Invariance Kernels of Polygonal Differential Inclusions

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Abstract

Polygonal hybrid systems are a subclass of planar hybrid automata which can be represented by piecewise constant differential inclusions. Here, we identify and compute an important object of such systems' phase portrait, namely *invariance kernels*. An *invariant set* is a set of initial points of trajectories which keep rotating in a cycle forever and the *invariance kernel* is the largest of such sets. We show that this kernel is a non-convex polygon and we give a non-iterative algorithm for computing the coordinates of its vertexes and edges.

1 Introduction

A *hybrid system* is a system where both continuous and discrete behaviors interact with each other. A typical example is given by a discrete program that interacts with (controls, monitor, supervises) a continuous physical environment. In the last decade many (un)decidability results for a variety of problems concerning classes of hybrid systems have been given [ACH+95, ABDM00, BT00], [DM98, GM99, KV00]. One of the main research areas in hybrid systems is reachability analysis. Most of the proved decidability results are based on the existence of a finite and computable partition of the state space into classes of states which are equivalent with respect to reachability. This is the case for timed automata [AD94], and classes of rectangular automata [HKPV95] and hybrid automata with linear vector fields [LPY99]. For some particular classes of two-dimensional dynamical systems a *geometrical* method, which relies on the analysis of topological properties of the plane, has been developed. This approach has been proposed in [MP93]. There, it is shown that the reachability problem for two-dimensional systems with piece-wise constant derivatives (PCD) is decidable. This result has been extended in [CV96] for planar piecewise Hamiltonian systems and in [ASY01] for polygonal hybrid systems, a class

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of nondeterministic systems that correspond to piecewise constant differential inclusions on the plane, see Figure 1. For historical reasons we call such a system an SPDI [Sch02]. In [AMP95] it has been shown that the reachability problem for PCD is undecidable for dimensions higher than two.

Another important issue in the analysis of a (hybrid) dynamical system is the study of its qualitative behavior, namely the construction of its *phase portrait*. Some typical questions on this sense are "does every trajectory except the equilibrium point in the origin converges to a limit cycle which is the unit circle?", or "what is the biggest set such that any point on it is reachable from any other point on the set?". There have been very few results on the qualitative properties of trajectories of hybrid systems [ASY02, Aub01, DV95, KV96, KdB01, MS00, SJSL00]. In particular, the question of defining and constructing phase portraits of hybrid systems has not been directly addressed except in [MS00], where phase portraits of deterministic systems with piecewise constant derivatives are explored and in [ASY02] where viability and controllability kernels for polygonal differential inclusion systems have been computed.

In this paper we show how to compute another important object of phase portraits of SPDIs, namely the *invariance kernel*. In general, an *invariant set* is a set of points such that for any point in the set, every trajectory starting in such point remains in the set forever and the *invariance kernel* is the largest of such sets. We show that, for SPDIs, this kernel is a non-convex polygon and we give a non-iterative algorithm for computing the coordinates of its vertexes and edges.[1]. Clearly, the invariance kernel provides useful insight about the behavior of the SPDI around simple cycles.

The paper is organized as follows. In section 2 we recall the definition as well as some important results concerning SPDIs. Section 3 is concerned with the computation of invariance kernel for simple cycles of SPDIs. In the last section we present some concluding remarks.

2 Preliminaries

2.1 Truncated affine multivalued functions

Definition 1 *A (positive)* affine *function* $f : \mathbb{R} \to \mathbb{R}$ *is such that* $f(x) = ax + b$ *with* $a > 0$. *An* affine multivalued *function* $F : \mathbb{R} \to 2^{\mathbb{R}}$*, denoted* $F = \langle f_l, f_u \rangle$*, is defined by* $F(x) = \langle f_l(x), f_u(x) \rangle$ *where* f_l *and* f_u *are affine and* $\langle \cdot, \cdot \rangle$ *denotes an interval.* ■

In what follows we will consider only well-formed intervals, i.e. $\langle l, u \rangle$ is an interval iff $l \leq u$. For notational convenience, we do not make explicit whether intervals are open, closed, left-open or right-open, unless required for comprehension. For an interval $I = \langle l, u \rangle$ we have that $F(\langle l, u \rangle) = \langle f_l(l), f_u(u) \rangle$.

[1] Since SPDIs are partially defined over the plane, their invariance kernels are in general different from the whole plane.

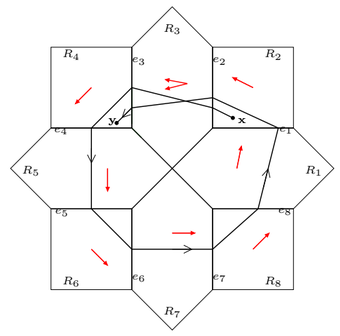

Figure 1: An SPDI and its trajectory segment.

Definition 2 *The* inverse *of F is defined by $F^{-1}(x) = \{y \mid x \in F(y)\}$. It is not difficult to show that $F^{-1} = \langle f_u^{-1}, f_l^{-1} \rangle$. The* universal inverse *of F is defined by $\tilde{F}^{-1}(I) = I'$ iff I' is the greatest non-empty interval such that for all $x \in I'$, $F(x) \subseteq I$.* ■

Notice that if I is a singleton then $\tilde{F}^{-1}$ is defined only if $f_l = f_u$. These classes of functions are closed under composition.

Definition 3 *A* truncated affine multivalued *function (TAMF) $\mathcal{F} : \mathbb{R} \to 2^{\mathbb{R}}$ is defined by an affine multivalued function F and intervals $S \subseteq \mathbb{R}^+$ and $J \subseteq \mathbb{R}^+$ as follows: $\mathcal{F}(x) = F(x) \cap J$ if $x \in S$, otherwise $\mathcal{F}(x) = \emptyset$.* ■

For convenience we write $\mathcal{F}(x) = F(\{x\} \cap S) \cap J$. For an interval I, $\mathcal{F}(I) = F(I \cap S) \cap J$ and $\mathcal{F}^{-1}(I) = F^{-1}(I \cap J) \cap S$. We say that $\mathcal{F}$ is *normalized* if $S = \mathsf{Dom}\mathcal{F} = \{x \mid F(x) \cap J \neq \emptyset\}$ (thus, $S \subseteq F^{-1}(J)$) and $J = \mathsf{Im}\mathcal{F} = \mathcal{F}(S)$. In what follows we only consider normalized TAMFs.

Definition 4 *The* universal inverse *of $\mathcal{F}$ is defined by $\tilde{\mathcal{F}}^{-1}(I) = I'$ iff I' is the greatest non-empty interval such that for all $x \in I'$, $F(x) \subseteq I$ and $F(x) = \mathcal{F}(x)$.* ■

The following theorem states that TAMFs are closed under composition [ASY01].

Theorem 1 The composition of two TAMFs $\mathcal{F}_1(I) = F_1(I \cap S_1) \cap J_1$ and $\mathcal{F}_2(I) = F_2(I \cap S_2) \cap J_2$, is the TAMF $(\mathcal{F}_2 \circ \mathcal{F}_1)(I) = \mathcal{F}(I) = F(I \cap S) \cap J$, where $F = F_2 \circ F_1$, $S = S_1 \cap F_1^{-1}(J_1 \cap S_2)$ and $J = J_2 \cap F_2(J_1 \cap S_2)$. □

2.2 SPDI

Let $\mathbf{a} = (a_1, a_2), \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. The *inner product* of two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{x} = (x_1, x_2)$ is defined as $\mathbf{a}\ \mathbf{x} = a_1x_1 + a_2x_2$, while $\lambda\mathbf{x} = (\lambda x_1, \lambda x_2)$ is the product of a vector by an scalar λ and $|\mathbf{x}| = \sqrt{\mathbf{x}\ \mathbf{x}}$ is the *2-norm*. We denote by $\hat{\mathbf{x}}$ the vector $(x_2, -x_1)$ obtained from $\mathbf{x}$ by rotating clockwise by the angle $\pi/2$. Notice that $\mathbf{x}\ \hat{\mathbf{x}} = 0$.

The *distance* between two points $\mathbf{x}$ and $\mathbf{y}$ is defined to be $|\mathbf{x}-\mathbf{y}|$. For $\epsilon > 0$, the *ϵ-neighborhood* of $\mathbf{x}$ is $B_\epsilon(\mathbf{x}) = \{\mathbf{y} \mid |\mathbf{x} - \mathbf{y}| < \epsilon\}$. The *interior* of $X \subseteq \mathbb{R}^2$, denoted by $\mathtt{int}(X)$, is the set of $\mathbf{x} \in X$ for which there exists $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subseteq X$.

For $\mathbf{x}_1, \ldots \mathbf{x}_n \in \mathbb{R}^2$ a *linear* combination is a vector $\mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i$ for some $\lambda_i \in \mathbb{R}$. A *positive combination* is a linear combination with $\lambda_i \geq 0$ for every i. The positive *hull* of a set $X \subseteq \mathbb{R}^2$ is the set of all positive combinations of points in X. A (closed) *half-space* is the set of all points $\mathbf{x}$ satisfying $\mathbf{a}\ \mathbf{x} \leq b$. A *convex closed polygonal set* P is the intersection of finitely many half-spaces. An *edge* e is a segment of line in $\mathbb{R}^2$.

An *angle* $\angle^{\mathbf{b}}_{\mathbf{a}}$ on the plane, defined by two non-zero vectors $\mathbf{a}, \mathbf{b}$ is the set of all positive linear combinations $\mathbf{x} = \alpha\ \mathbf{a} + \beta\ \mathbf{b}$, with $\alpha, \beta \geq 0$, and $\alpha + \beta > 0$. We can always assume that $\mathbf{b}$ is situated in the counter-clockwise direction from $\mathbf{a}$. That is, given $A = \{\mathbf{a}, \mathbf{b}\}$, with $\hat{\mathbf{a}}\ \mathbf{b} < 0$, $\angle^{\mathbf{b}}_{\mathbf{a}}$ is the positive hull of A.

Let S be a finite index set and $\mathbb{P} = \{P_s\}_{s \in S}$ be a finite set of convex closed polygonal sets, called *regions*, such that:

1. For all $s \in S$, $\mathtt{int}(P_s) \neq \emptyset$;
2. For all $s \neq r \in S$, $\mathtt{int}(P_s \cap P_r) = \emptyset$;
3. $\bigcup_{s \in S} P_s = \mathbb{R}^2$.

Condition 1 states that regions are full dimensional. Condition 2 says that the intersection between two regions is either empty, an edge, or a point, whereas the third condition states that the regions covers the whole space. Thus, $\mathbb{P}$ is a *partition*[2].

Let $\mathbb{F} = \{\phi_s\}_{s \in S}$ be such that ϕ_s is the positive hull of two vectors $\mathbf{a}_s$ and $\mathbf{b}_s$ with $\hat{\mathbf{a}}_s\ \mathbf{b}_s < 0$ and $\mathbb{P}$ be a partition of the plane.

A *polygonal differential inclusion system* (SPDI) consists of a partition of the plane into convex polygonal regions, together with a differential inclusion associated with each region. More formally,

Definition 5 *A* polygonal differential inclusion system *(SPDI) is a pair $\mathcal{H} = (\mathbb{P}, \mathbb{F})$. Each region P_s has dynamics $\dot{\mathbf{x}} \in \phi_s$ for $\mathbf{x} \in P_s$ (given a generic region P we also use the notation $\phi(P)$).* ■

Let $E(P)$ be the set of edges of P. We say that $e \in E(P)$ is an *entry* of P if for all $\mathbf{x} \in e$ and for all $\mathbf{c} \in \phi(P)$, $\mathbf{x} + \mathbf{c}\epsilon \in P$ for some $\epsilon > 0$. We say that e is an *exit* of P if the same condition holds for some $\epsilon < 0$. We denote by

[2] Condition (3) can be relaxed, and then we can consider partition of a subset of the plane.

$In(P) \subseteq E(P)$ the set of all entries of P and by $Out(P) \subseteq E(P)$ the set of all exits of P.

Assumption 1 *All the edges in $E(P)$ are either entries or exits, that is, $E(P) = In(P) \cup Out(P)$.*

Example 1 Consider the SPDI illustrated in Figure 1. For each region R_i, $1 \leq i \leq 8$, there is a pair of vectors $(\mathbf{a}_i, \mathbf{b}_i)$, where:

$$\begin{array}{l}
\mathbf{a}_1 = \mathbf{b}_1 = (1, 5), \\
\mathbf{a}_2 = \mathbf{b}_2 = (-1, \frac{1}{2}), \\
\mathbf{a}_3 = (-1, \frac{11}{60}) \text{ and } \mathbf{b}_3 = (-1, -\frac{1}{10}), \\
\mathbf{a}_4 = \mathbf{b}_4 = (-1, -1), \\
\mathbf{a}_5 = \mathbf{b}_5 = (0, -1), \\
\mathbf{a}_6 = \mathbf{b}_6 = (1, -1), \\
\mathbf{a}_7 = \mathbf{b}_7 = (1, 0), \\
\mathbf{a}_8 = \mathbf{b}_8 = (1, 1). \qquad \blacksquare
\end{array}$$

Definition 6 *A* trajectory segment *of an SPDI is a continuous function $\xi : [0, T] \to \mathbb{R}^2$ which is smooth everywhere except in a discrete set of points, and such that for all $t \in [0, T]$, if $\xi(t) \in P$ and $\dot{\xi}(t)$ is defined then $\dot{\xi}(t) \in \phi(P)$. The* signature*, denoted $\mathsf{Sig}(\xi)$, is the ordered sequence of edges traversed by the trajectory segment, that is, $e_1, e_2, \ldots$, where $\xi(t_i) \in e_i$ and $t_i < t_{i+1}$. If $T = \infty$, a trajectory segment is called a* trajectory. $\blacksquare$

Assumption 2 *We will only consider trajectories with infinite signatures.*

2.3 Successors and predecessors

Given an SPDI, we fix a one-dimensional coordinate system on each edge to represent points laying on edges [ASY01]. For notational convenience, we indistinctly use letter e to denote the edge or its one-dimensional representation. Accordingly, we write $\mathbf{x} \in e$ or $x \in e$, to mean "point $\mathbf{x}$ in edge e with coordinate x in the one-dimensional coordinate system of e". The same convention is applied to sets of points of e represented as intervals (e.g., $\mathbf{x} \in I$ or $x \in I$, where $I \subseteq e$) and to trajectories (e.g., "ξ starting in x" or "ξ starting in $\mathbf{x}$"). Now, let $P \in \mathbb{P}$, $e \in In(P)$ and $e' \in Out(P)$.

Definition 7 *For $I \subseteq e$, $\mathsf{Succ}_{e,e'}(I)$ is the set of all points in e' reachable from some point in I by a trajectory segment $\xi : [0, t] \to \mathbb{R}^2$ in P (i.e., $\xi(0) \in I \wedge \xi(t) \in e' \wedge \mathsf{Sig}(\xi) = ee'$).* $\blacksquare$

We have shown in [ASY01] that $\mathsf{Succ}_{e,e'}$ is a TAMF[3].

[3] In [ASY01] we explain how to choose the positive direction on every edge in order to guarantee positive coefficients in the TAMF.

Example 2 Let $e_1, \ldots, e_8$ be as in Figure 1 and $I = [l, u]$. We assume a one-dimensional coordinate system such that $e_i = S_i = J_i = (0, 1)$. We have that:

$$F_{e_1e_2}(I) = \left[\frac{l}{2}, \frac{u}{2}\right] \qquad F_{e_2e_3}(I) = \left[l - \frac{1}{10}, u + \frac{11}{60}\right]$$
$$F_{e_ie_{i+1}}(I) = I \quad 3 \leq i \leq 7 \qquad F_{e_8e_1}(I) = \left[l + \frac{1}{5}, u + \frac{1}{5}\right]$$

with $\mathsf{Succ}_{e_ie_{i+1}}(I) = F_{e_ie_{i+1}}(I \cap S_i) \cap J_{i+1}$, for $1 \leq i \leq 7$, and $\mathsf{Succ}_{e_8e_1}(I) = F_{e_8e_1}(I \cap S_8) \cap J_1$. ■

Given a sequence $w = e_1, e_2, \ldots, e_n$, Theorem 1 implies that the successor of I along w defined as $\mathsf{Succ}_w(I) = \mathsf{Succ}_{e_{n-1},e_n} \circ \ldots \circ \mathsf{Succ}_{e_1,e_2}(I)$ is a TAMF.

Example 3 Let $\sigma = e_1 \cdots e_8e_1$. We have that $\mathsf{Succ}_\sigma(I) = F(I \cap S) \cap J$, where:

$$F(I) \quad = \quad \left[\frac{l}{2} + \frac{1}{10}, \frac{u}{2} + \frac{23}{60}\right] \tag{1}$$

$S = (0, 1)$ and $J = (\frac{1}{5}, \frac{53}{60})$ are computed using Theorem 1. ■

We can define in a similar way the edge-to-edge predecessor of an interval.

Definition 8 *For $I \subseteq e'$, $\mathsf{Pre}_{e,e'}(I)$ is the set of points in e that can reach a point in I by a trajectory segment in P.* ■

We have that [ASY01]: $\mathsf{Pre}_{e,e'} = \mathsf{Succ}^{-1}_{e,e'}$ and $\mathsf{Pre}_\sigma = \mathsf{Succ}^{-1}_\sigma$.

Example 4 Let $\sigma = e_1 \ldots e_8e_1$ be as in Figure 1 and $I = [l, u]$. We have that $\mathsf{Pre}_{e_ie_{i+1}}(I) = F^{-1}_{e_ie_{i+1}}(I \cap J_{i+1}) \cap S_i$, for $1 \leq i \leq 7$, and $\mathsf{Pre}_{e_8e_1}(I) = F^{-1}_{e_8e_1}(I \cap J_1) \cap S_8$, where:

$$F^{-1}_{e_1e_2}(I) = [2l, 2u] \qquad F^{-1}_{e_2e_3}(I) = \left[l - \frac{11}{60}, u + \frac{1}{10}\right]$$
$$F^{-1}_{e_ie_{i+1}}(I) = I \quad 3 \leq i \leq 7 \qquad F^{-1}_{e_8e_1}(I) = \left[l - \frac{1}{5}, u - \frac{1}{5}\right]$$

Besides, $\mathsf{Pre}_\sigma(I) = F^{-1}(I \cap J) \cap S$, where $F^{-1}(I) = [2l - \frac{23}{30}, 2u - \frac{1}{5}]$. ■

2.4 Qualitative analysis of simple edge-cycles

Let $\sigma = e_1 \cdots e_ke_1$ be a simple edge-cycle, i.e., $e_i \neq e_j$ for all $1 \leq i \neq j \leq k$. Let $\mathsf{Succ}_\sigma(I) = F(I \cap S) \cap J$ with $F = \langle f_l, f_u \rangle$ (we suppose that this representation is normalized). We denote by $\mathcal{D}_\sigma$ the one-dimensional discrete-time dynamical system defined by Succ_σ, that is $x_{n+1} \in \mathsf{Succ}_\sigma(x_n)$. For $x \in \mathbb{R}$ and $I \subset \mathbb{R}$, $x < I$ means that $x < y$ for all $y \in I$.

Assumption 3 *None of the two functions f_l, f_u is the identity.*

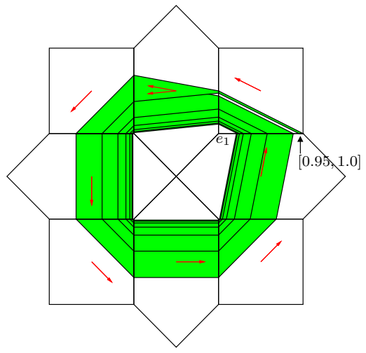

Figure 2: Reachability analysis.

Let l^* and u^* be the fixpoints[4] of f_l and f_u, respectively, and $S \cap J = \langle L, U \rangle$. We have shown in [ASY01] that a simple cycle is of one of the following types:

STAY. The cycle is not abandoned neither by the leftmost nor the rightmost trajectory, that is, $L \leq l^* \leq u^* \leq U$.

DIE. The rightmost trajectory exits the cycle through the left (consequently the leftmost one also exits) or the leftmost trajectory exits the cycle through the right (consequently the rightmost one also exits), that is, $u^* < L \vee l^* > U$.

EXIT-BOTH. The leftmost trajectory exits the cycle through the left and the rightmost one through the right, that is, $l^* < L \wedge u^* > U$.

EXIT-LEFT. The leftmost trajectory exits the cycle (through the left) but the rightmost one stays inside, that is, $l^* < L \leq u^* \leq U$.

EXIT-RIGHT. The rightmost trajectory exits the cycle (through the right) but the leftmost one stays inside, that is, $L \leq l^* \leq U < u^*$.

Example 5 Let $\sigma = e_1 \cdots e_8 e_1$. We have that $S \cap J = \langle L, U \rangle = (\frac{1}{5}, \frac{53}{60})$. The fixpoints of Eq. (1) are such that $L = l^* = \frac{1}{5} < u^* = \frac{23}{30} < U$. Thus, σ is STAY. ■

The classification above gives us some information about the qualitative behavior of trajectories. Any trajectory that enters a cycle of type DIE will

[4]Obviously, the fixpoint x^* is computed by solving a linear equation $f(x^*) = x^*$, which can be finite or infinite (see Lemma 6, page 45 of [Sch02]).

eventually quit it after a finite number of turns. If the cycle is of type STAY, all trajectories that happen to enter it will keep turning inside it forever. In all other cases, some trajectories will turn for a while and then exit, and others will continue turning forever. This information is very useful for solving the reachability problem [ASY01].

Example 6 Consider again the cycle $\sigma = e_1 \cdots e_8 e_1$. Figure 2 shows the reach set of the interval $[0.95, 1.0] \subset e_1$. Notice that the leftmost trajectory "converges to" the limit $l^* = \frac{1}{5}$. Figure 2 has been automatically generated by the SPeeDi toolbox [APSY02] we have developed for reachability analysis of SPDIs based on the results of [ASY01]. ■

The above result does not allow us to directly answer other questions about the behavior of the SPDI such as determine for a given point (or set of points) whether any trajectory (if it exists) starting in the point remains in the cycle forever. In order to do this, we need to further study the properties of the system around simple edge-cycles and in particular STAY cycles.

3 Invariance Kernel

In this section we define the notion of *invariance kernel* and we show how to compute it. In general, an *invariant set* is a set of points such that for any point in the set, every trajectory starting in such point remains in the set forever and the *invariance kernel* is the largest of such sets.

In particular, for SPDI, given a cyclic signature, an *invariant set* is a set of points which keep rotating in the cycle forever and the *invariance kernel* is the largest of such sets. We show that this kernel is a non-convex polygon (often with a hole in the middle) and we give a non-iterative algorithm for computing the coordinates of its vertexes and edges.

3.1 Preliminaries on TAMFs

Before showing how to compute the invariance kernel of SPDIs, we need some results about TAMFs and STAY cycles. This section is rather technical and it can be skipped in a first reading.

The following lemma gives a characterization of $\tilde{F}^{-1}$:

Lemma 2 *If I' is non empty, then $\tilde{F}^{-1}(I) = I'$ iff $F(I') = I$.*

Proof: That $F(I') = I$ implies $\tilde{F}^{-1}(I) = I'$ follows directly from the definition of $\tilde{F}^{-1}$.

Assume now that $\tilde{F}^{-1}(I) = I'$, then by definition I' is the greatest non-empty interval such that for all $x \in I'$, $F(x) \subseteq I$. Hence $F(I') \subseteq I$. We prove now that $I \subseteq \tilde{F}^{-1}(I')$. Let $x \in I$, then it exists $y \in \tilde{F}^{-1}(x)$, i.e. $x \in F(y)$.

Thus, $x \in F(I')$ and then $I \subseteq \tilde{F}^{-1}(I')$. We have then prove that $\tilde{F}^{-1}(I) = I'$ implies $F(I') = I$. $\square$

It is not difficult to show that $\tilde{F}^{-1} = \langle f_l^{-1}, f_u^{-1} \rangle$:

Lemma 3 *Given an AMF $F = \langle f_l, f_u \rangle$, then $\tilde{F}^{-1} = \langle f_l^{-1}, f_u^{-1} \rangle$.*

Proof: Let $I = \langle l_I, u_I \rangle$ and suppose that $I' = \langle l_{I'}, u_{I'} \rangle$ is a non-empty interval.

$$
\begin{array}{lll}
\tilde{F}^{-1}(I) = I' & \Longleftrightarrow F(I') = I & \text{(by Lemma 2)} \\
& \Longleftrightarrow \langle f_l(l_{I'}), f_u(u_{I'}) \rangle = \langle l_I, u_I \rangle \rangle \\
& \Longleftrightarrow f_l(l_{I'}) = l_I \ \wedge \ f_u(u_{I'}) = u_I \\
& \Longleftrightarrow l_{I'} = f_l^{-1}(l_I) \ \wedge \ u_{I'} = f_u^{-1}(u_I) \\
& \Longleftrightarrow \langle l_{I'}, u_{I'} \rangle = \langle f_l^{-1}(l_I) = f_u^{-1}(u_I) \rangle \\
& \Longleftrightarrow I' = \langle f_l^{-1}(l_I) = f_u^{-1}(u_I) \rangle \\
& \Longleftrightarrow \tilde{F}^{-1}(I) = \langle f_l^{-1}(l_I) = f_u^{-1}(u_I) \rangle
\end{array}
$$

$\square$

The following Lemma relate $\tilde{F}^{-1}$ with $\tilde{\mathcal{F}}^{-1}$:

Lemma 4 *Given a TAMF $\mathcal{F}(I) = F(I \cap S) \cap J$, then $\tilde{\mathcal{F}}^{-1}(I) = \tilde{F}^{-1}(I \cap J) \cap S$.*

Proof: Similar to Lemma 1, page 41 of [Sch02]. $\square$

The following lemmas state some properties of STAY cycles.

Lemma 5 *For STAY cycles, $F(S \cap J) \subseteq S \cap J$.*

Proof: Direct from the definition of STAY cycles. $\square$

Lemma 6 *For STAY cycles, $\mathcal{F}(S \cap J) = F(S \cap J)$.*

Proof: Suppose that $x \in \mathcal{F}(S \cap J)$, then $x \in F(S \cap J) \cap J$ by definition of $\mathcal{F}$, thus $x \in F(S \cap J)$. Let's prove now that $F(S \cap J) \subseteq \mathcal{F}(S \cap J)$. Let $x \in F(S \cap J)$, then $x \in S \cap J$ by Lemma 5. Hence $x \in F(S \cap J) \ \wedge \ x \in S \cap J$, i.e. $x \in F(S \cap J) \cap (S \cap J)$ or what is the same, $x \in (F(S \cap J) \cap J) \ \wedge \ x \in S$. Thus $x \in \mathcal{F}(S \cap J)$. $\square$

Lemma 7 *If $\mathcal{F}(I) = F(I)$ then for all interval $H \subseteq I$, $\mathcal{F}(H) = F(H)$.*

Proof: Direct by monotonicity of $\mathcal{F}$ and F. $\square$

Lemma 8 *For STAY cycles, $J \subseteq S$.*

Proof: By normalization, $J = \mathcal{F}(S)$ and by definition of STAY $\mathcal{F}(S) \subseteq S$, thus $J \subseteq S$. $\square$

Lemma 9 *For STAY cycles, the greatest interval I_K such that $\mathcal{F}(I_K) = F(I_K)$ is bounded by $S \cap J$ and S, i.e. $S \cap J \subseteq I_K \subseteq S$.*

Proof: That $S \cap J \subseteq I_K$ follows immediately from Lemma 6 and the hypothesis. We prove now that $I_K \subseteq S$. By hypothesis and by definition of $\mathcal{F}$, $F(I_K) = F(I_K \cap S) \cap J$, then $F(I_K) \subseteq F(I_K \cap S)$, but by monotonicity of F this is only possible if $I_K \subseteq I_K \cap S$ that implies that $I_K \subseteq S$. □

Lemma 10 *For STAY cycles, $\tilde{F}^{-1}(J) \subseteq S$.*

Proof: By Lemma 8, $J \subseteq S$ and by Lemma 5 $F(J) \subseteq J$. On the other hand $\mathcal{F}(S) = J$ and then $J \subseteq F(S)$ (since $\mathcal{F}(H) \subseteq F(H)$ for any H). Applying $\tilde{F}^{-1}$ to both sides we obtain $\tilde{F}^{-1}(J) \subseteq \tilde{F}^{-1}(F(S))$ and by Lemma 11, $\tilde{F}^{-1}(F(S)) = S$ and then $\tilde{F}^{-1}(J) \subseteq S$. □

Lemma 11 *If $\tilde{F}^{-1}(I) \neq \emptyset$, then $\tilde{F}^{-1}(F(I)) = F(\tilde{F}^{-1}(I))$.*

Proof: Follows directly from the definition of $\tilde{F}^{-1}$. □

Lemma 12 *For STAY cycles, $\mathcal{F}(\tilde{\mathcal{F}}^{-1}(J)) = J$.*

Proof:

$$
\begin{array}{lll}
J & = \mathcal{F}(\tilde{\mathcal{F}}^{-1}(J)) & \\
& = \mathcal{F}(\tilde{F}^{-1}(J) \cap S) & \text{(by definition of } \tilde{\mathcal{F}}^{-1}) \\
& = F(\tilde{F}^{-1}(J) \cap S) \cap J & \text{(by definition of } \mathcal{F}) \\
& = F(\tilde{F}^{-1}(J)) \cap J & \text{(by Lemma 10)} \\
& = J \cap J & \text{(by Lemma 11)} \\
& = J &
\end{array}
$$

□

Lemma 13 *For STAY cycles, $F(\tilde{\mathcal{F}}^{-1}(J)) = \mathcal{F}(\tilde{\mathcal{F}}^{-1}(J))$.*

Proof: That $\mathcal{F}(\tilde{\mathcal{F}}^{-1}(J)) \subseteq F(\tilde{\mathcal{F}}^{-1}(J))$ follows directly from the definition of F and $\mathcal{F}$. We prove now that $F(\tilde{\mathcal{F}}^{-1}(J)) \subseteq \mathcal{F}(\tilde{\mathcal{F}}^{-1}(J))$. Let $x \in F(\tilde{\mathcal{F}}^{-1}(J))$, then by Lemma 4, $x \in F(\tilde{F}^{-1}(J) \cap S)$. But by Lemma 10 $\tilde{F}^{-1}(J) \subseteq S$ and then $\tilde{F}^{-1}(J) \cap S \neq \emptyset$. We have then that $x \in F(\tilde{F}^{-1}(J)) \cap F(S)$ and by Lemma 12 $x \in J \ \wedge \ x \in F(S)$. On the other hand, from $x \in F(\tilde{F}^{-1}(J) \cap S)$ and $x \in J$, $x \in F(\tilde{\mathcal{F}}^{-1}(J) \cap S) \cap J$ and by definition of $\mathcal{F}$, $x \in \mathcal{F}(\tilde{\mathcal{F}}^{-1}(J))$. □

3.2 Invariance kernel computation

In what follows, let $K \subset \mathbb{R}^2$. We recall the definition of *viable* trajectory.

Definition 9 *A trajectory ξ is* viable *in K if $\xi(t) \in K$ for all $t \geq 0$. K is a* viability domain *if for every $\mathbf{x} \in K$, there exists at least one trajectory ξ, with $\xi(0) = \mathbf{x}$, which is viable in K.* ■

We define now invariance kernels.

Definition 10 *We say that a set K is* invariant *if for any $x \in K$ such that there exists at least one trajectory starting in it, every trajectory starting in x is viable in K. Given a set K, its largest invariant subset is called the* invariance kernel *of K and is denoted by $\mathsf{Inv}(K_\sigma)$.* ■

Recall that $\mathcal{D}_\sigma$ denotes the one-dimensional discrete-time dynamical system defined by Succ_σ, that is $x_{n+1} \in \mathsf{Succ}_\sigma(x_n)$. The concepts above can be defined for $\mathcal{D}_\sigma$, by setting that a trajectory $x_0x_1\ldots$ of $\mathcal{D}_\sigma$ is viable in an interval $I \subseteq \mathbb{R}$, if $x_i \in I$ for all $i \geq 0$. Similarly we say that an interval I in an edge e is invariant if *any* trajectory starting on $x_0 \in I$ is viable in I.

Before showing how to compute the invariance kernel of a cycle, we give a characterization of one-dimensional discrete-time invariant.

Lemma 14 *For $\mathcal{D}_\sigma$ and σ a STAY cycle, the following is valid. If I is such that $F(I) \subseteq I$ and $F(I) = \mathcal{F}(I)$ then I is invariant. On the other hand if I is invariant then $F(I) = \mathcal{F}(I)$.*

Proof: Suppose that $F(I) = \mathcal{F}(I)$ and $F(I) \subseteq I$, then $\mathcal{F}(I) \subseteq I$, thus by definition of STAY and monotonicity of $\mathcal{F}$, we know that for all n, $\mathcal{F}^n(I) \subseteq I$. Hence I is invariant. Let suppose now that I is invariant, then for any trajectory starting on $x_0 \in I$, $x_0x_1\ldots$ is in I and trivially $F(I) = \mathcal{F}(I)$. □

Given two edges e and e' and an interval $I \subseteq e'$ we define the $\forall$-predecessor $\widetilde{\mathsf{Pre}}(I)$ in a similar way to $\mathsf{Pre}(I)$ using the universal inverse instead of just the inverse.

Definition 11 *For $I \subseteq e'$, $\widetilde{\mathsf{Pre}}_{e,e'}(I)$ is the set of points in e such that any successor of such points are in I by a trajectory segment in P.* ■

We have that $\widetilde{\mathsf{Pre}}_{e,e'} = \widetilde{\mathsf{Suc}}^{-1}_{e,e'}$ and, given a signature $\sigma = e_1e_2\ldots e_m$, $\widetilde{\mathsf{Pre}}_\sigma = \widetilde{\mathsf{Suc}}^{-1}_\sigma$.

Theorem 15 *For $\mathcal{D}_\sigma$, if $\sigma = e_1\ldots e_ne_1$ is STAY then $\mathsf{Inv}(e_1) = \widetilde{\mathsf{Pre}}_\sigma(J)$, otherwise $\mathsf{Inv}(e_1) = \emptyset$.*

Proof: That $\mathsf{Inv}(e_1) = \emptyset$ for any type of cycle but STAY follows directly from the definition of each type of cycle.

Let's consider a STAY cycle with signature σ. Let $I_K = \tilde{\mathcal{F}}^{-1}(J) = \widetilde{\mathsf{Pre}}_\sigma(J)$. By Lemma 13, $F(\tilde{\mathcal{F}}^{-1}(J)) = \mathcal{F}(\tilde{\mathcal{F}}^{-1}(J))$ and by STAY property, $F(\tilde{\mathcal{F}}^{-1}(J)) \subseteq \tilde{\mathcal{F}}^{-1}(J)$, thus by Lemma 14 we have that I_K is invariant. We prove now that I_K is indeed the greatest invariant. Let suppose that there exists an invariant $H \subseteq S$ strictly greater than I_K. By assumption we have that $I_K = \tilde{\mathcal{F}}^{-1}(J) \subset H$, then by monotonicity of $\mathcal{F}$, $\mathcal{F}(\tilde{\mathcal{F}}^{-1}(J)) \subset \mathcal{F}(H)$ and by Lemma 12 we have that $J \subset \mathcal{F}(H)$, but this contradicts the monotonicity of $\mathcal{F}$ since $J = \mathcal{F}(S) \subset \mathcal{F}(H)$

and then $S \subset H$ which contradicts the hypothesis that $H \subseteq S$. Hence, $\mathsf{Inv}(e_1) = \widetilde{\mathsf{Pre}}_\sigma(J)$. □

The invariance kernel for the continuous-time system can be now found by propagating $\widetilde{\mathsf{Pre}}(J)$ from e_1 using the following operator. The *extended* $\forall$*-predecessor* of an output edge e of a region R is the set of points in R such that every trajectory segment starting in such point reaches e without traversing any other edge. More formally,

Definition 12 *Let R be a region and e be an edge in $Out(R)$. The* e-extended $\forall$-predecessor *of I, $\widetilde{\overline{\mathsf{Pre}}}_e(I)$ is defined as:*

$$\widetilde{\overline{\mathsf{Pre}}}_e(I) = \{\mathbf{x} \mid \forall \xi \,.\, (\xi(0) = \mathbf{x} \Rightarrow \exists t \geq 0 \,.\, (\xi(t) \in I \wedge \mathsf{Sig}(\xi[0,t]) = e))\}.\blacksquare$$

The above notion can be extended to cyclic signatures (and so to edge-signatures) as follows. Let $\sigma = e_1 \dots e_k$ be a cyclic signature. For $I \subseteq e_1$, the *σ-extended $\forall$-predecessor* of I, $\widetilde{\overline{\mathsf{Pre}}}_\sigma(I)$ is the set of all $\mathbf{x} \in \mathbb{R}^2$ for which any trajectory segment ξ starting in $\mathbf{x}$, reaches some point in I, such that $\mathsf{Sig}(\xi)$ is a suffix of $e_2 \dots e_k e_1$.

It is easy to see that $\widetilde{\overline{\mathsf{Pre}}}_\sigma(I)$ is a polygonal subset of the plane which can be calculated using the following procedure. First compute $\widetilde{\overline{\mathsf{Pre}}}_{e_i}(I)$ for all $1 \leq i \leq n$ and then apply this operation k times:

$$\widetilde{\overline{\mathsf{Pre}}}_\sigma(I) = \bigcup_{i=1}^{k} \widetilde{\overline{\mathsf{Pre}}}_{e_i}(I_i)$$

with

$$\begin{array}{l} I_1 = I, \\ I_k = \widetilde{\mathsf{Pre}}_{e_k e_1}(I_1) \text{ and} \\ I_i = \widetilde{\mathsf{Pre}}_{e_i e_{i+1}}(I_{i+1}), \text{ for } 2 \leq i \leq k-1. \end{array}$$

Now, let define the following set:

$$K_\sigma \quad = \quad \bigcup_{i=1}^{k} (\mathtt{int}(P_i) \cup e_i) \tag{2}$$

where P_i is such that $e_{i-1} \in In(P_i)$, $e_i \in Out(P_i)$ and $\mathtt{int}(P_i)$ is the interior of P_i. We can now compute the invariance kernel of K_σ.

Theorem 16 *If σ is STAY then* $\mathsf{Inv}(K_\sigma) = \widetilde{\overline{\mathsf{Pre}}}_\sigma(\widetilde{\mathsf{Pre}}_\sigma(J))$*, otherwise* $\mathsf{Inv}(K_\sigma) = \emptyset$.

Proof: Trivially $\mathsf{Inv}(K_\sigma) = \emptyset$ for any type of cycle but STAY. That $\mathsf{Inv}(K_\sigma) = \widetilde{\overline{\mathsf{Pre}}}_\sigma(\widetilde{\mathsf{Pre}}_\sigma(J))$ for STAY cycles, follows directly from Theorem 15 and definition of $\widetilde{\overline{\mathsf{Pre}}}$. □

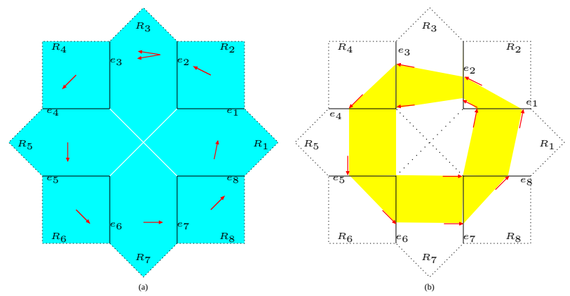

Figure 3: Invariance kernel.

Example 7 Let $\sigma = e_1 \dots e_8 e_1$. Figure 3 depicts: (a) K_σ, and (b) $\widetilde{\overline{\mathsf{Pre}}}_\sigma(\widetilde{\mathsf{Pre}}_\sigma(J))$. ■

4 Conclusion

The main contribution of this paper is an automatic procedure to obtain an important object of the phase portrait of polygonal differential inclusions systems (SPDIs [Sch02]), namely invariance kernels.

Besides the inherent importance of invariance kernels, they play a key role for proving termination of a breadth-first search reachability algorithm for SPDIs (to be published).

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References

[ABDM00] E. Asarin, O. Bournez, T. Dang, and O. Maler. Approximate reachability analysis of piecewise-linear dynamical systems. In *HSCC'00*, number 1790 in LNCS. Springer Verlag, 2000.

[ACH+95] R. Alur, C. Courcoubetis, N. Halbwachs, T. Henzinger, P. Ho, X. Nicollin, A. Olivero, J. Sifakis, and S. Yovine. The algorithmic analysis of hybrid systems. *Theoretical Computer Science*, 138:3–34, 1995.

[AD94] R. Alur and D.L. Dill. A theory of timed automata. *Theoretical Computer Science*, 126:183–235, 1994.

[AMP95] E. Asarin, O. Maler, and A. Pnueli. Reachability analysis of dynamical systems having piecewise-constant derivatives. *Theoretical Computer Science*, 138:35–65, 1995.

[APSY02] E. Asarin, G. Pace, G. Schneider, and S. Yovine. Speedi: a verification tool for polygonal hybrid systems. In E. Brinksma and K.G. Larsen, editors, *CAV'2002*, volume 2404 of *LNCS*, pages 354–358, Copenhagen, Denmark, July 2002. Springer-Verlag.

[ASY01] E. Asarin, G. Schneider, and S. Yovine. On the decidability of the reachability problem for planar differential inclusions. In *HSCC'2001*, number 2034 in LNCS, pages 89–104, Rome, Italy, 2001. Springer-Verlag.

[ASY02] E. Asarin, G. Schneider, and S. Yovine. Towards computing phase portraits of polygonal differential inclusions. In *HSCC'2002*, number 2289 in LNCS, pages 49–61, Stanford, USA, March 2002. Springer-Verlag.

[Aub01] J.-P. Aubin. The substratum of impulse and hybrid control systems. In *HSCC'01*, volume 2034 of *LNCS*, pages 105–118. Springer, 2001.

[BT00] O. Botchkarev and S. Tripakis. Verification of hybrid systems with linear differential inclusions using ellipsoidal approximations. In *HSCC'00*, number 1790 in LNCS. Springer Verlag, 2000.

[CV96] K. Cerāns and J. Vīksna. Deciding reachability for planar multi-polynomial systems. In *Hybrid Systems III*, number 1066 in LNCS. Springer Verlag, 1996.

[DM98] T. Dang and O. Maler. Reachability analysis via face lifting. In *HSCC'98*, number 1386 in LNCS, pages 96–109. Springer Verlag, 1998.

[DV95] A. Deshpande and P. Varaiya. Viable control of hybrid systems. In *Hybrid Systems II*, number 999 in LNCS, pages 128–147, 1995.

[GM99] M. R. Greenstreet and I. Mitchell. Reachability analysis using polygonal projections. In *HSCC'99*, number 1569 in LNCS, pages 103–116. Springer Verlag, 1999.

[HKPV95] T.A. Henzinger, P.W. Kopke, A. Puri, and P. Varaiya. What's decidable about hybrid automata? In *27th Annual Symposium on Theory of Computing*, pages 373–382. ACM Press, 1995.

[KdB01] P. Kowalczyk and M. di Bernardo. On a novel class of bifurcations in hybrid dynamical systems. In *HSCC'01*, number 2034 in LNCS. Springer, 2001.

[KV96] M. Kourjanski and P. Varaiya. Stability of hybrid systems. In *Hybrid Systems III*, number 1066 in LNCS, pages 413–423. Springer, 1996.

[KV00] A.B. Kurzhanski and P. Varaiya. Ellipsoidal techniques for reachability analysis. In *HSCC'00*, number 1790 in LNCS. Springer Verlag, 2000.

[LPY99] G. Lafferriere, G. J. Pappas, and S. Yovine. A new class of decidable hybrid systems. In *Hybrid Systems : Computation and Control*, volume 1569 of *Lecture Notes in Computer Science*, pages 137–151. Springer Verlag, 1999.

[MP93] O. Maler and A. Pnueli. Reachability analysis of planar multilinear systems. In C. Courcoubetis, editor, *Proceedings of the 4th Computer-Aided Verification*. LNCS 697, Springer Verlag, July 1993.

[MS00] A. Matveev and A. Savkin. *Qualitative theory of hybrid dynamical systems*. Birkhäuser Boston, 2000.

[Sch02] Gerardo Schneider. *Algorithmic Analysis of Polygonal Hybrid Systems*. PhD thesis, VERIMAG – UJF, Grenoble, France, July 2002.

[SJSL00] S. Simić, K. Johansson, S. Sastry, and J. Lygeros. Towards a geometric theory of hybrid systems. In *HSCC'00*, number 1790 in LNCS. Springer, 2000.

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