu^2_{dec,in}}{\sigma^2}, \frac{4\mu^2_{dec,in}}{\sigma^2})$, for a given $\mu_{dec,in} = \frac{\sigma}{\sigma_{dec,in}}$. Figure 4 shows $\mu^2_{dec,out}$ versus $\mu^2_{dec,in}$. The dotted curves are obtained by simulations for different values of the SNR. The solid curve is the theoretical curve obtained using (11). Notice that as for the equalizer, the theoretical curve does not depend on the SNR, however the curves obtained by simulations vary with the SNR. Simulations show that the analysis of the decoder holds for high SNR values and is less accurate for low SNR values. Hence, in the convergence analysis of the iterative receiver, we will use it for high SNR values. For low SNR values, we will perform a simulation of the decoder.

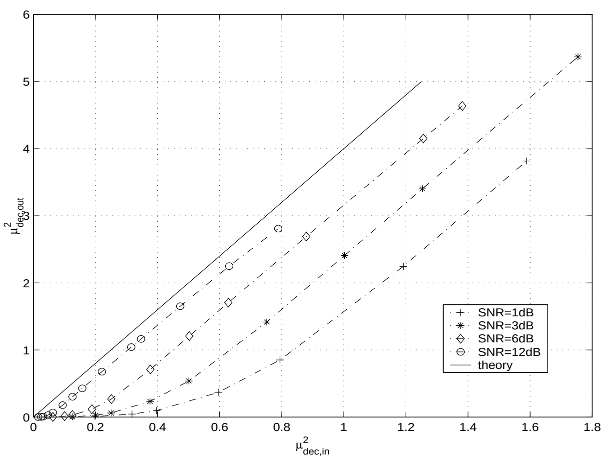

Figure 4: Decoder analysis : $\mu^2_{dec,out}$ versus $\mu^2_{dec,in}$.

6. CONVERGENCE ANALYSIS

In our turbo equalizer, equalization and decoding steps are iterated by passing the extrinsic LLRs $LLR_{ext\_eq}(s_k)$ and $LLR_{ext\_dec}(s_k)$ between the equalizer and the decoder. The mechanism of turbo equalization is completely described by the evolution of the distribution of the LLRs $LLR_{ext\_eq}(s_k)$ and $LLR_{ext\_dec}(s_k)$. This density evolution can be approximated by the changes of $\mu^2_{eq,in}$ to $\mu^2_{eq,out}$ and $\mu^2_{dec,in}$ to $\mu^2_{dec,out}$. At the first iteration, there is no *a priori* information at the input of the equalizer, thus $\mu^2_{eq,in} = 0$. Then, the output LLRs $LLR_{ext\_eq}(s_k)$ described by $\mu^2_{eq,out} = \mu^2_{dec,in}$ are fed into the decoder yielding LLRs $LLR_{ext\_dec}(s_k)$ described by $\mu^2_{dec,out} = \mu^2_{eq,in}$ which are fed back to the equalizer and so forth.

We propose here to perform an analysis based on the evolution of the parameters $\mu^2_{eq,out} = \mu^2_{dec,in}$ and $\mu^2_{dec,out} = \mu^2_{eq,in}$. We notice that the relations (8) and (11) between $\mu^2_{eq,in}$, $\mu^2_{eq,out}$ and $\mu^2_{dec,in}$, $\mu^2_{dec,out}$ are independent of the SNR, which is an interesting characteristic of this analysis. The fixed point of $f_{eq} \circ f_{dec}$ represents the asymptotic convergence point of the turbo detector.

Under mild condition on the code properties and on the channel, the performance of the turbo equalizer converges to the AWGN performance at high SNR.

Mild condition: To have the convergence of the turbo detector performance to the AWGN performance, the fixed point needs to be in the region of reliable *a priori* information at the equalizer input s.t. (7) is valid. It means that the value of $\mu^2_{dec,out}$ in (11) such that $\mu^2_{dec,in} = 1$ must be greater than $1 - \frac{d^2_{min}}{4}$. This condition can be rewritten as:

$$\mu^2_{dec,out} = \mu^2_{dec,in}(d_{free} - 1)|_{\mu^2_{dec,in}=1} = d_{free} - 1 > 1 - \frac{d^2_{min}}{4}.$$

This leads to the **condition** at high SNR

$$d_{free} > 2 - \frac{d^2_{min}}{4}. \qquad (12)$$

7. SIMULATION RESULTS

In this section, we propose to test for the validity of the convergence analysis of the turbo equalizer performed in section 6. We consider the whole system with the channel coding at the transmitter and the turbo equalizer at the receiver. In the simulations, the modulation used is the BPSK. We consider the channel with the impulse response (0.29;0.50;0.58;0.50;0.29). The information data are encoded using the rate $R_c = 1/2$ convolutional code with generator polynomials $(7,5)$. The interleaver size is 2048. Figures 5 and 6 represent $\mu^2_{eq,out} = \mu^2_{dec,in}$ versus $\mu^2_{dec,out} = \mu^2_{eq,in}$ respectively at SNR=6dB and SNR=14dB. The dotted curves (eq-turbo and dec-turbo) are obtained by using simulations when the turbo equalizer of figure 2 is used. The solid curve for the equalizer (eq-th) is obtained by using the theoretical analysis (equation (8)). In figure 5, the solid curve (dec-simul) for the decoder is obtained by simulations using artificial Gaussian *a priori* LLRs with the pdf $\mathcal{N}(\frac{2s_k\mu^2_{dec,in}}{\sigma^2}, \frac{4\mu^2_{dec,in}}{\sigma^2})$ at its input, as assumed in section 5. We use simulations since the analysis of the decoder performed in section 5 is valid for high SNR. It is worth mentioning that this simulation of the decoder has to be done only once. Figure 5 shows that the curve obtained using artificial Gaussian *a priori* LLRs at the input of the decoder (dec-simul) approximates well the curve obtained by simulating the turbo detector (dec-turbo). As in figure 3, we notice that the analysis is accurate for $\mu_{eq,in} >> \mu_{eq-\lim}$ and $\mu_{eq,in} << \mu_{eq-\lim}$ and less accurate around the limit value $\mu_{eq\_\lim}$.

In figure 6, the solid curve (dec-th) for the decoder is obtained by using the analysis of section 5, which becomes accurate at high SNR (14 dB). The analysis of the decoder and the equalizer becomes here more accurate than when SNR=6dB. The simulations show that the performance of the turbo equalizer converges to the AWGN performance. This is predicted by the analysis, since the condition (12) is satisfied.

8. CONCLUSION

In this paper, we consider a coded transmission over a frequency selective channel. Based on the results of [1, 2], we propose to study the convergence of the turbo equalizer using a Maximum *a posteriori* equalizer and a MAP decoder. We give a new representation space for the convergence analysis of the turbo equalizer, which is independent of the SNR for the equalizer. We show that, at high SNR, under a certain condition on the channel and the code, the performance of the turbo equalizer converges to the AWGN performance.

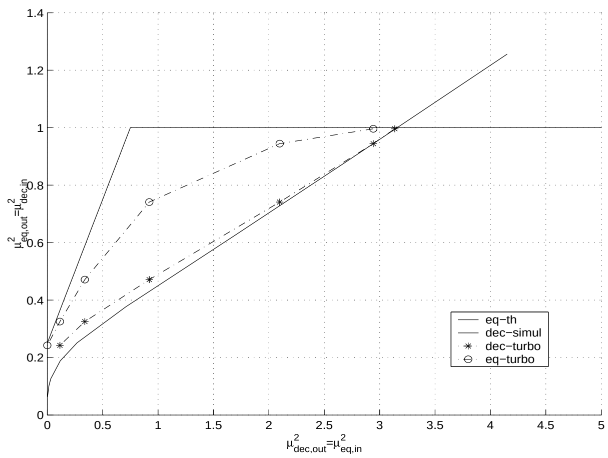

Figure 5: Convergence analysis at SNR=6dB, $\mu^2_{eq,out} = \mu^2_{dec,in}$ versus $\mu^2_{dec,out} = \mu^2_{eq,in}$: Semi-analytical analysis (solid curves), simulations (dotted curves).

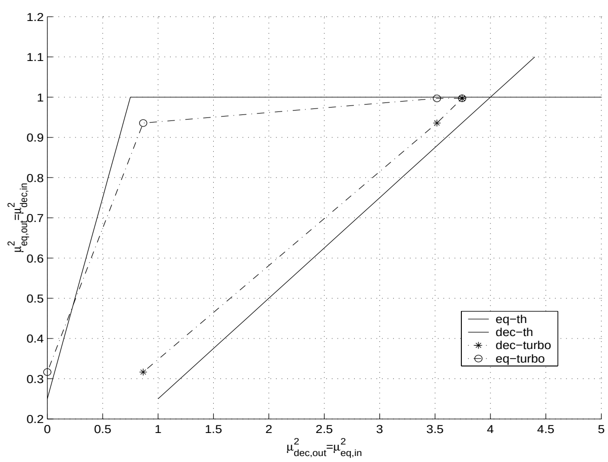

Figure 6: Convergence analysis at SNR=14dB, $\mu^2_{eq,out} = \mu^2_{dec,in}$ versus $\mu^2_{dec,out} = \mu^2_{eq,in}$: Analytical analysis (solid curves), simulations (dotted curves).

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