Linear Transceivers for Sending Correlated Sources Over the Gaussian MAC

Onkar Dabeer
School of Technology and Computer Science
Tata Institute for Fundamental Research
Mumbai, India
Email: onkar@tcs.tifr.res.in

Aline Roumy, Christine Guillemot
IRISA-INRIA
35042 Rennes Cedex
France

***Abstract*— We consider the problem of sending two correlated random sources over a Gaussian multiple-access channel. The sources are assumed to be temporally memoryless. The performance criterion is the MSE and we seek linear transceivers that minimize it. When the bandwidth expansion factor is unity, it is shown that uncoded transmission is the best linear code for any SNR. When the bandwidth expansion factor is two, we present a scheme that involves uncoded transmission of one source with appropriate power adaptation and uncoded transmission of the other source and its negative with appropriate power adaptation. This scheme is shown to strictly ourperform coding strategies based on TDMA.**

I. Introduction

Sensor networks have great potential in applications such as environmental monitoring and currently there is a vibrant research activity in this area. In one design of such networks, the sensors organize themselves into clusters and the data collected is sent to a cluster head for further processing. Since the sensors observe the same physical phenomenon, their observations are correlated. Moreover, due to their proximity to each other, they share the same wireless communication channel to the cluster head. This naturally leads to the problem of communicating correlated information sources over a multiple-access channel (MAC). This is a challenging problem for which the source-channel separation theorem does not hold in general ([1]) and a single-letter characterization of the capacity region is not known. In fact, it is known that source-channel separation can be substantially worse than joint source-channel coding ([2]). In this paper, we look at linear (over the real field) joint source-channel codes for transmitting two correlated memoryless sources over a Gaussian MAC. One motivation for this is the simplicity of linear processing which makes it attractive for building low-cost sensors. Another motivation is a recent result in [3], which shows that for transmitting memoryless, bivariate Gaussian sources over the Gaussian MAC, uncoded transmission is optimal below a certain SNR. It is then natural to ask if linear codes can improve over uncoded transmission for SNR greater than the threshold given in [3]. In this paper, we study the best linear transceivers for this problem.

This paper is organized as follows. The problem statement is given in Section II. In Section III-A, we present an upper-bound on the MSE achievable with block linear codes, while in Section III-B we present a lower bound. In Section III-C we prove that when the source and channel bandwidths are the same, uncoded transmission is the best linear code for any SNR. In Section III-D we show that when the bandwidth expansion factor is two, then the linear code corresponding to the upper-bound in Section III-A uses uncoded transmission of one source with power adaptation, and uncoded transmission of the other source and its negative with power adaptation. This code is better than TDMA based codes.

Notation: Bold small-case letters denote vectors while bold capital letters denote matrices. $\mathbf{1}$ denotes the all ones vector, $\mathbf{0}$ denotes the zero vector/matrix, $\boldsymbol{I}_n$ denotes the $n \times n$ identity matrix, $\mathrm{diag}(\boldsymbol{x})$ denotes the diagonal matrix with diagonal $\boldsymbol{x}$. We use := and =: to define terms; for example, $2+x^2 =: 2+y$ implicitly defines $y := x^2$. The least achievable MSE using linear codes is denoted by MSE_*.

II. Problem Statement

Suppose we have two correlated sources $\{s_{1,t}, s_{2,t}\}_{t=1}^N$ that are i.i.d. with zero mean and covariance matrix $\left(\begin{smallmatrix} 1 & \rho \\ \rho & 1 \end{smallmatrix}\right)$, $\rho \in [0,1]$. These sources are to be transmitted over the Gaussian MAC channel. For simplicity, we write the source symbols in vector notation $\boldsymbol{s}_k$, $k = 1, 2$. The source vectors $\boldsymbol{s}_k$ are to be transmitted using M uses of the channel. The bandwidth expansion factor is defined to be $B = M/N$, and in this paper, for simplicity we only consider $B = 1, 2$. Each source is encoded separately and has a power constraint P. Let $\boldsymbol{x}_k = \boldsymbol{H}_k \boldsymbol{s}_k$ be the encoded signal, where $\boldsymbol{H}_k$ is a real $M \times N$ encoding matrix. To ensure the power constraint we need:

$$E\left[\|\boldsymbol{x}_k\|^2\right] = \mathrm{tr}\left\{\boldsymbol{H}_k \boldsymbol{H}_k^T\right\} \leq MP, \quad k = 1, 2 \qquad (1)$$

where we have used the fact that the covariance matrix of $\boldsymbol{s}_k$ is the identity $\boldsymbol{I}_N$. The output of the Gaussian MAC channel is:

$$\boldsymbol{y} = \sum_{k=1}^{K} \boldsymbol{x}_k + \boldsymbol{w}.$$

Let

$$\boldsymbol{H} := [\boldsymbol{H}_1 \;\; \boldsymbol{H}_2]$$

and

$$\boldsymbol{s} := \left[\boldsymbol{s}_1^T \ \boldsymbol{s}_2^T\right]^T.$$

Then

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{s} + \boldsymbol{w}.$$

We note that $\boldsymbol{s}$ has zero mean and covariance matrix

$$\boldsymbol{C} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \otimes \boldsymbol{I}_N.$$

The receiver generates an estimate $\hat{\boldsymbol{s}}$ and the metric of interest to us is

$$\text{MSE} = \frac{1}{2N} E\left[\|\hat{\boldsymbol{s}} - \boldsymbol{s}\|^2\right].$$

For a fixed transmitter, the MMSE linear receiver can be obtained by finding the MMSE estimate for each s_i, which is the i^{th} component of $\boldsymbol{s}$. Let

$$\boldsymbol{r}_i = E[s_i \boldsymbol{y}] = \boldsymbol{H}\boldsymbol{C}\boldsymbol{e}_i \tag{2}$$

where $\boldsymbol{e}_i$ has 1 in the i^{th} position and zero entries otherwise. Let

$$\boldsymbol{R} = E[\boldsymbol{y}\boldsymbol{y}^T] = \boldsymbol{H}\boldsymbol{C}\boldsymbol{H}^T + \sigma^2 \boldsymbol{I}_M. \tag{3}$$

Then for the MMSE estimate $\hat{s}_i$:

$$E[(s_i - \hat{s}_i)^2] = 1 - \boldsymbol{r}_i^T \boldsymbol{R}^{-1} \boldsymbol{r}_i = 1 - \text{tr}\left\{\boldsymbol{R}^{-1}\boldsymbol{r}_i\boldsymbol{r}_i^T\right\}.$$

Then the MSE is

$$\begin{aligned}
\text{MSE} &= \sum_{i=1}^{2N} E[(s_i - \hat{s}_i)^2]/(2N) \\
&= 1 - \text{tr}\left\{\boldsymbol{R}^{-1}\left(\frac{1}{2N}\sum_{i=1}^{2N}\boldsymbol{r}_i\boldsymbol{r}_i^T\right)\right\} \\
&= 1 - \text{tr}\left\{\boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{C}\left(\frac{1}{2N}\sum_{i=1}^{2N}\boldsymbol{e}_i\boldsymbol{e}_i^T\right)\boldsymbol{C}\boldsymbol{H}^T\right\} \\
&= 1 - \frac{1}{2N}\text{tr}\left\{\boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{C}^2\boldsymbol{H}^T\right\}. \qquad (4)
\end{aligned}$$

Problem Statement: Minimize the MSE (4) over all encoding matrices $\boldsymbol{H}_1, \boldsymbol{H}_2$ satisfying the power conditions (1). (We recall that the least achievable MSE is denoted by MSE$_*$.)

To address this problem, it is convenient to express the MSE and the power constraints in a different form. Let $\boldsymbol{G} := \boldsymbol{H}\sqrt{\boldsymbol{C}}$; since $\boldsymbol{C}$ is invertible we work with $\boldsymbol{G}$ instead of $\boldsymbol{H}$. Using $\text{tr}\{AB\} = \text{tr}\{BA\}$ and (3), we get

$$\text{MSE} = 1 - \frac{1}{2N}\text{tr}\left\{\boldsymbol{G}^T(\boldsymbol{G}\boldsymbol{G}^T + \sigma^2\boldsymbol{I}_m)^{-1}\boldsymbol{G}\boldsymbol{C}\right\}. \tag{5}$$

From the singular value decomposition we have $\boldsymbol{G} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$, where $\boldsymbol{U}$ is an $M \times M$ orthogonal matrix, $\boldsymbol{V}$ is an $2N \times 2N$ orthogonal matrix, and $\boldsymbol{\Sigma}$ is an $M \times 2N$ matrix with zeros outside the main diagonal and the singular values of $\boldsymbol{G}$ along the main diagonal. Let $r = \min\{M, 2N\}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ be the singular values of $\boldsymbol{G}$. We note that $r = N$ for $B = 1$ and $r = 2N$ for $B = 2$. Let $\theta_i := \lambda_i^2$ and for a vector $\boldsymbol{x}$, $\|\boldsymbol{x}\|_{\boldsymbol{C}}^2 := \boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x}$. Then it is shown in Appendix I that

$$\text{MSE} = 1 - \frac{1}{2N}\sum_{i=1}^{r}\frac{\theta_i}{\sigma^2 + \theta_i}\|\boldsymbol{v}_i\|_{\boldsymbol{C}}^2 \tag{6}$$

where $\boldsymbol{v}_i$ is the i^{th} column of $\boldsymbol{V}$.

We note that $\boldsymbol{C}$ has eigenvalue $(1+\rho)$ with eigenvectors

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \boldsymbol{e}_i = \frac{1}{\sqrt{2}}\begin{bmatrix}\boldsymbol{e}_i\\\boldsymbol{e}_i\end{bmatrix}, \quad i = 1, ..., N$$

and eigenvalue $(1-\rho)$ with eigenvectors

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \otimes \boldsymbol{e}_i = \frac{1}{\sqrt{2}}\begin{bmatrix}\boldsymbol{e}_i\\-\boldsymbol{e}_i\end{bmatrix}, \quad i = 1, ..., N.$$

Hence if $\boldsymbol{v}_i$ is partitioned into two $N \times 1$ vectors $\boldsymbol{v}_i^{(j)}$, $j = 1, 2$, then we can write

$$\|\boldsymbol{v}_i\|_{\boldsymbol{C}}^2 = \frac{(1+\rho)}{2}\|\boldsymbol{v}_i^{(1)} + \boldsymbol{v}_i^{(2)}\|^2 + \frac{(1-\rho)}{2}\|\boldsymbol{v}_i^{(1)} - \boldsymbol{v}_i^{(2)}\|^2 \tag{7}$$

in (6). Further it is shown in Appendix II that (1) becomes

$$\sum_{i=1}^{r}\left\|\frac{\boldsymbol{v}_i^{(1)} + \boldsymbol{v}_i^{(2)}}{2\sqrt{(1+\rho)}} \pm \frac{\boldsymbol{v}_i^{(1)} - \boldsymbol{v}_i^{(2)}}{2\sqrt{(1-\rho)}}\right\|^2 \theta_i \leq MP. \tag{8}$$

Equivalent Problem: Find an orthogonal matrix $\boldsymbol{V}$ and non-negative $\{\theta_i\}$ such that (6) is minimized subject to (8). (We note that $\boldsymbol{U}$ does not influence the MSE or the power constraints, and hence we take it to be the identity.)

III. Main Results

A. An Upper Bound on MSE$_*$

One way to obtain an upper-bound is to fix some orthogonal matrix $\boldsymbol{V}$ and then minimize with respect to $\{\theta_i\}$, which is the minimization of a convex function (6) under the linear constraints (8). Below we provide a way to choose a good $\boldsymbol{V}$.

For simplicity let $a_i = \|\boldsymbol{v}_i\|_{\boldsymbol{C}}^2$ and let $b_i := \sum_{j=1}^{i} a_j$, $i = 1, ..., r$, $b_0 := 0$. From (6) we get

$$\begin{aligned}
\text{MSE} &= 1 - \frac{1}{2N}\sum_{i=1}^{r}\left[1 - \frac{\sigma^2}{\sigma^2 + \theta_i}\right]a_i \\
&= 1 - \frac{b_r}{2N} + \frac{\sigma^2}{2N}\sum_{i=1}^{r}\frac{(b_i - b_{i-1})}{\sigma^2 + \theta_i} \\
&= 1 - \left[1 - \frac{1}{\sigma^2 + \theta_r}\right]\frac{b_r}{2N} \\
&\quad - \frac{\sigma^2}{2N}\sum_{i=1}^{r-1}\left[\frac{1}{\sigma^2 + \theta_{i+1}} - \frac{1}{\sigma^2 + \theta_i}\right]b_i.
\end{aligned}$$

Since $\theta_i \geq \theta_{i+1} \geq 0$, the coefficients of b_i are non-positive. Thus in order to minimize the MSE we want to choose b_i as large as possible under the conditions (8).

Define $\boldsymbol{V}_i$ to be the $2N \times i$ matrix with columns $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_i$. Then we note that $\boldsymbol{V}_i^T\boldsymbol{V}_i = \boldsymbol{I}_i$ and

$$b_i = \text{tr}\left\{\boldsymbol{V}_i^T\boldsymbol{C}\boldsymbol{V}_i\right\}.$$

Then from [4, (4), pp. 72] we get that

$$b_i \leq \xi_1 + \cdots + \xi_i$$

where $\xi_1 = \xi_2 = \cdots = \xi_N = (1+\rho)$, $\xi_{N+1} = \xi_{N+2} = \cdots = \xi_{2N} = (1-\rho)$ are the eigenvalues of $\boldsymbol{C}$. We see that the upper bound on b_i is achieved if we choose $\boldsymbol{v}_i$ to be the eigenvector of $\boldsymbol{C}$ corresponding to the eigenvalue ξ_i, $i = 1, ..., r$. Let $\boldsymbol{V}_C$ denote the orthogonal matrix whose columns are the eigenvectors of $\boldsymbol{C}$. For the choice of $\boldsymbol{V} = \boldsymbol{V}_C$, we have

$$\text{MSE} = 1 - \frac{1}{2N}\sum_{i=1}^{r}\xi_i + \frac{\sigma^2}{2N}\sum_{i=1}^{r}\frac{\xi_i}{\sigma^2+\theta_i}. \tag{9}$$

If we now minimize (9) over $\{\theta_i\}$ subject to (8), then we obtain an upper bound on the least achievable MSE. We note that this is a convex optimization problem. It is discussed in detail in subsequent sections for $B = 1, 2$.

Remark: The matrix $\boldsymbol{V}_C$ is obtained by ignoring the power constraints. It therefore leads to a lower bound on the MSE for any given $\{\theta_i\}$. However, minimizing this lower bound over $\{\theta_i\}$ subject to (8) does not lead to a lower bound on MSE$_*$, since this excludes those $\{\theta_i\}$ that do not satisfy (8) when $\boldsymbol{V}_C$ is used.

B. A Lower Bound on MSE$_*$

One way to obtain a lower bound is to minimize (6) under a constraint weaker than (8). We do so now. By expanding the norm and using $\|\boldsymbol{v}_i\| = 1$ it is easy to show that

$$\frac{1}{2}\left\|\frac{\boldsymbol{v}_i^{(1)}+\boldsymbol{v}_i^{(2)}}{2\sqrt{(1+\rho)}}+\frac{\boldsymbol{v}_i^{(1)}-\boldsymbol{v}_i^{(2)}}{2\sqrt{(1-\rho)}}\right\|^2 + \frac{1}{2}\left\|\frac{\boldsymbol{v}_i^{(1)}+\boldsymbol{v}_i^{(2)}}{2\sqrt{(1+\rho)}}-\frac{\boldsymbol{v}_i^{(1)}-\boldsymbol{v}_i^{(2)}}{2\sqrt{(1-\rho)}}\right\|^2 \geq \frac{1}{2(1+\rho)}.$$

Hence the power constraints (8) lead to the following weaker constraint

$$\sum_{i=1}^{r}\theta_i \leq 2(1+\rho)MP. \tag{10}$$

Since this constraint does not depend upon $\boldsymbol{V}$, minimization over $\boldsymbol{V}$ once again gives us (9). If we now minimize (9) under (10), then we obtain a lower bound on MSE$_*$. We note that this is a convex optimization problem and we analyze it in subsequent sections.

C. No Bandwidth Expansion

In this case $M = N$ and $r = N$. We first consider the upper bound derived in Section III-A. From (9), substituting for the eigenvalues and eigenvectors of $\boldsymbol{C}$ we get that

$$\text{MSE} = 1 - \frac{(1+\rho)}{2} + \frac{1+\rho}{2N}\sum_{i=1}^{N}\frac{\sigma^2}{\sigma^2+\theta_i}. \tag{11}$$

Also the power constraints (8) become

$$\sum_{i=1}^{N}\theta_i \leq 2(1+\rho)NP. \tag{12}$$

We note that this coincides with the weaker constraint (10) derived in Section III-B. Thus the convex optimization problems for finding the upper and lower bound on MSE$_*$ coincide, and solving this problem gives us MSE$_*$. To obtain MSE$_*$, we minimize (11) under (12) using the fact that the harmonic mean is bounded above by the arithmetic mean with equality iff all the numbers are equal. We get that $\theta_i^* = 2(1+\rho)P$, $i = 1, ..., N$. This gives

$$\begin{aligned}\text{MSE}_* &= 1 - \frac{(1+\rho)}{2}\left[1 - \frac{\sigma^2}{\sigma^2+2(1+\rho)P}\right] \\ &= \frac{1+(1-\rho^2)\gamma}{1+2(1+\rho)\gamma}\end{aligned} \tag{13}$$

where $\gamma := P/\sigma^2$ is the SNR at each of the transmitters. The corresponding encoding matrix is given by

$$\boldsymbol{H}_k = \sqrt{P}\boldsymbol{I}_N.$$

Thus for $B = 1$ uncoded transmission is optimal.

D. Bandwidth Expansion of Two

Now consider $B = 2$, $r = 2N$. We first consider the upper bound derived in Section III-A. Substituting for the eigenvalues and eigenvectors of $\boldsymbol{C}$, from (9) we get that

$$\begin{aligned}\text{MSE} = &\frac{(1+\rho)\sigma^2}{2}\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\sigma^2+\theta_i} \\ &+ \frac{(1-\rho)\sigma^2}{2}\frac{1}{N}\sum_{i=N+1}^{2N}\frac{1}{\sigma^2+\theta_i}\end{aligned} \tag{14}$$

and the power constraints (8) are now given by

$$\sum_{i=1}^{N}\frac{\theta_i}{1+\rho} + \sum_{i=N+1}^{M}\frac{\theta_i}{1-\rho} \leq 2MP = 4NP. \tag{15}$$

To find an upper bound on MSE$_*$ we have to minimize (14) under (15). Due to space constraints we do not give the details of this minimization, but the solution is given by

$$\theta_i^* = 2(1+\rho)Q_1^*,\ i = 1, ..., N;$$
$$\theta_i^* = 2(1-\rho)Q_2^*,\ i = N+1, ..., 2N$$

where for $\gamma \geq \frac{\rho}{2(1-\rho^2)}$

$$Q_1^* = P + \frac{\rho\sigma^2}{2(1-\rho^2)}, \quad Q_2^* = P - \frac{\rho\sigma^2}{2(1-\rho^2)},$$

and for $\gamma < \frac{\rho}{2(1-\rho^2)}$

$$Q_1^* = 2P, \quad Q_2^* = 0.$$

The corresponding upper bound is

$$\begin{aligned}\text{MSE}_* &\leq \frac{1}{\frac{1}{1-\rho^2}+2\gamma} \text{ if } \gamma \geq \frac{\rho}{2(1-\rho^2)} \\ &\leq \frac{1+2(1-\rho^2)\gamma}{1+4(1+\rho)\gamma} \text{ otherwise.}\end{aligned} \tag{16}$$

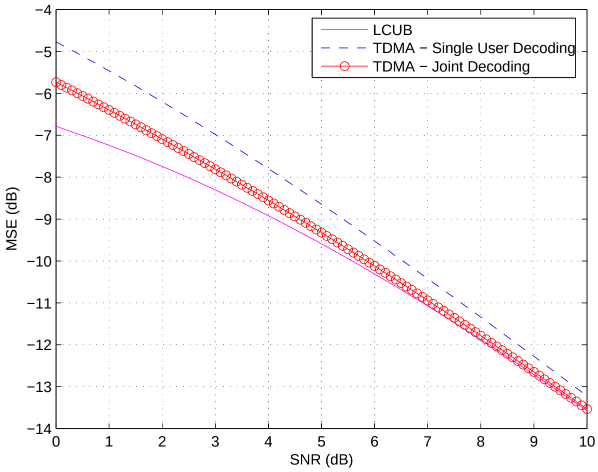

Fig. 1. The linear code corresponding to the upper bound (LCUB) outperforms the TDMA based scheme. Here $\rho = 0.8$.

The corresponding transmission scheme is as follows:

1) Source 1 uses uncoded transmission for $t = 1, ..., N$ with power Q_1^* and repeats the uncoded transmission with power Q_2^* for $t = N+1, ..., 2N$.
2) Source 2 uses uncoded transmission for $t = 1, ..., N$ with power Q_1^* and transmits the negative of the source at power Q_2^* for $t = N+1, ..., 2N$.

Unfortunately, the above upper bound does not match with the lower bound (which we do not present here due to space constraints). However we next compare this code with TDMA. For ease of reference, we refer to the above code as LCUB (linear code corresponding to the upper bound). For $M = 2N$, in the TDMA scheme, source 1 transmits for $t = 1, ..., N$ with power $2P$ and source 2 for $t = N+1, ..., 2N$ with power $2P$. Assume now that the sources are jointly Gaussian. For the transmission of a Gaussian source over a Gaussian channel with number of source symbols equal to the number of channel uses, it is well-known that uncoded transmission is the best amongst the class of all possible codes ([5]). Thus for the TDMA scheme with single-user decoding, uncoded transmission is optimal and in this case

$$\mathrm{MSE}_{TDMA,1} = \frac{1}{1+2\gamma}.$$

If joint decoding is used, then we get

$$\mathrm{MSE}_{TDMA,2} = \frac{1+2(1-\rho^2)\gamma}{1+4\gamma+4(1-\rho^2)\gamma^2}.$$

The TDMA scheme is also a linear transmitter, but it can be checked that it is worse than LCUB. In Figure 1 we compare LCUB and the TDMA scheme for $\rho = 0.8$; it is evident that the loss of TDMA is high for low SNR.

IV. Conclusions

In this paper we presented lower and upper bounds for the least MSE obtainable using linear joint source-channel codes for transmitting correlated sources over a MAC channel. These bounds coincide for $B = 1$ and uncoded transmission is optimal in this case. For $B = 2$, the transmission scheme corresponding to our upper bound performs superior to any scheme based on TDMA. We do not know if this scheme is the optimal linear code.

Appendix I
Alternate Expressions for MSE

We first prove (6). We note that

$$\boldsymbol{G}\boldsymbol{G}^T = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\boldsymbol{U}^T.$$

Since $\boldsymbol{U}\boldsymbol{U}^T = \boldsymbol{I}_M$, we get that

$$\left(\boldsymbol{G}\boldsymbol{G}^T + \sigma^2\boldsymbol{I}_M\right)^{-1} = \boldsymbol{U}\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T + \sigma^2\boldsymbol{I}_M\right)^{-1}\boldsymbol{U}^T.$$

Therefore

$$\boldsymbol{G}^T\left(\boldsymbol{G}\boldsymbol{G}^T + \sigma^2\boldsymbol{I}_M\right)^{-1}\boldsymbol{G} = \boldsymbol{V}\boldsymbol{\Sigma}^T\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T + \sigma^2\boldsymbol{I}_M\right)^{-1}\boldsymbol{\Sigma}\boldsymbol{V}^T.$$

We note that $\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T$ is a diagonal matrix with eigenvalues λ_i^2, $i = 1, ..., r$ with eigenvectors $\boldsymbol{e}_i$. Hence, using the spectral decomposition, we get

$$\begin{aligned}
\boldsymbol{G}^T\left(\boldsymbol{G}\boldsymbol{G}^T + \sigma^2\boldsymbol{I}_M\right)^{-1}\boldsymbol{G} &= \boldsymbol{V}\boldsymbol{\Sigma}^T\left[\sum_{i=1}^{r}\frac{\boldsymbol{e}_i\boldsymbol{e}_i^T}{\lambda_i^2+\sigma^2}\right]\boldsymbol{\Sigma}\boldsymbol{V}^T \\
&= \boldsymbol{V}\left[\sum_{i=1}^{r}\frac{\lambda_i^2}{\lambda_i^2+\sigma^2}\boldsymbol{e}_i\boldsymbol{e}_i^T\right]\boldsymbol{V}^T \\
&= \sum_{i=1}^{r}\frac{\lambda_i^2}{\lambda_i^2+\sigma^2}\boldsymbol{v}_i\boldsymbol{v}_i^T
\end{aligned}$$

where $\boldsymbol{v}_i$ is the i^{th} column of $\boldsymbol{V}$. Then substituting in (5), we get

$$\begin{aligned}
\mathrm{MSE} &= 1 - \frac{1}{2N}\sum_{i=1}^{r}\frac{\lambda_i^2}{\lambda_i^2+\sigma^2}\mathrm{tr}\left\{\boldsymbol{v}_i\boldsymbol{v}_i^T\boldsymbol{C}\right\} \\
&= 1 - \frac{1}{2N}\sum_{i=1}^{r}\frac{\theta_i}{\sigma^2+\theta_i}\|\boldsymbol{v}_i\|_{\boldsymbol{C}}^2
\end{aligned}$$

which is the desired expression (6).

Appendix II
Alternate Expressions for Power Conditions

We now obtain the power constraints in terms of $\boldsymbol{v}_i$, θ_i. We see that the MSE does not depend on $\boldsymbol{U}$ and hence we just take it to be the identity. Thus

$$\boldsymbol{G} = \begin{bmatrix} \sqrt{\theta_1}\boldsymbol{v}_1^T \\ \sqrt{\theta_2}\boldsymbol{v}_2^T \\ \vdots \\ \sqrt{\theta_r}\boldsymbol{v}_r^T \end{bmatrix}$$

and

$$\boldsymbol{H} = \boldsymbol{G}\sqrt{\boldsymbol{C}^{-1}} = \begin{bmatrix} \sqrt{\theta_1}\boldsymbol{v}_1^T\sqrt{\boldsymbol{C}^{-1}} \\ \sqrt{\theta_2}\boldsymbol{v}_2^T\sqrt{\boldsymbol{C}^{-1}} \\ \vdots \\ \sqrt{\theta_r}\boldsymbol{v}_r^T\sqrt{\boldsymbol{C}^{-1}} \end{bmatrix}.$$

We note that $\sqrt{\boldsymbol{C}^{-1}}$ has eigenvalues $1/\sqrt{1+\rho}$, $1/\sqrt{1-\rho}$ with the same eigenvectors as the corresponding eigenvectors of $\boldsymbol{C}$. Therefore partitioning $\boldsymbol{v}_i$ as in (7),

$$\sqrt{\boldsymbol{C}^{-1}}\boldsymbol{v}_i = \frac{1}{2\sqrt{1+\rho}}\begin{bmatrix}\boldsymbol{v}_i^{(1)}+\boldsymbol{v}_i^{(2)}\\ \boldsymbol{v}_i^{(1)}+\boldsymbol{v}_i^{(2)}\end{bmatrix}+\frac{1}{2\sqrt{1-\rho}}\begin{bmatrix}\boldsymbol{v}_i^{(1)}-\boldsymbol{v}_i^{(2)}\\ -(\boldsymbol{v}_i^{(1)}-\boldsymbol{v}_i^{(2)})\end{bmatrix}.$$

Thus we get that

$$\begin{aligned}\boldsymbol{H}_1 = \frac{1}{2\sqrt{1+\rho}}&\begin{bmatrix}\sqrt{\theta_1}(\boldsymbol{v}_1^{(1)}+\boldsymbol{v}_1^{(2)})^T\\ \sqrt{\theta_2}(\boldsymbol{v}_2^{(1)}+\boldsymbol{v}_2^{(2)})^T\\ \vdots\\ \sqrt{\theta_r}(\boldsymbol{v}_r^{(1)}+\boldsymbol{v}_r^{(2)})^T\end{bmatrix}\\ &+\frac{1}{2\sqrt{1-\rho}}\begin{bmatrix}\sqrt{\theta_1}(\boldsymbol{v}_1^{(1)}-\boldsymbol{v}_1^{(2)})^T\\ \sqrt{\theta_2}(\boldsymbol{v}_2^{(1)}-\boldsymbol{v}_2^{(2)})^T\\ \vdots\\ \sqrt{\theta_r}(\boldsymbol{v}_r^{(1)}-\boldsymbol{v}_r^{(2)})^T\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{H}_2 = \frac{1}{2\sqrt{1+\rho}}&\begin{bmatrix}\sqrt{\theta_1}(\boldsymbol{v}_1^{(1)}+\boldsymbol{v}_1^{(2)})^T\\ \sqrt{\theta_2}(\boldsymbol{v}_2^{(1)}+\boldsymbol{v}_2^{(2)})^T\\ \vdots\\ \sqrt{\theta_r}(\boldsymbol{v}_r^{(1)}+\boldsymbol{v}_r^{(2)})^T\end{bmatrix}\\ &-\frac{1}{2\sqrt{1-\rho}}\begin{bmatrix}\sqrt{\theta_1}(\boldsymbol{v}_1^{(1)}-\boldsymbol{v}_1^{(2)})^T\\ \sqrt{\theta_2}(\boldsymbol{v}_2^{(1)}-\boldsymbol{v}_2^{(2)})^T\\ \vdots\\ \sqrt{\theta_r}(\boldsymbol{v}_r^{(1)}-\boldsymbol{v}_r^{(2)})^T\end{bmatrix}.\end{aligned}$$

Substituting in (1), the power constraints become

$$\sum_{i=1}^{r}\left\|\frac{\boldsymbol{v}_i^{(1)}+\boldsymbol{v}_i^{(2)}}{2\sqrt{(1+\rho)}}\pm\frac{\boldsymbol{v}_i^{(1)}-\boldsymbol{v}_i^{(2)}}{2\sqrt{(1-\rho)}}\right\|^2\theta_i \le MP$$

which is the desired result.

References

[1] T. Cover, A. E. Gamal, and M. Salehi, “Multiple access channels with arbitrarily correlated sources,” *IEEE Transactions on Information Theory*, vol. 26, no. 6, pp. 648–657, November 1980.

[2] M. Gastpar and M. Vetterli, “Power, spatio-temporal bandwidth, and distortion in large sensor networks,” *IEEE Journal on Selected Areas in Communications*, vol. 23, no. 4, pp. 745–754, April 2005.

[3] A. Lapidoth and S. Tinguely, “Sending a bivariate Gaussian source over a Gaussian MAC,” arXiv:cs.IT/061029, May 2006.

[4] H. Lütkepohl, *Handbook of Matrices*, John Wiley and Sons, New York, 1996.

[5] T. Cover and J. A. Thomas, *Elements of Information Theory*, John Wiley and Sons, New York, 1991.