HIGHLY-OSCILLATORY PROBLEMS WITH TIME-DEPENDENT VANISHING FREQUENCY

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Abstract. In the analysis of highly-oscillatory evolution problems, it is commonly assumed that a single frequency is present and that it is either constant or, at least, bounded from below by a strictly positive constant uniformly in time. Allowing for the possibility that the frequency actually depends on time and vanishes at some instants introduces additional difficulties from both the asymptotic analysis and numerical simulation points of view. This work is a first step towards the resolution of these difficulties. In particular, we show that it is still possible in this situation to infer the asymptotic behaviour of the solution at the price of more intricate computations and we derive a second order uniformly accurate numerical method.

Keywords: highly-oscillatory problems, time-dependent vanishing frequency, asymptotic expansion, uniform accuracy.

AMS subject classification (2010): 74Q10, 65L20.

1. Introduction.

1.1. Context.

Highly-oscillatory evolution equations of the form

$$\dot{U}^\varepsilon(t) := \frac{d}{dt}U^\varepsilon(t) = \frac{1}{\varepsilon}AU^\varepsilon(t) + f\Big(U^\varepsilon(t)\Big), \quad U^\varepsilon(0) = U_0, \quad 0 \le t \le T,$$

where T is a strictly positive fixed time, independent of ε, and where the operator A is supposed to be diagonalizable and to have all its eigenvalues in $i\mathbb{Z}$ (equivalently $\exp(2\pi A) = I$), have received considerable attention in the literature, from both the point of view of asymptotic analysis [Per69, SV85, HLW06, CMSS12, CMSS15, CLM17] and the point of view of numerical methods [CCMSS11, CMMV14, CCMM15]. However, allowing the parameter ε to take values in a whole interval of the form $]0, 1]$, prevents the use of numerical methods constructed for specific regimes. As a matter of fact, standard methods[1] from the litterature [HNrW93, HW10] typically have error bounds expressed as powers of the step-size h of the form[2]

$$\text{error } \le C\frac{h^p}{\varepsilon^q}, \quad p > 0, \quad q > 0,$$

where p is the order of the method and q is equal to p or $p-1$: while suitable for the regime ε close to 1, they require formidable computational power for small values of ε. At the other end of the spectrum, methods based on averaging and designed for small values of ε (see for instance [CMSS10]) typically admit error bounds of the form

$$\text{error } \le C(h^p + \varepsilon^q), \quad p > 0, \quad q > 0,$$

where p is the order of the method and q is the order of averaging: they thus encompass an incompressible error for larger values of ε. In contrast, *uniformly accurate methods*

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[1]Such as, for instance, the Runge-Kutta method used in the Matlab routine ODE45 (see the "Numerical experiments" Section 3.4).

[2]The constant C here is independent of ε and h.

[CLM13, CCLM15, CLMV18] are robust schemes that are able to deliver numerical approximations with an error (and at a cost) independent of the value of $\varepsilon \in]0,1]$,

$$\text{error } \leq C h^p.$$

In this paper, our objective is to construct uniformly accurate methods for equations whose frequency of oscillation *depends on time*. More precisely, we consider systems of differential equations of the form

$$\dot{U}^\varepsilon(t) = \frac{\gamma(t)}{\varepsilon} A U^\varepsilon(t) + f\Big(U^\varepsilon(t)\Big) \in \mathbb{R}^d, \quad U^\varepsilon(0) = U_0 \in \mathbb{R}^d, \quad 0 \leq t \leq T, \tag{1.1}$$

where $A \in \mathbb{R}^{d\times d}$ and where the function f is assumed to be sufficiently smooth. The parameter ε again lies in the whole interval $(0,1]$ and the real-valued function γ is assumed to be continuous on $[0,+\infty)$.

Many semi-classical models for quantum dynamics also assume the form of highly oscillatory PDEs with a varying frequency (which, once discretized in space, obey equation (1.1)), e.g. quantum models for surface hopping [CJLM15], graphene models [MS11], or quantum dynamics in periodic lattice [Mor09]. In such applications, the frequency γ may depend on time (and sometimes also on U^ε and measures the gap between different energy bands, while the parameter ε is nothing but the Planck constant. We emphasize that the case of a varying frequency with a positive lower bound has been studied in [CL17] for surface hopping, in [CJLM] for graphene, and also in [HL16] where the long-term preservation of adiabatic quantities is established in a situation where the right-hand-side of equation (1.1) is Hamiltonian. However, the case where the frequency may become small (e.g. of the order of ε) or even vanish is more delicate and requires special attention from both analysis and numerical points of view. This is the reason why *the main novel assumption* in this article is that the *function γ vanishes at some instant* t_0, or more precisely, that there exists (a unique) $t_0 \in [0,T]$ such that $\gamma(t_0) = 0$.

Our goal is to investigate problem (1.1) under these new circumstances, from both the asymptotic analysis (when $\varepsilon \to 0$) and the numerical approximation viewpoints. For the sake of simplicity in this introductory paper, we assume that $\gamma(t)$ is of the form [3]

$$\exists p \in \mathbb{N}^*, \quad \forall t \geq 0, \quad \gamma(t) = (p+1)(t-t_0)^p.$$

We emphasize that this situation is not covered by the standard theory of averaging as considered e.g. in [Per69, SV85, HLW06, CMSS10, CMSS15, CLM17], and that recent numerical approaches [CCMSS11, CLM13, CCLM15, CLMV18] are ineffective. All techniques therein indeed rely fundamentally on the assumption that $\gamma(t) \geq \gamma_0$ uniformly in time, for some constant $\gamma_0 > 0$, and cannot be transposed to the context under consideration here[4].

[3] Note that applying an analytic time-transformation to (1.1) allows to consider more general analytic functions $\gamma(t)$ and our analysis is not restricted to the polynomial case.

[4] As a related recent work, we also mention the study [AD18] for the uniformly accurate approximation of the stationary Schrödinger equation in the presence of turning points which are spatial points used in quantum tunnelling models and where the spatial oscillatory frequency vanishes (analogously to our assumption $\gamma(t_0) = 0$ with $t_0 = 0$). However, the equation under consideration is *linear and assumed to have an explicit solution* on $[t_0, t_1]$ for some $t_1 > 0$ independent of ε. Beyond t_1, the problem can be handled with a Wentzel-Kramers-Brillouin expansion, since the frequency is then lower bounded by positive constant.

1.2. Formulation as a periodic non-autonomous problem and main results. Upon defining $u^\varepsilon(t) = \exp\left(-\frac{(t-t_0)^{p+1}}{\varepsilon}A\right)U^\varepsilon(t)$, the original equation (1.1) may be rewritten

$$\text{(1.2)}\quad \dot{u}^\varepsilon(t) = F\left(\frac{(t-t_0)^{p+1}}{\varepsilon}, u^\varepsilon(t)\right), \quad u^\varepsilon(0) = u_0^\varepsilon := \exp\left(-\frac{(-t_0)^{p+1}}{\varepsilon}A\right)U_0,$$

where $F(\theta, u) = e^{-\theta A} f(e^{\theta A}u)$ is 2π-periodic w.r.t. θ and smooth in (θ, u). We make the following assumption:

ASSUMPTION 1.1. *The function f is twice continuously differentiable on $\mathbb{R}^d$ and there exists $M > 0$ such that for all $0 < \varepsilon \leq 1$, equation (1.2) with $t_0 \in [0, T]$ has a unique solution on $[0, T]$, bounded by M, uniformly w.r.t. ε.*

In the sequel, C will denote a *generic constant* that only depends on t_0 and on the bounds of $\partial_2^\alpha F$, $\alpha = 0, 1, 2, 3$, on the set $\{(\theta, u), \theta \in \mathbb{T}, |u| \leq 2M\}$, where $\mathbb{T} = [0, 2\pi]$.

The aim of this work is now twofold. On the one hand, we show that, under mild and standard assumptions, an averaged equation (for (1.2) of the form

$$\text{(1.3)}\qquad \forall t \in [0, T], \quad \dot{\underline{u}}^\varepsilon(t) = \langle F\rangle\left(\underline{u}^\varepsilon(t)\right), \qquad \underline{u}^\varepsilon(0) = u_0^\varepsilon$$

persists (in $\langle F\rangle$, function F is averaged w.r.t. the time variable).[5] More precisely, we have the following theorem (see the proof in Section 2.2), which can be refined with the next-order asymptotic term (see Section 2.3).

THEOREM 1.2. *Suppose that Assumption 1.1 is satisfied and consider the solutions $u^\varepsilon(t), \underline{u}^\varepsilon(t)$ of problems (1.2), (1.3), respectively, on the time interval $[0, T]$. Then, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, and all $t \in [0, T]$,*

$$\text{(1.4)}\qquad |u^\varepsilon(t) - \underline{u}^\varepsilon(t)| \leq C\varepsilon^{\frac{1}{p+1}}.$$

Note that the bound $|u^\varepsilon(t) - \underline{u}^\varepsilon(t)| \leq C\varepsilon$ obtained in the classical case [SV85] of a constant frequency (that is to say in the case where $\gamma(t) = 1$ in equation (1.1)), is degraded to (1.4) for $p \geq 1$. For $p = 0$, both estimates coincide.

On the other hand, we construct in the case $p = 1$ a second-order *uniformly accurate* scheme for the approximation of u^ε, that is to say a method for which the error and the computational cost remain independent of the value of ε (for more details on uniformly accurate methods, refer for instance to [CCLM15, CLMV18]).

2. Averaging results. We introduce the following function $\Gamma : [0, T] \to [0, S]$ with $S = (T - t_0)^{p+1} + t_0^{p+1}$,

$$\Gamma(t) := \int_0^t |\gamma(\xi)| d\xi = t_0^{p+1} + \mu_t\,(t - t_0)^{p+1}, \quad \mu_t = \operatorname{sign}(t - t_0)^p = \pm 1,$$

and notice right away that Γ is invertible with inverse $\Gamma^{-1} : [0, S] \to [0, T]$ given by

$$\Gamma^{-1}(s) = s_0^{\frac{1}{p+1}} + \operatorname{sign}(s - s_0)\,|s - s_0|^{\frac{1}{p+1}}, \quad s_0 = t_0^{p+1}.$$

[5] Note that here as in the sequel, we denote the average of a function $\omega : \mathbb{T} \mapsto \mathbb{R}^d$ by

$$\langle\omega\rangle = \frac{1}{2\pi}\int_0^{2\pi} \omega(\theta) d\theta.$$

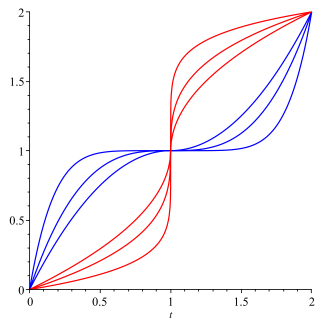

FIG. 1. *The functions* Γ *(in blue) and* Γ^{-1} *(in red) with* $t_0 = 1$ *and* $T = 2$ *for* $p = 1, 2, 5$.

Let us now consider for $s = \Gamma(t)$ the function $v^\varepsilon(s) = u^\varepsilon(t)$, which, for $s \neq s_0$, satisfies

$$\frac{d}{ds}v^\varepsilon(s) = \frac{1}{\Gamma' \circ \Gamma^{-1}(s)}\,\dot{u}^\varepsilon\left(\Gamma^{-1}(s)\right) = \frac{1}{(p+1)|s-s_0|^{\frac{p}{p+1}}}\,F_{\mu_s}\left(\frac{s-s_0}{\varepsilon}, v^\varepsilon(s)\right) \tag{2.1}$$

with initial condition $v^\varepsilon(0) = v_0^\varepsilon := u_0^\varepsilon$,

$$\mu_s = \begin{cases} 1 & \text{if } (s-s_0)^p \geq 0 \\ -1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad F_{\pm 1}(\theta, u) := F(\pm\theta, u).$$

As an immediate consequence of Assumption 1.1, equation (2.1) has a unique solution on $[0, S]$, bounded by M uniformly in $0 < \varepsilon \leq 1$.

In this section, our aim is to show that there exists an averaged model for (2.1) of the form

$$\forall s \in [0, S], \quad \dot{\underline{v}}^\varepsilon(s) = \frac{1}{(p+1)|s-s_0|^{\frac{p}{p+1}}}\,\langle F\rangle\,(\underline{v}^\varepsilon(s)), \qquad \underline{v}^\varepsilon(0) = v_0^\varepsilon, \tag{2.2}$$

and then construct the first term of the asymptotic expansion of v^ε (see Section 2.3). Note that, despite the singularity at $s = s_0$ of the right-hand side of (2.2), its integral formulation clearly indicates the existence of a *continuous* solution on $[0, S]$.

2.1. Preliminaries. Let us introduce the following 2π-periodic zero-average functions

$$G_{\pm 1}(\theta, u) = \int_0^\theta (F_{\pm 1}(\sigma, u) - \langle F\rangle(u))d\sigma - \left\langle \int_0^s (F_{\pm 1}(\sigma, u) - \langle F\rangle(u))d\sigma \right\rangle,$$

and

$$H_{\pm 1}(\theta, u) = \int_0^\theta G_{\pm 1}(\sigma, u)d\sigma - \left\langle \int_0^s G_{\pm 1}(\sigma, u)d\sigma \right\rangle.$$

Note that

$$\frac{1}{2\pi}\int_0^{2\pi} F_1(\sigma, u)d\sigma = \frac{1}{2\pi}\int_0^{2\pi} F_{-1}(\sigma, u)d\sigma = \frac{1}{2\pi}\int_0^{2\pi} F(\sigma, u)d\sigma = \langle F\rangle(u)$$

which is the reason why $\langle F\rangle$ appears *in lieu of* $\langle F_{\pm 1}\rangle$ *in the definition of* $G_{\pm 1}$. It is clear that these functions and their derivatives in u are uniformly bounded: for $|u| \leq 2M$, $v \in \mathbb{R}^d$ and $s \in \mathbb{R}$, we have

$$|G_{\pm 1}(s, u)| + |H_{\pm 1}(s, u)| \leq C, \qquad |\partial_2 G_{\pm 1}(s, u)v| + |\partial_2 H_{\pm 1}(s, u)v| \leq C|v|, \tag{2.3}$$

$$(2.4) \qquad |\partial_2^2 G_{\pm 1}(s,u)(v,v)| + |\partial_2^2 H_{\pm 1}(s,u)(v,v)| \le C|v|^2,$$

where we have denoted ∂_2 the partial derivative with respect to the variable u. We eventually define the function

$$(2.5) \qquad \forall u \in \mathbb{R}^d, \forall s \in \mathbb{R}_+, \quad \Omega_{\pm 1}(s,u) = \int_s^{+\infty} \frac{1}{\sigma^{\frac{p}{p+1}}} (F_{\pm 1}(\sigma,u) - \langle F\rangle(u)) d\sigma.$$

The following two technical lemmas will be useful all along this article.

Lemma 2.1. *The function $\Omega_{\pm 1}$ is well-defined for all $s \in \mathbb{R}_+$ and $u \in \mathbb{R}^d$. Moreover, for all u satisfying $|u| \le 2M$, all $s \ge 0$ and all $v \in \mathbb{R}^d$, we have the estimates*

$$(2.6) \qquad |\Omega_{\pm 1}(s,u)| \le C, \quad |\partial_2 \Omega_{\pm 1}(s,u)v| \le C|v|, \quad |\partial_2^2 \Omega_{\pm 1}(s,u)(v,v)| \le C|v|^2.$$

Restricting to strictly positive values of s, i.e. $s > 0$, we have furthermore

$$(2.7) \qquad |\Omega_{\pm 1}(s,u)| \le \frac{C}{s^{\frac{p}{p+1}}}, \quad |\partial_2 \Omega_{\pm 1}(s,u)v| \le \frac{C|v|}{s^{\frac{p}{p+1}}},$$

and

$$(2.8) \qquad \left| \Omega_{\pm 1}(s,u) + \frac{G_{\pm 1}(s,u)}{s^{\frac{p}{p+1}}} \right| \le \frac{C}{s^{1+\frac{p}{p+1}}}, \quad \left| \partial_2 \Omega_{\pm 1}(s,u)v + \frac{\partial_2 G_{\pm 1}(s,u)v}{s^{\frac{p}{p+1}}} \right| \le \frac{C|v|}{s^{1+\frac{p}{p+1}}}.$$

Proof. We only prove the results for $\Omega_{\pm 1}$ as their adaptation to $\partial_2 \Omega_{\pm 1}$ and $\partial_2^2 \Omega_{\pm 1}$ is immediate. An integration by parts yields

$$\Omega_{\pm 1}(s,u) = -\frac{G_{\pm 1}(s,u)}{s^{\frac{p}{p+1}}} + \frac{p}{p+1} \int_s^{+\infty} \frac{1}{\sigma^{1+\frac{p}{p+1}}} G_{\pm 1}(\sigma,u) d\sigma,$$

where, from (2.3), the last integral is convergent and bounded by $\frac{C}{s^{\frac{p}{p+1}}}$. This yields the well-posedness of $\Omega_{\pm 1}$ for all $s > 0$ and (2.7). We now simply remark that for all $s \ge 0$

$$\Omega_{\pm 1}(s,u) = \int_s^1 \frac{1}{\sigma^{\frac{p}{p+1}}} (F_{\pm 1}(\sigma,u) - \langle F\rangle(u)) d\sigma + \Omega_{\pm 1}(1,u).$$

This gives the well-posedness for $s = 0$ and (2.6) can be deduced from (2.7) written for $s = 1$. A second integration by parts then gives

$$\Omega_{\pm 1}(s,u) = -\frac{G_{\pm 1}(s,u)}{s^{\frac{p}{p+1}}} - \frac{p}{p+1} \frac{H_{\pm 1}(s,u)}{s^{1+\frac{p}{p+1}}} + \frac{p}{p+1}\left(1 + \frac{p}{p+1}\right) \int_s^{+\infty} \frac{1}{\sigma^{2+\frac{p}{p+1}}} H_{\pm 1}(\sigma,u) d\sigma.$$

Previous integral is bounded by $\frac{C}{s^{1+\frac{p}{p+1}}}$ owing to (2.3) and this yields (2.8). ∎

Remark 2.2. *Since $\left(\frac{1+s}{s}\right)^{\frac{p}{p+1}} \le 2$ for $s \ge 1$, estimates (2.6) and (2.7) also imply for instance that for all $s \ge 0$,*

$$|\Omega_{\pm 1}(s,u)| \le \frac{C}{(1+s)^{\frac{p}{p+1}}} \quad \text{and} \quad |\partial_2 \Omega_{\pm 1}(s,u)v| \le \frac{C|v|}{(1+s)^{\frac{p}{p+1}}}.$$

In order to state next result, we now define, for any function $\phi : \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}^d$ and for $0 \le a \le b \le S$, the integral

$$\mathcal{I}^\varepsilon(a,b) = \frac{1}{p+1} \int_a^b \frac{1}{|\sigma - s_0|^{\frac{p}{p+1}}} \phi\left(\frac{|\sigma - s_0|}{\varepsilon}, v^\varepsilon(\sigma)\right) d\sigma \tag{2.9}$$

where v^ε is assumed to be the solution of equation (2.1).

Lemma 2.3. *For a given $p \in \mathbb{N}^*$, consider two smooth functions $\phi, \psi : \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying the estimates*

$$|\psi(\sigma,u)| \le C \quad \text{and} \quad \left|\phi(\sigma,u) + \frac{\psi(\sigma,u)}{(1+\sigma)^{\frac{p}{p+1}}}\right| \le \frac{C}{(1+\sigma)^{1+\frac{p}{p+1}}}, \tag{2.10}$$

for all $\theta \in \mathbb{T}$ and all $|u| \le M$. If $p = 1$, we have

$$\forall b \in [0, s_0], \quad \mathcal{I}^\varepsilon(0,b) = \frac{\sqrt{\varepsilon}}{2} \log\left(\frac{\varepsilon + s_0 - b}{s_0 + \varepsilon}\right) \langle\psi\rangle\,(v^\varepsilon(b)) + \mathcal{O}(\sqrt{\varepsilon}), \tag{2.11}$$

$$\forall b \in [s_0, S], \quad \mathcal{I}^\varepsilon(s_0,b) = \frac{\sqrt{\varepsilon}}{2} \log\left(\frac{\varepsilon}{b - s_0 + \varepsilon}\right) \langle\psi\rangle\,(v^\varepsilon(s_0)) + \mathcal{O}(\sqrt{\varepsilon}), \tag{2.12}$$

where averages are taken w.r.t. the first variable. If $p \ge 2$, we have the estimate

$$\forall 0 \le a \le b \le S, \quad |\mathcal{I}^\varepsilon(a,b)| \le C\varepsilon^{\frac{1}{p+1}}. \tag{2.13}$$

Proof. Consider $0 \le b \le s_0$. A change of variables allows to write $\mathcal{I}^\varepsilon(0,b)$ as

$$\mathcal{I}^\varepsilon(0,b) = \frac{\varepsilon^{\frac{1}{p+1}}}{p+1} \int_{\frac{s_0-b}{\varepsilon}}^{\frac{s_0}{\varepsilon}} \frac{1}{\sigma^{\frac{p}{p+1}}} \phi\left(\sigma, v^\varepsilon(s_0 - \varepsilon\sigma)\right) d\sigma.$$

Now, we split $(p+1)\varepsilon^{\frac{-1}{p+1}}\mathcal{I}^\varepsilon(0,b) = \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 - \mathcal{J}_1$ into the sum of the four terms

$$\mathcal{J}_2 = \int_{\frac{s_0-b}{\varepsilon}}^{\frac{s_0}{\varepsilon}} \left(\frac{1}{(1+\sigma)^{\frac{2p}{p+1}}} - \frac{1}{(\sigma(1+\sigma))^{\frac{p}{p+1}}}\right) \langle\psi\rangle\,(v^\varepsilon(s_0 - \varepsilon\sigma))\, d\sigma,$$

$$\mathcal{J}_3 = \int_{\frac{s_0-b}{\varepsilon}}^{\frac{s_0}{\varepsilon}} \frac{1}{(\sigma(1+\sigma))^{\frac{p}{p+1}}} \left(\langle\psi\rangle - \psi\right)(\sigma, v^\varepsilon(s_0 - \varepsilon\sigma))\, d\sigma,$$

$$\mathcal{J}_4 = \int_{\frac{s_0-b}{\varepsilon}}^{\frac{s_0}{\varepsilon}} \frac{1}{\sigma^{\frac{p}{p+1}}} r(\sigma, v^\varepsilon(s_0 - \varepsilon\sigma)) d\sigma, \quad \mathcal{J}_1 = \int_{\frac{s_0-b}{\varepsilon}}^{\frac{s_0}{\varepsilon}} \frac{1}{(1+\sigma)^{\frac{2p}{p+1}}} \langle\psi\rangle\,(v^\varepsilon(s_0 - \varepsilon\sigma))\, d\sigma,$$

where we have denoted $r(\sigma,u) = \phi(\sigma,u) + \frac{\psi(\sigma,u)}{(1+\sigma)^{\frac{p}{p+1}}}$. Owing to assumption (2.10) and

$$\frac{1}{(1+\sigma)^{\frac{2p}{p+1}}} - \frac{1}{\sigma^{\frac{p}{p+1}}(1+\sigma)^{\frac{p}{p+1}}} \sim -\frac{p}{p+1} \frac{1}{\sigma^{\frac{3p+1}{p+1}}}, \quad \sigma \to +\infty,$$

integrals $\mathcal{J}_2$ and $\mathcal{J}_4$ are absolutely convergent and uniformly bounded w.r.t. ε. As for $\mathcal{J}_3$, we use the relation

$$-\frac{(\psi - \langle\psi\rangle)(\sigma, v^\varepsilon(s_0 - \varepsilon\sigma))}{\sigma^{\frac{p}{p+1}}(1+\sigma)^{\frac{p}{p+1}}} = \frac{d}{d\sigma}\left(\kappa\left(\sigma, v^\varepsilon(s_0 - \varepsilon\sigma)\right)\right)$$
$$+ \frac{\varepsilon^{\frac{1}{p+1}}}{(p+1)\sigma^{\frac{p}{p+1}}} \left(\partial_2\kappa\, F_{-\mu}\right)(\sigma, v^\varepsilon(s_0 - \varepsilon\sigma))$$

where we have taken equation (2.1) into account with $\mu_s = \mu = (-1)^p$ and

$$\kappa(s,u) = \int_s^{+\infty} \frac{(\psi - \langle\psi\rangle)(\sigma,u)}{\sigma^{\frac{p}{p+1}}(1+\sigma)^{\frac{p}{p+1}}} d\sigma,$$

in order to write $\mathcal{J}_3$ as

$$\mathcal{J}_3 = \kappa\left(\frac{s_0}{\varepsilon}, v^\varepsilon(0)\right) - \kappa\left(\frac{s_0-b}{\varepsilon}, v^\varepsilon(b)\right) + \frac{\varepsilon^{\frac{1}{p+1}}}{(p+1)} \int_{\frac{s_0-b}{\varepsilon}}^{\frac{s_0}{\varepsilon}} \frac{1}{\sigma^{\frac{p}{p+1}}} \left(\partial_2\kappa\, F_{-\mu}\right)(\sigma, v^\varepsilon(s_0-\varepsilon\sigma))\, d\sigma$$

from which we may prove that $\mathcal{J}_3$ is bounded (note indeed that $\partial_2\kappa\, F_{-\mu}$ is bounded). For $p > 1$ it is clear that $\mathcal{J}_1$ is bounded owing to (2.10) and finally, that $\mathcal{I}^\varepsilon(0,b)$ is bounded. The contribution of $\mathcal{J}_1$ for $p = 1$ is more intricate and requires to be decomposed as follows

$$\begin{aligned}\mathcal{J}_1 &= \int_{\frac{s_0-b}{\varepsilon}}^{\frac{s_0}{\varepsilon}} \frac{1}{1+\sigma} \langle\psi\rangle(v^\varepsilon(b)) d\sigma + \int_{\frac{s_0-b}{\varepsilon}}^{\frac{s_0}{\varepsilon}} \frac{1}{1+\sigma}\Big(\langle\psi\rangle(v^\varepsilon(s_0-\varepsilon\sigma)) - \langle\psi\rangle(v^\varepsilon(b))\Big) d\sigma \\ &= \log\left(\frac{s_0+\varepsilon}{\varepsilon+s_0-b}\right)\langle\psi\rangle(v^\varepsilon(b)) + \int_{\frac{s_0-b}{\varepsilon}}^{\frac{s_0}{\varepsilon}} \frac{1}{1+\sigma}\left(\langle\psi\rangle(v^\varepsilon(s_0-\varepsilon\sigma)) - \langle\psi\rangle(v^\varepsilon(b))\right) d\sigma.\end{aligned}$$

To estimate the second term, we use (2.1) and $s_0 - \varepsilon\sigma \le b \le s_0$ to get

$$\left| \left[\langle\psi\rangle(v^\varepsilon(\tau))\right]_b^{s_0-\varepsilon\sigma} \right| \le \left| \int_{s_0-\varepsilon\sigma}^{b} \frac{1}{2\sqrt{s_0-\tau}} \left(\langle\partial_2\psi\rangle\, F_\mu\right)\left(\frac{\tau-s_0}{\varepsilon}, v^\varepsilon(\tau)\right) d\tau \right| \le C\sqrt{\varepsilon\sigma}$$

so that

$$\left| \int_{\frac{s_0-b}{\varepsilon}}^{\frac{s_0}{\varepsilon}} \frac{\left(\langle\psi\rangle(v^\varepsilon(s_0-\varepsilon\sigma)) - \langle\psi\rangle(v^\varepsilon(b))\right)}{1+\sigma} d\sigma \right| \le C\sqrt{\varepsilon} \int_0^{\frac{s_0}{\varepsilon}} \frac{\sqrt{\sigma}}{(1+\sigma)} d\sigma \le C\sqrt{s_0}.$$

We finally obtain that

$$\mathcal{I}^\varepsilon(0,b) = \frac{\sqrt{\varepsilon}}{2} \log\left(\frac{\varepsilon+s_0-b}{s_0+\varepsilon}\right) \langle\psi\rangle(v^\varepsilon(b)) + \mathcal{O}(\sqrt{\varepsilon}).$$

Mutatis mutandis, a similar conclusion holds true for the case $a = s_0$ and $b \ge s_0$ as can be seen by writing the new value of $\mathcal{J}_1$ as

$$\int_0^{\frac{b-s_0}{\varepsilon}} \frac{\langle\psi\rangle(v^\varepsilon(s_0)) + \langle\psi\rangle(v^\varepsilon(s_0+\varepsilon\sigma)) - \langle\psi\rangle(v^\varepsilon(s_0))}{1+\sigma} = \log\left(1 + \frac{b-s_0}{\varepsilon}\right)\langle\psi\rangle(v^\varepsilon(s_0)) + \mathcal{O}(1). \qquad \square$$

2.2. The averaged model. We are now in position to state the first averaging estimate, from which Theorem 1.2 follows by considering the change of variable Γ.

PROPOSITION 2.4. *Let v^ε be the solution of problem (2.1) on $[0,S]$, under Assumption 1.1. Then, for all $0 < \varepsilon < \varepsilon_0$ where ε_0 depends only on bounds on the derivatives of F, the solution $\underline{v}^\varepsilon$ of the averaged model (2.2) exists on $[0,S]$ and one has*

$$\forall s \in [0,S], \quad |v^\varepsilon(s) - \underline{v}^\varepsilon(s)| \le C\,\varepsilon^{\frac{1}{p+1}}. \tag{2.14}$$

Proof. The integral formulation of equation (2.1) reads

$$v^\varepsilon(s) = v_0^\varepsilon + \frac{1}{p+1}\int_0^s \frac{1}{|\sigma - s_0|^{\frac{p}{p+1}}} \left\langle F\right\rangle (v^\varepsilon(\sigma))d\sigma + R^\varepsilon(s), \tag{2.15}$$

where (with $\mu_\sigma = \operatorname{sign}(\sigma - s_0)^p$)

$$R^\varepsilon(s) = \frac{1}{p+1}\int_0^s \frac{1}{|\sigma - s_0|^{\frac{p}{p+1}}} \left(F_{\mu_\sigma}\left(\frac{\sigma - s_0}{\varepsilon}, v^\varepsilon(\sigma)\right) - \left\langle F\right\rangle (v^\varepsilon(\sigma))\right) d\sigma, \tag{2.16}$$

which is well-defined for all $s \in [0, S]$. From (2.5) with $\varsigma = \operatorname{sign}(\sigma - s_0)$, $\sigma \neq s_0$, we have

$$\begin{aligned}
\frac{d}{d\sigma}\Omega_\nu\left(\frac{|\sigma - s_0|}{\varepsilon}, v^\varepsilon(\sigma)\right) =& \frac{\varsigma}{\varepsilon}(\partial_1\Omega_\nu)\left(\frac{|\sigma - s_0|}{\varepsilon}, v^\varepsilon(\sigma)\right)\\
&+ (\partial_2\Omega_\nu)\left(\frac{|\sigma - s_0|}{\varepsilon}, v^\varepsilon(\sigma)\right)\dot{v}^\varepsilon(\sigma)\\
=& -\frac{\varsigma}{\varepsilon^{\frac{1}{p+1}}|\sigma - s_0|^{\frac{p}{p+1}}}\left(F_{\varsigma\nu}\left(\frac{\sigma - s_0}{\varepsilon}, v^\varepsilon(\sigma)\right) - \left\langle F\right\rangle (v^\varepsilon(\sigma))\right)\\
&+ \frac{|\sigma - s_0|^{\frac{-p}{p+1}}}{(p+1)}(\partial_2\Omega_\nu)\left(\frac{|\sigma - s_0|}{\varepsilon}, v^\varepsilon(\sigma)\right) F_{\mu_\sigma}\left(\frac{\sigma - s_0}{\varepsilon}, v^\varepsilon(\sigma)\right),
\end{aligned}$$

that is to say, taking $\nu = \varsigma\mu_\sigma$

$$\begin{aligned}
\frac{1}{|\sigma - s_0|^{\frac{p}{p+1}}}\left(F_{\mu_\sigma}\left(\frac{\sigma - s_0}{\varepsilon}, v^\varepsilon(\sigma)\right) - \left\langle F\right\rangle (v^\varepsilon(\sigma))\right) =& -\varsigma\varepsilon^{\frac{1}{p+1}}\frac{d}{d\sigma}\left(\Omega_{\varsigma\mu_\sigma}\left(\frac{|\sigma - s_0|}{\varepsilon}, v^\varepsilon(\sigma)\right)\right)\\
&+ \frac{\varsigma\varepsilon^{\frac{1}{p+1}}}{(p+1)|\sigma - s_0|^{\frac{p}{p+1}}}\partial_2\Omega_{\varsigma\mu_\sigma}\left(\frac{|\sigma - s_0|}{\varepsilon}, v^\varepsilon(\sigma)\right) F_{\mu_\sigma}\left(\frac{\sigma - s_0}{\varepsilon}, v^\varepsilon(\sigma)\right),
\end{aligned} \tag{2.17}$$

where we have used (2.1). For $\sigma \leq s \leq s_0$ we have $\mu_\sigma = (-1)^p = \mu_s$, $\varsigma = -1$ and therefore

$$\begin{aligned}
R^\varepsilon(s) =& \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\left(\Omega_{-\mu_s}\left(\frac{s_0 - s}{\varepsilon}, v^\varepsilon(s)\right) - \Omega_{-\mu_s}\left(\frac{s_0}{\varepsilon}, v_0^\varepsilon\right)\right)\\
&- \frac{\varepsilon^{\frac{1}{p+1}}}{(p+1)^2}\int_0^s \frac{1}{(s_0 - \sigma)^{\frac{p}{p+1}}}\partial_2\Omega_{-\mu_s}\left(\frac{s_0 - \sigma}{\varepsilon}, v^\varepsilon(\sigma)\right) F_{-\mu_s}\left(\frac{s_0 - \sigma}{\varepsilon}, v^\varepsilon(\sigma)\right) d\sigma
\end{aligned} \tag{2.18}$$

a relation from which we may deduce, using (2.6) and Assumption 1.1, that $|R^\varepsilon(s)| \leq C\varepsilon^{1/(p+1)}$. In particular, $|R^\varepsilon(s_0)| \leq C\varepsilon^{1/(p+1)}$. As for $s \geq s_0$, we have $\mu_\sigma = \varsigma = 1$ and thus

$$\begin{aligned}
R^\varepsilon(s) =& R^\varepsilon(s_0) + \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\left(\Omega_1\left(0, v^\varepsilon(s_0)\right) - \Omega_1\left(\frac{s - s_0}{\varepsilon}, v^\varepsilon(s)\right)\right)\\
&+ \frac{\varepsilon^{\frac{1}{p+1}}}{(p+1)^2}\int_{s_0}^s \frac{1}{(\sigma - s_0)^{\frac{p}{p+1}}}\partial_2\Omega_1\left(\frac{\sigma - s_0}{\varepsilon}, v^\varepsilon(\sigma)\right) F_1\left(\frac{\sigma - s_0}{\varepsilon}, v^\varepsilon(\sigma)\right) d\sigma
\end{aligned} \tag{2.19}$$

and we may again conclude from (2.6) and Assumption 1.1 that $|R^\varepsilon(s)| \le C\varepsilon^{\frac{1}{p+1}}$ for $s_0 \le s \le S$ and eventually for all $0 \le s \le S$. Finally, we have on the one hand,

$$v^\varepsilon(s) = v_0^\varepsilon + \frac{1}{p+1}\int_0^s \frac{1}{|\sigma - s_0|^{\frac{p}{p+1}}} \langle F \rangle (v^\varepsilon(\sigma)) d\sigma + \mathcal{O}(\varepsilon^{\frac{1}{p+1}}),$$

and on the other hand,

$$\underline{v}^\varepsilon(s) = v_0^\varepsilon + \frac{1}{p+1}\int_0^s \frac{1}{|\sigma - s_0|^{\frac{p}{p+1}}} \langle F \rangle (\underline{v}^\varepsilon(\sigma)) d\sigma,$$

as long as the solution of (2.2) exists. Assumption 1.1 and a standard bootstrap argument based on the Gronwall lemma then enable to conclude. ◻

2.3. Next term of the asymptotic expansion. This section now presents how the estimate of Proposition 2.4 (analogously Theorem 1.2) can be refined by introducing an additional term of higher order in ε, namely $\varepsilon^{\frac{2}{p+1}}$, in the asymptotic expansion.

PROPOSITION 2.5. *Let $\mu = (-1)^p$, and $\delta_p = 1$ if $p = 1$, $\delta_p = 0$ otherwise. Under Assumption 1.1, if we consider the solutions $\bar{v}^\varepsilon$ and $\bar{w}^\varepsilon$ of the averaged equation (2.2) respectively on $[0, s_0]$ and $[s_0, S]$ and with the respective initial conditions*

$$\bar{v}^\varepsilon(0) = v_0^\varepsilon - \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Omega_{-\mu}\left(\frac{s_0}{\varepsilon}, v_0^\varepsilon\right), \tag{2.20}$$

$$\bar{w}^\varepsilon(s_0) = \bar{v}^\varepsilon(s_0) + \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Big(\Omega_1\left(0, \bar{v}^\varepsilon(s_0)\right) + \Omega_{-\mu}\left(0, \bar{v}^\varepsilon(s_0)\right)\Big) - \frac{\delta_p \varepsilon}{4}\log\left(\frac{\varepsilon}{\varepsilon + s_0}\right)\langle \partial_2 G\, F\rangle (\bar{v}^\varepsilon(s_0)),$$

and $\tilde{v}^\varepsilon$ the continuous function defined by the following expressions:

$$s \le s_0,\ \widetilde{v}^\varepsilon(s) = \bar{v}^\varepsilon(s) + \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Omega_{-\mu}\left(\frac{s_0 - s}{\varepsilon}, \bar{v}^\varepsilon(s)\right) - \frac{\delta_p\varepsilon}{4}\log\left(\frac{\varepsilon + s_0 - s}{\varepsilon + s_0}\right)\langle \partial_2 G F\rangle (\bar{v}^\varepsilon(s)),$$

$$s_0 \le s,\ \widetilde{v}^\varepsilon(s) = \bar{w}^\varepsilon(s) - \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Omega_1\left(\frac{s - s_0}{\varepsilon}, \bar{w}^\varepsilon(s)\right) + \frac{\delta_p\varepsilon}{4}\log\left(\frac{\varepsilon + s - s_0}{\varepsilon}\right)\langle \partial_2 G F\rangle (\bar{w}^\varepsilon(s_0)) + \beta^\varepsilon$$

where

$$\beta^\varepsilon = \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Omega_1\left(0, \bar{w}^\varepsilon(s_0)\right) - \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Omega_1\left(0, \bar{v}^\varepsilon(s_0)\right),$$

then we have

$$\forall s \in [0, S], \quad |v^\varepsilon(s) - \widetilde{v}^\varepsilon(s)| \le C\,\varepsilon^{\frac{2}{p+1}}. \tag{2.21}$$

REMARK 2.6. *In classical averaging theory (i.e. for $\gamma(t) \equiv 1$ or equivalently for $p = 0$), the solution $v^\varepsilon(s)$ of (2.1) is obtained as the composition of three maps (see for instance [Per69] or [SV85]): (i) a change of variable of the form $v_0^\varepsilon + \varepsilon\varphi_0^\varepsilon(v_0^\varepsilon)$ applied at initial time, (ii) the flow map at time s of a smooth differential equation whose vector field is of the form $\langle F\rangle + \varepsilon\widetilde{F}^\varepsilon$ and (iii) a change of variable of the form $v_0^\varepsilon + \varepsilon\varphi_s^\varepsilon(v_0^\varepsilon)$ applied time s. The $\varepsilon^{\frac{1}{p+1}}$ and* log *terms in (2.20) and in $\tilde{v}^\varepsilon$ and $\tilde{w}^\varepsilon$ are the counterpart of $\varphi_0^\varepsilon(v_0^\varepsilon)$ and $\varphi_s^\varepsilon(v_0^\varepsilon)$ in this more intricate situation.*

Proof. In order to refine estimates (2.18) and (2.19) of $R^\varepsilon(s)$ obtained in the proof of Proposition 2.4, we rewrite them as

(2.22)
$$s \le s_0 : R^\varepsilon(s) = \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\left(\Omega_{-\mu}\left(\frac{s_0 - s}{\varepsilon}, v^\varepsilon(s)\right) - \Omega_{-\mu}\left(\frac{s_0}{\varepsilon}, v_0^\varepsilon\right) - \mathcal{I}_{-\mu}(0,s)\right),$$

(2.23)
$$s \ge s_0 : R^\varepsilon(s) = R^\varepsilon(s_0) + \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\left(\Omega_1\left(0, v^\varepsilon(s_0)\right) - \Omega_1\left(\frac{s - s_0}{\varepsilon}, v^\varepsilon(s)\right) + \mathcal{I}_1(s_0,s)\right),$$

where the expression of $\mathcal{I}_\nu^\varepsilon$ coincides with $\mathcal{I}^\varepsilon$ in Lemma 2.3 for $\phi(\sigma,u) = \partial_2\Omega_\nu F_\nu(\sigma,u)$ and $\psi(\sigma,u) = \partial_2 G_\nu F_\nu(\sigma,u)$. If x and $\underline{x}$ differ by an $\mathcal{O}(\varepsilon^{\frac{1}{p+1}})$, then, using (2.6)-(2.7), one has

$$\forall \nu = \pm 1, \quad \left|\Omega_\nu\left(\frac{s}{\varepsilon}, x\right) - \Omega_\nu\left(\frac{s}{\varepsilon}, \underline{x}\right)\right| \le C\varepsilon^{\frac{1}{p+1}}$$

and owing to (2.14), estimates $\bar{v}^\varepsilon(0) - \underline{v}^\varepsilon(0) = \mathcal{O}(\varepsilon^{\frac{1}{p+1}})$ and $\bar{w}^\varepsilon(s_0) - \bar{v}^\varepsilon(s_0) = \mathcal{O}(\varepsilon^{\frac{1}{p+1}})$, and the Gronwall lemma, it stems that

$$\forall 0 \le s \le s_0, \quad v^\varepsilon(s) - \bar{v}^\varepsilon(s) = \mathcal{O}(\varepsilon^{\frac{1}{p+1}}) \quad \text{and} \quad \forall s_0 \le s \le S, \quad \bar{w}^\varepsilon(s) - v^\varepsilon(s) = \mathcal{O}(\varepsilon^{\frac{1}{p+1}}).$$

As a consequence, $v^\varepsilon(s)$ can be replaced by $\bar{v}^\varepsilon(s)$ in (2.22) and by $\bar{w}^\varepsilon(s)$ in (2.23), up to $\mathcal{O}(\varepsilon^{\frac{2}{p+1}})$-terms.

Case $p > 1$**:** Lemma 2.3 shows that the terms $\frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\mathcal{I}_\nu^\varepsilon$ in (2.22) and (2.23) are of order $\mathcal{O}(\varepsilon^{\frac{2}{p+1}})$, we thus have for $s \le s_0$

$$v^\varepsilon(s) = v_0^\varepsilon + \frac{1}{p+1}\int_0^s \frac{\langle F\rangle\left(v^\varepsilon(\sigma)\right)}{|\sigma - s_0|^{\frac{p}{p+1}}} d\sigma + \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\left[\Omega_{-\mu}\left(\frac{s_0 - \sigma}{\varepsilon}, \bar{v}^\varepsilon(\sigma)\right)\right]_{\sigma=0}^{\sigma=s} + \mathcal{O}(\varepsilon^{\frac{2}{p+1}}),$$

that is to say, by denoting $V^\varepsilon(s) = v^\varepsilon(s) - \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Omega_{-\mu}\left(\frac{s_0 - s}{\varepsilon}, \bar{v}^\varepsilon(s)\right)$, the equation

$$\begin{aligned}
V^\varepsilon(s) - V^\varepsilon(0) &= \frac{1}{p+1}\int_0^s \frac{\langle F\rangle\Big(V^\varepsilon(\sigma) + \left(v^\varepsilon(\sigma) - V^\varepsilon(\sigma)\right)\Big)}{(s_0 - \sigma)^{\frac{p}{p+1}}} d\sigma + \mathcal{O}(\varepsilon^{\frac{2}{p+1}}) \\
&= \frac{1}{p+1}\int_0^s \frac{\langle F\rangle\left(V^\varepsilon(\sigma)\right)d\sigma + \langle \partial_2 F\rangle\left(V^\varepsilon(\sigma)\right)\left(v^\varepsilon(\sigma) - V^\varepsilon(\sigma)\right)}{(s_0 - \sigma)^{\frac{p}{p+1}}} d\sigma + \mathcal{O}(\varepsilon^{\frac{2}{p+1}}), \\
&= \frac{1}{p+1}\int_0^s \frac{1}{(s_0 - \sigma)^{\frac{p}{p+1}}}\langle F\rangle\left(V^\varepsilon(\sigma)\right) d\sigma + \mathcal{O}(\varepsilon^{\frac{2}{p+1}}),
\end{aligned}$$

where we have used Remark 2.2 to get the bound

$$\int_0^s \left|\frac{1}{(s_0 - \sigma)^{\frac{p}{p+1}}}\langle \partial_2 F\rangle\left(V^\varepsilon(\sigma)\right)\Omega_{-\mu}\left(\frac{s_0 - \sigma}{\varepsilon}, \bar{v}^\varepsilon(\sigma)\right)\right| d\sigma \le C\varepsilon^{\frac{1}{p+1}}\int_0^{+\infty} \frac{1}{(\sigma(1+\sigma))^{\frac{p}{p+1}}} d\sigma.$$

From $V^\varepsilon(0) - \bar{v}^\varepsilon(0) = \mathcal{O}(\varepsilon^{\frac{2}{p+1}})$ and equation (2.2), we obtain by the Gronwall lemma

$$\forall s \le s_0, \quad |\tilde{v}^\varepsilon(s) - v^\varepsilon(s)| = |V^\varepsilon(s) - \bar{v}^\varepsilon(s)| \le C\varepsilon^{\frac{2}{p+1}}.$$

For $s \geq s_0$, we write

$$v^\varepsilon(s) = v^\varepsilon(s_0) + \frac{1}{p+1}\int_{s_0}^{s} \frac{\langle F\rangle (v^\varepsilon(\sigma))}{(\sigma - s_0)^{\frac{p}{p+1}}} d\sigma + (R^\varepsilon(s) - R^\varepsilon(s_0))$$

$$= v^\varepsilon(s_0) + \frac{1}{p+1}\int_{s_0}^{s} \frac{\langle F\rangle (v^\varepsilon(\sigma))}{(\sigma - s_0)^{\frac{p}{p+1}}} d\sigma - \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\left[\Omega_1\left(\frac{\sigma - s_0}{\varepsilon}, \bar{w}^\varepsilon(\sigma)\right)\right]_{\sigma=s_0}^{\sigma=s}$$

$$+ \mathcal{O}(\varepsilon^{\frac{2}{p+1}}).$$

Denoting $W^\varepsilon(s) = v^\varepsilon(s) + \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Omega_1\left(\frac{s-s_0}{\varepsilon}, \bar{w}^\varepsilon(s)\right)$, we have the simple equation

$$W^\varepsilon(s) = W^\varepsilon(s_0) + \frac{1}{p+1}\int_{s_0}^{s} \frac{\langle F\rangle \Big(W^\varepsilon(\sigma)\Big)}{(\sigma - s_0)^{\frac{p}{p+1}}} d\sigma + \mathcal{O}(\varepsilon^{\frac{2}{p+1}}),$$

and by comparing with equation (2.2), Gronwall lemma enables to conclude that $W^\varepsilon(s) - \bar{w}^\varepsilon(s) = \mathcal{O}(\varepsilon^{\frac{2}{p+1}})$ given that $W^\varepsilon(s_0) - \bar{w}^\varepsilon(s_0) = \mathcal{O}(\varepsilon^{\frac{2}{p+1}})$ (by definition of $\bar{w}^\varepsilon(s_0)$ and $W^\varepsilon(s_0)$ and estimate (2.21) for $s = s_0$). The statement for $s \geq s_0$ now follows from $\beta^\varepsilon = \mathcal{O}(\varepsilon^{\frac{2}{p+1}})$.

Case $p = 1$**:** This case differs in that the terms $\frac{\sqrt{\varepsilon}}{2}\mathcal{I}_\nu^\varepsilon$ in (2.22) and (2.23) are now of order $\varepsilon \log(\varepsilon)$ for s close to s_0. This yields for $s \leq s_0$

$$v^\varepsilon(s) = v_0^\varepsilon + \frac{1}{2}\int_0^s \frac{\langle F\rangle (v^\varepsilon(\sigma))}{\sqrt{s_0 - \sigma}} d\sigma + \frac{\sqrt{\varepsilon}}{2}\Omega_{-\mu}\left[\left(\frac{s_0 - \sigma}{\varepsilon}, \bar{v}^\varepsilon(\sigma)\right)\right]_{\sigma=0}^{\sigma=s} - \frac{\sqrt{\varepsilon}}{2}\mathcal{I}_{-\mu}(0, s)$$

$$+ \mathcal{O}(\varepsilon),$$

that is to say, by denoting

$$V^\varepsilon(s) = v^\varepsilon(s) - \frac{\sqrt{\varepsilon}}{2}\Omega_{-\mu}\left(\frac{s_0 - s}{\varepsilon}, \bar{v}^\varepsilon(s)\right) + \frac{\varepsilon}{4}\log\left(\frac{\varepsilon + s_0 - s}{\varepsilon + s_0}\right)\langle \partial_2 G\, F\rangle (\bar{v}^\varepsilon(s)),$$

the equation

$$V^\varepsilon(s) = V_0^\varepsilon + \int_0^s \frac{\langle F\rangle \big(V^\varepsilon(\sigma)\big)}{2\sqrt{s_0 - \sigma}} d\sigma + \int_0^s \frac{\langle \partial_2 F\rangle (V^\varepsilon(\sigma))}{2\sqrt{s_0 - \sigma}} (v^\varepsilon(\sigma) - V^\varepsilon(\sigma)) d\sigma + \mathcal{O}(\varepsilon)$$

$$= V_0^\varepsilon + \int_0^s \frac{\langle F\rangle \big(V^\varepsilon(\sigma)\big)}{2\sqrt{s_0 - \sigma}} d\sigma + \frac{\sqrt{\varepsilon}}{4}\int_0^s \frac{\langle \partial_2 F\rangle (V^\varepsilon(\sigma))}{\sqrt{s_0 - \sigma}}\,\Omega_{-\mu}\left(\frac{s_0 - \sigma}{\varepsilon}, \bar{v}^\varepsilon(\sigma)\right) d\sigma$$

$$- \frac{\varepsilon}{8}\int_0^s \frac{\log\left(\frac{\varepsilon + s_0 - \sigma}{\varepsilon + s_0}\right)}{\sqrt{s_0 - \sigma}}\,\langle \partial_2 F\rangle (V^\varepsilon(\sigma))\,\langle \partial_2 G\, F\rangle (\bar{v}^\varepsilon(\sigma)) + \mathcal{O}(\varepsilon)$$

$$= V_0^\varepsilon + \int_0^s \frac{\langle F\rangle \big(V^\varepsilon(\sigma)\big)}{2\sqrt{s_0 - \sigma}} d\sigma + \mathcal{O}(\varepsilon),$$

where we have used Lemma 2.3 again now with $\phi(\sigma, u) = \langle \partial_2 F\rangle(u)\,\Omega_{-\mu}(\sigma, u)$ and $\psi(\sigma, u) = \langle \partial_2 F\rangle(u)\,G_{-\mu}(\sigma, u)$, and noticed that $\langle\psi\rangle = \langle \partial_2 F\rangle\,\langle G_{-\mu}\rangle = 0$, to get rid of the second term of the second line. The third term may be bounded through an integartion by parts. We finally conclude by Gronwall lemma. For $s \geq s_0$, we get

$$v^\varepsilon(s) = v^\varepsilon(s_0) + \frac{1}{2}\int_{s_0}^{s} \frac{\langle F\rangle (v^\varepsilon(\sigma))}{\sqrt{\sigma - s_0}} d\sigma - \frac{\sqrt{\varepsilon}}{2}\left[\Omega_1\left(\frac{\sigma - s_0}{\varepsilon}, \bar{w}^\varepsilon(\sigma)\right)\right]_{\sigma=s_0}^{\sigma=s} + \frac{\sqrt{\varepsilon}}{2}\mathcal{I}_1(s_0, s)$$

$$+ \mathcal{O}(\varepsilon),$$

∎

that is to say, by denoting

$$W^\varepsilon(s) = v^\varepsilon(s) + \frac{\sqrt{\varepsilon}}{2}\Omega_1\left(\frac{s-s_0}{\varepsilon}, \bar{w}^\varepsilon(s)\right) - \frac{\varepsilon}{4}\log\left(\frac{\varepsilon}{\varepsilon+s-s_0}\right)\langle \partial_2 G\, F\rangle\,(\bar{w}^\varepsilon(s_0)),$$

the equation

$$\begin{aligned}
W^\varepsilon(s) = W^\varepsilon(s_0) &+ \int_{s_0}^{s} \frac{\langle F\rangle\big(W^\varepsilon(\sigma)\big)}{2\sqrt{\sigma-s_0}}d\sigma - \frac{\sqrt{\varepsilon}}{4}\int_{s_0}^{s}\frac{\langle\partial_2 F\rangle\,(W^\varepsilon(\sigma))}{\sqrt{\sigma-s_0}}\,\Omega_1\left(\frac{\sigma-s_0}{\varepsilon},\bar{w}^\varepsilon(\sigma)\right)d\sigma \\
&+ \frac{\varepsilon}{8}\int_{s_0}^{s}\frac{\log\left(\frac{\varepsilon}{\varepsilon+s-s_0}\right)}{\sqrt{\sigma-s_0}}\,\langle\partial_2 F\rangle\,(W^\varepsilon(\sigma))\,\langle\partial_2 G\,F\rangle\,(\bar{w}^\varepsilon(s_0)) + \mathcal{O}(\varepsilon) \\
&= W^\varepsilon(s_0) + \int_{s_0}^{s}\frac{\langle F\rangle\big(W^\varepsilon(\sigma)\big)}{2\sqrt{\sigma-s_0}}d\sigma + \mathcal{O}(\varepsilon),
\end{aligned}$$

where we have used equation (2.12) of Lemma 2.3, and we may conclude as before. ∎

COROLLARY 2.7. *Let* $\mu = (-1)^p$, $\delta_p = 1$ *if* $p = 1$, $\delta_p = 0$ *otherwise and* $\tau_0 = \frac{t_0^{p+1}}{\varepsilon}$. *Under Assumption 1.1, consider* $\bar{u}_1^\varepsilon$ *and* $\bar{u}_2^\varepsilon$*, the solutions of*

$$\dot{\bar{u}}^\varepsilon(t) = \langle F\rangle(\bar{u}^\varepsilon(t)), \tag{2.24}$$

respectively on $[0,t_0]$ *and* $[t_0,T]$ *with respective initial conditions*

$$\bar{u}_1^\varepsilon(0) = u_0^\varepsilon - \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Omega_{-\mu}\left(\tau_0, u_0^\varepsilon\right)$$

and

$$\bar{u}_2^\varepsilon(t_0) = \bar{u}_1^\varepsilon(t_0) + \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Big(\Omega_1\left(0,\bar{u}_1^\varepsilon(t_0)\right) + \Omega_{-\mu}\left(0,\bar{u}_1^\varepsilon(t_0)\right)\Big) + \frac{\delta_p\varepsilon}{4}\log\left(1+\tau_0\right)\langle\partial_2 G\,F\rangle\,(\bar{u}_1^\varepsilon(t_0)).$$

Then we have

$$\forall t\in[0,T], \quad |u^\varepsilon(t) - \widetilde{u}^\varepsilon(t)| \le C\,\varepsilon^{\frac{2}{p+1}} \tag{2.25}$$

where $\tilde{u}^\varepsilon$ *is the continuous function defined on* $[0,T]$ *by the following expressions:*

$$0\le t\le t_0: \quad \widetilde{u}^\varepsilon(t) = \bar{u}_1^\varepsilon(t) + \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Omega_{-\mu}\left(\tau,\bar{u}_1^\varepsilon(t)\right) - \frac{\delta_p\varepsilon}{4}\log\left(\frac{1+\tau}{1+\tau_0}\right)\langle\partial_2 GF\rangle\,(\bar{u}_1^\varepsilon(t)),$$

$$t_0\le t\le T: \quad \widetilde{u}^\varepsilon(t) = \bar{u}_2^\varepsilon(t) - \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Omega_1\left(\tau,\bar{u}_2^\varepsilon(t)\right) + \frac{\delta_p\varepsilon}{4}\log\left(1+\tau\right)\langle\partial_2 GF\rangle\,(\bar{u}_2^\varepsilon(t_0)) + \beta^\varepsilon,$$

with $\tau = \frac{|t-t_0|^{p+1}}{\varepsilon}$ *and* $\beta^\varepsilon = \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Omega_1\left(0,\bar{u}_2^\varepsilon(t_0)\right) - \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Omega_1\left(0,\bar{u}_1^\varepsilon(t_0)\right)$.

3. A micro-macro method.

In this section, we suggest a micro-macro decomposition, analogous to the one introduced in [CLM17] and elaborated from the asymptotic analysis of Section 2. In a second step, we propose a *uniformly accurate* numerical method derived from this decomposition.

3.1. The decomposition method. Let $u^\varepsilon(t)$ be the solution of (1.2) and let $\widetilde{u}^\varepsilon(t)$ be the approximation defined in Corollary 2.7, and consider the defect function

$$\Delta^\varepsilon(t) = u^\varepsilon(t) - \widetilde{u}^\varepsilon(t), \qquad \text{for } t \in [0,T]. \tag{3.1}$$

Proposition 3.1. *Assume that f is of class C^2 and consider the solution $u^\varepsilon(t)$ of (1.2) on $[0,T]$. For $p \geq 1$, the function $\Delta^\varepsilon(t)$ defined by (3.1) satisfies*

$$\forall t \in [0,T], \qquad |\Delta^\varepsilon(t)| \leq C\varepsilon^{\frac{2}{p+1}}, \tag{3.2}$$

$$\forall t \in [0,t_0[\cup]t_0,T], \qquad \left|\dot{\Delta}^\varepsilon(t)\right| \leq C\varepsilon^{\frac{1}{p+1}} \quad \text{and if } p=1 \quad \left|\ddot{\Delta}^\varepsilon(t)\right| \leq C. \tag{3.3}$$

Proof. By construction, $\tilde{u}^\varepsilon$ is continuous on $[0,T]$ and estimate (3.2) is nothing but (2.25). However, its derivatives are not continuous at t_0. Hereafter, it is enough to consider t in $[0,t_0[$ as the same arguments can be repeated for values in $]t_0,T]$. From the expression of

$$\widetilde{u}^\varepsilon(t) = \bar{u}_1^\varepsilon(t) + \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\Omega_{-\mu}\left(\tau, \bar{u}_1^\varepsilon(t)\right) - \frac{\delta_p\varepsilon}{4}\log\left(\frac{1+\tau}{1+\tau_0}\right)\langle\partial_2 GF\rangle\left(\bar{u}_1^\varepsilon(t)\right),\ \tau = \frac{(t-t_0)^{p+1}}{\varepsilon},$$

it stems by definition of Ω (see (2.5)) that

$$\begin{aligned}\dot{\Delta}^\varepsilon(t) =& F_{-\mu}\left(\tau, u^\varepsilon\right) - F_{-\mu}\left(\tau, \bar{u}_1^\varepsilon\right) - \frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\partial_2\Omega_{-\mu}\left(\tau, \bar{u}_1^\varepsilon\right)\langle F\rangle(\bar{u}_1^\varepsilon) \\ & - \frac{\delta_p\sqrt{\varepsilon}}{2}\frac{\sqrt{\tau}}{1+\tau}\left\langle\partial_2 G\, F\right\rangle\left(\bar{u}_1^\varepsilon\right) + \frac{\delta_p\varepsilon}{4}\log\left(\frac{1+\tau}{1+\tau_0}\right)\frac{d}{dt}\Big(\left\langle\partial_2 G\, F\right\rangle\left(\bar{u}_1^\varepsilon\right)\Big),\end{aligned} \tag{3.4}$$

where we have omitted t in $u^\varepsilon(t)$ and $\bar{u}_1^\varepsilon(t)$. Since $|\bar{u}_1^\varepsilon(t) - \underline{u}^\varepsilon(t)| \leq C\varepsilon^{\frac{1}{p+1}}$ on $[0,t_0]$ (and $|\bar{u}_2^\varepsilon(t) - \underline{u}^\varepsilon(t)| \leq C\varepsilon^{\frac{1}{p+1}}$ on $[t_0,T]$), we have from Prop. 2.4 and Eq. (2.6), the following estimates

$$\left|F_{-\mu}\left(\tau, u^\varepsilon\right) - F_{-\mu}\left(\tau, \bar{u}_1^\varepsilon\right)\right| \leq C\varepsilon^{\frac{1}{p+1}} \quad \text{and} \quad \left|\frac{\varepsilon^{\frac{1}{p+1}}}{p+1}\partial_2\Omega_{-\mu}\left(\tau, \bar{u}_1^\varepsilon\right)\langle F\rangle(\bar{u}_1^\varepsilon)\right| \leq C\varepsilon^{\frac{1}{p+1}}.$$

Besides, $2\sqrt{\tau} \leq 1+\tau$, $|\varepsilon\log\varepsilon| \leq \sqrt{\varepsilon}$, and the first estimate of (3.3) is thus proven. Assuming now that $p=1$ and using again equations (1.2) and (2.2), a second derivation leads to

$$\begin{aligned}\ddot{\Delta}^\varepsilon(t) =& -\frac{2\sqrt{\tau}}{\sqrt{\varepsilon}}\Big(\partial_1 F\left(\tau, u^\varepsilon\right) - \partial_1 F\left(\tau, \bar{u}_1^\varepsilon\right)\Big) + \partial_2 F\left(\tau, u^\varepsilon\right)F\left(\tau, u^\varepsilon\right) \\ & - 2\partial_2 F\left(\tau, \bar{u}_1^\varepsilon\right)\langle F\rangle(\bar{u}_1^\varepsilon) + \langle\partial_2 F\rangle(\bar{u}_1^\varepsilon)\langle F\rangle(\bar{u}_1^\varepsilon) - \frac{\sqrt{\varepsilon}}{2}\partial_2^2\Omega_1\left(\tau, \bar{u}_1^\varepsilon\right)\Big(\langle F\rangle(\bar{u}_1^\varepsilon), \langle F\rangle(\bar{u}_1^\varepsilon)\Big) \\ & - \frac{\sqrt{\varepsilon}}{2}\partial_2\Omega_1\left(\tau, \bar{u}_1^\varepsilon\right)\langle\partial_2 F\rangle(\bar{u}_1^\varepsilon)\langle F\rangle(\bar{u}_1^\varepsilon) + \frac{1-\tau}{2(1+\tau)^2}\left\langle\partial_2 GF\right\rangle\left(\bar{u}_1^\varepsilon\right) \\ & - \frac{\sqrt{\varepsilon}}{2}\frac{\sqrt{\tau}}{1+\tau}\left\langle\partial_2 G\, F\right\rangle\left(\bar{u}_1^\varepsilon\right) + \frac{\varepsilon}{4}\log\left(\frac{1+\tau}{1+\tau_0}\right)\frac{d^2}{dt^2}\Big(\left\langle\partial_2 G\, F\right\rangle\left(\bar{u}_1^\varepsilon\right)\Big).\end{aligned}$$

Thanks to Assumption 1.1, Lemma 2.1 and (2.2), all the terms are clearly uniformly bounded, except the critical one in the first line, which requires more attention. We get

$$\left|\frac{\sqrt{\tau}}{\sqrt{\varepsilon}}\Big(\partial_1 F\left(\tau, u^{\varepsilon}\right)-\partial_1 F\left(\tau, \bar{u}_1^{\varepsilon}\right)\Big)\right| \leq C \frac{|t-t_0|}{\varepsilon}|u^{\varepsilon}-\bar{u}_1^{\varepsilon}| \leq C \frac{|t-t_0|}{\varepsilon}|\bar{u}_1^{\varepsilon}-\widetilde{u}^{\varepsilon}|+C,$$

where we have used the result of Proposition 2.5, i.e. $|u^{\varepsilon}-\widetilde{u}^{\varepsilon}| \leq C\varepsilon$. It remains, using the expression of $\widetilde{u}^{\varepsilon}$, to observe that for $t \neq t_0$, $0<\tau \leq \tau_0$ so that owing to (2.7), we obtain

$$\begin{aligned}
\frac{\sqrt{\tau}}{\sqrt{\varepsilon}}|\widetilde{u}^{\varepsilon}-\bar{u}_1^{\varepsilon}| &\leq \frac{\sqrt{\tau}}{\sqrt{\varepsilon}}\left(\frac{\sqrt{\varepsilon}}{2}\left|\Omega_1\left(\tau, \bar{u}_1^{\varepsilon}\right)\right|+\frac{\varepsilon}{4}\left|\log\left(\frac{1+\tau}{1+\tau_0}\right)\right|\left|\left\langle\partial_2 G F\right\rangle\left(\bar{u}_1^{\varepsilon}(t_0)\right)\right|\right) \\
&\leq C \frac{\sqrt{\tau}}{\sqrt{\varepsilon}} \frac{\sqrt{\varepsilon}}{\sqrt{\tau}}+C \sqrt{\frac{\tau}{\tau_0}}\left|\log\left(\frac{1+\tau}{1+\tau_0}\right)\right| \leq C.
\end{aligned}$$

This completes the proof. ∎

3.2. A uniformly accurate first order numerical method. We are now in position to introduce uniformly accurate numerical schemes for (1.2). In this Section, we derive a uniformly accurate first-order method for $p \geq 1$. Consider $0=t^0<\ldots<t^k<\cdots<t^N=T$ a subdivision of the interval $[0, T]$ containing the singularity t_0, with $h=\max_{k=1,\ldots,N}(t^k-t^{k-1})$. Inspired by the integral schemes in [CLMV18], we introduce the following method,

$$u^{k+1}=u^k+\int_{t^k}^{t^{k+1}} F\Big(\frac{(t-t_0)^{p+1}}{\varepsilon}, u^k\Big) dt \tag{3.5}$$

Thanks to estimate (3.2) and the first estimate of (3.3), we obtain the following proposition.

Proposition 3.2. *Assume that f is of class C^1. Consider the solution $u^{\varepsilon}(t)$ of (1.2) on $[0,T]$, and the numerical scheme u^k defined in (3.5). Then u^k yields a uniformly accurate approximation of order one of the solution $u^{\varepsilon}(t^k)$. Precisely, there exist $\varepsilon_0>0$ and $h_0>0$ such that for all $\varepsilon \leq \varepsilon_0$ and all $h \leq h_0$,*

$$|u^k-u^{\varepsilon}(t^k)| \leq Ch$$

for all $t^k \leq T$ and where C is independent of ε and h.

The method (3.5) can be efficiently implemented numerically by using the Fourier expansion of the vector field $F(\theta, u)$,

$$F(\tau, u)=\sum_{\ell \in \mathbb{Z}} e^{i\ell\tau} F_{\ell}(u).$$

The induction (3.5) then reads

$$u^{k+1}=u^k+(t^{k+1}-t^k) F_0(u^k)+\sum_{\ell \neq 0}\left(\frac{\varepsilon}{\ell}\right)^{1/(p+1)} F_{\ell}(u^k) \int_{t^k}^{t^{k+1}} e^{i\varepsilon^{-1}k(t-t_0)^{p+1}} dt.$$

Using the change of variables $s=\varepsilon^{-1}k(t-t_0)$ and introducing the notation

$$\Lambda_p(t)=\int_t^{+\infty} e^{is^{p+1}} ds,$$

we obtain the method (3.5) can be implemented numerically as

$$u^{k+1} = u^k + (t^{k+1} - t^k) F_0(u^k) + \sum_{\ell \neq 0} \left(\frac{\varepsilon}{\ell}\right)^{1/(p+1)} F_\ell(u^k) \left(\Lambda_p\Big(\Big(\frac{\ell}{\varepsilon}\Big)^{1/(p+1)}(t^k - t_0)\Big) - \Lambda_p\Big(\Big(\frac{\ell}{\varepsilon}\Big)^{1/(p+1)}(t^{k+1} - t_0)\Big)\right). \tag{3.6}$$

Observe that the function $\Lambda_p(t)$ can be evaluated using the incomplete complex Gamma function $\Gamma(\nu, z) = \int_z^{+\infty} t^{\nu-1} e^{-t} dt$ where $\nu = 1/(p+1)$ for which efficient numerical packages exist.

3.3. A uniformly accurate second order numerical method. In this section, we introduce a scheme of uniform order two. The new method provides approximations $(\bar{u}^k, \Delta^k)$ of the pair $(\bar{u}^\varepsilon(t^k), \Delta^\varepsilon(t^k))$. Assume that t_0 is one of the discretization points, i.e. $t_0 = t^{k_0}$ for some k_0. An approximation u^k of $u^\varepsilon(t^k)$ is then derived by assembling the approximation $\widetilde{u}^k$ of $\widetilde{u}^\varepsilon(t^k)$ from formulas in Corollary 2.7 and eventually by setting $u^k = \widetilde{u}^k + \Delta^k$. Given that problem (2.24) is nonstiff, any second-order numerical scheme is suitable for the computation of $\bar{u}^k$ and thus of $\widetilde{u}^k$, and we simply choose here the Heun method

$$\bar{u}^{k+1} = \bar{u}^k + \frac{h}{2}\langle F\rangle(\bar{u}^k) + \frac{h}{2}\langle F\rangle\left(\bar{u}^k + h\langle F\rangle(\bar{u}^k)\right).$$

As a consequence, we limit ourselves to the scheme for Δ^ε. Starting from

$$\Delta^\varepsilon(t^{k+1}) = \Delta^\varepsilon(t^k) + \int_{t^k}^{t^{k+1}} F\left(\tau(\xi), \widetilde{u}^\varepsilon(\xi) + \Delta^\varepsilon(\xi)\right) d\xi - (\widetilde{u}^\varepsilon(t^{k+1}) - \widetilde{u}^\varepsilon(t^k)), \tag{3.7}$$

where $\tau(\xi) = \frac{|\xi - t_0|^{p+1}}{\varepsilon}$, we consider at time $t^{k+1/2} = \frac{t^k + t^{k+1}}{2}$ the approximation

$$\Delta^{k+\frac{1}{2}} = \Delta^k + \int_{t^k}^{t^{k+\frac{1}{2}}} F\left(\tau(\xi), \widetilde{u}^k + \Delta^k\right) d\xi - (\widetilde{u}^{k+\frac{1}{2}} - \widetilde{u}^k).$$

Since the function $\widetilde{u}^\varepsilon + \Delta^\varepsilon = u^\varepsilon$ has a bounded first time-derivative, the error associated to this scheme is of order $\mathcal{O}(h^2)$. Expanding F in Fourier series, we see that the scheme necessitates the computation of integrals of terms of the form $e^{i\ell\xi^2}$ which may be easily computed numerically using the complex `erf` function. Now, for $k < k_0$ and $t \leq t_0$, we identify the smooth part of $u^\varepsilon(t)$ as

$$a^\varepsilon(t) = \bar{u}^\varepsilon(t) + \Delta^\varepsilon(t) - \frac{\varepsilon}{4}\log\left(\frac{1+\tau}{1+\tau_0}\right)\langle \partial_2 GF\rangle(\bar{u}^\varepsilon(t)),$$

so that

$$u^\varepsilon(t) = \widetilde{u}^\varepsilon(t) + \Delta^\varepsilon(t) = a^\varepsilon(t) + \frac{\sqrt{\varepsilon}}{2}\Omega_1\left(\tau(t), \bar{u}^\varepsilon(t)\right)$$

and, by Proposition 3.1 and its proof, it is clear that the second time-derivative of a^ε is uniformly bounded. In order to approximate (3.7), we remark that

$$a^\varepsilon(\xi) = a^k + \frac{a^{k+1/2} - a^k}{t^{k+1/2} - t^k}\left(\xi - t^k\right) + \mathcal{O}(h^2),$$

where setting $\bar{u}^{k+1/2} = \bar{u}^k + \frac{h}{2}\langle F\rangle(\bar{u}^k)$, we define for $\tau^{k+1/2} = \tau(t^{k+1/2})$,

$$a^{k+1/2} = \bar{u}^{k+1/2} + \Delta^{k+1/2} - \frac{\varepsilon}{4}\log\left(\frac{1+\tau^{k+1/2}}{1+\tau_0}\right)\langle \partial_2 GF\rangle(\bar{u}^{k+1/2}).$$

Moreover, we have

$$\forall (s,\hat{s}) \in \mathbb{R}_+^2, \qquad \left|\Omega_1(s,\bar{u}^k) - \Omega_1(\hat{s},\bar{u}^k)\right| = \left|\int_s^{\hat{s}} \frac{F(\sigma,\bar{u}^k) - \langle F\rangle(\bar{u}^k)}{\sqrt{\sigma}} d\sigma\right| \le C|\sqrt{\hat{s}} - \sqrt{s}|$$

so that

$$\left|\frac{\sqrt{\varepsilon}}{2}\Omega_1\left(\tau(\xi),\bar{u}^k\right) - \frac{\sqrt{\varepsilon}}{2}\Omega_1\left(\tau(t^k),\bar{u}^k\right)\right| \le Ch$$

and

$$\frac{\sqrt{\varepsilon}}{2}\Omega_1\left(\tau(\xi),\bar{u}^\varepsilon(\xi)\right) = \frac{\sqrt{\varepsilon}}{2}\Omega_1\left(\tau(\xi),\bar{u}^k\right) + \frac{\sqrt{\varepsilon}}{2}\left(\xi - t^k\right)\partial_2\Omega_1\left(\tau(\xi),\bar{u}^k\right)\langle F\rangle(\bar{u}^k) + \mathcal{O}(h^2).$$

Therefore, denoting

$$b^k = a^k + \frac{\sqrt{\varepsilon}}{2}\Omega_1\left(\tau(t^k),\bar{u}^k\right),$$

our numerical scheme takes the form

$$\begin{aligned}
\Delta^{k+1} =& \Delta^k + \int_{t^k}^{t^{k+1}} F\left(\tau(\xi),b^k\right) d\xi + \int_{t^k}^{t^{k+1}} \left(\xi - t^k\right)\partial_2 F\left(\tau(\xi),b^k\right)\frac{a^{k+1/2}-a^k}{t^{k+1/2}-t^k} d\xi \\
&+ \int_{t^k}^{t^{k+1}} \frac{\sqrt{\varepsilon}}{2}\left(\xi - t^k\right)\partial_2 F\left(\tau(\xi),b^k\right)\partial_2\Omega_1\left(\tau(\xi),\bar{u}^k\right)\langle F\rangle(\bar{u}^k) d\xi \\
&+ \int_{t^k}^{t^{k+1}} \frac{\sqrt{\varepsilon}}{2}\partial_2 F\left(\tau(\xi),b^k\right)\left(\Omega_1\left(\tau(\xi),\bar{u}^k\right) - \Omega_1\left(\tau(t^k),\bar{u}^k\right)\right) d\xi \\
&+ \widetilde{u}^\varepsilon(t^{k+1}) - \widetilde{u}^\varepsilon(t^k),
\end{aligned}$$

and has a truncation error of size $\mathcal{O}(h^3)$, uniformly in ε. As for $k \ge k_0$, we have

$$a^k = \bar{u}^k + \Delta^k + \frac{\varepsilon}{4}\log\left(\frac{1+\tau}{1+\tau_0}\right)\langle\partial_2 GF\rangle\left(\bar{u}^\varepsilon(t_0)\right) + \beta^\varepsilon, \;\; b^k = a^k - \frac{\varepsilon^{1/2}}{2}\Omega_1\left(\tau(t^k),\bar{u}^k\right),$$

and

$$\begin{aligned}
\Delta^{k+1} =& \Delta^k + \int_{t^k}^{t^{k+1}} F\left(\tau(\xi),b^k\right) d\xi + \int_{t^k}^{t^{k+1}} \left(\xi - t^k\right)\partial_2 F\left(\tau(\xi),b^k\right)\frac{a^{k+1/2}-a^k}{t^{k+1/2}-t^k} d\xi \\
&- \int_{t^k}^{t^{k+1}} \frac{\sqrt{\varepsilon}}{2}\left(\xi - t^k\right)\partial_2 F\left(\tau(\xi),b^k\right)\partial_2\Omega_1\left(\tau(t^k),\bar{u}^k\right)\langle F\rangle(\bar{u}^k) d\xi \\
&- \int_{t^k}^{t^{k+1}} \frac{\sqrt{\varepsilon}}{2}\partial_2 F\left(\frac{\Gamma(\tau)}{\varepsilon},b^k\right)\left(\Omega_1\left(\tau(\xi),\bar{u}^k\right) - \Omega_1\left(\tau(t^k),\bar{u}^k\right)\right) d\tau \\
&+ \widetilde{u}^\varepsilon(t^{k+1}) - \widetilde{u}^\varepsilon(t^k).
\end{aligned}$$

According to the above computations, the uniform accuracy with second order of the proposed scheme may now be stated:

PROPOSITION 3.3. *Assume that f is of class C^2. Consider the solution $u^\varepsilon(t)$ of (1.2) on $[0,T]$, and the numerical scheme $(\widetilde{u}^k,\Delta^k)$ defined above. Then $u^k = \widetilde{u}^k + \Delta^k$ yields a uniformly accurate approximation of order two of the solution $u^\varepsilon(t^k)$. Precisely, there exist $\varepsilon_0 > 0$ and $h_0 > 0$ such that for all $\varepsilon \le \varepsilon_0$ and all $h \le h_0$,*

$$|u^k - u^\varepsilon(t^k)| \le Ch^2$$

for all $t^k \le T$ and where C is independent of ε and h.

3.4. Numerical experiments. We test our method on the Hénon-Heiles system with solution $U^\varepsilon = (q_1, q_2, p_1, p_2)$,

$$\dot{U}^\varepsilon(t) = \Big(\frac{\gamma(t)}{\varepsilon} p_1, p_2, -\frac{\gamma(t)}{\varepsilon} q_1 - 2q_1q_2, -q_2 - q_1^2 + q_2^2\Big),\ U^\varepsilon(0) = (0.9, 0.6, 0.8, 0.5),$$

with a time-varying parameter $\gamma(t) = (p+1)(t-t_0)^p$ where t_0 is a zero of multiplicity p. The associated filtered system, satisfied by the variable $u^\varepsilon(t) \in \mathbb{R}^4$ defined by

$$u^\varepsilon(t) = (\cos(\theta)q_1(t) - \sin(\theta)\, p_1(t), q_2(t), \sin(\theta)q_1(t) + \cos(\theta)p_1(t), p_2(t))\,,$$

with $\theta = \frac{(t-t_0)^2}{\varepsilon}$, takes the form (1.2) with

$$F_1(\theta, u) = 2\sin\theta\big(u_1\cos\theta + u_3\sin\theta\big)u_2, \quad F_2(\theta, u) = u_4,$$
$$F_3(\theta, u) = -2\cos\theta\big(u_1\cos\theta + u_3\sin\theta\big)u_2,\ F_4(\theta, u) = -\big(u_1\cos\theta + u_3\sin\theta\big)^2 + u_2^2 - u_2.$$

We consider a time interval of length $T = 1$ and take $t_0 = 1/3$ as time where the oscillatory frequency vanishes. The reference solution is obtained using the matlab `ode45` routine with a tiny tolerance. In Figures 2 and 3, we have represented the error versus the stepsize of the numerical solution u^k in (3.5) (uniform order 1) in cases where $\gamma(t)$ has multiplicities $p = 1$ and $p = 2$ respectively. In Figure 4, we consider the method of Section 3.3 (uniform order 2) for $p = 1$. On the left pictures, the error is plotted as a function of the stepsize h, for fixed values $\varepsilon \in \{2^{-k},\ k = 0, \cdots, 11\}$ where lines of slope 1 (Fig. 2 and 3) and slope 2 (Fig. 4) can be observed. On the right pictures, the error is plotted as a function of ε, for fixed values $h \in \{0.1/2^k,\ k = 0, \cdots, 9\}$, which illustrates the uniform accuracy of the schemes with respect to ε. All curves are in perfect agreement with Propositions 3.2 and 3.3.

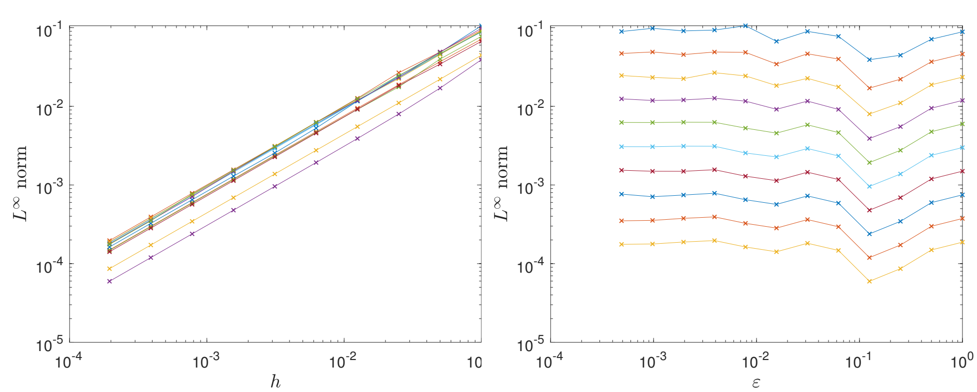

Fig. 2. *Method (3.5) (uniform order 1) for multiplicity $p = 1$. Error as a function of h for $\varepsilon \in \{2^{-k},\ k = 0, \cdots, 11\}$ (left) and error as a function of ε for $h \in \{0.1/2^k,\ k = 0, \cdots, 9\}$ (right).*

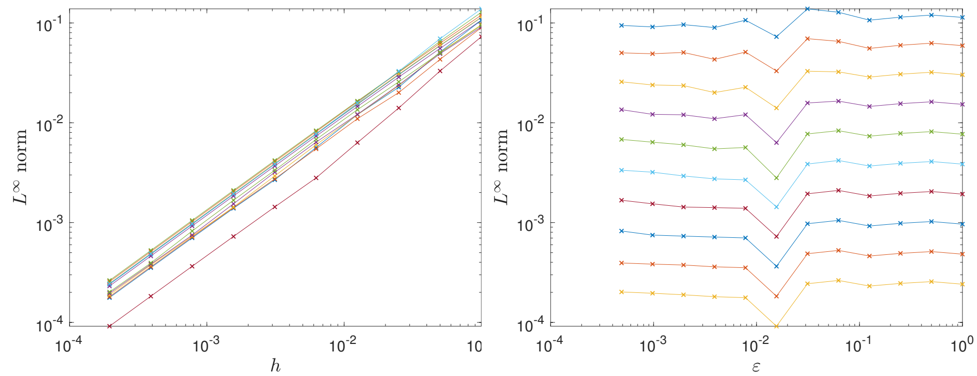

FIG. 3. *Method* (3.5) *(uniform order* 1*) for multiplicity* $p = 2$*. Error as a function of* h *for* $\varepsilon \in \{2^{-k},\, k = 0, \cdots, 11\}$ *(left) and error as a function of* ε *for* $h \in \{0.1/2^k,\, k = 0, \cdots, 9\}$ *(right).*

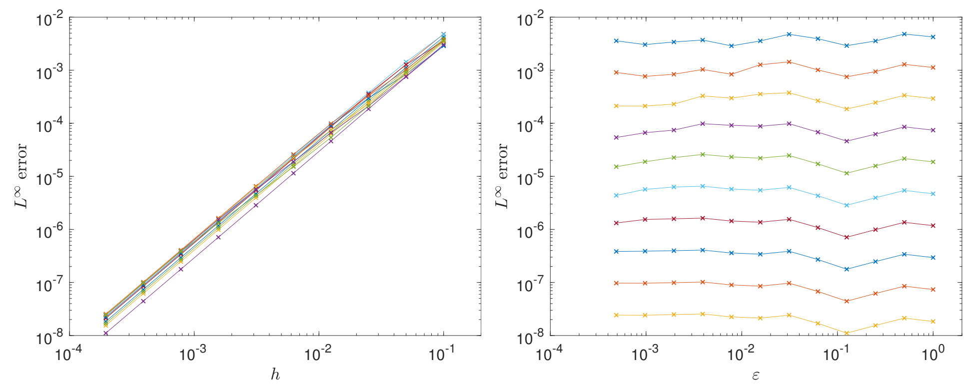

FIG. 4. *Method with uniform order* 2 *for multiplicity* $p = 1$*. Error as a function of* h *for* $\varepsilon \in \{2^{-k},\, k = 0, \cdots, 11\}$ *(left) and error as a function of* ε *for* $h \in \{0.1/2^k,\, k = 0, \cdots, 9\}$ *(right).*

4. Conclusion. In this work, we have derived the first terms of the asymptotic expansion in ε of the exact solution of equation (1.2). As compared to standard averaging where γ is assumed to be bounded from below by a strictly positive constant, convergence towards the so-called averaged model is severally deteriorated for large values of p. For $p = 1$, the next term in the asymptotic expansion behaves quite unexpectedly as $\varepsilon \log(\varepsilon)$ when ε goes to zero and this seems to be the first time such a behaviour is revealed. Based on this asymptotic expansion, we have shown that it is possible to construct uniformly accurate numerical schemes of orders 1 for all $p \geq 1$ and 2 for $p = 1$. Whether one may envisage to construct a uniformly accurate second-order method for $p > 1$ remains an open question and will be the subject of further investigations.

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