Density evolution of Orthogonal Matching Pursuit

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***Abstract*—This paper studies the Orthogonal Matching Pursuit (OMP) algorithm as a sparsity pattern recovery problem. We analyze the performance of the algorithm in an information theoretic framework, where both the signal to be recovered and the observation matrix are random and when the sizes of the signal and the matrix tend to infinity. We derive the joint densities of all the statistics involved throughout the process. This allows us to show that any strict fraction of the sparsity pattern can be recovered provided that the number of measurements is strictly greater than the length of the sparsity pattern.**

***Index Terms*—Compressive sensing, support recovery, Orthogonal Matching Pursuit.**

Let $x \in \mathbb{R}^N$ be a sparse signal whose support Λ is of cardinality $K \ll N$. x is not directly observed. Instead, M measurements are taken such that each measurement is a linear combination of the coordinates of x. This model is equivalent to a linear regression model

$$y = \phi x \tag{1}$$

where $y \in \mathbb{R}^M$ is the observed vector and $\phi \in \mathbb{R}^{M \times N}$ a sensing matrix. The sparsity pattern recovery problem aims at reconstructing the support Λ of x, knowing only y, ϕ and eventually the size of the support. This problem arises in several problems such as sparse approximation, compressive sensing, or subset selection [1]. Many greedy algorithms have been proposed which reconstruct the support iteratively, choosing one element of the support at each iteration. Among these greedy algorithms, we focus on orthogonal matching pursuit (OMP) as it includes several key steps used in many other greedy algorithms (e.g. decision based on scalar products, orthogonal projection).

At each iteration t, the algorithm selects the index λ_t of the most correlated atom with the residue vector r_{t-1} at the previous iteration ($r_0 = y$ is the observed vector and $r_t = Q_t y$ where Q_t is the orthogonal projector on the orthogonal of the space generated by the columns of ϕ_{Λ_t})

$$\lambda_t = \underset{i \in [\![1,N]\!] / \Lambda_{t-1}}{\arg\max} \ |\langle r_{t-1}, \phi_i \rangle| \tag{2}$$

$$\Lambda_t = \Lambda_{t-1} \cup \{\lambda_t\} \tag{3}$$

And computes an estimate x_t of the unknown x

$$x_t = \underset{z \in \mathbb{R}^n \ : \ \mathrm{supp}(z) \subset \Lambda_t}{\arg\min} ||y - \phi z||_2 \tag{4}$$

$$\Leftrightarrow \begin{cases} x_{t\ (\Lambda_t)} & = \left(\phi_{\Lambda_t}^T \phi_{\Lambda_t}\right)^{-1} \phi_{\Lambda_t}^T y \\ x_{t\ (\Lambda_t^c)} & = 0 \end{cases}$$

Several worst-case performance analyses of OMP exist. Worst-case analyses consider a deterministic signal x and look for properties on ϕ that guarantee necessary and/or sufficient conditions for perfect reconstruction of *any possible* sparse vector x. The analysis is either performed for a given ϕ matrix [2], [3], [4], [5], [6] or averaged over a matrix ensemble [7], [8], [9]. But, this leads to pessimistic conditions as there might exist sparse vectors x, that can be recovered even if the condition is not satisfied.

Instead, average analyses consider a random sparse signal and determine the probability to estimate the correct support, averaged over all matrices and signals. More precisely, conditions are computed such that perfect recovery of a strict fraction of the support occurs with high probability. First, the approach is more optimistic. Second, it is compliant with an information theoretical setup. Such average analyses have been performed for reconstruction algorithms, where the reconstructed signal at convergence can be characterized. However, no such analysis exists for OMP, and this is due to the statistical dependencies between iterations, which make exact analysis difficult. The statistical dependency between iterations has been tackled for deterministic signal x in [7] where upper bounds based on the union bound have been derived, and in [9] with a genie aided algorithm that knows the true sparsity pattern. Instead, we analyze the true OMP algorithm and derive the joint probability of all statistics used during the whole process. This allows us to give an asymptotic equivalent of the probability of recovering an atom of the right support at a given iteration, and a lower bound for the probability that the algorithm performs well and recovers the whole support.

More precisely, we show that, if ϕ and x are Gaussian, the joint distribution of the scalar products $W_j(t) = \langle \phi_j, r_t \rangle$ implied in the OMP algorithm can be expressed in terms of known distributions (Bessel and Chi). Then, we prove that these scalar products asymptotically converge toward a Gaussian sequence. At each iteration, we give an equivalent (when N, K, M, tend to infinity with $M = \mu K$ and $K = \gamma N$) of the probability of recovering an atom from the support and we derive a lower bound for the probability of recovering any strict subset of the whole support. This lower bound allows us to prove that, in the asymptotic regime defined above and when the overmeasuring rate μ satisfies $\mu > 1$, any strict subset of the support can be recovered with a probability tending to 1 when N tends to infinity.

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