COHERENCE SPACES AS EVENT STRUCTURES AND CONCRETE DATA STRUCTURES[1]

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Abstract

In [ZHA 89] Zhang presents a relation between coherence spaces (that were introduced by Girard [GIR 89] for modelling the system F and to give a semantic for linear logic) and event domains [WIN 81] using theory of categories. In this paper we give a characterization of the relation between coherence spaces, event domains and concrete domains [KAH 93], as well as a relation between the "concrete counterparts" of these domains: webs of coherence spaces, event structures and concrete data structures (cds).

According to Zhang [ZHA 89, p. 156], a coherence space can be seen as a particular case of dI-domains[2] and of event structures too. With respect to the second "characterization" of coherence spaces (related with event structures), we show that the former are related with event domains (the abstract counterpart of the event structures) and not with the event structures as asserts Zhang. The results are presented by twelve propositions and theorems

1 Introduction

In this paper, we study the relationship between coherence spaces, event domains and concrete domains, as well as between their "concrete" counterparts: webs of coherence spaces, event structures and concrete data structures (cds). Zhang [ZHA 89] shows how different categories are related, and in particular Zhang shows the relationship between

[1] This work was supported by UCPel and FAPERGS under QaP-For project (Aplicação de Métodos Formais para a melhoria da Qualidade e Produtividade em Software).

[2] A dI-domain is a consistently complete domain D that satisfies the axiom d (distributivity) and the axiom I:

axiom d) $\forall x,y,z \in D,\ y \uparrow z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

axiom I) $\forall d \in D^0,\ |\{x \mid x \leq d\}| < \infty$.

coherence spaces and event structures as categories, but a characterization of this relationship is not given.
Zhang [ZHA 89, p. 156] says that a coherence space can be seen as a particular case of a *dI*-domain and of event structure too. As a *dI*-domain, a coherence space is a coherent domain D. As a stable event structure, a coherence space is a structure (E, Con, $\{\varnothing \vdash e \mid e \in E\}$), where Con is defined by a *conflict relation*.

With respect to the second "characterization" of the coherence spaces (i.e. related with event structure) we show in this paper that coherence spaces are related with the event domains (the abstract counterpart of the event structures) and not with the event structures as assert Zhang.

2 Preliminary Concepts

We will see many useful definitions in this section [SCH 96 chapter 2].

A *partial order* (*po*) is a pair $(D, \leq)$ where D is a non empty set and $\leq$ is binary relation that satisfies: $\forall x \in D$, $x \leq x$ (reflexivity), $\forall x, y \in D$, $x \leq y$, $y \leq x \Rightarrow x = y$ (anti-symmetry) and $\forall x,y,z \in D$, $x \leq y$, $y \leq z \Rightarrow x \leq z$ (transitivity). We say that two elements are *comparable* when $x \leq y$ or $y \leq x$. If this doesn't happen then we say that the elements are *incomparable* ($x \parallel y$). A partial order where any two elements are comparable is a *chain*. Let $(D, \leq)$ be a *po*, H a subset of D and x an element of D. We say that x is *upper bound* of H iff $\forall y \in H$, $y \leq x$ (x is a *lower bound* of H iff $\forall y \in H$, $x \leq y$). The element x is a *least upper bound* (or *supremum*) of H iff it is an upper bound of H and $\forall z$ upper bound of H, $x \leq z$ (x is a *great lower bound* (or *infimum*) of H iff it is an lower bound of H and $\forall z$ upper bound of H, $z \leq x$). An element x in H is a *maximum* iff x is the supremum of H and is the *minimum* iff it is the infimum of H. We denote the supremum (minimum) x of H by $x = \sup(H)$ ($x = \inf(H)$) or $x = \bigcup H$ ($x = \bigcap H$). If $H = \{a, b\}$, then they are respectively denoted as $x = a \vee b$ ($x = a \cup b$) and $x = a \wedge b$ ($x = a \cap b$).

We say that two elements x and y of D are *compatible* if $\{x, y\}$ has an upper bound. This relationship are denoted as $x \uparrow y$ and its complement, the relation of *incompatibility*, is written $x \# y$. In a po $(D, \leq)$, a subset X of D is *directed* iff X is non empty and

$\forall x,y \in X$, $\exists z \in X$ such that $x \leq z$, $y \leq z$. A po $(D, \leq)$ is *complete* (or *cpo*) iff D has a "bottom" element $\perp$, and any directed subset X of D has a supremum. A po $(D, \leq)$ is *conditional complete* iff any subset X of D that has an upper bound, has a supremum. In a po $(D, \leq)$ a subset X of D is *consistent* iff any two elements of X are compatible. We say that a po $(D, \leq)$ is *consistently complete (coherent)* iff every consistent subset X of D has a supremum. A cpo $(D, \leq)$ is consistently complete iff every consistent 3-tuple $\{x, y, z\}$ has a supremum. In a po $(D, \leq)$ an element x é *isolated* (or *compact*) iff in any directed set that dominates[3] x, we can find an element y that dominates x ($\forall X \subset D$, X directed and $x \leq \bigcup X \Rightarrow \exists y \in X, x \leq y$).
Note: The set of compact elements less (with respect to $\leq$) than x are denoted $\mathcal{A}(x)$. An element in $\mathcal{A}(x)$ is called an *approximation* of x. The set of all the compact elements in $(D, \leq)$ is written $\mathcal{A}(D)$ or D^0. An element x is compact iff $x \in \mathcal{A}(x)$, so $\mathcal{A}(D) = \bigcup_{x \in D} \mathcal{A}(x)$.

[3]We say that an element x *dominates* y in a po $(D, \leq)$ if $y \leq x$.

We say that a po $(D, \leq)$ is *algebraic* iff for all x in D the set $\mathcal{A}(x)$ is directed and $x = \bigcup\mathcal{A}(x)$. If $\mathcal{A}(D)$ is numerable, then $(D, \leq)$ is *ω-algebraic*. A consistently complete and ω-algebraic partial order is a *computational domain*.

Let $(D, \leq)$ a po, then we define:

- a *cover chain* (or *chain*) from x to y is a sequence $x = x_0,..., x_n = y$, denoted by $x_0 \prec ... \prec x_n$, such that $\forall 0\leq i<n, x_i \prec x_{I+1}$; if C is a chain $x_0 \prec ... \prec x_n$ from x to y, then $x' \prec x_0 \prec ... \prec x_n$ is denoted by $x' \prec C$ (and by $C \prec y'$ if $y \prec y'$)
- a *prime interval* of D, denoted by $[x, x']$, is such that x, x' are compact elements and $x \prec x'$; (i.e., a prime interval from x to x' contains only x and x')
- the *cover* relation $\prec$ between prime intervals by $[x, x'] \prec [y, y']$ iff $x \prec y$, $y\neq x'$ and $x' \prec y'$
- if s, t are elements (prime intervals) of D, we call *zigzag* from s to t, a sequence $s = s_0,..., s_n = t$ of elements (prime intervals) of D such that $\forall i<n, s_i\prec s_{i+1}$ or $s_{i+1}\prec s_i$
- we call *equivalence* (denoted by $\succ\!\prec$) the reflexive, symmetric and transitive closure of $\prec$ in prime intervals, i.e., $[x, x'] \succ\!\prec [y, y']$ iff exists a zigzag from $[x,x']$ to $[y, y']$; the equivalence class of $[x, x']$ is denoted by $[x, x']_{\succ\prec}$. Two intervals $[x, x']$ and $[y, y']$ such that $[x, x'] \succ\!\prec [y, y']$ are said to be *equivalent* or *projective*.

A *concrete domain* is a ω-algebraic cpo $(D, \leq)$ that satisfies the following axioms for all compact elements $x, x', y, y'\in D$:

(F) $\{y\in D \mid y \leq x\}$ is finite;

(C) if $x \prec y$, $x \prec z$, $z \neq y$ and $y\uparrow z$, then exists $y\vee z$, and $y \prec y\vee z$, $z \prec y\vee z$;

(R) $[x, y] \succ\!\prec [x, z] \Rightarrow y = z$.

(Q) if z, x, y are compact elements such that $z \prec x$, $z \prec y$ and $x\#y$, then exists only one x' such that $z \prec x'\leq y$ and $x\#x'$. A *concrete data structure* (*cds*), $(C, V, E, \vdash)$, is given by three sets C, V, E of *cells*, *values* and *events* respectively, such that: 1) $E \subseteq C \times V$ and $\forall c\in C, \exists v\in V$ such that $(c, v)\in E$ ("any cell can be filled"), 2) a relation $\vdash$, called an *enabled relation* between finite parts of E and elements of C. We say that $\{e_1,..., e_n\}$ is an *enabling* of c if $\{e_1,..., e_n\} \vdash c$, and we write $e_1,..., e_n \vdash c$. A cell c such that $\vdash c$, is called *initial*. C and V are considered countable. A *state* is a subset x of E such that: (1) $(c, v_1), (c, v_2) \in x \Rightarrow v_1 = v_2$, and (2) if $(c, v) \in x$, then exists a sequence of events $e_1,..., e_n = (c, v)$ such that $e_i = (c_i, v_i) \in x$ and $\{ e_j \mid j < i\}$ contains an enabling for c_i for all $i \leq n$.

An *event structure* E = (E, #, $\vdash$) is a countable set E of *events*, a symmetric binary relation "#" in E of *conflict* and an *enabling* relation "$\vdash$" between finite parts of E and elements of E. A *state* of (E, #, $\vdash$) is a set x of events that verifies the following conditions of *consistence* and *safety*: 1) $e_1 \in x \wedge e_1 \# e_2 \Rightarrow e_2 \notin x$; 2) $e \in x \Rightarrow \exists\, e_0, e_1, ..., e_n \in x$ s.t. $(\forall i\leq n, \exists X_i \subseteq \{ e_j / j<i \}; X_i \vdash e_j)$. The set of states E ordered by inclusion, denoted by $(D(E), \leq)$, is the *canonical event domain associated with E* [DRO 89]. The main difference between these structures and cds is the absent of cells and that the notion of conflict is given "a priori".

An *event domain* is a ω-algebraic cpo $(D, \leq)$ that satisfies the following conditions for any compact elements $x, x', y, y' \in D$:

(F) $\{y\in D \mid y \leq x\}$ is finite;

(C) if $x \prec y$, $x \prec z$, $z \neq y$ and $y\uparrow z$, then exists $y\vee z$, and $y \prec y\vee z$, $z \prec y\vee z$;

(R) $[x, y] \succ\!\prec [x, z] \Rightarrow y = z$.

(V) $[x, x'] \succ\prec [y, y'], [x, x''] \succ\prec [y, y'']$ and $x'\uparrow x'' \Rightarrow y'\uparrow y''$.

A *Droste's event domain*[4] is an algebraic cpo $(D, \leq)$ that satisfies the following conditions for all $x, x', y, y', z \in D^0$:

(F) $\{y\in D \mid y \leq x\}$ is finite;

(C) if $x \prec y$, $x \prec z$, $z \neq y$ and $y\uparrow z$, then exists $y\vee z$, and $y \prec y\vee z$, $z \prec y\vee z$;

(I) $[x, x'] \succ\prec [y, y'] \wedge x \leq y \Rightarrow x' \leq y'$.

A *web* $\boldsymbol{A} \equiv (A, \approx_A)$[5] is a pair formed by a set A in which is defined a symmetric and reflexive relation $\approx_A$. A subset x of a web $\boldsymbol{A}$ is coherence iff $\forall\alpha, \beta \in x, (\alpha \approx_A \beta)$. The set of all the *coherence subsets* of $\boldsymbol{A}$ is denoted by $Coh(\boldsymbol{A})$. We have that $Coh(\boldsymbol{A}) = Coh(A, \approx_A) = \{ x \subseteq A \mid \forall\ \alpha, \beta \in x, (\alpha \approx_A \beta) \}$.

A *coherence space* (or *Girard's domain*) $\mathcal{A} = (Coh(A, \approx_A), \subset)$ is a collection of coherence subsets of a web $\boldsymbol{A}$ partially ordered by inclusion. The elements of A are the *tokens* (or *atoms*) of the domain $\mathcal{A}$.

3 Coherence Spaces, Event Domains and Concrete Domains

In this section we give a characterization of coherence spaces as event domains and concrete domains. As a natural consequence, webs of coherence spaces will be characterized as event structures and cds.

Droste [DRO 89] gives the following definition of event domains:

3.1 Proposition

Every event domain is a Droste's event domain. ■

Droste gives a proof of this proposition [DRO 89, p. 43].

3.2 Proposition

Let $(E, \#, \vdash)$ be an event structure such that $\forall e\in E, \vdash e$. Then $(E, \approx)$ is a web of a coherence space where

$\forall e_1, e_2 \in E, e_1 \approx e_2$ iff $\neg(e_1 \# e_2)$.

Proof)

A web is defined as a set together with a reflexive and symmetric relation, which implies that the corresponding coherence space has as atoms all the singletons. The canonical event domain associated with the event structure $(E, \#, \vdash)$ not contains necessary all the singletons $\{e\}$ (for all event $e\in E$), because it depends on the enabled relation $\vdash$ (this is the reason why we settled the condition $\forall e\in E, \vdash e$). So, given a event structure $(E, \#, \vdash)$, the only thing we have to prove is that "$\approx$" is a reflexive and symmetric relation.

Reflexivity: $\forall e_1\in E, \neg(e_1 \# e_1)$, because if $e_1\in x$ and $e_1 \# e_1$, then, by the definition of #, $e_1\notin x$ and we have a contradiction. So $e_1 \approx e_1$.

Symmetry: $\forall e_1, e_2 \in E, e_1 \approx e_2$ iff $\neg(e_1 \# e_2)$ iff $\neg(e_2 \# e_1)$ iff $e_2 \approx e_1$.

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[4]What we call here *Droste´s event domain*, Droste [DRO 89, p. 40] calls *event domain*, and what we call *event domain*, Droste calls *conflict event domain* (see [CUR 86, p. 133] and [DRO 89, p. 43]).

[5]Girard [GIR 89, p. 56] defines a web as $|\mathcal{A}| \equiv \cup\mathcal{A} = \{e; \{e\}\in\mathcal{A}\}$, where $\mathcal{A}$ is a coherence space and $\alpha \approx_A \beta$ iff $\{\alpha, \beta\}\in \mathcal{A}$.

From the results above we have that $(E, \approx)$ is a web of a coherence space. ■

3.3 Proposition

Let $(D, \subseteq)$ be an event domain. Then, $(D, \subseteq)$ is a coherence space if and only if D satisfies the following property (of down closure):

(dc) $x \in D, x' \subseteq x \Rightarrow x' \in D$.

In other words: (F), (C), (R), (V), (dc) $\Rightarrow$ (bc).

***Proof*)**

Intuitively, for all $e \in E$ an event structure must satisfies the condition $\vdash e$ to be a web of a coherence space, so the event domain must satisfies $\forall e$, $\{e\} \in D$. The (dc) property implies that $\vdash e$, so we could deduce the (bc) property from D. ■

As a corollary of the above proposition, we have that D must have all the singletons (atoms of the coherence space).

3.4 Theorem

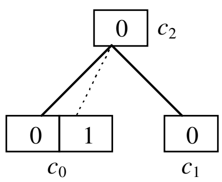
Let D be an event domain such that it is a coherence space. Then, the canonical event structure obtained by the Winskel's Theorem is a web of a coherence space.

***Proof*)**

Let $[x, x']_{\succ\prec}$ be the equivalence class of x, so $x' = x \cup \{e\}$, and by the (dc) property of coherence spaces, $\{e\} \in D$. This implies that $\vdash e$, where e is any event. But $\forall e$, $e = [x, x']_{\succ\prec}$, for some $x, x' \in D^0$, and then we have that $\forall e$, $\vdash e$. We conclude, by proposition 3.2, that an event structure generated by the event domain (coherence space) is a web. ■

Note: This result is intuitive because the event domains (in particular the concrete domains) that do not satisfy $\forall e$, $\vdash e$, do not have the (dc) property (down closure) of the coherence spaces (used for prove the proposition above). We can see this in the following example (over concrete domains) given by Curien & Berry in [BER 82, p. 271]:

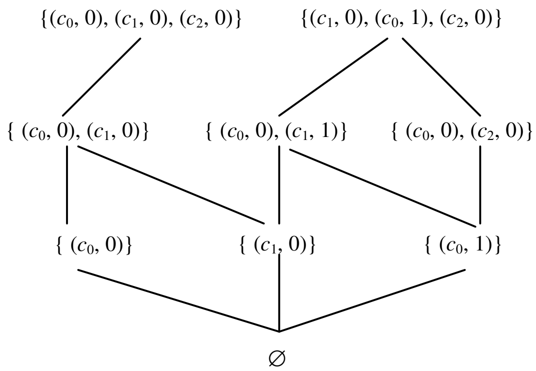
Let $\mathbb{Q}$ be the cds:



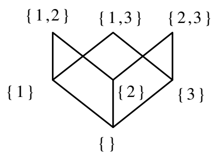
where

$$\vdash c_0$$
$$\vdash c_1$$
$$(c_0, 0), (c_1, 0) \vdash c_2$$
$$(c_0\ 1) \vdash c_2.$$

The set of states is the following concrete domain:



This domain isn't a coherence space because it doesn't satisfy the (dc) property (the subset {(c_1, 0), (c_2, 0)} should be a state) because c_2 is enabled by sets of events different than ∅ ((c_0, 0), (c_1, 0) $\vdash c_2$ and (c_0 1) $\vdash c_2$). An example of a domain that doesn't satisfy the binary completeness (bc) property is the following: $E = \{1,2,3\}$, with $\forall e \in E, \vdash e$:



It isn't a coherence space because it doesn't satisfy the (bc) property: we have that $M = \{1,2,3\} \subseteq E$, $\{1\} \cup \{2\} \in E$, $\{2\} \cup \{3\} \in E$ and $\{1\} \cup \{3\} \in E$, but $\cup M = \{1,2,3\} \notin E$, then the proposition 3.3 can't be applied. By proposition 3.2, we can obtain a web because $\forall e \in E, \vdash e$ but this lead us to a contradiction. In this domain we have 3 equivalence classes of prime intervals:

$1 \equiv \{[\varnothing,\{1\}], [\{2\},\{1,2\}], [\{3\},\{1,3\}]\} = [\varnothing,\{1\}]_{\succ\prec}$
$2 \equiv \{[\varnothing,\{2\}], [\{1\},\{1,2\}], [\{3\},\{2,3\}]\} = [\varnothing,\{2\}]_{\succ\prec}$
$3 \equiv \{[\varnothing,\{3\}], [\{2\},\{2,3\}], [\{1\},\{1,3\}]\} = [\varnothing,\{3\}]_{\succ\prec}$

Clearly this is a Droste's event domain (it holds properties (F), (C) and (I)). But it isn't an event domain, because for example $[\varnothing,\{1\}] \succ\prec [\{2\},\{1,2\}]$, $[\varnothing,\{3\}] \succ\prec [\{2\},\{2,3\}]$ and $\{1\} \uparrow \{3\}$. However, this doesn't imply that $\{1,2\} \uparrow (\{2,3\}$, which means that property (V) doesn't hold.

The example above shows why property (dc) is necessary in proposition 3.3 for an event domain to be a coherence space.

3.5 Proposition

(C) ∧ (Q) ∧ (dc) ⇒ (bc).

Proof)

Khan & Plotkin [KHA 93, p. 216] claim that the following domain doesn't satisfy property (Q):

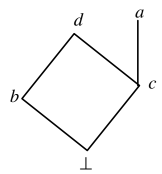
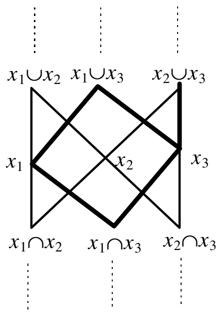
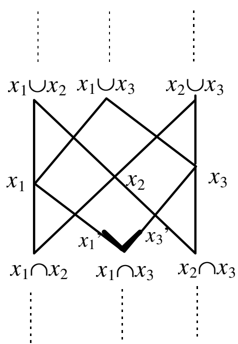


Fig. 1: Domain that not satisfy property (Q)

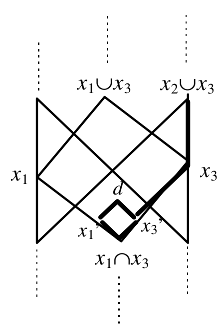
Let's suppose that property (dc) doesn't hold, i.e. if $(E, \subseteq)$ is a domain such that $M \subseteq E$, $\forall x_1, x_2 \in M$, $x_1 \cup x_2 \in E$ and $\bigcup M \notin E$. Let's see how this domain looks like. For example, let $M = \{x_1, x_2, x_3\}$ be a set of states. By property (dc), exist $x_1 \cap x_2$, $x_1 \cap x_3$ and $x_2 \cap x_3$, so the domain has the following form:



where the part in bold remember Fig. 1. For the hypothesis became valid, we must have x_1', x_3' such that $x_1' \cap x_3 \prec x_1'$, where $x_1 \cap x_3 = x_1' \cap x_3 \prec x_1' \subseteq x_1$. In the same way, we have that $x_1 \cap x_3 \prec x_3' \subseteq x_3$, i.e., we have



By condition (C), because we know that $x_1 \cap x_3 \prec x_1'$, $x_1 \cap x_3 \prec x_3'$, $x_1' \neq x_3$ and $x_1' \uparrow x_3'$,we have that $\exists\ d = x_1' \cup x_3'$ and $x_1' \prec x_1' \cup x_3'$, $x_3' \prec x_1' \cup x_3'$:



So, we have x_1' # $x_2 \cup x_3$ and $x_1' \cap (x_2 \cup x_3) = x_1 \cap x_3 \prec x_1'$, but it doesn't exist an element t such that t # x_1' and $x_1' \cap (x_2 \cup x_3) \prec t \leq x_2 \cup x_3$, because $t = x_3'$ and $x_1' \uparrow x_3'$, which contradicts the property. Such situation is possible because $\bigcup M \notin E$ and then $x_1 \cup x_2 \cup x_3 = (x_1 \cup x_3) \cup (x_2 \cup x_3) \notin E$, and this implies that $(x_1 \cup x_3)$ must be compatible with $(x_2 \cup x_3)$, i.e. $x_1 \cup x_2 \cup x_3 \in E$.

Clearly, this proof can be extended to subsets of E with cardinality greater than three. Following this construction we will always find a structure like the one on Fig. 1. ■

We will see now when a domain concrete become a coherence space. The domains in Fig. 2 are concrete domains, but they aren't coherence spaces. The domain of Fig. 2(a) doesn't hold property (dc) and the one of Fig. 2(b) doesn't hold property (bc). A concrete domain that holds property (dc) will be a coherence space.

(a) (b)

Fig. 2: Concrete domains that are not coherence spaces

3.6 Corollary

Let D a concrete domain such that

(dc) $x \in D, x' \subseteq x \Rightarrow x' \in D$

holds. Then D is a coherence space.

***Proof*)**

A concrete domain satisfies properties (F), (C), (R) and (Q), and a coherence space must satisfy (dc) and (bc). D satisfies the assumption of proposition 3.5, so D satisfies (bc) which implies that D is a coherence space. ■

3.7 Proposition

Every coherence space E is an event domain.

***Proof*)**

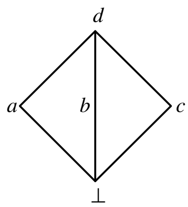
We must prove that E is an algebraic cpo and that it satisfy properties (F), (C), (R) and (V).

<u>E is a cpo</u>: We have to show that:

1. $\subseteq$ is a complete partial order: obvious, because the relation is the inclusion between sets;
2. exists an element "bottom": obvious, let $\emptyset = \perp$ be the bottom element;
3. given a directed set $A \subseteq E$ (i.e. if $A \neq \emptyset$ and if $x,y \in A$, $\exists z \in A$ such that $x \subseteq z$ and $y \subseteq z$), then $\exists$ sup(A) in E: Given a directed set $A \subseteq E$ clearly sup(A) = $\bigcup A$ and by property (bc), $\bigcup A \in E$, so exists sup(A) in E.

From 1., 2. and 3. we have that E is a cpo.

$\underline{E \text{ is algebraic}}$: We have to prove that $\forall x \in E$ the set of the approximations $\mathcal{A}(x)$ of x (the set of the compact elements less or equal than x) is directed and $x = \bigcup \mathcal{A}(x)$.
Let $x \in E$ and by definition $\mathcal{A}(x) \subseteq x$, by property (dc), $\mathcal{A}(x) \in E$. But $A \neq \varnothing$ because $\varnothing \subseteq x$. Let $y, z \in \mathcal{A}(x)$, so by definition we have that $y \subseteq x$ e $z \subseteq x$, which implies that $\mathcal{A}(x)$ is directed.
We show now that $x = \bigcup \mathcal{A}(x)$. Let's suppose that $x \neq \bigcup \mathcal{A}(x)$, then we have that:
1. $x \supset \bigcup \mathcal{A}(x)$: then $\exists e \in x$ such that $e \notin \bigcup \mathcal{A}(x)$, i.e., e doesn't belong to any approximation of x ($\forall y \in \bigcup \mathcal{A}(x)$, $e \notin y$), but $\{e\} \subseteq x$, and by property (dc), $\{e\} \in E$, which implies that $\{e\} \in \mathcal{A}(x)$. In other words, we have that $\exists y \in \mathcal{A}(x)$ such that $e \in y$, what is a contradiction!
2. $x \subset \bigcup \mathcal{A}(x)$: then $\exists e \in \bigcup \mathcal{A}(x)$ such that $e \notin x$, i.e., $\exists y \in \mathcal{A}(x)$ such that $e \in y$, which implies that $\neg(y \subseteq x)$, i.e., $y \notin \mathcal{A}(x)$, which is a contradiction!
From 1. and 2. we have that $x = \bigcup \mathcal{A}(x)$.
$\underline{E \text{ satisfies property (F)}}$: We have to prove that $\{d \in E \mid d \leq x\}$ is finite, with $x \in E^0$.
We know that the web of E is an event structure (see proposition 3.2) and that the set of states of E ($D(E)$), ordered by inclusion, form an event domain ($D(E)$, $\subseteq$), where the compact elements are the finite states (sets of finite states), which is a finite set.
$\underline{E \text{ satisfies property (C)}}$: We have to prove that if $x \prec y$, $x \prec z$, $z \neq y$ and $y \uparrow z$, then exists $y \vee z$, and $y \prec y \vee z$, $z \prec y \vee z$, for all $x, y, z \in E^0$.
Let $x \prec y$, $x \prec z$, $z \neq y$ and $y \uparrow z$, then $\exists a_1, a_2, a_3 \in E$ such that $y = x \cup \{a_1\}$, $z = x \cup \{a_2\}$ and $a_1 \neq a_2$. If $y \uparrow z$ then $\exists w \in E$ such that $y \subseteq w$ and $z \subseteq w$, so $y \cup z \subseteq w$, and by property (dc) we have that $y \cup z \in E$.
By the other hand, we know that $y \cup z = y \cup (x \cup \{a_2\}) = y \cup \{a_2\}$, i.e., $y \prec y \cup z$. In the same way we have that: $y \cup z = (x \cup \{a_2\}) \cup z = \{a_1\} \cup z$, i.e., $z \prec y \cup z$.
$\underline{E \text{ satisfies property (R)}}$: We have to prove that $[x, y] \succ\!\prec [x, z] \Rightarrow y = z$, for all $x, y, z \in E^0$. Let's consider property (R_F), that is equivalent to (R), that claims that there is no sub-lattice isomorphic to the following lattice:



This is immediate because E is a coherence space and then all the "singletons" (unitary sets) belong to E, which implies that $\perp$ must be $\varnothing$, a must be $\{a\}$, b must be $\{b\}$, c must be $\{c\}$ and d must be $\{a,b,c\}$, which it isn't a coherence space because it doesn't satisfy property (dc).
$\underline{E \text{ satisfies property (V)}}$: We have to show that $[x, x'] \succ\!\prec [y, y']$, $[x, x''] \succ\!\prec [y, y'']$ and $x' \uparrow x'' \Rightarrow y' \uparrow y''$, for all $x, x', x'', y, y', y'' \in E^0$:
If $[x, x'] \succ\!\prec [y, y']$, then $x' = x \cup \{e_1\}$ and $y' = y \cup \{e_1\}$. Similarly we have that $[x, x'']$ $\succ\!\prec [y, y'']$ $x'' = x \cup \{e_2\}$ and $y'' = y \cup \{e_2\}$.
Let's suppose, by absurd, that y' isn't compatible with y'' ($y' \# y''$). Then exist $e' \in y'$, $e'' \in y''$, such that $\neg(e' \approx e'')$. If $e' \approx e''$ then $y' \cup y'' = y \cup \{e_1\} \cup \{e_2\}$ should be an object of E, because E is a coherence space and by definition it contains all the coherent subsets of the web. But $y' = y \cup \{e_1\}$ and $y'' = y \cup \{e_2\}$, and then we have that $e' = e_1$

and $e'' = e_2$, because y is a consistent set of events (because E is a coherence space). Then it couldn't exist $x' \cup x'' = x \cup \{e_1\} \cup \{e_2\}$, which would imply that x' is incompatible with x'', which contradicts the hypothesis. Then E holds property (V).
We have then showed that every coherence space E is an event domain. ■

3.8 Proposição

Every coherence space E is a Droste's event domain.
Proof)
Direct from propositions 3.1 and 3.7. ■

3.9 Proposition

Let E be a coherence space. E is a concrete domain if it satisfies the following:

(Q) if z, x, y are compact elements such that $z \prec x$, $z \prec y$ and $x\#y$, then exists a unique x' such that $z \prec x' \leq y$ and $x\#x'$.

Proof)
Immediate. From proposition 3.8 we know that a coherence space E satisfies (F), (C) and (R), and we have to prove only that (Q) is satisfied, which is satisfied by hypothesis.■

3.10 Theorem

Let E be a coherence space and $|E| = (E, \approx)$ its web. Then $(E, \#, \vdash)$ is an event structure, where:

$$\forall e_1, e_2 \in E, e_1 \# e_2 \text{ if and only if } \neg(e_1 \approx e_2),$$

and

$$\forall e \in |E|, \vdash e.$$

We can say more: the canonical event domain associated with $(E, \#, \vdash)$ is a coherence space.
Proof)
Immediate, because $(E, \#, \vdash)$ defined as above is an event structure. On the other hand, the canonical event domain is the set of states of $(E, \#, \vdash)$, which by hypothesis ($\forall e \in |E|$, $\vdash e$) implies that it contains all the sets $\{e\}$ for all e. It's easy to see that it satisfies (dc) and (bc). ■

3.11 Theorem

Let $M = (C, V, E, \vdash)$ be a cds such that the set of states (concrete domain) be a coherence space. The $(E', \approx)$ is the web of the coherence space, where
1. $E' = E$
2. $\forall e_1, e_2 \in E'$, $e_1 \approx e_2$ iff $\exists c \in C$, $\exists v_1, v_2 \in V$ s. t. $e_1 = (c, v_1)$, $e_2 = (c, v_2)$ and $v_1 \neq v_2$. ■

3.12 Theorem

Let $\boldsymbol{E}$ be a coherence space and its web $|\boldsymbol{E}| = (E, \approx)$. Then it is possible to obtain a cds M such that $(\boldsymbol{E}, \subseteq)$ is isomorphic to $(D(M), \subseteq)$, in the following way:

1. $\forall e \in E, \vdash e$
2. the transitive closure of the negation if $\approx$ is transitive
3. $\forall X, e_1, e_2, X \vdash e_1, \neg(e_1 \approx e_2) \Rightarrow X \vdash e_2$.

***Proof*)**
Given $(E, \approx)$ we can obtain (by theorem 3.11) an event structure $(E, \#, \vdash)$, and by Curien's assertion [CUR 86, section 2.2 p. 139] we have that, given an event structure $(E, \#, \vdash)$ we can obtain a cds such that $D(M)$ and $D(E)$ be isomorphic, given the following conditions:
- the transitive closure of the negation of $\approx$ is transitive,
- $\forall X, e_1, e_2, X \vdash e_1,\ e_1 \# e_2 \Rightarrow X \vdash e_2$. ■

4 Conclusion

We have shown the relationship between coherence spaces, concrete domains and event domains, and between their concrete counterparts.

In [SCH 96] the author defined the *linear algorithms* in a similar way of Curien's sequential algorithms [CUR 86]. Curien defined the sequential algorithms as states of concrete data structures. On the other hand, the "of course" operator of linear logic was used to define linear algorithms, this operator was applied to coherence spaces, which generates a new coherence space. The results of this paper were important to the definition of linear algorithms.

Independently of their application to linear algorithms, the results of this work are important by themselves, because coherence spaces are simpler and easier domains to work than the other ones as each object of a coherence space is a set which must satisfy only symmetry and reflexivity.

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