Nonlinear compressible vortex sheets 
in two space dimensions

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Abstract

We consider supersonic compressible vortex sheets for the isentropic Euler equations of 
gas dynamics in two space dimensions. The problem is a free boundary nonlinear hyperbolic 
problem with two main difficulties: the free boundary is characteristic, and the so-called 
Lopatinskii condition holds only in a weak sense, which yields losses of derivatives. Neve-
ertheless, we prove the local in time existence of such piecewise smooth solutions to the 
Euler equations. Since the a priori estimates for the linearized equations exhibit a loss of 
regularity, our existence result is proved by using a suitable modification of the Nash-Moser 
iteration scheme. We also show how a similar analysis yields the existence of weakly sta-
ble shock waves in isentropic gas dynamics, and the existence of weakly stable liquid/vapor 
phase transitions.

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loss of derivatives, Nash-Moser iteration scheme.

1 Introduction

The Cauchy problem for the compressible Euler equations in several space dimensions is a 
major challenge in the domain of hyperbolic conservation laws. The (local in time) existence 
of smooth solutions away from vacuum follows from a general Theorem by Kato [20], while the 
existence of smooth solutions with vacuum is proved by Chemin in [8]. Due to the finite time 
blow-up of smooth solutions, see [36] for an example, it is natural to look for weak solutions 
to the Euler equations. The construction of (local in time) piecewise smooth solutions is a 
preliminary step in this direction. The first breakthrough in this direction is the existence of 
one multidimensional uniformly stable shock wave, that was obtained by Majda in [24, 23], see 
also [6] and the references therein for a different approach. The existence of two uniformly stable 
shock waves was shown by Métivier in [27]. Then the existence of multidimensional rarefaction 
waves was obtained by Alinhac in [1]. More recently, Francheteau and Métivier [14] have studied 
the asymptotic behavior of multidimensional shock waves when the strength of the shock tends 
to zero. The limit of such weak shock waves are sonic waves, whose existence is proved in 
[28]. All these works are based on an appropriate iterative scheme (either a standard Picard
iteration or a Nash-Moser iteration), that is proved to converge thanks to a tame estimate on the linearized equations. In this work, we show the existence of contact discontinuities in two space dimensions for the isentropic Euler equations. A similar analysis could be done for the nonisentropic Euler equations, since the stability properties of contact discontinuities for the isentropic Euler equations, and for the nonisentropic Euler equations are quite similar\(^1\).

Let us recall briefly the important features of Majda’s work on shock waves. The existence result of [23] was obtained under a uniform stability assumption, that ensures a good a priori estimate for the linearized equations. By “good” a priori estimate, we mean an estimate where there is no loss of regularity from the source terms to the solution. However, this uniform stability condition is not satisfied by all shock waves in gas dynamics\(^2\). Furthermore, this uniform stability condition (or more precisely the analogue of this condition for characteristic discontinuities), is never satisfied by contact discontinuities in two or three space dimensions, see e.g. [30, 13] or [35, page 222]. As a matter of fact, in three space dimensions, every contact discontinuity is violently unstable (this violent instability is the analogue of the Kelvin-Helmholtz instability for incompressible fluids), while in two space dimensions, a large jump of the tangential velocity makes the contact discontinuity weakly stable. A precise study of this weak stability has been performed by the authors in [12], where it was shown that for such weakly stable contact discontinuities, the linearized equations satisfy an a priori estimate with a loss of one derivative. In this case, one cannot hope to prove the existence of solutions to the nonlinear problem by means of a Picard iteration. In this paper, we shall show that a suitable Nash-Moser iteration converges toward a contact discontinuity solution to the Euler equations.

At the end of the paper, we give two other examples where our analysis applies. More precisely, we can apply the same type of iteration scheme to show the existence of weakly stable shock waves in two or three space dimensions, and the existence of liquid/vapor phase transitions in two or three space dimensions. Roughly speaking, our work shows that the weak Lopatinskii condition, that is known to be sufficient for linear well-posedness, see [10], is also sufficient for nonlinear well-posedness (even when the verification of the weak Lopatinskii condition is submitted to nonlinear constraints). However, we prefer not to give the proof of such an abstract result, and we shall focus on the problem of contact discontinuities for the Euler equations since it gathers the two main difficulties, namely a characteristic free boundary, and the weak Lopatinskii condition under nonlinear constraints.

2 The nonlinear equations

We consider the isentropic Euler equations in the whole plane \(\mathbb{R}^2\). Denoting by \(u \in \mathbb{R}^2\) the velocity of the fluid, and by \(\rho\) its density, the equations read:

\[
\begin{align*}
\partial_t \rho + \nabla_x \cdot (\rho \ u) &= 0, \\
\partial_t (\rho \ u) + \nabla_x \cdot (\rho \ u \otimes u) + \nabla_x p &= 0,
\end{align*}
\]

where \(p = p(\rho)\) is the pressure law. In all this paper, \(p\) is a \(C^\infty\) function of \(\rho\), defined on \([0, +\infty[\), and such that \(p'(\rho) > 0\) for all \(\rho\). The speed of sound \(c(\rho)\) in the fluid is defined by the relation

\[
\forall \rho > 0, \quad c(\rho) := \sqrt{p'(\rho)}.
\]

It is a well-known fact that, for such a pressure law, (1) is a strictly hyperbolic system of conservation laws in the region \([0, +\infty[ \times \mathbb{R}^2\), and is endowed with a strictly convex entropy. (The system is thus symmetrizable).

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\(^1\)We refer the reader to [30, 13, 11] for the stability criteria in the nonisentropic case.

\(^2\)The stability of shock waves heavily depends on the pressure law, but the general idea is that shock waves of moderate strength are uniformly stable, while large shock waves may be only weakly stable.

\(^3\)In particular, one may choose the so-called \(\gamma\)-law, \(p(\rho) = C \rho^\gamma\).
In all what follows, the first and second coordinates of the velocity field are denoted respectively \(v\) and \(u\), that is, \(\mathbf{u} = (v, u) \in \mathbb{R}^2\). Then, for all \(U = (\rho, \mathbf{u}) \in ]0, +\infty[ \times \mathbb{R}^2\), we define the following matrices:

\[
A_1(U) := \begin{pmatrix}
v & \rho & 0 \\
p'(\rho) & v & 0 \\
\rho & v & 0 \end{pmatrix}, \quad A_2(U) := \begin{pmatrix}
u & 0 & \rho \\
0 & u & 0 \\
p'(\rho) & 0 & v \end{pmatrix}.
\]

In the region where \((\rho, \mathbf{u})\) is smooth (say, differentiable), \((1)\) is equivalent to its quasilinear version:

\[
\partial_t U + A_1(U) \partial_{x_1} U + A_2(U) \partial_{x_2} U = 0.
\]

In this paper, we are interested in solutions to \((1)\) that are smooth on either side of a surface \(\Gamma := \{x_2 = \varphi(t, x_1), t \in [0, T], x_1 \in \mathbb{R}\}\), and such that, at each time \(t \in [0, T]\), the tangential velocity is the only quantity that experiments a jump across the curve \(\Gamma(t)\). (Tangential should be understood as tangential with respect to \(\Gamma(t)\).) The density, and the normal velocity should be continuous across \(\Gamma(t)\). For such solutions, the jump conditions across \(\Gamma\) read:

\[
\partial_t \varphi = -v^+ \partial_{x_1} \varphi + u^+ = -v^- \partial_{x_1} \varphi + u^-, \quad \rho^+ = \rho^-.
\]

As detailed in [12], for the isentropic Euler equations \((1)\), these solutions are exactly the contact discontinuities, in the sense of Lax [21]. (Recall that the second characteristic field of \((1)\) is linearly degenerate, and thus, gives rise to contact discontinuities). Observe that for such discontinuous solutions, there is no mass transfer from one side of \(\Gamma(t)\) to the other. (Recall that shock waves are exactly the opposite situation where there is a mass transfer from one side to the other).

The discontinuity surface \(\Gamma\) is part of the unknowns, and it is convenient to reformulate the problem in the fixed domain \(\{t \in [0, T], x_1 \in \mathbb{R}, x_2 \geq 0\}\), by introducing a change of variables. This change of variables is detailed in [12, section 2], see also [1, 24, 29]. After fixing the unknown front, we are led to constructing smooth solutions \(U^\pm = (\rho^\pm, v^\pm, u^\pm)\), \(\Phi^\pm\), to the following system of equations:

\[
\partial_t U^+ + A_1(U^+) \partial_{x_1} U^+ + \frac{1}{\partial x_2 \Phi^+} (A_2(U^+) - \partial_t \Phi^+ - \partial_{x_1} \Phi^+ A_1(U^+)) \partial_{x_2} U^+ = 0, \tag{3a}
\]

\[
\partial_t U^- + A_1(U^-) \partial_{x_1} U^- + \frac{1}{\partial x_2 \Phi^-} (A_2(U^-) - \partial_t \Phi^- - \partial_{x_1} \Phi^- A_1(U^-)) \partial_{x_2} U^- = 0, \tag{3b}
\]

in the interior domain \(\{t \in [0, T], x_1 \in \mathbb{R}, x_2 > 0\}\), with the boundary conditions:

\[
\Phi^+|_{x_2=0} = \Phi^-|_{x_2=0} = \varphi, \tag{4a}
\]

\[
(v^+ - v^-)|_{x_2=0} \partial_{x_1} \varphi - (u^+ - u^-)|_{x_2=0} = 0, \tag{4b}
\]

\[
\partial_t \varphi + v^+|_{x_2=0} \partial_{x_1} \varphi - u^+|_{x_2=0} = 0, \tag{4c}
\]

\[
(r^+ - r^-)|_{x_2=0} = 0. \tag{4d}
\]

We will also consider the initial conditions

\[
(\rho^\pm, v^\pm, u^\pm)|_{t=0} = (\rho^0_0, v^0_0, u^0_0)(x_1, x_2), \quad \varphi|_{t=0} = \varphi_0(x_1), \tag{5}
\]

in the space domain \(\mathbb{R}^2_+ = \{x_1 \in \mathbb{R}, x_2 > 0\}\). The functions \(\Phi^+\) and \(\Phi^-\) should also satisfy the constraints

\[
\forall (t, x_1, x_2) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+, \quad \partial_{x_2} \Phi^+(t, x) \geq \kappa, \quad \text{and} \quad \partial_{x_2} \Phi^-(t, x) \leq -\kappa, \tag{6}
\]
for a suitable constant $\kappa > 0$, as well as the eikonal equations:

$$
\begin{align*}
\partial_t \Phi^+ + v^+ \partial_{x_1} \Phi^+ - u^+ &= 0, \\
\partial_t \Phi^- + v^- \partial_{x_1} \Phi^- - u^- &= 0,
\end{align*}
$$

in the whole domain $\{ t \in [0, T], x_1 \in \mathbb{R}, x_2 > 0 \}$.

Before going on, let us make a few remarks:

**Remark 1.** The interior equations (3a), and (3b), are decoupled. The coupling between the “right” and “left” states arises in the boundary conditions (4).

The constraint (6) ensures that the mapping

$$(t, x_1, x_2) \rightarrow \begin{cases} 
(t, x_1, \Phi^+(t, x_1, x_2)), & \text{if } x_2 > 0, \\
(t, x_1, \Phi^-(t, x_1, -x_2)), & \text{if } x_2 < 0,
\end{cases}
$$

is a change of variables that straightens the unknown front.

The eikonal equations (7), that are clearly imposed on the boundary $\{ x_2 = 0 \}$ by (4a)-(4b)-(4c), ensure that the matrices $A_2(U^+) - \partial_t \Phi^+ - \partial_{x_1} \Phi^+ A_1(U^+)$ have a constant rank in the whole domain $\{ x_2 \geq 0 \}$, and not only on the boundary. Indeed, when (7) is satisfied, (2) gives

$$
A_2(U^+) - \partial_t \Phi^+ - \partial_{x_1} \Phi^+ A_1(U^+) = \begin{pmatrix} 0 & -\rho^+ \partial_{x_1} \Phi^+ & \rho^+ \\
-p'(\rho^+) \partial_{x_1} \Phi^+ / \rho^+ & 0 & 0 \\
p'(\rho^+) / \rho^+ & 0 & 0
\end{pmatrix},
$$

so the rank of these matrices is 2. This constant rank property was crucial in [12] to perform a Kreiss’ type symmetrizers construction and to derive a priori estimates. We refer for instance to [16, 26, 33, 34] for various aspects of this constant rank condition in hyperbolic characteristic boundary value problems.

With an obvious definition for the nonlinear operator $\mathbb{L}$, the equations (3a)-(3b) can be rewritten in the compact form:

$$
\mathbb{L}(U^+, \Phi^+) = 0, \quad \mathbb{L}(U^-, \Phi^-) = 0.
$$

For later use, it is also convenient to write the nonlinear operator $\mathbb{L}$ under the form

$$
\mathbb{L}(U, \Phi) = L(U, \Phi)U.
$$

In other words, we have set

$$
L(U, \Phi)V := \partial_t V + A_1(U)\partial_{x_1} V + \frac{1}{\partial_{x_2} \Phi} \left( A_2(U) - \partial_t \Phi - \partial_{x_1} \Phi A_1(U) \right) \partial_{x_2} V.
$$

In the same way, the boundary conditions (4) can be rewritten in the compact form:

$$
\Phi^+_|_{x_2=0} = \Phi^-|_{x_2=0} = \varphi, \\
\mathbb{B}(U^+_|_{x_2=0}, U^-|_{x_2=0}, \varphi) = 0.
$$

The reader should keep in mind that the nonlinear equations (8), and (10), are supplemented with the initial conditions (5), and with the constraints (6), and (7).

There exist many simple solutions of (8), (10), (6), and (7), that correspond (for the Euler equations (1) in the original variables) to stationary rectilinear vortex sheets:

$$
(\rho, u) = \begin{cases} 
(\rho, v, 0), & \text{if } x_2 > 0, \\
(\rho, -v, 0), & \text{if } x_2 < 0,
\end{cases}
$$
where \( \bar{\rho}, \bar{\tau} \in \mathbb{R} \), \( \bar{\rho} > 0 \). Up to Galilean transformations, every rectilinear vortex sheet has this form. In the straightened variables, these stationary vortex sheets correspond to the following smooth (stationary) solution to (8), (10), (6), (7):

\[
U^\pm \equiv \begin{pmatrix} \bar{\rho} \\ \pm \bar{\tau} \\ 0 \end{pmatrix}, \quad \Phi^\pm (t, x) \equiv \pm x_2, \quad \varphi \equiv 0. \tag{11}
\]

The corresponding constant \( \kappa \) in (6) equals 1. In this paper, we shall assume \( \bar{\tau} > 0 \), but the opposite case can be dealt with in the same way.

Our goal is to construct local in time solutions to the nonlinear system (8), (10), (6), (7), with initial data (5) that are close to the stationary solution (11). (We expect that the solution remains close to the constant stationary solution.) This is a nonlinear stability problem, and we wish to solve the nonlinear equations by solving a sequence of linearized problems. As detailed in the introduction, in the noncharacteristic uniformly stable case, that was first treated by Majda [24, 23] (see also [29, 31] for a refined version, or [6] for an alternative approach), a standard Picard iteration is sufficient to solve the nonlinear problem. In the case of compressible vortex sheets, the so-called uniform stability condition is never satisfied, therefore one cannot prove a maximal estimate in Sobolev spaces for the linearized equations. In [12], we have proved that the supersonic condition \( \bar{\tau} > \sqrt{2} c(\bar{\rho}) \) implies an a priori estimate for the linearized equations. (See section 3 for a precise statement). The a priori estimate indicates a loss of one tangential derivative from the source terms to the solution. The loss is fixed, and we can thus expect to solve the nonlinear problem by a Nash-Moser iteration scheme, see, e.g., [2, 17]. Recall that the Nash-Moser procedure was already used to construct other types of waves for multidimensional systems of conservation laws, see, e.g., [1, 14]. However, the Nash-Moser procedure we shall use here is not completely standard, since the tame estimate for the linearized equations will need to make sure that these constraints are satisfied at each iteration step. The convergence of the Nash-Moser iteration together with the fulfillment of such nonlinear constraints (at each iteration step) is the major contribution of the present work. Let us now state our main result:

**Theorem 1.** Let \( T > 0 \), and let \( \mu \in \mathbb{N} \), with \( \mu \geq 6 \). Assume that the stationary solution defined by (11) satisfies the “supersonic” condition:

\[
\bar{\tau} > \sqrt{2} c(\bar{\rho}). \tag{12}
\]

Assume that the initial data \((U_0^\pm, \varphi_0)\) have the form

\[
U_0^\pm = U^\pm + \hat{U}_0^\pm,
\]

with \( \hat{U}_0^\pm \in H^{\mu+15/2}(\mathbb{R}_+^2), \varphi_0 \in H^{\mu+8}(\mathbb{R}), \) and that they are compatible up to order \( \mu + 7 \) in the sense of Definition 1 (see section 4). Assume also that \((U_0^\pm, \varphi_0)\) have a compact support. Then, there exists \( \delta > 0 \) such that, if \( \|U_0^\pm\|_{H^{\mu+15/2}(\mathbb{R}_+^2)} + \|\varphi_0\|_{H^{\mu+8}(\mathbb{R})} \leq \delta \), then there exists a solution \nolinebreak\( U^\pm = U^\pm + \hat{U}^\pm, \Phi^\pm = \pm x_2 + \hat{\Phi}^\pm, \varphi \) of (3), (4), (5), (6), (7), on the time interval \([0, T]\). This solution satisfies \((\hat{U}^\pm, \hat{\Phi}^\pm) \in H^\mu([0, T] \times \mathbb{R}_+^2), \) and \( \varphi \in H^{\mu+1}([0, T] \times \mathbb{R}). \)

**Remark 2.** The linear stability of planar compressible vortex sheets (11) has been analyzed a long time ago, see [30, 13], see also [3, 35]. In three space dimensions, planar vortex sheets are known to be violently unstable. In the two dimensional case, subsonic vortex sheets (when \( \bar{\tau} < \sqrt{2} c(\bar{\rho}) \)) are also violently unstable, while supersonic vortex sheets under condition (12) are weakly linearly stable. This result formally agrees with the theory of incompressible vortex sheets.
In fact, in the incompressible limit, the speed of sound tends to infinity, with the result that two-dimensional vortex sheets are always unstable, the well-known Kelvin-Helmholtz instability.

Theorem 1 shows that condition (12) is also sufficient for nonlinear well-posedness. We prove that, if the initial vortex sheet at time zero is a sufficiently small perturbation of (11), (12), the solution of the nonlinear vortex sheet problem exists on a given time interval. The smallness condition ensures that singularities cannot form on this time interval. In particular, the formation of kink modes (as detailed in [3]) can occur only after the time $T$. Recall that the “supersonic” condition (12) also plays a crucial role in the analysis of [3].

The compatibility conditions (see Definition 1 in section 4) are strong conditions imposed in order that only the desired contact discontinuity is generated by the initial discontinuity, similarly to [23, 1, 14]. As far as we know the general Riemann problem, with all kinds of singularities (shocks, rarefaction waves, contact discontinuities . . . ) coming out from the initial discontinuity, is still an open problem. (See however [18] for the case of analytic data.)

For the sake of simplicity, we shall detail the proof of Theorem 1 when the support of $\varphi_0$ is included in $[-1, 1]$, and the support of $U_0^\pm$ is included in $\{x_2 \geq 0, \sqrt{x_1^2 + x_2^2} \leq 1\}$. Thanks to the finite speed of propagation of the Euler equations and the eikonal equation, the corresponding solution $U^\pm, \Phi^\pm, \varphi$ will have a compact support:

$$\text{Supp } (U^\pm, \Phi^\pm) \subset \{t \in [0,T], x_2 \geq 0, \sqrt{x_1^2 + x_2^2} \leq 1 + \lambda_{\text{max}} T\},$$

$$\text{Supp } \varphi \subset \{t \in [0,T], |x_1| \leq 1 + \lambda_{\text{max}} T\},$$

where $\lambda_{\text{max}} := \tau + c(\overline{\rho})$.

The rest of the paper is organized as follows: in section 3, we recall the basic a priori $L^2$ estimate that was derived in [12], and we show an a priori tame estimate in the Sobolev spaces. In section 4, we detail the compatibility conditions. Starting from compatible initial data, we detail the construction of an approximate solution. We are then reduced to solving a nonlinear system with zero initial data. In section 5, we describe the iteration scheme that will be used to solve this reduced problem. In section 6, we shall collect the main estimates that are needed to prove the convergence of the iterative scheme. This convergence will be proved in section 7. Appendices A and B are devoted to the application of our technique to similar problems, that is, the existence of weakly stable shock waves in isentropic gas dynamics, and the existence of isothermal liquid-vapor phase transitions in a van der Waals fluid.

### 3 Tame estimates for the linearized equations

#### 3.1 Weighted spaces and norms

To prove Theorem 1, we are going to use weighted Sobolev spaces that we introduce right now. The notations of this paragraph will be used throughout all the rest of this paper.

First we define the half-space

$$\Omega := \{(t, x_1, x_2) \in \mathbb{R}^3 \text{ such that } x_2 > 0\} = \mathbb{R}^2 \times \mathbb{R}^+.$$

The boundary $\omega := \partial \Omega$ is identified to $\mathbb{R}^2$. We denote the usual norm of $L^2(\mathbb{R}^2)$, resp. $L^2(\Omega)$, by $\| \cdot \|_{L^2(\mathbb{R}^2)}$, resp. $\| \cdot \|_{L^2(\Omega)}$.

For all real number $s$ and all $\gamma \geq 1$, we define the space

$$H^s_\gamma(\mathbb{R}^2) := \{u \in \mathcal{D}'(\mathbb{R}^2) \text{ such that } \exp(-\gamma t) u \in H^s(\mathbb{R}^2)\}.$$

It is equipped with the norm

$$\|u\|_{H^s_\gamma(\mathbb{R}^2)} := \| \exp(-\gamma t) u \|_{H^s(\mathbb{R}^2)}.$$
The space $L^2(\mathbb{R}^+; H^s_T(\mathbb{R}^2))$ is equipped with the obvious norm
\[
\|u\|_{L^2(\mathbb{R}^+; H^s_T(\mathbb{R}^2))}^2 := \int_0^{+\infty} \|u(t, x_2)\|^2_{H^s_T(\mathbb{R}^2)} \, dx_2.
\]

For all real number $T$ (in particular when $T > 0$), we let $\Omega_T$, and $\omega_T$ denote the sets
\[
\omega_T := [0, T]\times \mathbb{R}, \quad \Omega_T := [0, T]\times \mathbb{R}, \quad \omega_T := [0, T]\times \mathbb{R}^+.
\]

For all integer $m$, and all $\gamma \geq 1$, we define the weighted Sobolev space $H^m_\gamma(\Omega_T)$ as
\[
H^m_\gamma(\Omega_T) := \left\{ u \in D'(\Omega_T) \text{ such that } \exp(-\gamma t)u \in H^m(\Omega_T) \right\}.
\]

The definition of $H^m_\gamma(\omega_T)$ is similar. The norm on $H^m_\gamma(\Omega_T)$ may be defined by
\[
\|u\|_{H^m_\gamma(\Omega_T)} := \sum_{|\alpha| \leq m} \gamma^{m-|\alpha|} \|e^{-\gamma t} \partial^\alpha u\|_{L^2(\Omega_T)},
\]
which is equivalent to the norm $\|e^{-\gamma t} u\|_{H^m(\Omega_T)}$. The constant in the equivalence is independent of $\gamma \geq 1$, and $T$. The norm on $H^m_\gamma(\omega_T)$ is defined in the same way. Thus, the space $L^2(\mathbb{R}^+; H^s_\gamma(\omega_T))$ is equipped with the norm
\[
\|u\|_{L^2(\mathbb{R}^+; H^s_\gamma(\omega_T))} := \sum_{\alpha_0 + \alpha_1 \leq m} \gamma^{m-\alpha_0 - \alpha_1} \|e^{-\gamma t} \partial^\alpha t_1 \partial^\alpha x_1 u\|_{L^2(\omega_T)}.
\]

This is an anisotropic Sobolev space where one measures only the tangential regularity (tangential means tangential with respect to the boundary $\{x_2 = 0\}$).

**Remark 3.** For functions $u(t, x)$ that are supported in a strip $\{ t \in [T_1, T_2]\}$, $T_1 < T_2$, one has $u \in H^m_\gamma(\Omega_{T_2})$ if and only if $u \in H^m(\Omega_{T_2})$, and the norms are equivalent. However, the constant in the equivalence of the norms heavily depends on $\gamma$. This is the reason why it will be more convenient to derive a tame estimate in the $H^m_\gamma$ norm, that is defined by (13).

Eventually, for $1 \leq p < +\infty$, and $T \in \mathbb{R}$, the space $L^p_T(\Omega_T)$ is the set of measurable functions $u$ such that $e^{-2\gamma t/p} u$ belongs to $L^p(\Omega_T)$. The norm in $L^p_T(\Omega_T)$ is the obvious one. (Of course, there is a similar definition with $\omega_T$ instead of $\Omega_T$).

### 3.2 The (effective) linearized equations

We introduce the linearized equations around a state that is given as a perturbation of the stationary solution (11). More precisely, let us consider some functions
\[
U_r = \begin{pmatrix} \rho \cr \nu \cr 0 \end{pmatrix} + \tilde{U}_r(t, x), \quad U_l = \begin{pmatrix} \rho \cr -\nu \cr 0 \end{pmatrix} + \tilde{U}_l(t, x), \quad \Phi_r = x_2 + \tilde{\Phi}_r(t, x), \quad \Phi_l = -x_2 + \tilde{\Phi}_l(t, x),
\]
where
\[
U_{r,l}(t, x) = \begin{pmatrix} \rho_{r,l}(t, x) \\ \nu_{r,l}(t, x) \\ 0 \\ \tilde{u}_{r,l}(t, x) \\ \tilde{v}_{r,l}(t, x) \end{pmatrix}, \quad \tilde{U}_{r,l}(t, x) = \begin{pmatrix} \tilde{\rho}_{r,l}(t, x) \\ \tilde{\nu}_{r,l}(t, x) \end{pmatrix}.
\]

The index $r$ (resp. $l$) denotes the state on the right (resp. on the left) of the interface before the change of variables. We assume that the perturbations $\tilde{U}_{r,l}, \tilde{\Phi}_{r,l}$ have a compact support:
\[
\text{Supp } (\tilde{U}_{r,l}, \tilde{\Phi}_{r,l}) \subset \{ t \in [-T, 2T], x_2 \geq 0, \sqrt{x_1^2 + x_2^2} \leq 1 + 2\lambda_{max} T \}.
\]
The smoothness of these perturbations will be chosen later on. We also assume that these quantities satisfy the Rankine-Hugoniot conditions and the continuity condition for the functions \( \Phi_{r,l} \):

\[
\Phi_{r}|_{x_2=0} = \Phi_{l}|_{x_2=0} = \varphi,
\]

\[
(v_r - v_l)|_{x_2=0} \partial_{x_1} \varphi - (u_r - u_l)|_{x_2=0} = 0,
\]

\[
\partial_t \varphi + v_r|_{x_2=0} \partial_{x_1} \varphi - u_r|_{x_2=0} = 0,
\]

\[
(\rho_r - \rho_l)|_{x_2=0} = 0.
\]

Eventually, we assume that the functions \( \Phi_r \) and \( \Phi_l \) satisfy the eikonal equations:

\[
\partial_t \Phi_r + v_r \partial_{x_1} \Phi_r - u_r = 0,
\]

\[
\partial_t \Phi_l + v_l \partial_{x_1} \Phi_l - u_l = 0,
\]

(17)

together with

\[
\partial_{x_2} \Phi_r \geq \kappa_0, \quad \partial_{x_2} \Phi_l \leq -\kappa_0,
\]

(18)

for a suitable constant \( \kappa_0 \in [0, 1[ \), in the whole closed half-space \( \{x_2 \geq 0\} \). Note that (18) can be deduced from a “smallness” condition on the perturbations \( \dot{\Phi}_{r,l} \). More precisely, (18) can be derived from an estimate of the type

\[
\|\dot{\Phi}_{r,l}\|_{W^{1,\infty}(\mathbb{R}^2 \times \mathbb{R}^+)} \leq 1 - \kappa_0, \quad \kappa_0 \in [0, 1[.
\]

This will be useful in the construction of nonlinear vortex sheets.

Let us consider some families \( U^\pm_s = U_{r,l} + sV_\pm, \Phi^\pm_s = \Phi_{r,l} + s\Psi_\pm \), where \( s \) is a small parameter. Recalling that the nonlinear operator \( \mathbb{L} \) is defined in (8), we compute the linearized equations around the state \( U_{r,l}, \Phi_{r,l} \):

\[
\mathbb{L}^\prime(U_{r,l}, \Phi_{r,l})(V_\pm, \Psi_\pm) := \frac{d}{ds} \mathbb{L}(U^\pm_s, \Phi^\pm_s)|_{s=0} = f_\pm.
\]

We first compute

\[
\mathbb{L}^\prime(U_r, \Phi_r)(V_+, \Psi_+) = \partial_t V_+ + A_1(U_r) \partial_{x_1} V_+ + \frac{1}{\partial_{x_2} \Phi_r} (A_2(U_r) - \partial_t \Phi_r - \partial_{x_1} \Phi_r A_1(U_r)) \partial_{x_2} V_+
\]

\[
+ [dA_1(U_r)V_+] \partial_{x_1} U_r - \frac{\partial_{x_2} \Psi_+}{(\partial_{x_2} \Phi_r)^2} (A_2(U_r) - \partial_t \Phi_r - \partial_{x_1} \Phi_r A_1(U_r)) \partial_{x_2} U_r
\]

\[
+ \frac{1}{\partial_{x_2} \Phi_r} [dA_2(U_r)V_+] \partial_t \Psi_+ - \partial_{x_1} \Psi_+ A_1(U_r) - \partial_{x_1} \Phi_r dA_1(U_r)V_+] \partial_{x_2} U_r = f_+
\]

(19)

in the domain \( \{x_2 > 0\} \), and we obtain a similar equation with \( V_-, \Psi_-, U_l, \Phi_l, f_- \) instead of \( V_+, \Psi_+, U_r, \Phi_r, f_+ \).

Observe that the principal part (in \( V_\pm \)) of the linearized operator \( \mathbb{L}^\prime(U_{r,l}, \Phi_{r,l}) \), is nothing but the hyperbolic operator \( L(U_{r,l}, \Phi_{r,l}) \) defined in (9), see the first line in (19). The remaining terms in (19) are either zero order terms in \( V_+ \), or first order terms in \( \Psi_+ \).

The linearized equation (19) and the corresponding one for \( V_-, \Psi_- \) may be simplified by following [1] and introducing the “good unknown”:

\[
\dot{V}_+ := V_+ - \frac{\Psi_+}{\partial_{x_2} \Phi_r} \partial_{x_2} U_r, \quad \dot{V}_- := V_- - \frac{\Psi_-}{\partial_{x_2} \Phi_l} \partial_{x_2} U_l,
\]

(20)
A direct calculation shows that $\dot{V}_+$ and $\dot{V}_-$ satisfy
\begin{equation}
\begin{aligned}
\mathbb{L}'(U_r, \Phi_r)(V_+, \Psi_+) &= L(U_r, \Phi_r)\dot{V}_+ + C(U_r, \Phi_r)\dot{V}_+ + \frac{\Psi_+}{\partial x_2 \Phi_r} \partial x_2 \{ L(U_r, \Phi_r) \} = f_+, \\
\mathbb{L}'(U_l, \Phi_l)(V_-, \Psi_-) &= L(U_l, \Phi_l)\dot{V}_- + C(U_l, \Phi_l)\dot{V}_- + \frac{\Psi_-}{\partial x_2 \Phi_l} \partial x_2 \{ L(U_l, \Phi_l) \} = f_-, 
\end{aligned}
\end{equation}
where, for all smooth functions $(U, \Phi)$, the matrix $C(U, \Phi)$ is defined as follows:
\begin{equation}
C(U, \Phi) X := [dA_1(U) X] \partial_{x_1} U + \frac{1}{\partial x_2 \Phi} \left\{ [dA_2(U) X] - \partial_{x_1} \Phi [dA_1(U) X] \right\} \partial_{x_2} U.
\end{equation}
In particular, the matrices $C(U_{r,l}, \Phi_{r,l})$ are $C^\infty$ functions of $(\dot{U}_{r,l}, \nabla \dot{U}_{r,l}, \nabla \Phi_{r,l})$ that vanish at the origin (that is, when $(\dot{U}_{r,l}, \nabla \dot{U}_{r,l}, \nabla \Phi_{r,l}) = 0$).

In view of the results proved in [1, 14], we neglect the zero order term in $\Psi_+, \Psi_-$ in the linearized equations (21), and we thus consider the effective linear operators
\begin{equation}
\begin{aligned}
\mathbb{L}'(U_r, \Phi_r)\dot{V}_+ := L(U_r, \Phi_r)\dot{V}_+ + C(U_r, \Phi_r)\dot{V}_+ = f_+, \\
\mathbb{L}'(U_l, \Phi_l)\dot{V}_- := L(U_l, \Phi_l)\dot{V}_- + C(U_l, \Phi_l)\dot{V}_- = f_-. 
\end{aligned}
\end{equation}
In the next sections, we shall show how to solve the nonlinear problem (8), (10), (6), (7) by means of a sequence of linear equations of the form (23). The remaining terms in (21) will be considered as error terms at each iteration step. We thus focus on (23) from now on. (Observe that the linearized front $\Psi_{\pm}$ does not appear anymore in these equations.)

It is clear that the effective linearized equations (23) form a symmetrizable hyperbolic system. For instance, a Friedrichs symmetrizer for the operator $L'(U_r, \Phi_r)$ is given by
\begin{equation}
S_r(t, x) := \begin{pmatrix}
p'(\rho_r) & 0 & 0 \\
0 & \rho_r & 0 \\
0 & 0 & \rho_r 
\end{pmatrix} (t, x).
\end{equation}
Using the eikonal equations (17), we compute
\begin{equation}
\frac{S_r}{\partial x_2 \Phi_r} \left( A_2(U_r) - \partial_t \Phi_r - \partial_{x_1} \Phi_r A_1(U_r) \right) = \frac{1}{\partial x_2 \Phi_r} \begin{pmatrix}
0 & -p'(\rho_r)\partial_{x_1} \Phi_r & p'(\rho_r) \\
-p'(\rho_r)\partial_{x_1} \Phi_r & 0 & 0 \\
p'(\rho_r) & 0 & 0 
\end{pmatrix},
\end{equation}
and we thus expect to control the traces of the components $\dot{V}_{+,1}$, and $(\dot{V}_{+,3} - \partial_{x_1} \Phi_r \dot{V}_{+,2})$ on the boundary $\{x_2 = 0\}$. In the same way, we expect to control the traces of the components $\dot{V}_{-,1}$, and $(\dot{V}_{-,3} - \partial_{x_1} \Phi_l \dot{V}_{-,2})$ on the boundary. These preliminary considerations motivate the introduction of the following trace operator on the boundary:
\begin{equation}
\mathbb{P}(\varphi)\dot{V}_{\pm,2} := \begin{pmatrix}
\dot{V}_{\pm,1} \\ \dot{V}_{\pm,3} - \partial_{x_1} \varphi \dot{V}_{\pm,2}
\end{pmatrix} \mid_{x_2 = 0}.
\end{equation}
This operator will be used in the energy estimates for the linearized equations. Its image can be understood as the “noncharacteristic” part of the vector $\dot{V}$ (or more precisely, of its trace on $\{x_2 = 0\}$).

We now turn to the linearized boundary conditions. The linearization of (4) gives
\begin{equation}
\begin{aligned}
\Psi_{+|x_2=0} &= \Psi_{-|x_2=0} = \psi, \\
(v_r - v_l) \partial_{x_1} \psi + (v_+ - v_-) \partial_{x_1} \varphi - (u_+ - u_-) &= g_1, \\
\partial_t \psi + v_r \partial_{x_1} \psi + v_+ \partial_{x_1} \varphi - u_+ &= g_2, \\
\rho_+ - \rho_- &= g_3,
\end{aligned}
\end{equation}
on the boundary \( \{ x_2 = 0 \} \). Let us introduce the matrices
\[
b(t, x_1) := \begin{pmatrix} 0 & (v_r - v_l)|_{x_2=0} \\ 1 & v_r|_{x_2=0} \\ 0 & 0 \end{pmatrix},
\]
\[
M(t, x_1) := \begin{pmatrix} 0 & \partial_{x_1} \varphi & -1 & 0 & -\partial_{x_1} \varphi & 1 \\ 0 & \partial_{x_1} \varphi & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.
\]

Let us also denote \( V = (V_+, V_-)^T, \nabla \psi = (\partial_\psi, \partial_{x_1} \psi)^T \) and \( g = (g_1, g_2, g_3)^T \). Then the linearized boundary conditions equivalently read
\[
\Psi_+|_{x_2=0} = \Psi_-|_{x_2=0} = \psi, \\
\mathbb{B}'(U_{r,l}, \Phi_{r,l})(V|_{x_2=0}, \psi) = b \nabla \psi + M V|_{x_2=0} = g.
\]

In terms of the good unknown \( \dot{V} = (\dot{V}_+, \dot{V}_-)^T \) defined by (20), the linearized boundary conditions read
\[
\Psi_+|_{x_2=0} = \Psi_-|_{x_2=0} = \psi, \\
\mathbb{B}'(U_{r,l}, \Phi_{r,l})(\dot{V}|_{x_2=0}, \psi) := b \nabla \psi + M \left( \frac{\partial_{x_2} U_r}{\partial_{x_2} \Phi_r} \right) |_{x_2=0} \psi + M \dot{V}|_{x_2=0} = g.
\]

We observe that the linearized boundary conditions only involve \( \mathbb{P}(\varphi) \dot{V}|_{x_2=0} \), where \( \mathbb{P}(\varphi) \) is defined by (24), see the expression of the matrix \( M \) in (25).

### 3.3 The basic \( L^2 \) a priori estimate

We recall the \( L^2 \) a priori estimate for (23), (26) that we have derived in [12]. We assume that the perturbations satisfy
\[
\dot{U}_r, \dot{U}_l \in W^{2,\infty}(\Omega), \quad \dot{\Phi}_r, \dot{\Phi}_l \in W^{3,\infty}(\Omega), \\
||\dot{U}_r||_{W^{2,\infty}(\Omega)} + ||\dot{U}_l||_{W^{2,\infty}(\Omega)} + ||\dot{\Phi}_r||_{W^{3,\infty}(\Omega)} + ||\dot{\Phi}_l||_{W^{3,\infty}(\Omega)} \leq K,
\]
where \( K \) is a positive constant. Then the following result holds:

**Theorem 2** ([12]). Assume that the particular solution defined by (11) satisfies the “supersonic”condition (12), and that the perturbations \( \dot{U}_{r,l}, \dot{\Phi}_{r,l} \) satisfy (15), (16), (17), (18), and (27). There exist some positive constants \( K_0 > 0, C_0 > 0, \) and \( \gamma_0 \geq 1, \) such that if \( K \leq K_0, \) and \( \gamma \geq \gamma_0, \) then for all \( (\dot{V}, \psi) \in H^2_1(\Omega) \times H^2_2(\mathbb{R}^2) \) the following a priori estimate holds:
\[
\gamma \|
\dot{V}\|_{L^2_2(\Omega)}^2 + ||\mathbb{P}(\varphi) \dot{V}|_{x_2=0}||_{L^2_2(\mathbb{R}^2)}^2 + ||\psi||_{H^2_1(\mathbb{R}^2)}^2
\[
\leq C_0 \left( \frac{1}{\gamma^3} \| L'_e(U_r, \Phi_r) \dot{V}_+ \|_{L^2_2(H^1_1)}^2 + \frac{1}{\gamma^3} \| L'_e(U_l, \Phi_l) \dot{V}_- \|_{L^2_2(H^1_1)}^2 + \frac{1}{\gamma^2} \| \mathbb{B}'(U_{r,l}, \Phi_{r,l})(\dot{V}|_{x_2=0}, \psi) \|_{H^2_1(\mathbb{R}^2)}^2 \right). 
\]

The operators \( \mathbb{P}(\varphi), L'_e, \) and \( \mathbb{B}' \) are defined in (24), (23), and (26).

Before going on, we make a few comments:
The result of Theorem 2 is independent of the lower order term in the linearized equations. More precisely, if we consider two matrices $D_{r,l} \in \mathbb{W}^{1, \infty}(\Omega)$, then the same result as in Theorem 2 holds where the matrices $C(U_{r,l}, \Phi_{r,l})$, see (22)-(23), are replaced by the matrices $D_{r,l}$. The corresponding constants will only depend on the $W^{1, \infty}$ norm of $D_{r,l}$.

There is no loss of generality in assuming $K_0 \leq 1/2$ in Theorem 2. In particular, (18) is satisfied with $\kappa_0 = 1/2$. Therefore, from now on we shall assume $K_0 \leq 1/2$, and we forget the constraint (18).

One requirement on $K_0$ is that at each point $(t, x_1, 0)$ of the boundary $\{x_2 = 0\}$, the frozen state $(U_{r,l}, \Phi_{r,l})(t, x_1, 0)$ defines a weakly stable rectilinear vortex sheet (that is, it satisfies (12) up to an appropriate change of Galilean frame). This ensures that at each point of the boundary, the weak Lopatinskii condition is satisfied. Such a restriction is possible, because the domain of weak stability is defined by an “open” condition on the states that satisfy the Rankine-Hugoniot conditions.

### 3.4 Well-posedness of the linearized equations

In this section, we show how to apply the well-posedness result of [10] to the linearized equations (23), (26). In view of the result of [10], we only need to check that there exists a dual problem that satisfies an a priori estimate with a loss of one tangential derivative. (This is because the coefficients of the linearized operators satisfy the symmetrizability and regularity assumptions of [10]). The first task consists in defining a dual problem for (23), (26).

On the boundary $\{x_2 = 0\}$, the matrix in front of the normal derivative $\partial_{x_2}$ in the operator $L^r_\omega(U_{r,l}, \Phi_{r,l})$ equals

$$\frac{1}{\partial_{x_2} \Phi_{r,l}} \left\{ A_2(U_{r,l}) - \partial_t \Phi_{r,l} - \partial_{x_1} \Phi_{r,l} A_1(U_{r,l}) \right\}_{|x_2=0} = \frac{1}{\partial_{x_2} \Phi_{r,l}}|_{x_2=0} \begin{pmatrix} 0 & -\rho \partial_{x_1} \varphi & \rho \\ -\frac{c^2}{\rho} \partial_{x_1} \varphi & 0 & 0 \\ \frac{c^2}{\rho} & 0 & 0 \end{pmatrix},$$

where $\rho(t, x_1)$ denotes the common trace of $\rho_r$ and $\rho_l$, see (16), $c$ is a short notation for the sound speed $c(\rho(t, x_1))$, and $\varphi$ is the common trace of $\Phi_r$ and $\Phi_l$. Recall that the matrix $M(t, x_1)$ is defined by (25). In particular, one has $M \in W^{2, \infty}(\mathbb{R}^2)$. We now define the following matrices:

$$N(t, x_1) := \begin{pmatrix} 0 & \rho \partial_{x_1} \varphi & -1 & 0 & \partial_{x_1} \varphi & -1 \\ 0 & \varphi & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$M_1(t, x_1) := \begin{pmatrix} 0 & 0 & 0 & \rho \partial_{x_1} \varphi & 0 & 0 \\ 0 & 0 & \rho \partial_{x_1} \varphi & 0 & 0 & 0 \\ \rho \partial_{x_1} \varphi & 0 & 0 & 0 & 0 \\ -\frac{c^2}{2 \rho \partial_{x_2} \Phi_r} & 0 & \rho \partial_{x_1} \varphi & 0 & -\frac{c^2}{2 \rho \partial_{x_2} \Phi_l} \\ 0 & \frac{c^2}{2 \rho \partial_{x_2} \Phi_r} & 0 & 0 & \rho \partial_{x_1} \varphi \\ 0 & 0 & \frac{c^2}{2 \rho \partial_{x_2} \Phi_l} & 0 & \rho \partial_{x_1} \varphi \end{pmatrix},$$

$$N_1(t, x_1) := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{c^2}{2 \rho \partial_{x_2} \Phi_r} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{c^2}{2 \rho \partial_{x_2} \Phi_r} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{c^2}{2 \rho \partial_{x_2} \Phi_l} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{c^2}{2 \rho \partial_{x_2} \Phi_l} & 0 & 0 \end{pmatrix}.$$
Moreover, using (27), one has one has

\[ \frac{1}{\partial_x \Phi_{r,l}} (A_2(U_{r,l}) - \partial_t \Phi_{r,l} - \partial_x \Phi_{r,l} A_1(U_{r,l})) |_{x_2 = 0} = M_1^T M + N_1^T N. \]  

Moreover, using (27), one has one has \( M_1, N_1, N \in W^{2,\infty}(\mathbb{R}^2). \)

Following [29, page 61], we can define a dual problem for (23), (26), in the following way:

\[
\begin{align*}
L'_e(U_r, \Phi_r)^* U_+ &= \tilde{f}_+, \quad x_2 > 0, \\
L'_e(U_l, \Phi_l)^* U_- &= \tilde{f}_-, \quad x_2 > 0, \\
N_1 U|_{x_2=0} &= 0, \\
div (b^T M_1 U|_{x_2=0}) - b_1^T M_1 U|_{x_2=0} &= 0,
\end{align*}
\]

where \( N_1, M_1 \) are defined in (29), \( b \) is defined in (25), the vector \( b_t \) is defined in (26), \( div \) denotes the divergence operator in \( \mathbb{R}^2 \) with respect to the variables \((t, x_1)\), and the dual operators \( L'_e(U_{r,l}, \Phi_{r,l})^* \) are the formal adjoints of \( L'_e(U_{r,l}, \Phi_{r,l}) \), that is:

\[
L'_e(U_{r,l}, \Phi_{r,l})^* := -\partial_t - A_1(U_{r,l})^T \partial_{x_1} - \frac{1}{\partial_{x_2} \Phi_{r,l}} (A_2(U_{r,l}) - \partial_t \Phi_{r,l} - \partial_x \Phi_{r,l} A_1(U_{r,l}))^T \partial_{x_2} \\
+ C(U_{r,l}, \Phi_{r,l})^T - \partial_{x_1} A_1(U_{r,l})^T - \partial_{x_2} \left\{ \frac{1}{\partial_{x_2} \Phi_{r,l}} (A_2(U_{r,l}) - \partial_t \Phi_{r,l} - \partial_x \Phi_{r,l} A_1(U_{r,l}))^T \right\}.
\]

Recall that the dual problem is derived by integration by parts, and by using the relation (30) for the boundary terms. We refer to [7, 29] for the details.

Let us note that the dual problem (31) has exactly two independent scalar boundary conditions, since the two first rows of the matrix \( N_1 \) are zero, see (29). This is compatible with the number of incoming characteristics, because each of the boundary matrices of the operators \( L'_e(U_{r,l}, \Phi_{r,l})^* \) have one positive, one negative, and one null eigenvalue. The hyperbolic system (31) thus has two incoming characteristics, and two outgoing characteristics.

In view of the result of [10], we only need to check that the dual problem (31) satisfies an a priori estimate with a loss of one tangential derivative. First of all, we can derive the so-called Lopatinskii determinant associated with the boundary conditions in (31). This amounts to freezing the coefficients at one point of the boundary, and to performing a normal modes analysis. We shall not give the details of this analysis, since the calculations are really similar to those that can be found in [35, page 222], or in [12, page 956]. The result is the following:

**Lemma 1.** The dual problem (31) satisfies the backward weak Lopatinskii condition. Moreover, the associated Lopatinskii determinant is smooth (that is, \( C^\infty \)) near its zeroes, and its zeroes are simple. (The zeroes coincide with the zeroes of the Lopatinskii determinant associated with the original problem.)

As a matter of fact, the Lopatinskii determinant is exactly equal to the Lopatinskii determinant of the original problem (23), (26) (provided that the stable subspaces that are involved in the calculation are parametrized in a suitable way). Consequently, one can reproduce exactly the same analysis as done in [12], and show that the dual problem satisfies an a priori estimate with a loss of one tangential derivative (both on the boundary, and in the interior domain). The linearized equations (23), (26) thus satisfy all the symmetrizability, regularity and weak stability assumptions of [10]. We therefore have the following well-posedness result:

**Theorem 3.** Let \( T > 0 \). Assume that the stationary solution (11) satisfies (12), and that the perturbations \( \dot{U}_{r,l}, \dot{\Phi}_{r,l} \) satisfy (15), (16), (17), and (27). There exist three positive constants
$K_0 > 0$, $\gamma_0 \geq 1$, and $C_0 > 0$, that do not depend on $T$, such that if $K \leq K_0$, then for all source terms $f_+, f_-$, and $g$ that satisfy

$$f_\pm \in L^2(\mathbb{R}^+; H^1(\omega_T)), \quad g \in H^1(\omega_T),$$

and that vanish for $t < 0$, there exists a unique $(\dot{V}_+, \dot{V}_-, \psi) \in L^2(\Omega_T) \times L^2(\Omega_T) \times H^1(\omega_T)$ such that $\mathcal{P}(\varphi)\dot{V}_{|r=0} \in L^2(\omega_T)$, $(\dot{V}_+, \dot{V}_-, \psi)$ is a solution to

$$\begin{cases}
L'_e(U_r, \Phi_r) \dot{V}_+ = f_+, & t < T, \ x_2 > 0, \\
L'_e(U_l, \Phi_l) \dot{V}_- = f_-, & t < T, \ x_2 > 0, \\
\mathcal{E}'(U_{r,l}, \Phi_{r,l})(\dot{V}_{|r=0}, \psi) = g, & t < T,
\end{cases}$$

and $(\dot{V}_+, \dot{V}_-, \psi)$ vanishes for $t < 0$. In addition, $(\dot{V}_+, \dot{V}_-) \in C([0,T]; L^2(\mathbb{R}_+^2))$, and we have the following inequality for all $\gamma \geq \gamma_0$, and for all $t \in [0,T]$:

$$e^{-\gamma t}\Vert \dot{V}(t) \Vert_{L^2(\mathbb{R}_+^2)} + \sqrt{\gamma} \Vert \dot{V} \Vert_{L^2(\Omega_t)} + \Vert \mathcal{P}(\varphi)\dot{V}_{|r=0} \Vert_{L^2(\omega_t)} + \Vert \psi \Vert_{H^1_t(\omega_t)} \leq C_0 \left( \frac{1}{\gamma^{3/2}} \Vert (f_+, f_-) \Vert_{L^2(H^1_t(\omega_t))} + \frac{1}{\gamma} \Vert g \Vert_{H^1_t(\omega_t)} \right).$$

Theorem 3 states the well-posedness of (23), (26) in $L^2$ when the source terms belong to $L^2(H^1)$. When dealing with quasilinear problems, this type of estimate is clearly not sufficient and we need to obtain a similar well-posedness result in a Sobolev space $H^m$, with $m$ large enough. As a matter of fact, when dealing with a Nash-Moser iteration, one needs a well-posedness result in the Sobolev space $H^m$, where $m$ is arbitrarily large. The crucial point is to obtain also a tame estimate in $H^m$ (roughly speaking, there is a fixed loss of derivatives with respect to the source terms and to the coefficients, and high norms are multiplied by low norms). The remaining of this section is devoted to obtaining this tame estimate.

The method is the following: we consider the linear equations (23), (26). We first change the equations in order to deal with a boundary matrix of the form

$$I_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

This is possible because the boundary matrix has a constant rank in the whole domain $\Omega$. Then we estimate the tangential derivatives, and we can deduce an estimate for some of the normal derivatives. Observe that the boundary is characteristic so that the equations (23) enable us to control only 4 normal derivatives in terms of tangential derivatives. Actually, the missing normal derivatives are those of the tangential velocity (both on the right and on the left of the interface). These normal derivatives can be estimated by computing the equation for the vorticity.

Of course, the derivation of energy estimates in Sobolev spaces is possible only when the coefficients have more regularity than what is stated in (27). This is of little consequence since, in a Nash-Moser iteration, the coefficients are very smooth at each iteration step.

To estimate the tangential derivatives of the solution to (23), (26), one commutes the equation with a tangential derivative and applies the Gagliardo-Nirenberg inequalities to estimate the commutator. These estimates are recalled in Appendix C at the end of this paper, together with other useful nonlinear estimates.
3.5 An equivalent formulation of the linearized equations

We want to derive an estimate similar to that of Theorem 3, in the Sobolev space $H^m_\gamma$, with $m$ arbitrarily large. To achieve this goal, it is necessary to transform the interior equations (23) in order to deal with a hyperbolic operator that has a constant and diagonal boundary matrix (i.e., the matrix coefficient of $\partial x_2$ in the differential operators $L(U_{r,l}, \Phi_{r,l})$). Let us consider the coefficients of $\partial x_1 \dot{V}_\pm$ in (23), and forget for a moment the indices $r, l$. Using (17), this coefficient reduces to the matrix

$$A'_2(U, \Phi) = \frac{1}{\partial x_2} \begin{pmatrix} 0 & -\rho \partial x_1 \Phi & \rho \\ -\frac{\rho'}{\rho} \partial x_2 \Phi & 0 & 0 \\ \frac{\rho'}{\rho} & 0 & 0 \end{pmatrix}. $$

Let us now define the following invertible matrices

$$T(U, \Phi) := \begin{pmatrix} 0 & \langle \partial x_1, \Phi \rangle & \langle \partial x_1, \Phi \rangle \\ 1 & -\frac{c(\rho)}{\rho} \partial x_1 \Phi & \frac{c(\rho)}{\rho} \partial x_2 \Phi \\ \partial x_1 \Phi & \frac{c(\rho)}{\rho} & -\frac{c(\rho)}{\rho} \end{pmatrix},$$  

$$A_0(U, \Phi) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \partial x_2 \Phi & 0 \\ 0 & 0 & -\frac{\partial x_2 \Phi}{c(\rho)(\partial x_1 \Phi)} \end{pmatrix},$$

where we have introduced the notation $\langle \partial x_1 \Phi \rangle := \sqrt{1 + (\partial x_1 \Phi)^2}$. Those matrices satisfy

$$A_0 T^{-1} A'_2 T = I_2 := \text{diag} (0, 1, 1).$$

Let us define the new vectors:

$$W_+ := T(U_r, \Phi_r)^{-1} \dot{V}_+, \quad W_- := T(U_l, \Phi_l)^{-1} \dot{V}_-, $$

and set

$$\forall (t,x) \in \Omega, \quad T_{r,l}(t,x) := T(U_{r,l}, \Phi_{r,l})(t,x), \quad A_{0,l}^r(t,x) := A_0(U_{r,l}, \Phi_{r,l})(t,x).$$

After multiplication on the left side of the equations in (23) by $A_{0,l}^r T_{r,l}^{-1}$, we see that the vectors $W_\pm$ solve the system

$$A_0^r \partial_t W_+ + A_1^r \partial x_1 W_+ + I_2 \partial x_2 W_+ + C^r W_+ = F_+, $$

$$A_0^l \partial_t W_- + A_1^l \partial x_1 W_- + I_2 \partial x_2 W_- + C^l W_- = F_-, $$

where we have defined

$$\forall (t,x) \in \Omega, \quad A_{1,l}^r(t,x) := A_{0,l}^r T_{r,l}^{-1} A_1(U_{r,l}) T_{r,l}(t,x), $n

$$C_{r,l}(t,x) := A_{0,l}^r T_{r,l}^{-1} \partial_t T_{r,l} + A_1(U_{r,l}) \partial x_1 T_{r,l} + A_2(U_{r,l}, \Phi_{r,l}) \partial x_2 T_{r,l} + C(U_{r,l}, \Phi_{r,l}) T_{r,l}(t,x),$$

$$F_\pm(t,x) = A_{0,l}^r T_{r,l}^{-1} f_\pm(t,x).$$

The above equations (34) are equivalent to (23). Observe that the source terms in (34) slightly differ from the source terms in (23). In the remaining of this section, we always keep the notation $F_\pm$ for the source terms in (34) (that is, for the equation satisfied by $W_\pm$), while $f_\pm$ denote the source terms in (23) (that is, for the equation satisfied by $V_\pm$).
Remark 5. The matrices $A^{r,l}_0, A^{r,l}_1$ belong to $W^{2,\infty}(\Omega)$, and the matrix $C^{r,l}$ belongs to $W^{1,\infty}(\Omega)$. Moreover $A^{r,l}_0$, and $A^{r,l}_1$ are $C^\infty$ functions of $(\bar{U}_{r,l}, \nabla \Phi_{r,l})$ that do not vanish at the origin, while $C^{r,l}$ is a $C^\infty$ function of $(\bar{U}_{r,l}, \nabla \bar{U}_{r,l}, \nabla^2 \Phi_{r,l})$ that vanishes at the origin.

Using the vector $W = (W_+, W_-)^T$ as defined in (33), the linearized boundary conditions (26) become equivalent to
\begin{equation}
\begin{align*}
\Psi_+|_{x_2=0} &= \Psi_-|_{x_2=0} = \psi, \\
 b \nabla \psi + b_2 \psi + M W|_{x_2=0} &= g,
\end{align*}
\end{equation}
where we have set
\[ M := M(t, x_1) \begin{pmatrix} T_r & 0 \\ 0 & T_l \end{pmatrix} (t, x_1, 0), \]
and $M$ is defined in (25).

Remark 6. The matrices $b$ and $M$ belong to $W^2,\infty(\mathbb{R}^2)$, and the vector $b_2$ belongs to $W^1,\infty(\mathbb{R}^2)$. Moreover, $b$ is a $C^\infty$ function of $(\bar{U}_{r,l}|_{x_2=0}, \bar{U}_{r,l})$, $M$ is a $C^\infty$ function of $(\bar{U}_{r,l}|_{x_2=0}, \partial_x \varphi, \partial_x \Phi_{r,l}|_{x_2=0})$ that vanishes at the origin.

Observe that the new boundary conditions (36) only involve the “noncharacteristic” part of the vector $W = (W_+, W_-)^T$, that is, the sub-vector $W^{nc} := (W_{+2}, W_{+3}, W_{-2}, W_{-3})^T$. These are the components whose trace can be controlled on the boundary $\{x_2 = 0\}$. Consequently, we shall feel free to rewrite the product $M W|_{x_2=0}$ as $M W^{nc}|_{x_2=0}$, even though the dimensions of the matrices do not agree (but they would if one just omitted the two columns of $M$ that identically vanish, see [12, page 984]).

3.6 A priori estimate of tangential derivatives

In the new formulation (34), (36), the estimate of Theorem 3 gives the following estimate, where $W = (W_+, W_-)$, and $F = (F_+, F_-)$:
\begin{equation}
\sqrt{\gamma} \|W\|_{L^2(\Omega_T)} + \|W^{nc}|_{x_2=0}\|_{L^2(\omega_T)} + \|\psi\|_{H^1(\omega_T)} \leq C_0 \left( \frac{1}{\gamma^{3/2}} \|F\|_{L^2(H^1(\omega_T))} + \frac{1}{\gamma} \|g\|_{H^1(\omega_T)} \right),
\end{equation}
whenever $\gamma \geq \gamma_0$, and $K \leq K_0$. Until the end of this section, we shall always assume $K \leq K_0$, with $K_0$ given by Theorem 3.

From now on, we assume that the perturbations $\bar{U}_{r,l}$, and $\Phi_{r,l}$ not only satisfy (27), but also belong to the space $H^k_T(\Omega)$ for all integer $k$, and all $\gamma \geq 1$. Moreover, to avoid overloaded expressions, we shall feel free to write $\bar{U}, \Phi$, instead of $\bar{U}_{r,l}, \Phi_{r,l}$.

The aim of this paragraph is to prove the following result:

Proposition 1. Let $m \in \mathbb{N}$, $m \geq 1$, and let $T > 0$. There exist two constants $C_m > 0$ and $\gamma_m \geq 1$, that do not depend on $T$, such that for all $\gamma \geq \gamma_m$ and for all $(W, \psi) \in H^{m+2}(\Omega_T) \times H^{m+2}(\omega_T)$ solution to (34), (36), the following estimate holds:
\begin{equation}
\sqrt{\gamma} \|W\|_{L^2(H^m_T(\omega_T))} + \|W^{nc}|_{x_2=0}\|_{H^m_T(\omega_T)} + \|\psi\|_{H^{m+1}_T(\omega_T)} \\
\leq C_m \left\{ \frac{1}{\gamma^{3/2}} \|F\|_{L^2(H^{m+1}_T(\omega_T))} + \frac{1}{\gamma} \|g\|_{H^{m+1}_T(\omega_T)} + \frac{1}{\gamma^{3/2}} \|W\|_{W^{1,\infty}(\Omega_T)} \|\bar{U}, \nabla \Phi\|_{H^{m+2}_T(\omega_T)} \right. \\
\left. + \frac{1}{\gamma} \left( \|W^{nc}|_{x_2=0}\|_{L^\infty(\omega_T)} + \|\psi\|_{W^{1,\infty}(\omega_T)} \right) \|\bar{U}, \partial_2 \bar{U}, \nabla \Phi|_{x_2=0}\|_{H^{m+1}_T(\omega_T)} \right\}.
\end{equation}
Proof. We consider an integer $\ell$ such that $1 \leq \ell \leq m$, and a tangential derivative $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}$, with $\ell = |\alpha|$. Starting from (34), we compute

$$A_0 \partial_t \partial^\alpha W_+ + A_1 \partial_{x_1} \partial^\alpha W_+ + L_2 \partial_{x_2} \partial^\alpha W_+ + C' \partial^\alpha W_+$$

$$+ \sum_{|\beta|=1, \beta \leq \alpha} \star \left[ \partial^\beta A_0^\beta \partial^{\alpha-\beta} \partial_t W_+ + \partial^\beta A_1^\beta \partial^{\alpha-\beta} \partial_{x_1} W_+ \right] = \partial^\alpha F_+$$

$$+ \sum_{|\beta| \geq 2, \beta \leq \alpha} \star \left[ \partial^\beta A_0^\beta \partial^{\alpha-\beta} \partial_t W_+ + \partial^\beta A_1^\beta \partial^{\alpha-\beta} \partial_{x_1} W_+ \right] + \sum_{|\beta| \geq 1, \beta \leq \alpha} \star \left[ \partial^\beta C' \partial^{\alpha-\beta} W_+ \right]. \quad (39)$$

In (39), the $\star$ symbol denotes a constant coefficient that only depends on $\alpha$ and $\beta$ (which is harmless in the energy estimate). There is a similar equation for $\partial^\alpha W_-$ that we do not write. The difficulty that we are facing is that the “zero order” coefficient in (39) (that is, the terms that involve derivatives of order equal to $\ell$) does not only involve $\partial^\alpha W_+$. It also involves the derivatives $\partial^{\alpha-\beta} \partial_t W_+$ and $\partial^{\alpha-\beta} \partial_{x_1} W_+$, where $|\beta| = 1$. We thus write an enlarged system that is satisfied by all the tangential derivatives of order equal to $\ell$. Defining the vector

$$W^{(\ell)} := \left\{ \partial_x^{\alpha_0} \partial_{x_1}^{\alpha_1} W_+ , \quad \alpha_0 + \alpha_1 = \ell \right\},$$

the equation (39), and the corresponding one for $\partial^\alpha W_-$, show that the vector $W^{(\ell)}$ satisfies an equation of the form

$$A_0 \partial_t W^{(\ell)} + A_1 \partial_{x_1} W^{(\ell)} + I \partial_{x_2} W^{(\ell)} + C W^{(\ell)} = F^{(\ell)}. \quad (40)$$

The matrices $A_{0,1}$ are block diagonal, and their blocks are either $A_{0,1}$ or $A_{1,1}$. Therefore, they belong to $W^{2,\infty}(\Omega)$. The matrix $C$ belongs to $W^{1,\infty}(\Omega)$, and it decouples into

$$C W^{(\ell)} = \begin{pmatrix} C' & 0 \\ 0 & C' \end{pmatrix} \begin{pmatrix} W_+^{(\ell)} \\ W_-^{(\ell)} \end{pmatrix},$$

with rather obvious notations. (Note the difference between $I$, and $\ell$). The matrix $I$ is block diagonal and its blocks are the matrix $I_2$. Eventually, the source term $F^{(\ell)}$ is a sum of terms that appear in the right-hand side of (39).

We now turn to the boundary conditions satisfied by $W^{(\ell)}$. Starting from (36), we compute

$$b \nabla \partial^\alpha \psi + b_2 \partial^\alpha \psi + M \partial^\alpha W^{nc}_{x_2=0} + \partial^\alpha g$$

$$+ \sum_{|\beta| \geq 1, \beta \leq \alpha} \star \left[ \partial^\beta M \partial^{\alpha-\beta} W^{nc}_{x_2=0} + \partial^\beta b \nabla \partial^{\alpha-\beta} \psi + \partial^\beta b_2 \partial^{\alpha-\beta} \psi \right]. \quad (41)$$

Following what we have done for the interior equations, the collection of all the equations (41) can be rewritten as

$$B \nabla \psi^{(\ell)} + B_2 \psi^{(\ell)} + M W^{(\ell),nc}_{x_2=0} = G^{(\ell)}. \quad (42)$$

We claim that the enlarged system (40), (42) satisfies an energy estimate similar to (37). This is because the enlarged system satisfies the same regularity and stability properties as the original system (34), (36). As a matter of fact, we have only collected a certain number of copies of the original system, and modified the zero order coefficient in the interior equations. However, the a priori estimate in [12], and the well-posedness result of [10] are independent of the zero order coefficient, provided that it belongs to $W^{1,\infty}(\Omega)$ (which is the case). Therefore, we obtain an estimate for the tangential derivatives of order $\ell$. This estimate reads

$$\sqrt{\gamma} \| W^{(\ell)} \|_{L^2(\omega_T)} + \| W^{(\ell),nc}_{x_2=0} \|_{L^2(\omega_T)} + \| \psi^{(\ell)} \|_{H^1(\omega_T)}$$

$$\leq C_\ell \left( \frac{1}{\gamma^{1/2}} \| F^{(\ell)} \|_{L^2(H^1(\omega_T))} + \frac{1}{\gamma} \| G^{(\ell)} \|_{H^1(\omega_T)} \right). \quad (43)$$
The remaining of the proof consists in estimating the source terms $F^{(t)}$, and $G^{(t)}$, that is, in estimating the right-hand sides of (39), and (41).

First of all, we have
\[
\|\partial^\alpha F\|_{L^2(H^s_0(\omega_T))} \leq \gamma \|\partial^\alpha F\|_{L^2_3(\Omega_T)} + \|\nabla_t \partial^\alpha F\|_{L^2_3(\Omega_T)} \leq \|F\|_{L^2_3(H^{\alpha+1}_0(\omega_T))},
\]
and we also have
\[
\|\partial^\alpha g\|_{H^s_0(\omega_T)} \leq \|g\|_{H^{\alpha+1}_0(\omega_T)}.
\]

Omitting the superscripts $r, l$ or the subscripts $\pm$, we now turn to the estimate of the term $\partial^\beta A_0 \partial_t \partial^\alpha - \beta W$ in $L^2(H^s_0(\omega_T))$, where $\beta \leq \alpha$, and $|\beta| \geq 2$. For a fixed $x_2 > 0$, we apply Gagliardo-Nirenberg’s inequality (see Theorem 8, as well as Theorem 10 in Appendix C); then we integrate with respect to $x_2$. Decomposing $\beta = \beta' + \beta_1$, with $|\beta_1| = 1$, and recalling that $\partial^\beta A_0$ is a $C^\infty$ function of $(\dot{U}, \nabla \Phi, \nabla_{t,x_1} \dot{U}, \nabla_{t,x_1} \nabla \Phi)$ that vanishes at the origin, we obtain
\[
\|\partial^\beta A_0(x_2)\|_{H^s_0(\omega_T)} \leq C(K) \|\dot{U}, \nabla \Phi, \nabla_{t,x_1} \dot{U}, \nabla_{t,x_1} \nabla \Phi)(x_2)\|_{H^{s-1}(\omega_T)}^{2/p}, \quad \frac{2}{p} = \frac{|\beta| - 1}{|\alpha| - 1},
\]
\[
\|\partial^\alpha \partial_t W(x_2)\|_{L^2_0(\omega_T)} \leq C \|\partial_t W(x_2)\|_{H^{s-1}(\omega_T)}^{2/q}, \quad \frac{2}{q} = \frac{|\alpha| - |\beta|}{|\alpha| - 1}.
\]

Using Hölder’s inequality and integrating with respect to $x_2$, we obtain
\[
\|\partial^\beta A_0 \partial_t \partial^\alpha - \beta W\|_{L^2_3(\Omega_T)} \leq C(K) \left\{ \|W\|_{L^2(H^s_0(\omega_T))} + \|W\|_{H^{1,\infty}(\Omega_T)} \|\dot{U}, \nabla \Phi\|_{L^2(H^s_0(\omega_T))} \right\}.
\]

Decomposing $\beta = \beta'' + \beta_2$, with $|\beta_2| = 2$, one proves in a similar way the inequalities
\[
\|\partial_t (\partial^\beta A_0 \partial_t \partial^\alpha - \beta W)\|_{L^2_3(\Omega_T)} \leq C(K) \left\{ \|W\|_{L^2(H^s_0(\omega_T))} + \|W\|_{H^{1,\infty}(\Omega_T)} \|\dot{U}, \nabla \Phi\|_{L^2(H^{1+1}_0(\omega_T))} \right\},
\]
\[
\|\partial_{x_1} (\partial^\beta A_0 \partial_t \partial^\alpha - \beta W)\|_{L^2_3(\Omega_T)} \leq C(K) \left\{ \|W\|_{L^2(H^s_0(\omega_T))} + \|W\|_{H^{1,\infty}(\Omega_T)} \|\dot{U}, \nabla \Phi\|_{L^2(H^{1+1}_0(\omega_T))} \right\}.
\]

We have thus obtained
\[
\|\partial^\beta A_0 \partial_t \partial^\alpha - \beta W\|_{L^2(H^s_0(\omega_T))} \leq C(K) \left\{ \gamma \|W\|_{L^2(H^s_0(\omega_T))} + \|W\|_{H^{1,\infty}(\Omega_T)} \|\dot{U}, \nabla \Phi\|_{L^2(H^{1+1}_0(\omega_T))} \right\}.
\]

In the same way, we obtain
\[
\|\partial^\beta A_1 \partial_{x_1} \partial^\alpha - \beta W\|_{L^2(H^s_0(\omega_T))} \leq C(K) \left\{ \gamma \|W\|_{L^2(H^s_0(\omega_T))} + \|W\|_{H^{1,\infty}(\Omega_T)} \|\dot{U}, \nabla \Phi\|_{L^2(H^{1+1}_0(\omega_T))} \right\}.
\]

The estimate of the terms $\partial^\beta C \partial^\alpha - \beta W$, $|\beta| \geq 1$, can be carried out in a similar way. We simply recall that $C$ is a $C^\infty$ function of $(\dot{U}, \nabla \dot{U}, \nabla \Phi, \nabla^2 \Phi)$ that vanishes at the origin. We obtain
\[
\|\partial^\beta C \partial^\alpha - \beta W\|_{L^2(H^s_0(\omega_T))} \leq C(K) \left\{ \gamma \|W\|_{L^2(H^s_0(\omega_T))} + \|W\|_{H^{\infty}(\Omega_T)} \|\dot{U}, \nabla \Phi, \nabla^2 \Phi\|_{L^2(H^{1+1}_0(\omega_T))} \right\}.
\]

These preliminary estimates already enable us to obtain
\[
\|F^{(t)}\|_{L^2(H^s_0(\omega_T))} \leq C(K) \left\{ \|F\|_{L^2(H^{1+1}_0(\omega_T))} + \gamma \|W\|_{L^2(H^s_0(\omega_T))} + \|W\|_{H^{\infty}(\Omega_T)} \|\dot{U}, \nabla \Phi\|_{H^{1+2}(\Omega_T)} \right\}.
\]
The estimate of the right-hand side of \((41)\) is also carried out with the Theorems of Appendix C, and we shall not reproduce the details here. We only give the final estimate for \(G^\ell\):

\[
\|G^\ell\|_{H^2(\omega_T)} \leq C(K) \left\{ \|g\|_{H^{\ell+1}(\omega_T)} + \|W^{nc}_{|x=0}\|_{H^\ell(\omega_T)} + \|\psi\|_{H^{\ell+1}(\omega_T)} + \|\omega\|_{H^{\ell+1}(\omega_T)} \right\}.
\]

(45)

To finish the proof, we use \((44)\), and \((45)\) in \((43)\). Then we multiply this inequality by \(\gamma^{m-\ell}\), and sum over \(\ell = 0, \ldots, m\). Eventually, up to choosing \(\gamma\) large enough, we can absorb the term

\[
\frac{1}{\sqrt{\gamma}} \|W\|_{L^2(H^m(\omega_T))} + \frac{1}{\gamma} \left( \|W^{nc}_{|x=0}\|_{H^m(\omega_T)} + \|\psi\|_{H^{m+1}(\omega_T)} \right)
\]

in the left-hand side of the inequality. We have therefore obtained \((38)\).

\[ \square \]

### 3.7 The vorticity equation

Since the boundary matrix is singular at \(\omega_T\), all the normal derivatives of \(W\) cannot be estimated directly from the equations, as in the standard approach for noncharacteristic boundaries. However, this can be done for some of the normal derivatives, namely for those of the “non-characteristic” part of the solution. The remaining normal derivatives are estimated through a vorticity-type linearized equation. It is important to note that for our linearized problem \((23)\), \((26)\), there is no loss of derivatives when estimating the normal derivatives (see, e.g., \([16, 33, 34]\) for the general case where losses of normal derivatives occur).

In order to introduce the vorticity linearized equation, let us consider first of all the original Euler system \((1)\). If there exists a solution that is smooth on either side of an interface, this solution satisfies

\[
\rho \left( \partial_t u + (u \cdot \nabla x)u \right) + \nabla_x p(\rho) = 0, \quad u = \begin{pmatrix} v \\ u \end{pmatrix}.
\]

Hence the vorticity \(\xi := \partial_{x_1} u - \partial_{x_2} v\) satisfies (on either side of the interface):

\[
\partial_t \xi + u \cdot \nabla_x \xi + \xi (\nabla_x \cdot u) = 0.
\]

Recalling that the interface is a streamline, and that there is continuity of the normal velocity across the interface, this suggests the possibility of estimating the vorticity on either part of the front. Performing the change of variables by the introduction of the functions \(\Phi^\pm\) and taking account of the linearization leads to the definition of the “linearized vorticity” given below.

Let us now consider the system \((23)\) and in particular the equations satisfied by the components of the velocity. Writing the “good unknown” \((20)\) as \(\hat{V}_{\pm} = (\hat{\rho}_{\pm}, \hat{v}_{\pm}, \hat{u}_{\pm})^T\), we compute

\[
\begin{cases}
\partial_t \hat{v}_+ + \frac{p'(\rho_r)}{\rho_r} \partial_{x_1} \hat{\rho}_+ + v_r \partial_{x_1} \hat{v}_+ - \frac{p'(\rho_r)}{\rho_r} \partial_{x_2} \Phi_r \partial_{x_2} \hat{\rho}_+ = \mathcal{F}_1^+, \\
\partial_t \hat{u}_+ + v_r \partial_{x_1} \hat{u}_+ + \frac{p'(\rho_r)}{\rho_r} \partial_{x_2} \Phi_r \partial_{x_2} \hat{\rho}_+ = \mathcal{F}_2^+,
\end{cases}
\]

(46)

and similar equations for \((\hat{v}_-, \hat{u}_-)\), where we have let

\[
\mathcal{F}_1^\pm := (f_\pm - C(U_{r,l}, \Phi_{r,l}) \hat{V}_\pm)_2, \quad \mathcal{F}_2^\pm := (f_\pm - C(U_{r,l}, \Phi_{r,l}) \hat{V}_\pm)_3.
\]

(47)

Let us now define the “linearized vorticities” as follows:

\[
\dot{\xi}_+ := \partial_{x_1} \hat{u}_+ - \frac{1}{\partial_{x_2} \Phi_r} \left( \partial_{x_1} \Phi_r \partial_{x_2} \hat{u}_+ + \partial_{x_2} \hat{v}_+ \right),
\]

\[
\dot{\xi}_- := \partial_{x_1} \hat{u}_- - \frac{1}{\partial_{x_2} \Phi_l} \left( \partial_{x_1} \Phi_l \partial_{x_2} \hat{u}_- + \partial_{x_2} \hat{v}_- \right).
\]

(48)
We compute the “material” derivatives \( \partial_t \xi^\pm + v_r, l \partial_x \xi^\pm \). Using (46), the calculation gives

\[
\begin{align*}
\partial_t \xi^+ + v_r \partial_x \xi^+ &= \partial_x F_2^+ - \frac{1}{\partial_x \Phi} (\partial_x \Phi \partial_x F_2^+ + \partial_x F_1^+) + \Lambda_1^1 \partial_x \dot{V}^+ + \Lambda_2^1 \partial_x \dot{V}^+, \\
\partial_t \xi^- + v_l \partial_x \xi^- &= \partial_x F_2^- - \frac{1}{\partial_x \Phi} (\partial_x \Phi \partial_x F_2^- + \partial_x F_1^-) + \Lambda_1^1 \partial_x \dot{V}^- + \Lambda_2^1 \partial_x \dot{V}^-.
\end{align*}
\]

(49)

In (49), the vectors \( \Lambda_1^1, \Lambda_2^1 \) are \( C^\infty \) functions of \((\dot{U}_r, l, \nabla \dot{U}_r, l, \nabla^2 \dot{\Phi}, r, l)\) that vanish at the origin, and whose exact expression is of no interest. We have the following result:

**Proposition 2.** Let \( m \in \mathbb{N}, m \geq 1, \) and \( T > 0 \). There exist two constants \( C_m > 0 \) and \( \gamma_m \geq 1, \) that do not depend on \( T, \) such that for all \( \gamma \geq \gamma_m, \) the smooth solutions \( \xi^\pm \) to (49) satisfy the following a priori estimate:

\[
\sqrt{\gamma} \| \dot{\xi}^\pm \|_{H^{m-1}_\gamma(\Omega_T)} \leq \frac{C_m}{\sqrt{\gamma}} \left\{ \| \dot{V}^\pm \|_{H^m_\gamma(\Omega_T)} + \| \dot{F}^\pm \|_{L^\infty(\Omega_T)} \| \nabla \Phi \|_{H^m_\gamma(\Omega_T)} + \| \dot{V}^\pm \|_{W^{1,\infty}(\Omega_T)} \left[ \| \dot{U}_r, l \|_{H^{m+1}_\gamma(\Omega_T)} + \| \nabla \dot{\Phi}, r, l \|_{H^m_\gamma(\Omega_T)} \right] \right\}.
\]

(50)

**Proof.** We first rewrite (49) as

\[
\partial_t \xi^+ + v_r \partial_x \xi^+ = \mathcal{H}^+, \quad \partial_t \xi^- + v_l \partial_x \xi^- = \mathcal{H}^-,
\]

(51)

with obvious definitions for \( \mathcal{H}^\pm \). Integrating by parts, one first shows the basic \( L^2_\gamma \) estimate:

\[
\sqrt{\gamma} \| \dot{\xi}^\pm \|_{L^2(\Omega_T)} \leq \frac{C(K)}{\sqrt{\gamma}} \| \mathcal{H}^\pm \|_{L^2(\Omega_T)}.
\]

Then commuting (51) with all derivatives of the form \( \partial^\alpha = \partial_t^{\alpha_0} \partial_x^{\alpha_1} \partial_r^{\alpha_2} \), where \( 1 \leq |\alpha| \leq m - 1, \) and applying the Gagliardo-Nirenberg inequalities (see Theorem 8 in appendix C), we get

\[
\sqrt{\gamma} \| \dot{\xi}^\pm \|_{H^{m-1}_\gamma(\Omega_T)} \leq \frac{C_m(K)}{\sqrt{\gamma}} \left\{ \| \mathcal{H}^\pm \|_{H^{m-1}_\gamma(\Omega_T)} + \| \dot{V}^\pm \|_{H^{m-1}_\gamma(\Omega_T)} + \| \dot{\xi}^\pm \|_{L^\infty(\Omega_T)} \| \dot{U}_r, l \|_{H^m_\gamma(\Omega_T)} \right\}.
\]

The term \( \| \dot{\xi}^\pm \|_{H^{m-1}_\gamma(\Omega_T)} \) in the right-hand side of the inequality can be absorbed in the left-hand side by choosing \( \gamma \) large enough. Now we note that \( \dot{\xi}^\pm \) is a linear combination of derivatives of \( \dot{V}^\pm, \) see (48), so the \( L^\infty(\Omega_T) \) norm of \( \dot{\xi}^\pm \) is estimated by the \( W^{1,\infty}(\Omega_T) \) norm of \( \dot{V}^\pm \). Hence we get

\[
\sqrt{\gamma} \| \dot{\xi}^\pm \|_{H^{m-1}_\gamma(\Omega_T)} \leq \frac{C_m(K)}{\sqrt{\gamma}} \left\{ \| \mathcal{H}^\pm \|_{H^{m-1}_\gamma(\Omega_T)} + \| \dot{V}^\pm \|_{W^{1,\infty}(\Omega_T)} \| \dot{U}_r, l \|_{H^m_\gamma(\Omega_T)} \right\}.
\]

(52)

It only remains to estimate the source term \( \mathcal{H}^\pm, \) that is, the right-hand side of (49), in the space \( H^{m-1}_\gamma(\Omega_T). \) To derive such an estimate, we make use of the Theorems of Appendix C. We first obtain

\[
\| \mathcal{H}^\pm \|_{H^{m-1}_\gamma(\Omega_T)} \leq C_m(K) \left\{ \| \dot{F}^\pm \|_{H^m_\gamma(\Omega_T)} + \| \nabla \dot{\Phi}, r, l \|_{H^m_\gamma(\Omega_T)} \right\}.
\]

(53)

We apply (170) to estimate \( \dot{F}^\pm, \) see (47) for the definition. We recall that the matrices \( C(U_r, l, \dot{\Phi}, r, l) \) are \( C^\infty \) functions of \((\dot{U}_r, l, \nabla \dot{U}_r, l, \nabla^2 \dot{\Phi}, r, l)\) that vanish at the origin, see (22). Thus we
Using (47) and (22), it is also clear that we have order terms, as a linear combination of tangential derivatives of \( I \) in (56), we can also use the relation.

These two last equalities show that the normal derivative \( \partial_{x_1} \dot{u}_+ \) and \( \partial_{x_1} W_{+,1} \) can be expressed, up to lower order terms, as a linear combination of tangential derivatives of \( W_+ \) and the vorticity \( \xi_+ \). As regards the remaining normal derivatives \( \partial_{x_2} W_{+,2} \) and \( \partial_{x_2} W_{+,3} \), they are directly given by the equation (34):

\[
\mathbf{I}_2 \partial_{x_2} W_+ = \mathbf{I}_2 \left( F_+ - \mathbf{A}_0^r \partial_t W_+ - \mathbf{A}_1^r \partial_{x_1} W_+ - C^r W_+ \right) .
\]

We recall that \( \mathbf{I}_2 := \text{diag} \ (0, 1, 1) \). We may combine this latter relation with (56), and write

\[
\partial_{x_2} W_+ = \mathbf{I}_2 F_+ + \mathbf{A}_0^r \partial_t W_+ + \mathbf{A}_1^r \partial_{x_1} W_+ + C^r W_+ - \frac{\partial_x \Phi_r}{(\partial_{x_1} \Phi_r)^2} \begin{pmatrix} \xi_+ \\ 0 \\ 0 \end{pmatrix} ,
\]

with new matrices \( \mathbf{\tilde{A}}_0^r \) and \( \mathbf{\tilde{C}}^r \) whose exact expression is not useful. The only thing to keep in mind is that \( \mathbf{\tilde{A}}_0^r \) and \( \mathbf{\tilde{C}}^r \) are \( C^\infty \) functions of \( (U_r, \nabla \Phi_r) \), and that \( \mathbf{\tilde{C}}^r \) is a \( C^\infty \) function of \( (U_r, \nabla U_r, \nabla \Phi_r, \nabla^2 \Phi_r) \) that vanishes at the origin. These matrices include the contribution of the term \( \partial_{x_1} \dot{u}_+ \) and the contribution of the zero order terms in (56). Of course, there is a similar equation for \( \partial_{x_2} W_- \).

The equation (57) shows that, though the linearized problem (34), (36) is characteristic, all the normal derivatives are given as a linear combination of tangential derivatives, just as for noncharacteristic problems. (The only novelty here is the term that involves the vorticity).

Following the analysis of the noncharacteristic case, see [29, page 84], we are going to prove the following result:
Proposition 3. Let \( m \in \mathbb{N}, m \geq 1 \), and let \( T > 0 \). Then there exist some constants \( C_m \) and \( \gamma_m \), that do not depend on \( T \), such that the following a priori estimate holds for all \( \gamma \geq \gamma_m \), and all \( k = 1, \ldots, m \):

\[
\| \partial_x \mathcal{H}^k W_+ \|_{L^2(H^{m-k}(\omega_T))} \leq C_m \left\{ \| F_+ \|_{H^{m-1}(\Omega_T)} + \| \xi_+ \|_{H^{m-1}(\Omega_T)} \\
+ \| \xi_+ \|_{L^\infty(\Omega_T)} \| \nabla \Phi_{r,l} \|_{H^{m-1}(\Omega_T)} + \| W_+ \|_{L^2(H^{m}(\omega_T))} + \| W_+ \|_{H^{m-1}(\Omega_T)} \\
+ \| W_+ \|_{L^\infty(\Omega_T)} \| (U_{r,l}, \nabla \Phi_{r,l}) \|_{H^m(\Omega_T)} \right\}.
\]  

(58)

Proof. The proof follows from an induction argument. (We only give the proof for \( k = 1 \)). We now turn to the estimate of \( \mathcal{H}^1 W_+ \). Using Theorem 9 for a fixed \( x_2 \) and integrating with respect to \( x_2 \), we already obtain

\[
\| I_2 F_+ \|_{L^2(H^{m-1}(\omega_T))} \leq C \| F_+ \|_{L^2(H^{m-1}(\omega_T))} \leq C \| F_+ \|_{H^{m-1}(\Omega_T)},
\]

\[
\| \mathcal{C}^r W_+ \|_{L^2(H^{m-1}(\omega_T))} \leq C(K) \left\{ \| W_+ \|_{L^2(H^{m-1}(\omega_T))} + \| W_+ \|_{L^\infty(\Omega_T)} \| (U_{r,l}, \nabla \Phi_{r,l}) \|_{L^2(H^m(\omega_T))} \right\}.
\]

Decomposing

\[
\frac{\partial x_2 \Phi_r}{(\partial x_1 \Phi_r)^2} \xi_+ = \dot{\xi}_+ + H(\nabla \Phi_{r,l}) \dot{\xi}_+,
\]

where \( H \) is a \( C^\infty \) function that vanishes at the origin, we also get

\[
\left\| \frac{\partial x_2 \Phi_r}{(\partial x_1 \Phi_r)^2} \xi_+ \right\|_{L^2(H^{m-1}(\omega_T))} \leq C(K) \left\{ \| \dot{\xi}_+ \|_{L^2(H^{m-1}(\omega_T))} + \| \dot{\xi}_+ \|_{L^\infty(\Omega_T)} \| \nabla \Phi_{r,l} \|_{L^2(H^m(\omega_T))} \right\}.
\]

We now turn to the estimate of \( \mathcal{A}^0_0 \partial_t W_+ \). Applying a tangential derivative \( \partial^\alpha = \partial^\alpha_0 \partial \alpha_1 \) to this product, where \( \ell = \alpha_0 + \alpha_1 \leq m - 1 \), we get

\[
\partial^\alpha \left[ \mathcal{A}^0_0 \partial_t W_+ \right] = \mathcal{A}^0_0 \partial_t \partial^\alpha W_+ + \sum_{|\beta| \geq 1, \beta \leq \alpha} \left[ \partial^\beta \mathcal{A}^0_0 \partial^{\alpha-\beta} \partial_t W_+ \right].
\]

The first term is easily estimated in \( L^2(\Omega_T) \), while all the terms in the sum are estimated thanks to Gagliardo-Nirenberg’s inequality (Theorem 8), as for the estimates of the tangential derivatives of \( W_+ \). We obtain

\[
\| \partial^\alpha \left[ \mathcal{A}^0_0 \partial_t W_+ \right] \|_{L^2(\Omega_T)} \leq C(K) \left\{ \| W_+ \|_{L^2(H^{\ell+1}(\omega_T))} + \| \dot{W}_+ \|_{L^\infty(\Omega_T)} \| (U_{r,l}, \nabla \Phi_{r,l}) \|_{L^2(H^{\ell+1}(\omega_T))} \right\}.
\]

Multiplying by \( \gamma^{m-1-\ell} \), and summing over \( \ell = 0, \ldots, m - 1 \), we get

\[
\| \mathcal{A}^0_0 \partial_t W_+ \|_{L^2(H^{m-1}(\omega_T))} \leq C(K) \left\{ \| W_+ \|_{L^2(H^m(\omega_T))} + \| \dot{W}_+ \|_{L^\infty(\Omega_T)} \| (U_{r,l}, \nabla \Phi_{r,l}) \|_{L^2(H^m(\omega_T))} \right\}.
\]

A similar estimate holds for \( \mathcal{A}^1_1 \partial_{x_1} W_+ \). This completes the proof of (58) in the case \( k = 1 \).

From now on, we thus assume that (58) holds up to the order \( k < m, k \geq 1 \), and we are going to show that (58) holds at the order \( k+1 \leq m \). The starting point is again (57). Applying \( \partial_{x_2}^k \) yields

\[
\partial_{x_2}^{k+1} W_+ = I_2 \partial_{x_2}^k F_+ + \partial_{x_2}^k \left[ \mathcal{C}^r W_+ \right] - \partial_{x_2}^k \left\{ \frac{\partial x_2 \Phi_r}{(\partial x_1 \Phi_r)^2} \begin{pmatrix} \xi_+ \\ 0 \end{pmatrix} \right\} + \partial_{x_2}^k \left[ \mathcal{A}^0_0 \partial_t W_+ \right] + \partial_{x_2}^k \left[ \mathcal{A}^1_1 \partial_{x_1} W_+ \right].
\]

(59)
For the three first terms in the right-hand side of (59), we first use the obvious inequality
\[
\|\partial_{x_2}^k \theta\|_{L^2(H^{m-1-k}_T)} \leq \|\theta\|_{H^{m-1}_T},
\]
and then we apply Moser’s inequalities (Theorem 9). The three first terms in the right-hand side of (59) are thus estimated by
\[
C(K) \left\{ \|F_+\|_{H^{m-1}_T} + \|\tilde{\xi}_+\|_{H^{m-1}_T} + \|\tilde{\xi}_+\|_{L^\infty(T)} \|\nabla \phi_r\|_{H^{m-1}_T} \right. \\
\left. + \|W_+\|_{H^{m-1}_T} + \|W_+\|_{L^\infty(T)} \|\nabla \phi_r\|_{H^{m}_T} \right\},
\]
which is less than the right-hand side of (58).

We now turn to estimate \(\partial_{x_2}^k [\tilde{A}^*_0 \partial_t W_+]\) in \(L^2(H^{m-1-k}_T)\). We consider a tangential derivative \(\partial^n = \partial_{x_1}^{\alpha} \partial_{x_2}^j\), with \(\alpha_0 + \alpha_1 = \ell \leq m - 1 - k\), and we compute
\[
\partial^n \partial_{x_2}^k [\tilde{A}^*_0 \partial_t W_] = \tilde{A}^*_0 \partial_t \partial^n \partial_{x_2}^k W_+ + \sum_{|\beta| \geq 1, \beta \leq \alpha} [\partial^n \tilde{A}^*_0 \partial^{\alpha-\beta} \partial t \partial_{x_2}^k W_+] \\
+ \sum_{j=0}^{k-1} \sum_{\beta \leq \alpha} [\partial^{\alpha-\beta} \partial_{x_2}^{k-j} \tilde{A}^*_0 \partial^{\beta} \partial t \partial_{x_2}^j W_+].
\]
The estimate of the first term is trivial, namely
\[
\|\tilde{A}^*_0 \partial_t \partial^n \partial_{x_2}^k W_+\|_{L^2(T)} \leq C(K) \|\partial_{x_2}^k W_+\|_{L^2(H^{1-k+1}_T)},
\]
while all other terms in the sum can be estimated thanks to the Gagliardo-Nirenberg’s inequality in \(T\). If \(\beta \leq \alpha\), and \(|\beta| \geq 1\), we have
\[
\|\partial^n \tilde{A}^*_0 \partial^{\alpha-\beta} \partial t \partial_{x_2}^k W_+\|_{L^2(T)} \leq C(K) \left\{ \|W_+\|_{H^{1+k}_T} + \|W_+\|_{L^\infty(T)} \|\nabla \phi_r\|_{H^{1+k}_T} \right\},
\]
while, if \(\beta \leq \alpha\), and \(j \leq k - 1\), we still have
\[
\|\partial^n \tilde{A}^*_0 \partial^{\alpha-\beta} \partial t \partial_{x_2}^k W_+\|_{L^2(T)} \leq C(K) \left\{ \|W_+\|_{H^{1+k}_T} \right. \\
\left. + \|W_+\|_{L^\infty(T)} \|\nabla \phi_r\|_{H^{1+k}_T} \right\}.
\]
Multiplying by \(\gamma^{m-1-k-\ell}\), and summing over \(\ell = 0, \ldots, m - 1 - k\), we obtain
\[
\|\partial_{x_2}^k [\tilde{A}^*_0 \partial_t W_+]\|_{L^2(H^{m-1-k}_T)} \leq C(K) \left\{ \|\partial_{x_2}^k W_+\|_{L^2(H^{m-k}_T)} + \|W_+\|_{H^{m-1}_T} \right. \\
\left. + \|W_+\|_{L^\infty(T)} \|\nabla \phi_r\|_{H^{m}_T} \right\}.
\]
Applying the same arguments to \(\partial_{x_2}^k [\tilde{A}^*_0 \partial_{x_1} W_+]\) yields
\[
\|\partial_{x_2}^k [\tilde{A}^*_0 \partial_{x_1} W_+]\|_{L^2(H^{m-1-k}_T)} \leq C(K) \left\{ \|\partial_{x_2}^k W_+\|_{L^2(H^{m-k}_T)} + \|W_+\|_{H^{m-1}_T} \right. \\
\left. + \|W_+\|_{L^\infty(T)} \|\nabla \phi_r\|_{H^{m}_T} \right\}.
\]
Eventually, we apply the induction assumption to estimate the norm \(\|\partial_{x_2}^k W_+\|_{L^2(H^{m-k}_T)}\), and we have thus obtained (58) at the order \(k + 1\). The constant \(C_m\) depends on \(K\), and is therefore uniform as long as \(K \leq K_0\).
3.9 The a priori tame estimate

The results of the preceding paragraphs enable us to prove the following tame estimate in the $H^m_\gamma$ norm:

**Theorem 4.** Let $m \in \mathbb{N}$, and let $T > 0$. Assume that the stationary solution (11) satisfies (12). Assume also that the perturbations $\dot{U}, \dot{\Phi}$ in (14) satisfy (15), (16), (17), (27), and

$$\forall \gamma \geq 1, \quad (\dot{U}_r, \dot{\Phi}_r, \dot{\Phi}_t) \in H^{m+3}_\gamma(\Omega_T).$$

Then there exists a constant $K_0 > 0$, that does not depend on $m$ and $T$, and there exist two constants $C_m > 0$ and $\gamma_m \geq 1$, that depend on $m$ but not on $T$, such that, if $K \leq K_0$, $\gamma \geq \gamma_m$, and if $(\dot{V}_\pm, \psi) \in H^{m+2}_\gamma(\Omega_T) \times H^{m+2}_\gamma(\omega_T)$ is a solution to (23), (26), then one has

$$\sqrt{\gamma} \left\| \dot{V} \right\|_{H^m_\gamma(\Omega_T)} + \| \mathcal{P}(\varphi) \dot{V}_{x_2=0} \|_{H^m_\gamma(\omega_T)} + \| \psi \|_{H^{m+1}_\gamma(\omega_T)} \leq C_m \left\{ \frac{1}{\sqrt{\gamma}} \| f \|_{H^m_\gamma(\Omega_T)} + \frac{1}{\gamma^{3/2}} \| f \|_{L^2(H^{m+1}_\gamma(\omega_T))} + \frac{1}{\gamma^{3/2}} \| g \|_{H^{m+1}_\gamma(\omega_T)} + \frac{1}{\sqrt{\gamma}} \| \dot{\Phi} \|_{H^{m+1}_\gamma(\omega_T)} \right\}. \quad (60)$$

**Proof.** The first thing is to obtain a tame estimate for $W$ with respect to the source terms $F$, and $g$ in (34), (36). Then we shall obtain (60) by using (33), and (35).

The definition (13) gives

$$\| W \|_{H^m_\gamma(\Omega_T)} = \sum_{k=0}^{m} \| \partial_{x_2}^k W \|_{L^2(H^{m-k}_\gamma(\omega_T))},$$

and it is also clear that we have

$$\gamma \| \theta \|_{H^{m-1}_\gamma(\Omega_T)} \leq \| \theta \|_{H^m_\gamma(\Omega_T)}.$$

Then adding the estimates (38) and (58) that we have derived in the preceding paragraphs, we first obtain

$$\sqrt{\gamma} \left\| \dot{W} \right\|_{H^m_\gamma(\Omega_T)} + \left\| \dot{W}_{x_2=0} \right\|_{H^m_\gamma(\omega_T)} + \| \psi \|_{H^{m+1}_\gamma(\omega_T)} \leq C_m \left\{ \frac{1}{\sqrt{\gamma}} \| f \|_{H^m_\gamma(\Omega_T)} + \frac{1}{\gamma^{3/2}} \| f \|_{L^2(H^{m+1}_\gamma(\omega_T))} + \frac{1}{\gamma^{3/2}} \| g \|_{H^{m+1}_\gamma(\omega_T)} + \frac{1}{\sqrt{\gamma}} \| \dot{\Phi} \|_{H^{m+1}_\gamma(\omega_T)} \right\}. \quad (61)$$

The term $\| W \|_{H^m_\gamma(\Omega_T)} / \sqrt{\gamma}$ in the third line can be absorbed in the left-hand side of the inequality by choosing $\gamma$ large enough.

The definition (33) yields

$$\| W \|_{L^\infty(\Omega_T)} \leq C(K) \| \dot{V} \|_{W^{1,\infty}(\Omega_T)},$$

$$\| \dot{V} \|_{H^m_\gamma(\Omega_T)} \leq C(K) \left( \| W \|_{H^m_\gamma(\Omega_T)} + \| \dot{V} \|_{L^\infty(\Omega_T)} \right). \quad (62)$$
Moreover, the relation (35) yields

\[
\mathbb{P}(\varphi) \dot{V}_{|x_2=0} = \left( \frac{\partial_{x_1} \varphi}{\rho} - \frac{\rho}{\rho} (\partial_{x_1} \varphi)^2 \right) W_{nc}^{\varphi}_{|x_2=0},
\]

from which we deduce

\[
\|W_{nc}^{\varphi}_{|x_2=0}\|_{L^\infty(\Omega_T)} \leq C(K) \|\mathbb{P}(\varphi) \dot{V}_{|x_2=0}\|_{L^\infty(\omega_T)},
\]

\[
\|\mathbb{P}(\varphi) \dot{V}_{|x_2=0}\|_{H^m_\gamma(\omega_T)} \leq C(K) \left( \|W_{nc}^{\varphi}_{|x_2=0}\|_{H^m_\gamma(\omega_T)} + \|\mathbb{P}(\varphi) \dot{V}_{|x_2=0}\|_{L^\infty(\omega_T)} \|\dot{U}, \nabla \dot{\Phi}\|_{L^2(\omega_T)} \right).
\]

(63)

Moreover, the relation (35) yields

\[
\frac{1}{\gamma} \|F\|_{H^{m+1}_\gamma(\Omega_T)} + \frac{1}{\gamma^{3/2}} \|F\|_{L^2(\Omega_T)} \leq C(K) \left\{ \frac{1}{\sqrt{\gamma}} \|f\|_{H^{m}_\gamma(\Omega_T)} + \frac{1}{\gamma^{3/2}} \|f\|_{L^2(\Omega_T)} \right\}.
\]

(64)

Using (50), (62), (63), and (64) in (61), and choosing \(\gamma\) large enough, we obtain (60). The constants \(C_m\) and \(\gamma_m\) are uniform with respect to \(K\) as long as \(K \leq K_0\).

\[\square\]

**Remark 7.** The estimate (60) is not exactly a tame estimate in the usual sense, since the solution \(V, \psi\) appears in the right-hand side through \(L^\infty\) or \(W^{1,\infty}\) norms. However, these norms can be absorbed in the left-hand side of the inequality thanks to the classical Sobolev imbedding Theorem (provided that \(\gamma\) and \(T\) are fixed), see Proposition 6.

**Remark 8.** The estimate (60) is an a priori estimate, in the sense that it only gives an information for smooth solutions. However, we already know that the linearized problem (23), (26) is well-posed for source terms \((f, g)\) in \(L^2(H^1(\omega_T)) \times H^1(\omega_T)\) that vanish in the past. Following [7], Theorem 4 can be converted into a well-posedness result for source terms \((f, g)\) in \(H^{m+1}(\Omega_T) \times H^{m+1}(\omega_T)\) that vanish in the past. The corresponding solution \(\dot{V}, \psi\) belongs to \(H^m(\Omega_T) \times H^{m+1}(\omega_T)\), vanishes in the past, the trace \(\mathbb{P}(\varphi) \dot{V}_{|x_2=0}\) belongs to \(H^m(\omega_T)\), and (60) is satisfied for all \(\gamma \geq \gamma_m\).

4 Compatibility conditions for the initial data

4.1 The compatibility conditions

Let \(k \in \mathbb{N}\), with \(k \geq 3\). Given initial data \(U_0^\pm = (\rho_0^+, v_0^+, u_0^+)\) such that \(U_0^\pm = U^\pm + \hat{U}_0^\pm\), where \(\hat{U}_0^\pm \in H^{k+1/2}(\mathbb{R}_2^2)\), and \(\varphi_0 \in H^{k+1}(\mathbb{R})\), we need to prescribe the necessary compatibility conditions for the existence of a smooth solution \((U^\pm, \Phi^\pm)\) to (3), (4), (5), (6), and (7). As we will see, the choice of the functions \(\Phi^\pm\), which are required to satisfy (7), and (4a) makes the formulation of the compatibility conditions rather simple. We assume that the initial data \(\hat{U}_0^\pm\), and \(\varphi_0\) have compact support:

\[
\text{Supp } \hat{U}_0^\pm \subset \left\{ x_2 \geq 0, \sqrt{x_1^2 + x_2^2} \leq 1 \right\}, \quad \text{Supp } \varphi_0 \subset [-1, 1].
\]

(65)

Let us first extend \(\varphi_0\) to \(\mathbb{R}_+ = \mathbb{R}_1 \times \mathbb{R}_2^+\) by constructing \(\hat{\Phi}_0^+ = \hat{\Phi}_0^- \in H^{k+3/2}(\mathbb{R}_2^+),\) that satisfy \((\hat{\Phi}_0^\pm)_{x_2=0} = \varphi_0\), and the estimate

\[
\|\hat{\Phi}_0^\pm\|_{H^{k+3/2}(\mathbb{R}_2^+)} \leq C \|\varphi_0\|_{H^{k+1}(\mathbb{R})}.
\]

(66)
Up to multiplying \( \dot{\Phi}_0^\pm \) by a \( C^\infty \) function with compact support (whose choice only depends on the support of \( \varphi_0 \)), we may assume that \( \Phi_0^\pm \) satisfy
\[
\text{Supp } \Phi_0^\pm \subset \left\{ x_2 \geq 0, \sqrt{x_1^2 + x_2^2} \leq 1 + \frac{1}{2} \lambda_{max} T \right\}.
\] (67)

We define \( \Phi_0^\pm := \pm x_2 + \dot{\Phi}_0^\pm \). Because \( k + 3/2 > 3 \), the Sobolev’s imbedding Theorem yields
\[
\forall x \in \mathbb{R}_+^2, \quad \partial_{x_2} \Phi_0^\pm (x) \geq \frac{7}{8}, \quad \partial_{x_2} \Phi_0^\pm (x) \leq -\frac{7}{8},
\] (68)

provided that \( \varphi_0 \) is sufficiently small in \( H^{k+1}(\mathbb{R}) \), see (66).

Let us prescribe, for the eikonal equations (7) that must be satisfied by \( \Phi^\pm \), the initial conditions
\[
\Phi_{t=0}^\pm = \Phi_0^\pm,
\] (69)
in the space domain \( \mathbb{R}_+^2 = \{ x_1 \in \mathbb{R}, x_2 > 0 \} \). Of course, we look for a solution \( \Phi^\pm \) of the form \( \Phi^\pm = \pm x_2 + \dot{\Phi}^\pm \).

To derive the compatibility conditions, we follow the classical approach, see e.g. [32]. We first note that the equations (3), and (7), when evaluated at time \( t = 0 \), determine the traces \( \partial_t U_{t=0}^\pm \) and \( \partial_t \dot{\Phi}_{t=0}^\pm \) in terms of \( U_0^\pm \) and \( \dot{\Phi}_0^\pm \):
\[
\partial_t \Phi_{t=0}^\pm = -v_0^\pm \partial_{x_1} \Phi_0^\pm + \dot{u}_0^\pm,
\]
\[
\partial_t U_{t=0}^\pm = -A_1(U_0^\pm) \partial_{x_1} U_0^\pm - \frac{1}{\partial_{x_2} \Phi_0^\pm} \left( A_2(U_0^\pm) + v_0^\pm \partial_{x_2} \Phi_0^\pm - \dot{u}_0^\pm - \partial_{x_1} \Phi_0^\pm A_1(U_0^\pm) \right) \partial_{x_2} U_0^\pm.
\] (70)

These expressions can be generalized to higher order derivatives. Taking formally \( j \) time derivatives of (3), and (7) determines inductively the \((j+1)\)-th order derivatives \( \partial_t^{j+1} U^\pm, \partial_t^{j+1} \dot{\Phi}^\pm \) as functions of \( U^\pm, \dot{\Phi}^\pm \), and of their space derivatives up to order \((j+1)\). More precisely, let us denote the traces at time \( t = 0 \) by
\[
\dot{U}_+^\pm := \partial_t^1 U_{t=0}^\pm, \quad \dot{\Phi}_+^\pm := \partial_t^1 \dot{\Phi}_{t=0}^\pm, \quad \ell \geq 0.
\]

For \( j \geq 1 \), Leibniz’ rule yields the expression
\[
\dot{\Phi}_{j+1}^\pm = \dot{u}_j^\pm + \tau \partial_{x_1} \Phi_j^\pm - \sum_{\ell=0}^j C_{j, \ell}^\ell \dot{u}_\ell^\pm \partial_{x_1} \Phi_{j-\ell}^\pm.
\] (71)

Moreover, if we denote \( W^\pm := (U^\pm, \nabla_x U^\pm, \nabla_x \dot{\Phi}^\pm) \), we can rewrite (3) under the form
\[
\partial_t U^\pm = \mathbf{F}(W^\pm),
\]
where \( \mathbf{F} \) is a \( C^\infty \) function that vanishes at the origin. We take \( j - 1 \) time derivatives of this expression, then we take the trace at time \( t = 0 \). The Faà di Bruno’s formula yields
\[
\dot{U}_{j+1}^\pm = \sum_{\alpha \in \mathbb{N}, \alpha_1 + \ldots + \alpha_j = j} \frac{j!}{\alpha_1! \ldots \alpha_j!} D^{(\alpha)} \mathbf{F}(W_0) \left\{ \left( \frac{W_1}{1!} \right)^{\otimes \alpha_1}, \ldots, \left( \frac{W_j}{j!} \right)^{\otimes \alpha_j} \right\},
\] (72)

where we have used the notation
\[
W^\pm_\ell = (U_\ell^\pm, \nabla_x U_\ell^\pm, \nabla_x \dot{\Phi}_\ell^\pm), \quad \ell \geq 0.
\]

It is clear that the relations (70), (71), and (72) form an induction relation for the sequence \((U_j^\pm, \Phi_j^\pm)_{j \geq 0}\) that enables us to determine \( \dot{U}_j^\pm \), and \( \Phi_j^\pm \) in terms of \( U_0^\pm \), and \( \dot{\Phi}_0^\pm \).

With the help of (70), (71), and (72), we can prove the following result (that is the analogue of Lemma 4.2.1 in [29]):

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Lemma 2. Let $k \in \mathbb{N}$, $k \geq 3$. Let $U_j^\pm = U_j^\pm + U_0^\pm$, $U_0^\pm \in H^{k+1/2}(\mathbb{R}_+^2)$, and let $\varphi_0 \in H^{k+1}(\mathbb{R})$, where $U_0^\pm$, and $\varphi_0$ satisfy (65). Consider the functions $\Phi_0^\pm = \pm x_2 + \Phi_0^\pm$ that we have constructed above, and that satisfy (66), (67), and (68) if $\varphi_0$ is sufficiently small. Then the equations (71), and (72) determine $U_j^\pm \in H^{k+1/2-j}(\mathbb{R}_+^2)$, for $j = 1, \ldots, k$, and $\Phi_j^\pm = H^{k+3/2-\ell}(\mathbb{R}_+^2)$, for $\ell = 1, \ldots, k + 1$. Moreover, these functions satisfy:

$$\text{Supp } \Phi_j^\pm \subset \left\{ x_2 \geq 0, \sqrt{x_1^2 + x_2^2} \leq \frac{1}{2} \lambda_{\max} T \right\}, \quad \text{Supp } U_j^\pm \subset \left\{ x_2 \geq 0, \sqrt{x_1^2 + x_2^2} \leq 1 \right\},$$

and there exists a constant $C > 0$, that only depends on $k$, and on $\|\Phi_j^\pm\|_{W^{1,\infty}(\mathbb{R}_+^2)}$, such that

$$\sum_{j=1}^k \|U_j^\pm\|_{H^{k+1/2-j}(\mathbb{R}_+^2)} + \sum_{\ell=1}^{k+1} \|\Phi_\ell^\pm\|_{H^{k+3/2-\ell}(\mathbb{R}_+^2)} \leq C \left( \|U_0^\pm\|_{H^{k+1/2}(\mathbb{R}_+^2)} + \|\varphi_0\|_{H^{k+1}(\mathbb{R})} \right). \quad (73)$$

Proof. With an induction argument, we first prove that for $j = 1, \ldots, k$, $\Phi_j^\pm$, and $U_j^\pm$ can be written as

$$\Phi_j^\pm = F_j(U_0^\pm, \nabla_x U_0^\pm), \quad U_j^\pm = G_j(U_0^\pm, \nabla_x U_0^\pm), \quad \Phi_j^\pm = H_j(W_0^\pm). \quad (74)$$

where $F_j$, and $H_j$ are nonlinear functions of order $\leq j - 1$, and $G_j$ is a nonlinear function of order $\leq j$, in the sense of Definition 2 in appendix C.

We note immediately that if (74) holds, then $\Phi_{k+1}^\pm$ can also be written under the form

$$\Phi_{k+1}^\pm = F_{k+1}(U_0^\pm, \nabla_x U_0^\pm),$$

where $F_{k+1}$ is a nonlinear function of order $\leq k$, see (71).

The expressions (70) show that the decomposition (74) holds true for $j = 1$. Assume now that it holds true up to $j < k$. Then for all $\ell = 1, \ldots, j$, $(U_j^\pm, \nabla_x \Phi_j^\pm)$ is a nonlinear function of order $\leq \ell$ of $(U_0^\pm, \nabla_x U_0^\pm)$. Similarly, $W_\ell^\pm$ is a nonlinear function (of order $\leq \ell$) of $W_0^\pm$. Consequently, (71), and (72) show that $\Phi_{j+1}^\pm$ (resp. $U_{j+1}^\pm$) is a nonlinear function, of order $\leq j$, of $(U_0^\pm, \nabla_x U_0^\pm)$ (resp. $W_0^\pm$).

It only remains to show that $U_{j+1}^\pm$ is a nonlinear function of order $(j + 1)$ of $(U_0^\pm, \nabla_x U_0^\pm)$. To prove this, we rewrite (3) as

$$\partial_t U^\pm = G(U^\pm, \nabla_x U^\pm) \nabla_x U^\pm,$$

where $G(U^\pm, \nabla_x U^\pm)$ is linear. Using Leibniz’ rule, we compute

$$U_{j+1}^\pm = \sum_{\ell=0}^j C_j^\ell \partial_t^\ell G(U_0^\pm, \nabla_x U_0^\pm)|_{t=0} \nabla_x U_{j-\ell}^\pm. \quad (75)$$

Using once again the Faà di Bruno’s formula for the $\ell$-th derivative of a composed function, and exploiting the induction hypothesis, we can conclude that $U_{j+1}^\pm$ is a nonlinear function of order $\leq (j + 1)$ of $(U_0^\pm, \nabla_x U_0^\pm)$. This completes the induction.

The result of Lemma 2, that is the support property and the estimate (73), then follows from the relations (74), (75), and from Theorem 12 in appendix C. □

We introduce the following terminology:

Definition 1. Let $k \geq 3$. Let $U_0^\pm = (p_0^\pm, v_0^\pm, u_0^\pm)$ be such that $U_0^\pm = U_0^\pm + U_0^\pm$, with $U_0^\pm \in H^{k+1/2}(\mathbb{R}_+^2)$, let $\varphi_0 \in H^{k+1}(\mathbb{R})$, that satisfy (65). Consider the functions $\Phi_0^\pm = \pm x_2 + \Phi_0^\pm$ that we have constructed above, and that satisfy (66), (67), and (68), when $\varphi_0$ is sufficiently small.
The initial data \((\dot{U}_0^\pm, \varphi_0)\) are said to be compatible up to order \(k\) if the traces of the functions \(\dot{U}_1^\pm, \ldots, \dot{U}_k^\pm, \Phi_1^\pm, \ldots, \Phi_{k+1}^\pm\) satisfy
\[
\partial_{x_2}^\ell (\dot{\Phi}_j^+ - \dot{\Phi}_j^-)_{|x_2=0} = 0, \quad \text{for } j = 0, \ldots, k, \text{ and } \ell = 0, \ldots, k - j, \\
\partial_{x_2}^\ell (\dot{\rho}_j^+ - \dot{\rho}_j^-)_{|x_2=0} = 0, \quad \text{for } j = 0, \ldots, k - 1, \text{ and } \ell = 0, \ldots, k - 1 - j,
\]
and
\[
\int_{\mathbb{R}_+^2} |\partial_{x_2}^{k+1-j} (\dot{\Phi}_j^+ - \dot{\Phi}_j^-)|^2 \, dx_1 \frac{dx_2}{x_2} < +\infty, \quad \text{for } j = 0, \ldots, k + 1,
\]
\[
\int_{\mathbb{R}_+^2} |\partial_{x_2}^{j} (\dot{\rho}_j^+ - \dot{\rho}_j^-)|^2 \, dx_1 \frac{dx_2}{x_2} < +\infty, \quad \text{for } j = 0, \ldots, k.
\]

Observe that \(\dot{\rho}_0^\pm, \ldots, \dot{\rho}_{k-1}^\pm, \dot{\Phi}_0^\pm, \ldots, \dot{\Phi}_k^\pm \in H^{3/2}(\mathbb{R}_+^2)\), so it is legitimate to consider the traces of these functions on \(\{x_2 = 0\}\).

The second set of equalities in (76) obviously follows from (4d), since it requires the equality of time derivatives of \(\rho^\pm = p + \dot{\rho}^\pm\). The first set of equalities follows as well from (4a). We note that these equalities automatically yield the compatibility conditions for the last two boundary conditions (4b), and (4c). Namely, an equivalent formulation of (4b), and (4c) is
\[
\partial_t\varphi + v^+_1 \partial_{x_1}\varphi - u^+_1 = 0, \\
\partial_t\varphi + v^-_1 \partial_{x_1}\varphi - u^-_1 = 0,
\]
on the boundary \(\{x_2 = 0\}\). The extensions \(\dot{\Phi}^\pm\) of \(\varphi\) satisfy such equations everywhere in the space domain, see (7). Considering that the traces \(\dot{\Phi}^\pm := \partial_t^\ell \dot{\Phi}^\pm_{|x_2=0}\) are determined by (71), and that the equality \((\dot{\Phi}_j^+ - \dot{\Phi}_j^-)_{|x_2=0} = 0\) defines the common value \(\varphi_j := \partial_t^j \varphi_{|x_2=0} \in H^{k+1-j}(\mathbb{R})\), \(j = 0, \ldots, k\), it follows that the compatibility conditions (76), and the relations (71) yield
\[
\forall j = 1, \ldots, k, \quad \varphi_j + \sum_{\ell=0}^{j-1} C_j^\ell (v^+_1)_{|x_2=0} \partial_{x_1} \varphi_{j-\ell} - (u^+_1)_{|x_2=0} = 0,
\]
is nothing but the compatibility conditions
\[
\forall j = 0, \ldots, k - 1, \quad \partial_t^j \left( \partial_t \varphi + v^+_1 \partial_{x_1}\varphi - u^+_1 \right)_{|x_2=0} = 0. \tag{78}
\]

Definition 1 is not stated in an utterly rigorous way. Indeed, the functions \(\dot{\Phi}_0^\pm\) are not uniquely defined by \(\varphi_0\). More rigorously, one could say that the initial data \((U_0^\pm, \varphi_0)\) are compatible if there exist some functions \(\dot{\Phi}_0^\pm\) such that (66), (67), (76), and (77) hold. However, if one fixes once and for all a lifting operator, and a \(C_0^\infty\) function in order to force (67), then there is no more ambiguity. We shall thus assume that such operators and functions have been fixed once and for all.

### 4.2 Construction of an approximate solution

Let us consider the nonlinear equations (3), (4) written in the compact form (8), (10). With a slight abuse of notation, we shall write \(L(U, \Phi)\) for the pair \((L(U^+, \Phi^+), L(U^-, \Phi^-))\). We now introduce the following “approximate” solutions. These are solutions of (3), (4), (5), (6), (7) in the sense of Taylor’s series at \(t = 0\). When not explicitly written, it is understood that functions \((U, \Phi)\) defined in the interior domain have + and − states.

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In all what follows, \( \varepsilon(\cdot) \) denotes a function that tends to 0 when its argument tends to 0. For instance, the estimate (66), together with Sobolev’s imbedding Theorem enables us to rewrite (73) as

\[
\sum_{j=1}^{k} \| \dot{U}^+_j \|_{H^{k+1/2-j}(\mathbb{R}^2_+)} + \sum_{\ell=1}^{k+1} \| \dot{\Phi}^\pm_\ell \|_{H^{k+3/2-\ell}(\mathbb{R}^2_+)} \leq \varepsilon(\| \dot{U}^+_0 \|_{H^{k+1/2}(\mathbb{R}^2_+)} + \| \varphi_0 \|_{H^{k+1}(\mathbb{R})}).
\]

We consider the integer \( \mu \) of Theorem 1, some compatible initial data \((\dot{U}^0_0, \varphi_0)\), and we use the results of the previous section with \( k = \mu + 7 \). In particular, the initial data are compatible up to order \( \mu + 7 \). We have the following result:

**Lemma 3.** If \( \dot{U}_0^0 \) and \( \varphi_0 \) are sufficiently small, then there exist some functions \( U^a, \Phi^a, \varphi^a \), such that \( U^a - \overline{U} = \dot{U}^a \in H^\mu(\Omega) \), \( \Phi^a \pm \in \dot{x}_2 = \dot{\Phi}^a \pm \in H^{\mu+9}(\Omega), \varphi^a \in H^{\mu+17/2}(\omega) \), and such that

\[
\frac{\partial t^j L(U^a, \Phi^a)}{|t|_0 = 0}, \quad \text{for } j = 0, \ldots, \mu + 6, \quad (79)
\]
\[
\frac{\partial t \Phi^a + v^a \partial x_1 \Phi^a - u^a = 0}{\varphi^a = \Phi^a_{\pm}|_{x_2 = 0} = \Phi^a_{\pm}|_{x_2 = 0}}, \quad (80)
\]
\[
\mathbb{B}(U^a_{|x_2 = 0}, \varphi^a) = 0. \quad (81)
\]

Moreover, one has the estimates

\[
\| \dot{U}^a_0 \|_{H^{\mu+8}(\Omega)} + \| \Phi^a \|_{H^{\mu+9}(\Omega)} + \| \varphi^a \|_{H^{\mu+17/2}(\omega)} \leq \varepsilon(\| \dot{U}^+_0 \|_{H^{\mu+15/2}(\mathbb{R}^2_+)} + \| \varphi_0 \|_{H^{\mu+8}(\mathbb{R})}), \quad (82)
\]
\[
\forall (t,x) \in \Omega, \quad \partial x_2 \Phi^a(t,x) \geq \frac{3}{4}, \quad \partial x_2 \Phi^a(t,x) \leq -\frac{3}{4}, \quad (82)
\]

and the following compact supports:

\[
\text{Supp} (\dot{U}^a, \dot{\Phi}^a) \subset \left\{ t \in [-T, T], \ x_2 \geq 0, \ x_1^2 + x_2^2 \leq 1 + \lambda_{\text{max}} T \right\}, \quad (83)
\]
\[
\text{Supp} \varphi^a \subset \left\{ t \in [-T, T], \ x_1 \in [-1 - \lambda_{\text{max}} T, 1 + \lambda_{\text{max}} T] \right\}. \quad (84)
\]

**Proof.** Consider functions \( \dot{\rho}^a, \dot{v}^a \in H^{\mu+8}(\Omega) \), and \( \dot{\Phi}^a \in H^{\mu+9}(\Omega) \) such that

\[
\frac{\partial t^j (\dot{\rho}^a, \dot{v}^a)}{|t|_0 = 0} = (\dot{\rho}_j, \dot{v}_j) \in H^{\mu+15/2-j}(\mathbb{R}^2_+), \quad \text{for } j = 0, \ldots, \mu + 7, \quad (76)
\]
\[
\frac{\partial t^j (\dot{\Phi}^a)}{|t|_0 = 0} = \dot{\Phi}_\ell \in H^{\mu+17/2-\ell}(\mathbb{R}^2_+), \quad \text{for } \ell = 0, \ldots, \mu + 8, \quad (77)
\]

where the \( \dot{\rho}_j, \dot{v}_j \)'s and the \( \dot{\Phi}_\ell \)'s are given by Lemma 2. Thanks to the compatibility conditions (76), and (77), one can choose \( \dot{\rho}^a, \dot{v}^a \), and \( \dot{\Phi}^a \) that satisfy the additional requirement

\[
\dot{\rho}^a_{|x_2 = 0} = \dot{\rho}^a_{|x_2 = 0}, \quad \dot{\Phi}^a_{|x_2 = 0} = \dot{\Phi}^a_{|x_2 = 0}, \quad (85)
\]

see [22]. Because the \( \dot{U}_j \)'s, and the \( \dot{\Phi}_\ell \)'s have a compact support, see Lemma 2, one can choose \( \dot{\rho}^a, \dot{v}^a \), and \( \dot{\Phi}^a \) that satisfy (85), up to multiplying by a \( C^\infty_0 \) function. We define

\[
\varphi^a := \dot{\Phi}^a_{|x_2 = 0} = \dot{\Phi}^a_{|x_2 = 0} \in H^{\mu+17/2}(\mathbb{R}^2), \quad (86)
\]
\[
\dot{u}^a := \partial t \dot{\Phi}^a + v^a \partial x_1 \dot{\Phi}^a \in H^{\mu+8}(\Omega), \quad (87)
\]

and we immediately note that \( \dot{u}^a \), and \( \varphi^a \) also satisfy (85). With this definition of \( \dot{u}^a \), and \( \varphi^a \), and our choice of \( \dot{\rho}^a \), the relations (80), (82) are obvious. Moreover, the relations (71) show that \( \dot{u}^a \) satisfies

\[
\frac{\partial t^j (\dot{u}^a)}{|t|_0 = 0} = \dot{u}_j \in H^{\mu+15/2-j}(\mathbb{R}_+^2), \quad \text{for } j = 0, \ldots, \mu + 7. \quad (79)
\]

The equations (70), (71), and (72) yield (79). The estimate (83) follows from (73), and from the continuity of the lifting operators. Eventually, (84) is a direct consequence of (83), and the Sobolev’s imbedding Theorem, provided that the initial data are sufficiently small. \qed
Remark 9. In the proof of Lemma 3, it is crucial to construct \( \dot{u}^a \) in terms of \( \dot{\Phi}^a \), and not the other way. Indeed, if one first defines \( \dot{u}^a \) by lifting the traces \( \dot{u}_j \)'s, and then defines \( \dot{\Phi}^a \) as the solution to the linear transport equation

\[
\partial_t \dot{\Phi}^a + v^a \partial_{x_1} \dot{\Phi}^a - \dot{u}^a = 0,
\]

there is no reason why \( \dot{\Phi}^a \) should have a compact support in \((t, x)\), and there is also no reason why \( \dot{\Phi}^a \) should belong to \( H^{\mu+9}(\Omega) \), because the coefficients are only in \( H^{\mu+8}(\Omega) \).

The approximate solution \((U^a, \Phi^a)\) enables us to reformulate the original problem in a non-linear problem with zero initial data. Introduce

\[
\begin{align*}
    f^a &:= -\mathcal{L}(U^a, \Phi^a), & t > 0, \\
    f^a &:= 0, & t < 0.
\end{align*}
\]

Because \( \dot{U}^a \in H^{\mu+8}(\Omega) \), and \( \dot{\Phi}^a \in H^{\mu+9}(\Omega) \), (79) yields \( f^a \in H^{\mu+7}(\Omega) \), and using (85), we get

\[
\textrm{Supp} \ f^a \subset \left\{ t \in [0, T], x_2 \geq 0, \sqrt{x_1^2 + x_2^2} \leq 1 + \lambda_{\max} T \right\}.
\]

From (83), we also get the estimate

\[
\|f^a\|_{H^{\mu+7}(\Omega)} \leq \varepsilon(\|\dot{U}_0^+\|_{H^{\mu+15/2}(\mathbb{R}^2)} + \|\varphi_0\|_{H^{\mu+8}(\mathbb{R})})).
\]

For all real number \( T > 0 \), we let \( \Omega^+_T \), and \( \omega_T^+ \) denote the sets

\[
\omega_T^+ := [0, T] \times \mathbb{R}, \quad \Omega_T^+ := [0, T] \times \mathbb{R} \times [0, +\infty] = \omega_T^+ \times \mathbb{R}^+.
\]

Given the approximate solution \((U^a, \Phi^a)\) of Lemma 3, and \( f^a \) defined by (86), \((U, \Phi) = (U^a, \Phi^a) + (V, \Psi)\) is a solution on \( \Omega_T^+ \) of (3), (4), (5), (6), (7), if \( V = (V^+, V^-) \), \( \Psi = (\Psi^+, \Psi^-) \) satisfy the following system:

\[
\begin{align*}
    \mathcal{L}(V, \Psi) &= f^a, & \text{in } \Omega_T, \\
    \mathcal{E}(V, \Psi) &= \partial_t \Psi + (v^a + v) \partial_{x_1} \Psi - u + v \partial_{x_1} \Phi^a = 0, & \text{in } \omega_T, \\
    \Psi_{x_2=0}^+ = \Psi_{x_2=0}^- =: \psi, & \text{on } \omega_T, \\
    \mathcal{B}(V_{x_2=0}, \psi) &= 0, & \text{on } \omega_T, \\
    (V, \Psi) &= 0, & \text{for } t < 0,
\end{align*}
\]

where

\[
\begin{align*}
    \mathcal{L}(V, \Psi) &= \mathcal{L}(U^a + V, \Phi^a + \Psi) - \mathcal{L}(U^a, \Phi^a), \\
    \mathcal{B}(V_{x_2=0}, \psi) &= \mathcal{B}(U^a_{x_2=0} + V_{x_2=0}, \varphi^a + \psi).
\end{align*}
\]

We note that \((V, \Psi) = 0\) satisfy (89) for \( t < 0 \), because \( f^a = 0 \) for \( t < 0 \), and \( \mathcal{B}(U^a_{x_2=0}, \varphi^a) = 0 \) for all \( t \in \mathbb{R} \). Therefore the initial nonlinear problem on \( \Omega_T^+ \) is now substituted for a problem on \( \Omega_T \). The initial data (5) are absorbed into the equations by the introduction of the approximate solution \((U^a, \Phi^a, \varphi^a)\), and the problem has to be solved in the class of functions vanishing in the past (i.e., for \( t < 0 \)), which is exactly the class of functions in which we have a well-posedness result for the linearized problem.

Thanks to (83), and to the Sobolev imbedding Theorem, we see that for sufficiently small initial data, we have

\[
\|\dot{U}^a\|_{W^{2, \infty}(\Omega)} + \|\dot{\Phi}^a\|_{W^{3, \infty}(\Omega)} \leq K_0/2,
\]

where \( K_0 \) is given by Theorem 4.
5 Description of the iterative scheme

We solve problem (89) by a Nash-Moser type iteration. (We refer to [2, 17, 19] for a general description of the method). This method requires a family of smoothing operators whose construction is detailed below. The construction of these operators is inspired from [14]. Also in this section, even if not explicitly written, it is understood that functions defined in the interior domain have a +, and a − state.

5.1 The smoothing operators

We begin with a few notations. For $T > 0$, $s \geq 0$, and $\gamma \geq 1$, we let

$$\mathcal{F}_s^\gamma(\Omega_T) := \{ u \in H^s_0(\Omega_T) \mid u = 0 \text{ for } t < 0 \}.$$ 

This is a closed subspace of $H^s_0(\Omega_T)$, that we equip with the induced norm. The definition of $\mathcal{F}_s^\gamma(\omega_T)$ is entirely similar.

The following result follows from the method that was used in [1, section 4]:

**Lemma 4.** There exists a family $\{S^\theta_0\}_{\theta \geq 1}$ of operators

$$S^\theta_0 : \mathcal{F}_0^\gamma(\Omega_T) \longrightarrow \bigcap_{\beta \geq 0} \mathcal{F}_\beta^\gamma(\Omega_T),$$

such that

$$\|S^\theta_0 u\|_{H^\beta_0(\Omega_T)} \leq C \theta^{(\beta-\alpha)_+} \|u\|_{H^\alpha_0(\Omega_T)} \quad \forall \alpha, \beta \geq 0,$$

$$\|S^\theta_0 u - u\|_{H^\beta_0(\Omega_T)} \leq C \theta^{3-\alpha} \|u\|_{H^\alpha_0(\Omega_T)} \quad 0 \leq \beta \leq \alpha,$$

$$\|\frac{d}{d\theta} S^\theta_0 u\|_{H^\beta_0(\Omega_T)} \leq C \theta^{3-\alpha-1} \|u\|_{H^\alpha_0(\Omega_T)} \quad \forall \alpha, \beta \geq 0.$$

Here, and in all what follows, we use the classical notation $(\beta - \alpha)_+ := \max(0, \beta - \alpha)$. The constants in the inequalities are uniform with respect to $\alpha, \beta$ when $\alpha, \beta$ belong to some bounded interval.

As noted in [14], the problem with the operators $S^\theta_0$ is that they do not respect the traces. More precisely, if $u$ and $v$ have the same trace on $\omega_T$ (when this trace is well-defined), there is no reason why $S^\theta_0 u$ and $S^\theta_0 v$ should have the same trace on $\omega_T$. To correct this defect, we are going to introduce a lifting operator from the boundary to the interior:

**Lemma 5.** Let $T > 0$, $\gamma \geq 1$, and let $M \in \mathbb{N}$, $M \geq 1$. There exists an operator $R_T$ such that

- For all $s \in [1, M]$, $R_T$ is continuous from $\mathcal{F}^s_\gamma(\omega_T)$ to $\mathcal{F}^{s+1/2}_\gamma(\Omega_T)$,

- If $s \geq 1$, and $u \in \mathcal{F}^s_\gamma(\omega_T)$, then $(R_T u)|_{x_2=0} = u$.

Extending first the functions of $\mathcal{F}^s_\gamma(\omega_T)$ to functions of $\mathcal{F}^s_\gamma(\omega)$, we are reduced to defining a lifting operator $R$ from $\omega$ to $\Omega$. In this case, one can follow the construction of [14, chapter 5]. With the lifting operator $R_T$, we are going to prove the following (note that we now restrict Sobolev spaces of integer index):

**Proposition 4.** Let $T > 0$, $\gamma \geq 1$, and let $M \in \mathbb{N}$, with $M \geq 4$. There exists a family $\{S^\theta\}_{\theta \geq 1}$ of operators

$$S^\theta : \mathcal{F}^3_\gamma(\Omega_T) \times \mathcal{F}^3_\gamma(\Omega_T) \longrightarrow \bigcap_{\beta \geq 3} \mathcal{F}_\beta^\gamma(\Omega_T) \times \mathcal{F}_\beta^\gamma(\Omega_T),$$
and a constant $C > 0$ (depending on $M$), such that
\begin{align}
\| S_\theta U \|_{H^1_0(\Omega_T)} & \leq C \theta^{(d-\alpha)+} \| U \|_{H^\alpha_T(\Omega_T)}, \quad \forall \alpha, \beta \in \{1, \ldots, M\}, \\
\| S_\theta U - U \|_{H^1_0(\Omega_T)} & \leq C \theta^{\beta-\alpha} \| U \|_{H^\alpha_T(\Omega_T)}, \quad 1 \leq \beta \leq \alpha \leq M, \\
\| \frac{d}{d\theta} S_\theta U \|_{H^1_0(\Omega_T)} & \leq C \theta^{\beta-\alpha-1} \| U \|_{H^\alpha_T(\Omega_T)}, \quad \forall \alpha, \beta \in \{1, \ldots, M\}.
\end{align}

Moreover, (i) if $U = (u^+, u^-)$ satisfies $u^+ = u^-$ on $\omega_T$, then $S_\theta u^+ = S_\theta u^-$ on $\omega_T$, (ii) the following estimate holds:
\[ \| (S_\theta u^+ - S_\theta u^-) \|_{L^2(\Omega_T)} \leq C \theta^{(d+1-\alpha)+} \| (u^+ - u^-) \|_{L^2(\Omega_T)}, \quad \forall \alpha, \beta \in \{1, \ldots, M\}. \]

There is another family of operators, still denoted $S_\theta$, that acts on functions that are defined on the boundary $\omega_T$, and that enjoy the properties (92), with the norms $\| \cdot \|_{H^\gamma_T(\omega_T)}$.

**Proof.** For $U = (u^+, u^-)$, we first define the “projection”:
\[ \pi U := \left( u^+ - \frac{1}{2} R_T(u^+ - u^-) \right)_{x_2=0}, u^- + \frac{1}{2} R_T(u^+ - u^-) \right)_{x_2=0}, \]
so that $\pi$ projects the pair $(u^+, u^-)$ onto the set of pairs that have no jump on the boundary.

If $U = (u^+, u^-)$ has no jump on the boundary, that is if $\pi U = U$, then we set:
\[ S^\delta_\theta U := \pi S^\delta_\theta U, \]
where $S^\delta_\theta$ is given in Lemma 4. It is shown in [14] that the family $S^\delta_\theta$ enjoys the properties (92). The family $S_\theta$ is now defined by:
\[ S_\theta U := S^\delta_\theta(\pi U) + S^\delta_\theta(U - \pi U). \]

One easily checks that $S_\theta$ satisfies all the required properties. \[ \square \]

In Proposition 4, the smoothing operators are defined on pairs of functions. However, we shall use the same notation $S_\theta$ for smoothing operators that act on vector valued functions of the type $(\rho^\pm, v^\pm, u^\pm)$.

Furthermore, one can follow a similar construction, and define a family of smoothing operators on $F^\delta_\theta(\Omega_T)$, still denoted $S_\theta$, that satisfies (92), and such that if the trace of $u$ on $\omega_T$ is zero, then the trace of $S_\theta u$ on $\omega_T$ is zero. The construction is similar to the construction of Proposition 4, except that here we deal with a single scalar function.

In Proposition 4, we have not specified what the integer $M$ is. As a matter of fact, it will appear at the end of section 7 that the choice $M := \mu + 9$ is convenient.

### 5.2 Description of the iterative scheme

Let us describe the iterative scheme. The scheme starts from $V_0 = 0, \Psi_0 = 0, \psi_0 = 0$. Assume that $V_k, \Psi_k, \psi_k$ are already given for $k = 0, \ldots, n$ and verify
\[ \begin{cases}
(V_k, \Psi_k, \psi_k) = 0, & \text{for } t < 0, \\
(\Psi^+_k)_{x_2=0} = (\Psi^-_k)_{x_2=0} = \psi_k, & \text{on } \omega_T.
\end{cases} \]

As in [2], we consider
\[ V_{n+1} = V_n + \delta V_n, \quad \Psi_{n+1} = \Psi_n + \delta \Psi_n, \quad \psi_{n+1} = \psi_n + \delta \psi_n, \]

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where the differences $\delta V_n, \delta \Psi_n, \delta \psi_n$ will be specified later on. Given $\theta_0 \geq 1$, let us set $\theta_n := (\theta_0^2 + n)^{1/2}$, and consider the smoothing operators $S_{\theta_n}$. We decompose

$$\mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n) = \mathcal{L}'(U^n + V_{n+1}, \Phi^n + \Psi_{n+1}) - \mathcal{L}(U^n + V_n, \Phi^n + \Psi_n)$$

$$= \mathcal{L}'(U^n + V_n, \Phi^n + \Psi_n)(\delta V_n, \delta \Psi_n) + \epsilon'_n$$

$$= \mathcal{L}'(U^n + S_{\theta_n} V_n, \Phi^n + S_{\theta_n} \Psi_n)(\delta V_n, \delta \Psi_n) + \epsilon'_n + \epsilon''_n,$$

where $\epsilon'_n$ denotes the usual “quadratic” error of Newton’s scheme, and $\epsilon''_n$ the “substitution” error. The operator $\mathcal{L}'$ is given explicitly in (19).

Thanks to the properties of the smoothing operators, we have $(S_{\theta_n} \Psi_n)_{|x=0} = (S_{\theta_n} \Psi_n)_{|x=0}$, see (93), and we denote $\psi_n$ the common trace of these two functions. With this notation, we have

$$\mathcal{B}((V_{n+1})_{|x=0}, \psi_n) - \mathcal{B}((V_n)_{|x=0}, \psi_n) = \mathcal{B}'((U^n + V_{n+1})_{|x=0}, \Phi^n + \Psi_{n+1})((\delta V_n)_{|x=0}, \delta \psi_n) + \epsilon'_n$$

$$= \mathcal{B}'((U^n + S_{\theta_n} V_n)_{|x=0}, \Phi^n + \Psi_n) (\delta V_n)_{|x=0}, \delta \psi_n) + \epsilon'_n + \epsilon''_n,$$

where $\epsilon'_n$ denotes the “quadratic” error, and $\epsilon''_n$ the “substitution” error. At this point a standard Nash-Moser iteration would require, as a main step, the resolution of a linearized problem of the form

$$\begin{cases}
\mathcal{L}'(U^n + S_{\theta_n} V_n, \Phi^n + S_{\theta_n} \Psi_n)(\delta V_n, \delta \Psi_n) = \ldots,
\mathcal{B}'((U^n + S_{\theta_n} V_n)_{|x=0}, \Phi^n + \Psi_n) (\delta V_n)_{|x=0}, \delta \psi_n) = \ldots,
\end{cases}$$

with smoothed coefficients, and suitable source terms on the right-hand side. However this is not possible in our case since inversion of the operator $(\mathcal{L}', \mathcal{B}')$, or more precisely of the effective linearized operator $(\mathcal{L}_e', \mathcal{B}_e')$, requires the linearization around a state satisfying the constraints (16), (17), and (18). We thus need to introduce a smooth modified state, denoted $V_{n+1/2}, \Psi_{n+1/2}, \psi_{n+1/2}$, that satisfies the above mentioned constraints. (The exact definition of this intermediate state is detailed in section 7). Accordingly, we introduce the decompositions

$$\mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n) = \mathcal{L}_e'(U^n + V_{n+1/2}, \Phi^n + \Psi_{n+1/2})(\delta V_n, \delta \Psi_n) + \epsilon'_n + \epsilon''_n + \epsilon'''_n,$$

and

$$\mathcal{B}((V_{n+1})_{|x=0}, \psi_n) - \mathcal{B}((V_n)_{|x=0}, \psi_n)$$

$$= \mathcal{B}_e'(U^n + V_{n+1/2})_{|x=0}, \Phi^n + \Psi_{n+1/2})(\delta V_n)_{|x=0}, \delta \psi_n) + \epsilon'_n + \epsilon''_n + \epsilon'''_n,$$

where $\epsilon''_n, \epsilon'''_n$ denote the second “substitution” errors.

The final step is the introduction of the “good unknown” (compare with (20)):

$$\delta \dot{V}_n := \delta V_n - \delta \Psi_n \frac{\partial x_2(U^n + V_{n+1/2})}{\partial x_2(\Phi^n + \Psi_{n+1/2})}.$$  

(95)

For the interior equations this leads to

$$\mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n) = \mathcal{L}_e'(U^n + V_{n+1/2}, \Phi^n + \Psi_{n+1/2})(\delta \dot{V}_n$$

$$+ \epsilon'_n + \epsilon''_n + \epsilon'''_n + \frac{\delta \Psi_n}{\partial x_2(\Phi^n + \Psi_{n+1/2})} \partial x_2 \{ \mathcal{L}(U^n + V_{n+1/2}, \Phi^n + \Psi_{n+1/2}) \},$$

(96)

recalling (21), and (23). For the boundary terms, we obtain

$$\mathcal{B}((V_{n+1})_{|x=0}, \psi_n) - \mathcal{B}((V_n)_{|x=0}, \psi_n)$$

$$= \mathcal{B}_e'(U^n + V_{n+1/2})_{|x=0}, \Phi^n + \Psi_{n+1/2})(\delta \dot{V}_n)_{|x=0}, \delta \psi_n) + \epsilon'_n + \epsilon''_n + \epsilon'''_n,$$

(97)

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where $E_e'$ is defined in (26). For the sake of brevity we set
\[
D_{n+1/2} := \frac{1}{\partial_{x_2}(\Phi^a + \Psi_{n+1/2})} \partial_{x_2} \left\{ \mathbb{L}(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2}) \right\},
\]
\[
E_{n+1/2}' := E_{e}'((U^a + V_{n+1/2})|_{x_2=0}, \varphi^a + \psi_{n+1/2}).
\]
Let us also set
\[
e_n := e'_n + e''_n + e'''_n + D_{n+1/2} \delta \Psi_n,
\]
\[
\tilde{e}_n := \tilde{e}'_n + \tilde{e}''_n + \tilde{e}'''_n.
\]
The iteration proceeds as follows. Given
\[
V_0 := 0, \quad \Psi_0 := 0, \quad \psi_0 := 0,
\]
\[
f_0 := S_{\theta_0} f^a, \quad g_0 := 0, \quad E_0 := 0, \quad \tilde{E}_0 := 0,
\]
\[
V_1, \ldots, V_n, \quad \Psi_1, \ldots, \Psi_n, \quad \psi_1, \ldots, \psi_n,
\]
\[
f_1, \ldots, f_{n-1}, \quad g_1, \ldots, g_{n-1},
\]
\[
e_0, \ldots, e_{n-1}, \quad \tilde{e}_0, \ldots, \tilde{e}_{n-1},
\]
we first compute for $n \geq 1$
\[
E_n := \sum_{k=0}^{n-1} e_k, \quad \tilde{E}_n := \sum_{k=0}^{n-1} \tilde{e}_k.
\]
These are the accumulated errors at the step $n$. Then we compute $f_n$, and $g_n$ from the equations:
\[
\sum_{k=0}^{n} f_k + S_{\theta_n} E_n = S_{\theta_n} f^a, \quad \sum_{k=0}^{n} g_k + S_{\theta_n} \tilde{E}_n = 0,
\]
and we solve the linear problem
\[
\mathbb{L}_e'(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2}) \delta V_n = f_n \quad \text{in } \Omega_T,
\]
\[
E_{n+1/2}'((\delta V_n)|_{x_2=0}, \delta \psi_n) = g_n \quad \text{on } \omega_T,
\]
\[
\delta V_n = 0, \quad \delta \psi_n = 0 \quad \text{for } t < 0,
\]
finding $(\delta V_n, \delta \psi_n)$. Now we need to construct $\delta \Psi_n = (\delta \Psi^+_n, \delta \Psi^-_n)$ that satisfies $(\delta \Psi^+_n)|_{x_2=0} = \delta \psi_n$. Using the explicit expression of the boundary conditions in (101), see (26) and (25), we first note that $\delta \psi_n$ solves the equation:
\[
\partial_t \delta \psi_n + (v^a + v^a_{n+1/2})|_{x_2=0} \partial_{x_1} \delta \psi_n
\]
\[
+ \left\{ \partial_{x_1}(v^a + v^a_{n+1/2}) \partial_{x_2}(v^a + v^a_{n+1/2})|_{x_2=0} - \partial_{x_2}(v^a + v^a_{n+1/2})|_{x_2=0} \partial_{x_1}(v^a + v^a_{n+1/2})|_{x_2=0} \right\} \delta \psi_n
\]
\[
+ \partial_{x_1}(v^a + v^a_{n+1/2}) (\delta v^a_n)|_{x_2=0} = g_{n,2},
\]
and the equation

\[ \partial_t \delta\psi_n + (v^a - v_{n+2}^+) \partial_{x_2} \delta\psi_n + \left\{ \partial_{x_1} (\varphi^a + \psi_{n+2}^+) \frac{\partial_x (v^a - v_{n+2}^+)}{\partial_{x_2} (\varphi^a + \psi_{n+2}^+))} - \frac{\partial_x (u^a - u_{n+2}^+)}{\partial_{x_2} (\varphi^a + \psi_{n+2}^+))} \right\} \delta\psi_n + \partial_{x_1} (\varphi^a + \psi_{n+2}^+) \delta\psi_n \]

We shall thus define \( \delta\Psi_n^+, \delta\Psi_n^- \) as the solutions to the following equations:

\[ \partial_t \delta\Psi_n^+ + (v^a + v_{n+1}^-) \partial_{x_2} \delta\Psi_n^+ \]

\[ + \left\{ \partial_{x_1} (\varphi^a + \psi_{n+1}^-) \frac{\partial_x (v^a + v_{n+1}^-)}{\partial_{x_2} (\varphi^a + \psi_{n+1}^-))} - \frac{\partial_x (u^a + u_{n+1}^-)}{\partial_{x_2} (\varphi^a + \psi_{n+1}^-))} \right\} \delta\Psi_n^+ \]

\[ + \partial_{x_1} (\varphi^a + \psi_{n+1}^-) \delta\psi_n^+ - \delta\psi_n^+ = \mathcal{R}_T g_{n+2} + h_n^+ \]  

(104)

and

\[ \partial_t \delta\Psi_n^- + (v^a - v_{n+1}^-) \partial_{x_2} \delta\Psi_n^- \]

\[ + \left\{ \partial_{x_1} (\varphi^a - \psi_{n+1}^-) \frac{\partial_x (v^a - v_{n+1}^-)}{\partial_{x_2} (\varphi^a - \psi_{n+1}^-))} - \frac{\partial_x (u^a - u_{n+1}^-)}{\partial_{x_2} (\varphi^a - \psi_{n+1}^-))} \right\} \delta\Psi_n^- \]

\[ + \partial_{x_1} (\varphi^a - \psi_{n+1}^-) \delta\psi_n^- - \delta\psi_n^- = \mathcal{R}_T (g_{n+2} - g_{n+1}) + h_n^- \]  

(105)

In (104), and (105), the source terms \( h_n^\pm \) vanish on the boundary \( \omega_T \), and in the past, so that the unique smooth solutions to (104), and (105) will vanish in the past, and will satisfy the continuity condition \( (\delta\Psi_n^\pm)|_{x_2=0} = \delta\psi_n \). The only remaining task is to determine the source terms \( h_n^\pm \). In order to compute these source terms, we use a decomposition that is similar to (96) for the operator \( \mathcal{E} \). (Recall that this operator is defined in (89)). We have:

\[ \mathcal{E}(V_{n+1}, \Psi_{n+1}) - \mathcal{E}(V_n, \Psi_n) = \mathcal{E}'(V_{n+1}, \Psi_{n+1})(\delta V_n, \delta \Psi_n) + \hat{\theta}_n + \hat{\theta}_n' + \hat{\theta}_n'' \]

(106)

where \( \hat{\theta}_n \) is the “quadratic” error, \( \hat{\theta}_n' \) is the first “substitution” error, and \( \hat{\theta}_n'' \) is the second “substitution” error. We denote

\[ \hat{\theta}_n := \hat{\theta}_n' + \hat{\theta}_n'' + \hat{\theta}_n''' \]

\[ \hat{E}_n := \sum_{k=0}^{n-1} \hat{\theta}_k \]

Using the good unknown (95), and omitting the \( \pm \) superscripts, we compute

\[ \mathcal{E}'(V_{n+1}, \Psi_{n+1})(\delta V_n, \delta \Psi_n) = \partial_t \delta\Psi_n + (v^a + v_{n+1}^-) \partial_{x_2} \delta\Psi_n \]

\[ + \left\{ \partial_{x_1} (\varphi^a + \psi_{n+1}) \frac{\partial_x (v^a + v_{n+1}^-)}{\partial_{x_2} (\varphi^a + \psi_{n+1})} - \frac{\partial_x (u^a + u_{n+1}^-)}{\partial_{x_2} (\varphi^a + \psi_{n+1})} \right\} \delta\Psi_n \]

\[ + \partial_{x_1} (\varphi^a + \psi_{n+1}) \delta\psi_n - \delta\psi_n = \mathcal{R}_T (g_{n+2} - g_{n+1}) + h_n \]

Consequently, (104) and (106) yield

\[ \mathcal{E}(V_{n+1}, \Psi_{n+1}) - \mathcal{E}(V_n, \Psi_n) = \mathcal{R}_T g_{n+2} + h_n^+ + \hat{\theta}_n^+ \]

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The definition of \( h_n \) see (90), and (10). The previous relations lead to the following definition of the source term \( h_n^+ \):

\[
\sum_{k=0}^{n} h_k^+ + S_{\theta_n} (\hat{E}_n^+ - R_T \hat{E}_{n,2}) = 0 .
\]

(107)

The definition of \( h_n^- \) is entirely similar, and we omit the detailed calculations:

\[
\sum_{k=0}^{n} h_k^- + S_{\theta_n} (\hat{E}_n^- - R_T \hat{E}_{n,2} + R_T \hat{E}_{n,1}) = 0 .
\]

(108)

As in [14], one can check that the source terms \( h_k^\pm \) vanish in the past, and that their trace on \( \omega_T \) also vanish.

Once \( \delta \Psi_n \) is computed, the function \( \delta V_n \) is obtained from (95), and the functions \( V_{n+1}, \Psi_{n+1}, \psi_{n+1} \) are obtained from (94).

Finally, we compute \( e_n, \tilde{e}_n, \check{e}_n \) from

\[
\mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n) = f_n + e_n ,
\]

\[
\mathcal{E}(V_{n+1}^+, \Psi_{n+1}^+) - \mathcal{E}(V_n^+, \Psi_n^+) = R_T g_{n,2} + h_n^+ + \hat{e}_n^+ ,
\]

\[
\mathcal{E}(V_{n+1}^-, \Psi_{n+1}^-) - \mathcal{E}(V_n^-, \Psi_n^-) = R_T (g_{n,2} - g_{n,1}) + h_n^- + \hat{e}_n^- ,
\]

\[
\mathcal{B}((V_{n+1})_{|x=0}, \psi_{n+1}) - \mathcal{B}(V_n_{|x=0}, \psi_n) = g_n + \check{e}_n .
\]

(109)

To compute \( V_1, \Psi_1, \psi_1 \) we only consider steps (101), (104), (105), (109) for \( n = 0 \).

Adding (109) from 0 to \( N \), and combining with (100) gives

\[
\mathcal{L}(V_{N+1}, \Psi_{N+1}) - f^a = (S_{\theta_N} - I) f^a + (I - S_{\theta_N}) E_N + e_N ,
\]

\[
\mathcal{E}(V_{N+1}^+, \Psi_{N+1}^+) = R_T \left( \mathcal{E}(V_{N+1}^+_{|x=0}, \psi_{N+1}) \right) + (I - S_{\theta_N}) (\hat{E}_N - R_T \hat{E}_{N,2}) + \hat{e}_N^- + R_T \hat{e}_{N,2} ,
\]

\[
\mathcal{E}(V_{N+1}^-, \Psi_{N+1}^-) = R_T \left( \mathcal{E}(V_{N+1}^-_{|x=0}, \psi_{N+1}) \right) + (I - S_{\theta_N}) (\hat{E}_N - R_T (\hat{E}_{N,2} - \hat{E}_{N,1})) + \hat{e}_N^- - R_T (\check{e}_{N,2} - \check{e}_{N,1}) ,
\]

\[
\mathcal{B}(V_{N+1}_{|x=0}, \psi_{N+1}) = (I - S_{\theta_N}) \hat{E}_N + \check{e}_N .
\]

Because \( S_{\theta_N} \to I \) as \( N \to +\infty \), and since we expect \( (e_N, \hat{e}_N, \check{e}_N) \to 0 \), we will formally obtain the solution of the problem (89) from \( \mathcal{L}(V_{N+1}, \Psi_{N+1}) \to f^a , \mathcal{B}(V_{N+1}_{|x=0}, \psi_{N+1}) \to 0 \), and \( \mathcal{E}(V_{N+1}, \Psi_{N+1}) \to 0 . \)

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6 More tame estimates

In this section, we are going to summarize the estimates that will be needed in the proof of Theorem 1. We recall that $\mu$ denotes the integer given in the assumptions of Theorem 1. Also in Propositions 5, and 6 below, we use an integer $\tilde{\alpha}$ that will be specified in section 7. (We shall see that a suitable choice for this integer is $\tilde{\alpha} = \mu + 5$).

6.1 Tame estimate for the second derivatives

For the control of the errors $e_n', e_n'', e_n'''$ etc. in the iteration scheme, we need to estimate the second derivative of the operators $L$, $E$, and $B$. These operators are defined in (8), (10), and (89). We are going to derive these estimates with some weaker smoothness assumption for the coefficients, than what was needed in section 3. More precisely, we consider a fixed time $T > 0$, and we assume that

$$\dot{U}_{r,l}, \Phi_{r,l} \in W^{1,\infty}(\Omega_T), \quad \|\dot{U}_{r,l}, \Phi_{r,l}\|_{W^{1,\infty}(\Omega_T)} \leq \bar{K}, \quad (110)$$

where $\bar{K}$ is a positive constant. Let $\tilde{\alpha}$ be a sufficiently large integer that will be chosen later on. We have the following result:

**Proposition 5.** Let $m \in \mathbb{N}, m \in [3, \tilde{\alpha} - 1]$, and let $T > 0$. Assume that the perturbations $\dot{U}_{r,l}, \Phi_{r,l}$ satisfy (110), and

$$\forall \gamma \geq 1, \quad (\dot{U}_{r,l}, \Phi_{r,l}) \in H^{\tilde{\alpha}}_{\gamma}(\Omega_T).$$

Then there exist two constants $\bar{K}_0 > 0$, and $C > 0$, that do not depend on $T$, and $\gamma$, such that, if $\bar{K} \leq \bar{K}_0$, $\gamma \geq 1$, and if $(V', \Psi'), (V'', \Psi'') \in H^{m+1}_{\gamma}(\Omega_T)$, then one has

$$\|L''(U_{r,l}, \Phi_{r,l})((V', \Psi'), (V'', \Psi''))\|_{H^{m}_{\gamma}(\Omega_T)} \leq C \left\{ \|\dot{U}, \Phi\|_{H^{m+1}_{\gamma}(\Omega_T)} \|\dot{V'}, \Psi'\|_{W^{1,\infty}(\Omega_T)} \|\dot{V''}, \Psi''\|_{W^{1,\infty}(\Omega_T)} + \|V', \Psi'\|_{H^{m+1}_{\gamma}(\Omega_T)} \|\dot{V''}, \Psi''\|_{W^{1,\infty}(\Omega_T)} \right\}, \quad (111)$$

and

$$\|E''((V', \Psi'), (V'', \Psi''))\|_{H^{m}_{\gamma}(\Omega_T)} \leq C \left\{ \|\dot{V'}, \Psi'\|_{H^{m+1}_{\gamma}(\Omega_T)} \|\dot{V''}, \Psi''\|_{W^{1,\infty}(\Omega_T)} + \|V', \Psi'\|_{H^{m+1}_{\gamma}(\Omega_T)} \|\dot{V''}, \Psi''\|_{W^{1,\infty}(\Omega_T)} + \|V''\|_{L^\infty(\Omega_T)} \|\dot{\Psi'}, \Psi'\|_{H^{m+1}_{\gamma}(\Omega_T)} \right\}. \quad (112)$$

If $(W', \psi'), (W'', \psi'') \in H^{m}_{\gamma}(\omega_T) \times H^{m+1}_{\gamma}(\omega_T)$, then one has

$$\|B''((W', \psi'), (W'', \psi''))\|_{H^{m}_{\gamma}(\omega_T)} \leq C \left\{ \|\dot{W'}, \psi'\|_{H^{m}_{\gamma}(\omega_T)} \|\dot{W''}, \psi''\|_{W^{1,\infty}(\omega_T)} + \|W', \psi'\|_{H^{m+1}_{\gamma}(\omega_T)} \|\dot{W''}, \psi''\|_{W^{1,\infty}(\omega_T)} + \|W''\|_{L^\infty(\omega_T)} \|\dot{\psi'}, \psi'\|_{H^{m+1}_{\gamma}(\omega_T)} \right\}. \quad (113)$$

**Proof.** The proof of (111) follows from the long, but straightforward calculation of the explicit expression of $L''$, from Theorem 9, and from Theorem 10. The calculations are left to the reader. The constant $\bar{K}_0$ is fixed so that under the constraint $\bar{K} \leq \bar{K}_0$, then $U_{r,l}$ takes its values in a fixed compact domain of the hyperbolicity region, and $\|\dot{\Phi}_{r,l}\|_{W^{1,\infty}(\Omega_T)} \leq 1/2$. 36
The estimates (112), and (113) follow immediately from the expressions
\[
\mathcal{E}''(V', \Psi'), (V'', \Psi'') = \psi'' \partial_{x_1} \Psi' + \psi' \partial_{x_1} \Psi'', \\
\mathcal{B}''((W', \psi'), (W'', \psi'')) = \begin{pmatrix}
(v_+ - v_0) \partial_{x_1} \psi'' + (v_0' - v''_0) \partial_{x_1} \psi' \\
v_+ \partial_{x_1} \psi'' + v''_0 \partial_{x_1} \psi' \\
0
\end{pmatrix},
\]
and from Theorem 9. Observe that \(\mathcal{E}''\) and \(\mathcal{B}''\) do not depend on the state \((\dot{U}_{r,l}, \dot{\Phi}_{r,l})\) about which we linearize the nonlinear equations.

The estimates (111), (112), (113) hold for every \(m\), with a constant \(C\) that may depend on \(m\). Since in Proposition 5, \(m\) is taken in a bounded interval, the constant \(C\) may be assumed to be independent of \(m\). The same remark holds for Proposition 6 in the next subsection.

### 6.2 Tame estimate for the linearized problem

We conclude this section by writing the tame estimate (60) in a more convenient form:

**Proposition 6.** Let \(T > 0\), and \(m \in \mathbb{N}, m \in [3, \hat{a}]\). Let us assume that the hypotheses of Theorem 4 hold and that the perturbations satisfy
\[
\|(\dot{U}, \nabla \Phi)\|_{H^\gamma_3(\Omega_T)} + \|(\dot{U}, \partial_{x_2} \dot{U}, \nabla \Phi)\|_{\gamma_2=0} \|H^\gamma_4(\omega_T)\| \leq K. \tag{114}
\]

Then there exist some constants \(C > 0\) and \(\gamma \geq 1\), that only depend on \(K_0\) (that is given by Theorem 4), such that, if \(K \leq K_0\), and if \((f, g) \in H^{m+1}(\Omega_T) \times H^{m+1}(\omega_T)\) vanish in the past, then there exists a unique solution \((\dot{V}, \psi) \in H^m(\Omega_T) \times H^m(\omega_T)\) to (23), (26) that vanishes in the past. Moreover, the following estimate holds:
\[
\|\dot{V}\|_{H^m(\Omega_T)} + \|\dot{P}(\phi)\|_{\gamma_2=0} \|H^\gamma(\omega_T)\| + \|\psi\|_{H^\gamma_3(\omega_T)} \leq C \left\{ \left( \frac{1}{\sqrt{\gamma}} \|f\|_{H^\gamma_3(\Omega_T)} + \frac{\gamma}{T} \|g\|_{H^\gamma_3(\omega_T)} \right) \right\}
\]

**Proof.** Theorem 3 gives the well-posedness in \(L^2\), and we shall use without proof that for source terms \((f, g) \in H^{m+1}(\Omega_T) \times H^{m+1}(\omega_T)\) vanishing in the past, the solution \((\dot{V}, \psi)\) belongs to \(H^m(\Omega_T) \times H^m(\omega_T)\) and satisfies (60) for \(\gamma \geq \gamma_m\).

We fix \(\gamma > \gamma_0\) greater than the maximum of \(\gamma_0, \ldots, \gamma_\hat{a}\). From (60) with \(m = 3\), and Theorem 11, we get
\[
\begin{align*}
\sqrt{T} \|\dot{V}\|_{H^\gamma_3(\Omega_T)} + \|\dot{P}(\phi)\|_{\gamma_2=0} \|H^\gamma_3(\omega_T)\| &+ \|\psi\|_{H^\gamma_3(\omega_T)} \leq C \left\{ \left( \frac{1}{\sqrt{\gamma}} \|f\|_{H^\gamma_3(\Omega_T)} \right. \\
&+ \frac{1}{\sqrt{\gamma}} \|g\|_{H^\gamma_3(\omega_T)} + \frac{\gamma}{T} \|f\|_{H^\gamma_3(\Omega_T)} \|\dot{U}, \nabla \Phi\|_{H^\gamma_3(\omega_T)} + \frac{\gamma}{T} \|\dot{V}\|_{H^\gamma_3(\omega_T)} \|\dot{U}, \nabla \Phi\|_{H^\gamma_3(\omega_T)} \\
&\left. + \frac{\gamma}{T} \left( \|\dot{P}(\phi)\|_{\gamma_2=0} \|H^\gamma_3(\omega_T)\| + \|\dot{\psi}\|_{H^\gamma_3(\omega_T)} \right) \left\|\dot{U}, \partial_{x_2} \dot{U}, \nabla \Phi\right\|_{\gamma_2=0} \right\} \tag{115}
\end{align*}
\]

Choosing \(K_0\) small enough (observe that this choice depends on \(T\), and on \(\gamma\) that has been fixed as above), we can absorb the norms \(\|\dot{V}\|_{H^\gamma_3(\Omega_T)}\), \(\|\dot{P}(\phi)\|_{\gamma_2=0} \|H^\gamma_3(\omega_T)\|\), and \(\|\dot{\psi}\|_{H^\gamma_3(\omega_T)}\) in the left-hand side of (115). Consequently, we obtain
\[
\|\dot{V}\|_{H^\gamma_3(\Omega_T)} + \|\dot{P}(\phi)\|_{\gamma_2=0} \|H^\gamma_3(\omega_T)\| + \|\dot{\psi}\|_{H^\gamma_3(\omega_T)} \leq C \left( \|f\|_{H^\gamma_3(\Omega_T)} + \|g\|_{H^\gamma_3(\omega_T)} \right).
\]

Applying the Sobolev imbedding Theorem once again, we have
\[
\|\dot{V}\|_{W^{1, \infty}(\Omega_T)} + \|\dot{P}(\phi)\|_{\gamma_2=0} \|L^\infty(\omega_T)\| + \|\dot{\psi}\|_{W^{1, \infty}(\omega_T)} \leq C \left( \|f\|_{H^\gamma_3(\Omega_T)} + \|g\|_{H^\gamma_3(\omega_T)} \right),
\]
and we can use this inequality in (60). Using a classical estimate for the traces, we obtain (115).

Notice that from now on $\gamma$ is fixed, as detailed in the proof of Proposition 6.

\section{Proof of the main result}

We recall that the sequence $(\theta_n)$ is defined by

$$
\theta_0 \geq 1, \quad \theta_n := \sqrt{\theta_0^2 + n},
$$

and that we denote $\Delta_n := \theta_{n+1} - \theta_n$. In particular, the sequence $(\Delta_n)$ is decreasing, and tends to zero. Moreover, one has

$$
\forall n \in \mathbb{N}, \quad \frac{1}{3\theta_n} \leq \Delta_n = \sqrt{\theta_0^2 + 1} - \theta_n \leq \frac{1}{2\theta_n}.
$$

\subsection{Introduction of the iterative scheme}

Given a small number $\delta > 0$, and an integer $\tilde{\alpha}$ that will be chosen later on, let us assume that the following estimate holds:

$$
\|\hat{U}^a\|_{H^{\tilde{\alpha}+3}(\Omega_T)} + \|\hat{\Phi}^a\|_{H^{\tilde{\alpha}+4}(\Omega_T)} + \|\varphi^a\|_{H^{\tilde{\alpha}+7/2}(\omega_T)} + \|f^a\|_{H^{\tilde{\alpha}+2}(\Omega_T)} \leq \delta. \quad (117)
$$

Given the integer $\alpha := \mu + 1$, our inductive assumption reads:

\begin{equation}
(H_{n-1}) \quad \begin{cases}
\text{a)} \forall k = 0, \ldots, n-1, \quad \forall s \in [3, \tilde{\alpha}] \cap \mathbb{N}, \\
\|\langle \delta V_k, \delta \Psi_k \rangle\|_{H^s(\Omega_T)} + \|\delta \psi_k\|_{H^{\tilde{\alpha}+1}(\omega_T)} \leq \delta \theta_k^{s-\alpha-1} \Delta_k,
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
\text{b)} \forall k = 0, \ldots, n-1, \quad \forall s \in [3, \tilde{\alpha} - 2] \cap \mathbb{N}, \\
\|C(V_k, \Psi_k) - f^a\|_{H^{\tilde{\alpha}+1}(\omega_T)} \leq 2 \delta \theta_k^{s-\alpha-1},
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
\text{c)} \forall k = 0, \ldots, n-1, \quad \forall s \in [4, \alpha] \cap \mathbb{N}, \\
\|B(V_k, \psi_k)\|_{H^s(\omega_T)} \leq \delta \theta_k^{s-\alpha-1},
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
\text{d)} \forall k = 0, \ldots, n-1, \\
\|\mathcal{E}(V_k, \Psi_k)\|_{H^{\tilde{\alpha}}(\Omega_T)} \leq \delta \theta_k^{2-\alpha}.
\end{cases}
\end{equation}

For $k = 0, \ldots, n$, the functions $V_k, \Psi_k, \psi_k$ are also assumed to satisfy (93).

The first task is to prove that for a suitable choice of the parameters $\theta_0 \geq 1$, and $\delta > 0$, and for $f^a$ small enough, $(H_{n-1})$ implies $(H_n)$. In the end, we shall prove that $(H_0)$ holds for sufficiently small initial data.

From now on, we assume that $(H_{n-1})$ holds. Let us show some basic consequences:

\textbf{Lemma 6.} If $\theta_0$ is big enough, then for every $k = 0, \ldots, n$, and for every integer $s \in [3, \tilde{\alpha}]$, we have

\begin{equation}
\|\langle V_k, \psi_k \rangle\|_{H^{\tilde{\alpha}}(\Omega_T)} + \|\psi_k\|_{H^{\tilde{\alpha}+1}(\omega_T)} \leq \delta \theta_k^{(s-\alpha)+}, \quad \alpha \neq s, \quad (118a)
\end{equation}

\begin{equation}
\|\langle V_k, \Psi_k \rangle\|_{H^{\tilde{\alpha}}(\Omega_T)} + \|\psi_k\|_{H^{\tilde{\alpha}+1}(\omega_T)} \leq \delta \log \theta_k. \quad (118b)
\end{equation}
The proof follows from the triangle inequality, and from the classical comparisons between series and integrals. Lemma 6 yields the following:

**Lemma 7.** If \( \theta_0 \) is big enough, then for every \( k = 0, \ldots, n \), and for every integer \( s \in [3, \tilde{\alpha} + 4] \), we have

\[
\| (S_{\theta_k} V_k, S_{\theta_k} \Psi_k) \|_{H^s_1(\Omega_T)} \leq C \delta \theta_k^{s-4}, \quad s \neq \alpha, \quad (119a)
\]

\[
\| (S_{\theta_k} V_k, S_{\theta_k} \Psi_k) \|_{H^s_\psi(\Omega_T)} \leq C \delta \log \theta_k. \quad (119b)
\]

For every \( k = 0, \ldots, n \), and for every integer \( s \in [3, \tilde{\alpha}] \), we have

\[
\| (I - S_{\theta_k}) V_k, (I - S_{\theta_k}) \Psi_k \|_{H^s_1(\Omega_T)} \leq C \delta \theta_k^{s-\alpha}. \quad (120)
\]

The proof follows easily from Lemma 6, and from the properties (92) of the smoothing operators.

### 7.2 Estimate of the quadratic errors

We start by proving an estimate for the quadratic errors \( \epsilon_k^l, \epsilon_k^l, \epsilon_k^l \) of the iterative scheme. Recall that these errors are defined by

\[
\epsilon_k^l := \mathcal{L}(V_{k+1}, \Psi_{k+1}) - \mathcal{L}(V_k, \Psi_k) - \mathcal{L}'(V_k, \Psi_k)(\delta V_k, \delta \Psi_k), \quad (121)
\]

\[
\epsilon_k^l := \mathcal{E}(V_{k+1}, \Psi_{k+1}) - \mathcal{E}(V_k, \Psi_k) - \mathcal{E}'(V_k, \Psi_k)(\delta V_k, \delta \Psi_k), \quad (122)
\]

\[
\epsilon_k^l := \mathcal{B}(V_{k+1}|_{x_2=0}, \psi_{k+1}) - \mathcal{B}(V_k|_{x_2=0}, \psi_k) - \mathcal{B}'(V_k|_{x_2=0}, \psi_k)((\delta V_k)|_{x_2=0}, \delta \psi_k), \quad (123)
\]

where \( \mathcal{L}, \mathcal{E}, \text{and} \mathcal{B} \) are defined by (89), and (90).

**Lemma 8.** Let \( \alpha \geq 4 \). There exist \( \delta > 0 \) sufficiently small, and \( \theta_0 \geq 1 \) sufficiently large, such that for all \( k = 0, \ldots, n-1 \), and for all integer \( s \in [3, \tilde{\alpha} - 1] \), one has

\[
\| \epsilon_k^l \|_{H^s_1(\Omega_T)} \leq C \delta^2 \theta_k^{L_1(s)-1} \Delta_k, \quad (124a)
\]

\[
\| \epsilon_k^l \|_{H^s_\psi(\Omega_T)} \leq C \delta^2 \theta_k^{s+1-2\alpha} \Delta_k, \quad (124b)
\]

\[
\| \epsilon_k^l \|_{H^s_\omega(\Omega_T)} \leq C \delta^2 \theta_k^{L_1(s)-1} \Delta_k, \quad (124c)
\]

where \( L_1(s) := \max\{(s + 1 - \alpha)+ 4 - 2\alpha; s + 2 - 2\alpha\} \).

**Proof.** The quadratic error given in (121) may be written as

\[
\epsilon_k^l = \int_0^1 (1 - \tau) \mathbb{L}''(U^n + V_k + \tau \delta V_k, \Phi^n + \Psi_k + \tau \delta \Psi_k)(\delta V_k, \delta \Psi_k, (\delta V_k, \delta \Psi_k)) \ d\tau.
\]

From Theorem 11, (117), and (118a), we have

\[
\sup_{\tau \in [0,1]} \| (U^n + V_k + \tau \delta V_k, \Phi^n + \Psi_k + \tau \delta \Psi_k) \|_{W^{1,\infty}(\Omega_T)} \leq C \delta,
\]

so for \( \delta \) sufficiently small, we can apply Proposition 5. Using (\( H_{n-1} \)), and (117), we obtain

\[
\| \epsilon_k^l \|_{H^s_1(\Omega_T)} \leq C \left( \delta^2 \theta_k^{4-2\alpha} \Delta_k^2 (\delta + \| (V_k, \Psi_k) \|_{H^{s+1}_1(\Omega_T)} + \| (\delta V_k, \delta \Psi_k) \|_{H^{s+1}_\omega(\Omega_T)}) + \delta^2 \theta_k^{s+2-2\alpha} \Delta_k^2 \right).
\]

When \( s + 1 \neq \alpha \), (118a) yields

\[
\| \epsilon_k^l \|_{H^s_1(\Omega_T)} \leq C \delta^2 \Delta_k^2 \left\{ \theta_k^{(s+1-\alpha)+4-2\alpha} + \theta_k^{s+2-2\alpha} \right\} \leq C \delta^2 \theta_k^{L_1(s)-1} \Delta_k,
\]

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where we have used the inequality $\theta_k \Delta_k \leq 1/2$. If $s + 1 = \alpha$, we obtain a similar result by using (118b), and $\alpha \geq 4$:

$$\|e_k^\prime\|_{H^{s+1}_\omega(\Omega_T)} \leq C \delta^2 \Delta_k^2 \left\{ (\delta + \delta \log \theta_k + \delta \theta_k^{-1} \Delta_k) \theta_k^{\alpha - 2} + \theta_k^{1-\alpha} \right\} \leq C \delta^2 \Delta_k^2 \theta_k^{1-\alpha} \leq C \delta^2 \theta_k^{L_1(\alpha - 1) - 1} \Delta_k.$$  

We have thus obtained (124a). The estimate (124b) of $\hat{e}_k^\prime$ is similar, and follows from (112).

The quadratic error $\tilde{e}_k^\prime$ given in (123) may be written as

$$\tilde{e}_k^\prime = \frac{1}{2} \mathbb{E}'\left( ((\delta V_k)|_{x_2=0}, \delta \psi_k), ((\delta V_k)|_{x_2=0}, \delta \psi_k) \right),$$

and again we estimate this error by applying Proposition 5. Using a classical trace estimate, and Sobolev imbedding Theorem, we prove

$$\|\tilde{e}_k^\prime\|_{H^s_\omega(\Omega_T)} \leq C \left\{ \|\delta V_k\|_{H^{s+1}_\omega(\Omega_T)} \|\delta \psi_k\|_{W^{1,\infty}(\Omega_T)} + \|\delta V_k\|_{L^\infty(\Omega_T)} \|\delta \psi_k\|_{H^{s+1}_\omega(\Omega_T)} \right\} \leq C \left\{ \delta \theta_k^{s-\alpha} \Delta_k \delta \theta_k^{2-\alpha} \Delta_k + \delta \theta_k^{2-\alpha} \Delta_k \delta \theta_k^{-\alpha-1} \Delta_k \right\} \leq C \delta^2 \theta_k^{L_1(s) - 2} \Delta_k \leq C \delta^2 \theta_k^{L_1(s) - 1} \Delta_k,$$

which is (124c).

\[\square\]

### 7.3 Estimate of the first substitution errors

Now we estimate the first substitution errors $e_k^\prime, e_k^\prime, e_k^\prime$ of the iterative scheme, defined by

$$e_k^\prime := \mathcal{L}'(V_k, \Psi_k)(\delta V_k, \delta \Psi_k) - \mathcal{L}'(S\theta_k V_k, S\theta_k \Psi_k)(\delta V_k, \delta \Psi_k), \quad (125)$$

$$e_k^\prime := \mathcal{E}'(V_k, \Psi_k)(\delta V_k, \delta \Psi_k) - \mathcal{E}'(S\theta_k V_k, S\theta_k \Psi_k)(\delta V_k, \delta \Psi_k), \quad (126)$$

$$\tilde{e}_k^\prime := \mathcal{B}'((V_k)|_{x_2=0}, \psi_k)((\delta V_k)|_{x_2=0}, \delta \psi_k) - \mathcal{B}'((S\theta_k V_k)|_{x_2=0}, \psi_k)((\delta V_k)|_{x_2=0}, \delta \psi_k), \quad (127)$$

where $\psi_k^\epsilon$ denotes the common trace of $S\theta_k \Psi_k$ on the boundary $\omega_T$. (Recall that these traces coincide thanks to the properties of the smoothing operators).

**Lemma 9.** Let $\alpha \geq 4$. There exist $\delta > 0$ sufficiently small, and $\theta_0 \geq 1$ sufficiently large, such that for all $k = 0, \ldots, n - 1$, and for all integer $s \in [3, \tilde{\alpha} - 2]$, one has

$$\|e_k^\prime\|_{H^s_\omega(\Omega_T)} \leq C \delta^2 \theta_k^{L_2(s) - 1} \Delta_k, \quad (128a)$$

$$\|e_k^\prime\|_{H^s_\omega(\Omega_T)} \leq C \delta^2 \theta_k^{L_3(s) - 1} \Delta_k, \quad (128b)$$

$$\|\tilde{e}_k^\prime\|_{H^s_\omega(\Omega_T)} \leq C \delta^2 \theta_k^{L_3(s) - 1} \Delta_k, \quad (128c)$$

where $L_2(s) := \max\{(s + 1 - \alpha)_+ + 6 - 2\alpha; s + 5 - 2\alpha\}$.

**Proof.** The substitution error given in (125) may be written as

$$e_k^\prime := \int_0^1 \mathbb{L}'\left( U^\alpha + S\theta_k V_k + \tau(I - S\theta_k)V_k, \Phi^\alpha + S\theta_k \Psi_k + \tau(I - S\theta_k)\Psi_k \right) \left( (\delta V_k, \delta \Psi_k), ((I - S\theta_k)V_k, (I - S\theta_k)\Psi_k) \right) d\tau.$$  

From Theorem 11, (117), (119a), and (120), we have

$$\sup_{\tau \in [0,1]} \|(U^\alpha + S\theta_k V_k + \tau(I - S\theta_k)V_k, \Phi^\alpha + S\theta_k \Psi_k + \tau(I - S\theta_k)\Psi_k)\|_{W^{1,\infty}(\Omega_T)} \leq C \delta,$$  

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so we can apply Proposition 5 for \( \delta \) sufficiently small. For \( s + 1 \neq \alpha \), and \( s + 1 \leq \tilde{\alpha} \), the estimates (117), (\( H_{n-1} \)), (119a), and (120) yield

\[
\| \epsilon_k'' \|_{H^5\Omega_T} \leq C \left\{ \delta \theta_k^{s+1-\alpha}, \delta \theta_k^{2-\alpha} \Delta_k + \delta \theta_k^{2-\alpha} \delta \theta_k^{s-\alpha} \Delta_k + \delta \theta_k^{s-\alpha} \Delta_k \delta \theta_k^{s+1-\alpha} \right\}^\prime \\
\leq C \delta^2 \theta_k^{L_2(\alpha)-1} \Delta_k.
\]

For \( s + 1 = \alpha \), we obtain

\[
\| \epsilon_k'' \|_{H^5\Omega_T} \leq C \left\{ (\delta + \delta \log \theta_k) \delta \theta_k^{2-\alpha} \Delta_k + \delta^2 \theta_k^{2-\alpha} \Delta_k \right\}^\prime \\
\leq C \delta^2 \Delta_k \left\{ \theta_k^{5-2\alpha} \log \theta_k + \theta_k^{2-\alpha} \right\} \leq C \delta^2 \theta_k^{L_2(\alpha)-1} \Delta_k.
\]

We have thus proved (128a), and (128b) follows in the same way.

The substitution error given in (127) may be written as

\[
\tilde{\epsilon}_k'' = \mathbb{B}' \left( (\delta V_k)_{x_2=0}, \delta \psi_k \right), ((V_k - S_{\theta_k} V_k)_{x_2=0}, \psi_k - \psi_k^\prime) \right).
\]

We recall that \( \psi_k^\prime \) denotes the common trace of \( S_{\theta_k} \Psi_K^+ \), and \( S_{\theta_k} \Psi_K^- \) on \( \omega_T \). Thanks to Proposition 5, when \( s \leq \tilde{\alpha} - 2 \), we obtain

\[
\| \tilde{\epsilon}_k'' \|_{H^5\Omega_T} \leq C \left\{ \| \delta V_k \|_{H^{s+1}\Omega_T}, (I - S_{\theta_k}) \Psi_k \|_{W^{1,\infty}(\Omega_T)} \right. \\
+ \| \delta V_k \|_{L^\infty(\Omega_T)} (I - S_{\theta_k}) \Psi_k \|_{H^{s+2}(\Omega_T)} + \| (I - S_{\theta_k}) V_k \|_{H^{s+1}(\Omega_T)} \| \delta \psi_k \|_{W^{1,\infty}(\omega_T)} \right. \\
\left. + \| (I - S_{\theta_k}) V_k \|_{L^\infty(\Omega_T)} \| \delta \psi_k \|_{H^{s+1}(\omega_T)} \right\}.
\]

Using (\( H_{n-1} \)), and (120), we get (128c). \( \square \)

### 7.4 Construction and estimate of the modified state

The next step requires the construction of the smooth modified state \( V_{n+1/2}, \Psi_{n+1/2}, \psi_{n+1/2} \) satisfying the constraints (16), and (17). The additional constraint (18) will be obtained by choosing \( \delta \) small enough, so we first focus on (16), and (17).

**Proposition 7.** Let \( \alpha \geq 4 \). There exist some functions \( V_{n+1/2}, \Psi_{n+1/2}, \psi_{n+1/2} \), that vanish in the past, and such that \( U^a + V_{n+1/2}, \Psi^a + \Psi_{n+1/2}, \varphi^a + \psi_{n+1/2} \) satisfy the constraints (16), and (17). Moreover, these functions satisfy:

\[
\Psi_{n+1/2}^+ = S_{\theta_n} \Psi_n^+, \quad \psi_{n+1/2}^+ := (S_{\theta_n} \Psi_n^+)_{x_2=0} \quad \text{(129a)}
\]

\[
\Psi_{n+1/2}^- = S_{\theta_n} \psi_n^+, \quad \text{(129b)}
\]

\[
\| V_{n+1/2} - S_{\theta_n} V_n \|_{H^s(\Omega_T)} \leq C \delta \theta_n^{s+1-\alpha}, \quad \text{for } s \in [3, \tilde{\alpha} + 3]. \quad \text{(129c)}
\]

**Proof.** We want to construct some functions \( V_{n+1/2}, \Psi_{n+1/2}, \psi_{n+1/2} \) that satisfy

\[
(V_{n+1/2})_{x_2=0} = (\Psi_{n+1/2})_{x_2=0} = \psi_{n+1/2},
\]

\[
\mathbb{B}((U^a + V_{n+1/2})_{x_2=0} + (\Psi^a + \psi_{n+1/2}) = 0,
\]

\[
\mathcal{E}(V_{n+1/2}, \Psi_{n+1/2}) = 0.
\]

We note that the eikonal equations on the boundary, that is the two first components of the operator \( \mathbb{B} \), can be deduced from the eikonal equations in the interior. In other words, it is
enough to construct some functions that satisfy:

\[
\begin{align*}
\left(\Psi_{n+1/2}^+\right)_{|x_2=0} &= \left(\Psi_{n+1/2}^-\right)_{|x_2=0} = \psi_{n+1/2}, \quad (130a) \\
\left(\rho_{n+1/2}^+\right)_{|x_2=0} &= \left(\rho_{n+1/2}^-\right)_{|x_2=0}, \quad (130b) \\
\mathcal{E}(V_{n+1/2}; \Psi_{n+1/2}) &= 0. \quad (130c)
\end{align*}
\]

Let the errors \( \varepsilon_{1,2}^n \) be defined by

\[
\begin{align*}
\varepsilon_1^n &:= (S_{\partial_\nu} \rho_n^+)|_{x_2=0} - (S_{\partial_\nu} \rho_n^-)|_{x_2=0}, \quad (131a) \\
\varepsilon_2^n &:= \mathcal{E}(V_n, \Psi_n). \quad (131b)
\end{align*}
\]

We define the modified states \( V_{n+1/2}, \Psi_{n+1/2}, \psi_{n+1/2} \) as follows:

\[
\begin{align*}
\Psi_{n+1/2}^\pm &:= S_{\partial_\nu} \Psi_n^\pm, \\
\rho_{n+1/2}^\pm &:= S_{\partial_\nu} \rho_n^\pm + \frac{1}{2} \mathcal{R}_T \varepsilon_1^n, \\
v_{n+1/2}^\pm &:= S_{\partial_\nu} v_n^\pm, \\
u_{n+1/2}^\pm &:= \frac{1}{2} \partial_\nu \Psi_{n+1/2}^\pm + \left( v_a^\pm + v_{n+1/2}^\pm \right) \partial_{x_1} \Psi_{n+1/2}^\pm + v_{n+1/2}^\pm \partial_{x_1} \Phi_a,
\end{align*}
\]

where \( \mathcal{R}_T \) is the lifting operator that was already used to construct the smoothing operators, see Lemma 5. It is easy to check that \( V_{n+1/2}, \Psi_{n+1/2}, \psi_{n+1/2} \) fulfill \((130)\), and vanish in the past. We thus only need to prove the estimate \((129c)\).

Recall that \( \varepsilon_1^n \) is defined by \((131a)\). First of all, we note that points a), and c) of the induction assumption yield

\[
\| (\rho_n^+ - \rho_n^-) \|_{H^s(\Omega_T)} \leq \| (\rho_{n-1}^+ - \rho_{n-1}^-) \|_{H^s(\Omega_T)} + \| (\delta \rho_{n-1}^+ - \delta \rho_{n-1}^-) \|_{H^s(\Omega_T)}
\]

\[
\leq \| \mathcal{R}(V_{n-1}) \|_{H^s(\Omega_T)} + C \| \delta V_{n-1} \|_{H^s(\Omega_T)} \leq C \delta \theta_n^{s-\alpha-1},
\]

for all integer \( s \in [4, \alpha] \). Then using the properties of the lifting operator \( \mathcal{R}_T \), and of the smoothing operators \( S_{\partial_\nu} \) (see Proposition 4), we have:

\[
\| \varepsilon_1^n \|_{H^s(\Omega_T)} \leq C \theta_n^{s+1-\alpha} \| (\rho_n^+ - \rho_n^-) \|_{H^s(\Omega_T)} 
\]

\[
\leq C \delta \theta_n^{s-\alpha-1} \delta \theta_n^{-1} \leq C \delta \theta_n^{s-\alpha}, \quad s \in [\alpha, \alpha + 3],
\]

while for \( s \in [3, \alpha - 1] \), we have

\[
\| \varepsilon_1^n \|_{H^s(\Omega_T)} \leq C \| \rho_n^+ - \rho_n^- \|_{H^s(\Omega_T)} \leq C \delta \theta_n^{s-\alpha}.
\]

For \( s \in [3, \alpha + 3] \), we thus obtain

\[
\| \rho_{n+1/2}^\pm - S_{\partial_\nu} \rho_n^\pm \|_{H^s(\Omega_T)} = \frac{1}{2} \| \mathcal{R}_T \varepsilon_1^n \|_{H^s(\Omega_T)} \leq C \| \varepsilon_1^n \|_{H^s(\Omega_T)} \leq C \delta \theta_n^{s-\alpha}.
\]

We now turn to the estimate of \( u_{n+1/2} - S_{\partial_\nu} u_n \). Using the definition \((131b)\) of \( \varepsilon_2^n \), we get

\[
S_{\partial_\nu} u_n + S_{\partial_\nu} \varepsilon_2^n = S_{\partial_\nu} \partial_\nu \Psi_n + S_{\partial_\nu} \left( (\partial_a + \nu_n) \partial_{x_1} \Psi_n \right) + S_{\partial_\nu} \left( (\varepsilon_a + \nu_n) \partial_{x_1} \Psi_n \right).
\]

Combining with the definition of \( u_{n+1/2} \), see \((132)\), we can thus compute the relation

\[
\begin{align*}
u_{n+1/2} - S_{\partial_\nu} u_n &= S_{\partial_\nu} \varepsilon_2^n + [\partial_\nu, S_{\partial_\nu}] \Psi_n + \overline{\partial_{x_1}} [S_{\partial_\nu}] \Psi_n \\
&+ \left( (\varepsilon_a + \nu_n) \partial_{x_1} \Psi_n - S_{\partial_\nu} \left( (\varepsilon_a + \nu_n) \partial_{x_1} \Psi_n \right) \right) + (S_{\partial_\nu} \nu_n) \partial_{x_1} \Phi_a - S_{\partial_\nu} \left( (\varepsilon_a + \nu_n) \partial_{x_1} \Phi_a \right).
\end{align*}
\]
We thus need to control the regularized error $S_{\theta_n} \varepsilon^2$, and a sum of commutators. We begin with the regularized error $S_{\theta_n} \varepsilon^2$. Using the decomposition:

$$\varepsilon^2_n = \mathcal{E}(V_{n-1}, \Psi_{n-1}) + \partial_1(\delta \Psi_{n-1}) + (\delta v^a + v_{n-1})\partial_1(\delta \Psi_{n-1}) + \delta v_{n-1}\partial_1(\Phi^a + \Psi_n) - \delta u_{n-1},$$

and exploiting point d) of $(H_n - 1)$, we can prove the estimate\(^5\)

$$\|\varepsilon^2_n\|_{H^1_0(\Omega_T)} \leq C \delta \theta_n^{2-\alpha}.$$

Then regularized error is controled thanks to the properties of the smoothing operators:

$$\|S_{\theta_n} \varepsilon^2_n\|_{H^*_2(\Omega_T)} \leq C \theta_n^{\varepsilon-3} \varepsilon^2_2\|_{H^*_2(\Omega_T)} \leq C \delta \theta_n^{\varepsilon-\alpha-1}, \quad s \in [3, \bar{\alpha} + 3].$$

We detail now the estimate of the most involved commutator. (The other commutators can be estimated by following the same approach). Assume first that $s \in [\alpha + 1, \bar{\alpha} + 3]$. First we have

$$\|((\dot{v}^a + S_{\theta_n}v_n) \partial_{x_1}, S_{\theta_n} \Psi_n)\|_{H^*_2(\Omega_T)} \leq C \|\dot{v}^a + S_{\theta_n}v_n\| L^\infty(\Omega_T) \|S_{\theta_n} \Psi_n\|_{H^*_2(\Omega_T)} + C \|S_{\theta_n} \Psi_n\| W^{1,\infty}(\Omega_T) \|\dot{v}^a + S_{\theta_n}v_n\|_{H^*_2(\Omega_T)}$$

For $s \in [\alpha + 1, \bar{\alpha} + 3]$, we also have

$$\|S_{\theta_n}((\dot{v}^a + v_n) \partial_{x_1}, \Psi_n)\|_{H^*_2(\Omega_T)} \leq C \theta_n^{\varepsilon-\alpha} \|((\dot{v}^a + v_n) \partial_{x_1}, \Psi_n)\| H^*_2(\Omega_T)$$

Consequently, the triangle inequality yields

$$\|((\dot{v}^a + S_{\theta_n}v_n) \partial_{x_1}, S_{\theta_n} \Psi_n - S_{\theta_n}((\dot{v}^a + v_n) \partial_{x_1}, \Psi_n))\|_{H^*_2(\Omega_T)} \leq C \delta^2 \theta_n^{s+1-\alpha}.$$

We are now going to prove that this estimate is also valid for $s \in [3, \alpha]$. To prove this, we use the triangle inequality:

$$\|((\dot{v}^a + S_{\theta_n}v_n) \partial_{x_1}, S_{\theta_n} \Psi_n - S_{\theta_n}((\dot{v}^a + v_n) \partial_{x_1}, \Psi_n))\|_{H^*_2(\Omega_T)} \leq \|((\dot{v}^a + S_{\theta_n}v_n) \partial_{x_1}, S_{\theta_n} \Psi_n))\|_{H^*_2(\Omega_T)}$$

For each term of the right-hand side, we use the properties of the smoothing operators, as well as the classical tame estimate for a product. This enables us to prove that (134) also holds for $s \in [3, \alpha]$. The details are similar to what was done before, so we omit them. Applying the same strategy to the other commutators in (133), we can complete the proof of (129c).

### 7.5 Estimate of the second substitution errors

Now we may estimate the second substitution errors $\epsilon^m_k$, and $\tilde{\epsilon}^m_k$ of the iterative scheme, that are defined by

$$\epsilon^m_k := \mathcal{L}'(S_{\theta_k} V_k, S_{\theta_k} \Psi_k, (\frac{\partial}{\partial x})^2 V_k, \delta \Psi_k) - \mathcal{L}'(V_{k+1/2}, \Psi_{k+1/2}, (\frac{\partial}{\partial x})^2 V_{k+1/2}, \delta \Psi_{k+1/2}),$$

$$\tilde{\epsilon}^m_k := \mathcal{E}'(S_{\theta_k} V_k, S_{\theta_k} \Psi_k, (\frac{\partial}{\partial x})^2 V_k, \delta \Psi_k) - \mathcal{E}'(V_{k+1/2}, \Psi_{k+1/2}, (\frac{\partial}{\partial x})^2 V_{k+1/2}, \delta \Psi_{k+1/2}),$$

$$\tilde{\epsilon}^{m'}_k := \mathcal{B}'((S_{\theta_k} V_k)_{\mid x_2=0}, \psi^2_k, ((\frac{\partial}{\partial x})^2 V_k)_{\mid x_2=0}, \delta \psi_k) - \mathcal{B}'((V_{k+1/2})_{\mid x_2=0}, \psi_{k+1/2})((\frac{\partial}{\partial x})^2 V_{k+1/2})_{\mid x_2=0}, \delta \psi_k).$$

\(^5\)Observe that the estimate does not imply point d) of $(H_n)$ since there is a constant $C > 1$ in front of $\delta$!
Lemma 10. Let \( \alpha \geq 4 \). There exist \( \delta > 0 \) sufficiently small, and \( \theta_0 \geq 1 \) sufficiently large such that for all \( k = 0, \ldots, n - 1 \), and for all integer \( s \in [3, \tilde{\alpha} - 1] \), one has \( \tilde{e}''_k = e''_k = 0 \), and
\[
\| e''_k \|_{H^2_k(O_T)} \leq C \delta^2 \theta_k^{L_3(s)-1} \Delta_k ,
\]
where \( L_3(s) := \max\{(s + 1 - \alpha)_+ + 8 - 2\alpha; s + 5 - 2\alpha\} \).

Proof. Using the expression (129a), the substitution error given in (135a) may be written as
\[
e''_k = \int_0^1 \mathcal{L}''(U^a + V_{k+1/2} + \tau(S\theta_k V_k - V_{k+1/2}), \Phi^a + S\theta_k \Psi_k)
\]
\[
= ((\delta V_k, \delta \Psi_k), (S\theta_k V_k - V_{k+1/2}, 0)) d\tau .
\]
From Sobolev’s imbedding Theorem, Lemma 7, and Proposition 7, we have
\[
\sup_{\tau \in [0,1]} \| (\dot{U}^a + V_{k+1/2} + \tau(S\theta_k V_k - V_{k+1/2}), \dot{\Phi}^a + S\theta_k \dot{\Psi}_k) \|_{W^{1,\infty}(O_T)} \leq C \delta ,
\]
so we can apply Proposition 5, provided that \( \delta \) is small enough. Using Lemma 7, and Proposition 7, we can first derive the bound
\[
\| (\dot{U}^a + V_{k+1/2} + \tau(S\theta_k V_k - V_{k+1/2}), \dot{\Phi}^a + S\theta_k \dot{\Psi}_k) \|_{H^{s+1}_k(O_T)} \leq C \delta \theta_k^{(s+1-\alpha)_+ + 1}, \quad s \in [3, \tilde{\alpha} - 1] .
\]
Consequently, for \( s \in [3, \tilde{\alpha} - 1] \), Proposition 5 yields
\[
\| e''_k \|_{H^2_k(O_T)} \leq C \left\{ \delta \theta_k^{(s+1-\alpha)_+ + 1} \delta \theta_k^{2-\alpha} \Delta_k \delta \theta_k^{2-\alpha} \alpha \Delta_k \right\} \leq C \delta^2 \theta_k^{L_3(s)-1} \Delta_k ,
\]
which is (136).

The substitution error given in (135c) may be written as
\[
\tilde{e}''_k = \mathbb{B}''((\delta V_k)_{x_2=0}, \delta \psi_k), ((S\theta_k V_k - V_{k+1/2})_{x_2=0}, 0)) .
\]
Using the exact expression of \( \mathbb{B}'' \), see Proposition 5, and using also the equality \( v_{k+1/2} = S\theta_k v_k \), we get \( \tilde{e}''_k = 0 \). The same argument applies to \( \tilde{e}''_k \).

\( \Box \)

\section{7.6 Estimate of the last error term}

In our iterative scheme we have a last error term to be estimated, namely
\[
D_{k+1/2} \delta \Psi_k := \frac{\delta \Psi_k}{\partial x_2} \left( \Phi^a + \Psi_{k+1/2} \right) \left\{ \mathcal{L}(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2}) \right\} ,
\]
which results from the introduction of the good unknown in the decomposition of the linearized equations, see (96). Let us set
\[
R_k := \partial x_2 \left\{ \mathcal{L}(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2}) \right\} .
\]
Since \( V_{k+1/2} \) and \( \Psi_{k+1/2} \) vanish in the past, \( R_k \) does not vanish in the past. However, \( \delta \Psi_k \) vanishes in the past, so the error term \( D_{k+1/2} \delta \Psi_k \) also vanishes in the past.

From (117), (119a), and (129a), we have
\[
\left| \partial x_2 (\Phi^a + \Psi_{k+1/2}) \right| \geq \frac{1}{2} ,
\]
44
provided that δ is small enough. Then Theorem 9 enables us to obtain:

\[ \|D_{k+1/2} \delta \Psi_k\|_{L^2(\Omega_T)} = \|D_{k+1/2} \delta \Psi_k\|_{L^2(\Omega_T^+)} \leq C \left\{ \|\delta \Psi_k\|_{H^1(\Omega_T^+)} \right\} \]

\[ + \|\delta \Psi_k\|_{L^\infty(\Omega_T^+)} \left\{ \|R_k\|_{L^2(\Omega_T^+)} + \|R_k\|_{L^\infty(\Omega_T^+)} \|\hat{\Phi}^a + \Psi_{k+1/2}\|_{L^2(\Omega_T^+)} \right\} \} \]  \quad (137)

In order to estimate \( R_k \) on \( \Omega_T^+ \), we proceed as in [1]. For \( t > 0 \), we have the following decomposition:

\[ L(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2}) = L(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2}) - L(U^a + V_k, \Phi^a + \Psi_k) \]

\[ + L(V_k, \Psi_k) - f^a. \]  \quad (138)

Using this decomposition, we are going to prove the following:

**Lemma 11.** Let \( \alpha \geq 4, \bar{\alpha} \geq \alpha + 2 \). For \( \delta > 0 \) sufficiently small, \( \theta_0 \geq 1 \) sufficiently large, for all \( k = 0, \ldots, n - 1 \), and for all integer \( s \in [3, \bar{\alpha} - 2] \), one has

\[ \|R_k\|_{H^2(\Omega_T^+)} \leq C \delta \left( \theta_{k}^{s+3+\alpha} + \theta_{k}^{s+2+\alpha} \right). \]  \quad (139)

**Proof.** If \( s \leq \bar{\alpha} - 3 \), the induction assumption \((H_{n-1})\) yields

\[ \|L(V_k, \Psi_k) - f^a\|_{H^{s+1}(\Omega_T^+)} \leq 2 \delta \theta_{k}^{s-\alpha}. \]  \quad (140)

For the remaining terms in the right-hand side of (138), we write:

\[ L(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2}) - L(U^a + V_k, \Phi^a + \Psi_k) = \]

\[ \int_0^1 L'(U^a + V_k + \tau(V_{k+1/2} - V_k), \Phi^a + \Psi_k + \tau(\Psi_{k+1/2} - \Psi_k))(V_{k+1/2} - V_k, \Psi_{k+1/2} - \Psi_k) d\tau. \]

We have

\[ \sup_{\tau \in [0,1]} \|(U^a + V_k + \tau(V_{k+1/2} - V_k), \Phi^a + \Psi_k + \tau(\Psi_{k+1/2} - \Psi_k))\|_{W^{1,\infty}(\Omega_T^+)} \leq C \delta, \]

so for \( \delta \) small enough, we have an estimate for \( L' \) that is entirely similar to the estimate for the second derivative \( L'' \), namely:

\[ \|L'(U^a + V_k + \tau(V_{k+1/2} - V_k), \Phi^a + \Psi_k + \tau(\Psi_{k+1/2} - \Psi_k))(V_{k+1/2} - V_k, \Psi_{k+1/2} - \Psi_k)\|_{H^{s+1}(\Omega_T^+)} \]

\[ \leq C \left\{ \|(V_{k+1/2} - V_k, \Psi_{k+1/2} - \Psi_k)\|_{H^{s+2}(\Omega_T^+)} + \|(V_{k+1/2} - V_k, \Psi_{k+1/2} - \Psi_k)\|_{W^{1,\infty}(\Omega_T^+)} \times \right\} \]

\[ \|(U^a + V_k + \tau(V_{k+1/2} - V_k), \Phi^a + \Psi_k + \tau(\Psi_{k+1/2} - \Psi_k))\|_{H^{s+2}(\Omega_T^+)} \} \].

Using Proposition 7, and Lemma 7, we obtain the following estimate for \( s \in [3, \bar{\alpha} - 3] \):

\[ \|L'(U^a + V_k + \tau(V_{k+1/2} - V_k), \Phi^a + \Psi_k + \tau(\Psi_{k+1/2} - \Psi_k))(V_{k+1/2} - V_k, \Psi_{k+1/2} - \Psi_k)\|_{H^{s+1}(\Omega_T^+)} \]

\[ \leq C \delta \left( \theta_{k}^{s+3+\alpha} + \theta_{k}^{s+2+\alpha} \right). \]

Combining with (140), and using the decomposition (138), we have already proved (139) for \( s \in [3, \bar{\alpha} - 3] \). We now treat the case \( s = \bar{\alpha} - 2 \). We have

\[ \|R_k\|_{H^1(\Omega_T^+)} \leq \|L(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2})\|_{H^{s+1}(\Omega_T^+)} \]

\[ \leq C \|(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2})\|_{H^{s+2}(\Omega_T^+)} \leq C \delta \theta_{k}^{s+3+\alpha}, \]

and the proof of (139) is complete. \( \square \)
We are ready to prove the estimate of the last error term:

**Lemma 12.** Let \( \alpha \geq 5, \bar{\alpha} \geq \alpha + 2 \). There exist \( \delta > 0 \) sufficiently small, and \( \theta_0 \geq 1 \) sufficiently large such that for all \( k = 0, \ldots, n - 1 \), and for all integer \( s \in [3, \bar{\alpha} - 2] \), one has

\[
\| D_{k+1/2} \delta \Psi_k \|_{H^s_\omega(\Omega_T)} \leq C \delta^2 \theta_k^{L(s)-1} \Delta_k ,
\]

where \( L(s) := \max\{(s + 2 - \alpha)_+ + 8 - 2\alpha; (s + 1 - \alpha)_+ + 9 - 2\alpha; s + 6 - 2\alpha\} \).

**Proof.** We first use Lemma 11 to derive the bound:

\[
\| R_k \|_{L^\infty(\Omega_T^+)} \leq C \delta \theta_k^{6-\alpha} .
\]

We combine this \( L^\infty \) bound and (139) in (137). The terms in \( \delta \Psi_k \) are estimated by the induction assumption \((H_{n-1})\), and the terms in \( \Psi_{k+1/2} = S_{\theta_k} \Psi_k \) are estimated by Lemma 7. Putting all these estimates together yields (141). \( \square \)

### 7.7 Convergence of the iteration scheme

We first estimate the errors \( e_k, \hat{e}_k, \) and \( \tilde{e}_k \):

**Lemma 13.** Let \( \alpha \geq 5 \). There exist \( \delta > 0 \) sufficiently small, and \( \theta_0 \geq 1 \) sufficiently large, such that for all \( k = 0, \ldots, n - 1 \) and all integer \( s \in [3, \bar{\alpha} - 2] \), one has

\[
\| e_k \|_{H^s_\omega(\Omega_T)} + \| \hat{e}_k \|_{H^s_\omega(\Omega_T)} \leq C \delta^2 \theta_k^{L(s)-1} \Delta_k ,
\]

\[
\| \tilde{e}_k \|_{H^s_\omega(\Omega_T)} \leq C \delta^2 \theta_k^{3-2\alpha} \Delta_k ,
\]

where \( L(s) \) is defined in Lemma 12.

**Proof.** We recall that \( e_k, \hat{e}_k, \tilde{e}_k \) are defined in (98) as the sum of all the error terms of the \( k \)-th step. Adding the estimates (124), (128), (136) and (141), we obtain (142). \( \square \)

The preceding Lemma immediately yields the estimate of the accumulated errors \( E_n \), and \( \hat{E}_n \):

**Lemma 14.** Let \( \alpha \geq 7, \bar{\alpha} = \alpha + 4, \) and \( r = \alpha + 1 \). There exist \( \delta > 0 \) sufficiently small, \( \theta_0 \geq 1 \) sufficiently large, such that

\[
\| E_n \|_{H^r_\omega(\Omega_T)} + \| \hat{E}_n \|_{H^r_\omega(\Omega_T)} \leq C \delta^2 \theta_n , \quad \| \hat{E}_n \|_{H^r_\omega(\Omega_T)} \leq \delta^2 .
\]

**Proof.** When \( s = r + 1 = \alpha + 2 \), we get \( L(r+1) \leq 1 \), so (142) yields

\[
\| E_n \|_{H^r_\omega(\Omega_T)} \leq \sum_{k=0}^{n-1} \| e_k \|_{H^r_\omega(\Omega_T)} \leq \sum_{k=0}^{n-1} C \delta^2 \Delta_k \leq C \delta^2 \theta_n .
\]

The estimate of \( \hat{E}_n \) is similar.

Using (142), we have

\[
\| \hat{E}_n \|_{H^r_\omega(\Omega_T)} \leq \sum_{k=0}^{n-1} \| \hat{e}_k \|_{H^r_\omega(\Omega_T)} \leq \sum_{k=0}^{n-1} C \delta^2 \theta_k^2 \Delta_k \leq \frac{C \delta^2}{\theta_0} \leq \delta^2 .
\]

\( \square \)
Going on with the iteration scheme, the next Lemma gives the estimates of the source terms $f_n, g_n, h_n^\pm$, defined by equations (100), (107), and (108):

**Lemma 15.** Let $\alpha \geq 7$, and let $\tilde{\alpha}, r$ be given as in Lemma 14. There exist $\delta > 0$ sufficiently small and $\theta_0 \geq 1$ sufficiently large, such that for all integer $s \in [3, \tilde{\alpha} + 1]$, one has

\[
\|f_n\|_{H^s_t(\Omega_T)} \leq C_n \Delta_n \left\{ \theta_n^{s-\alpha-2} \left( \|f^a\|_{H^{s+1}_x(\Omega_T)} + \delta^2 \right) + \delta^2 \theta_n^{L(s)-1} \right\},
\]

(144a)

\[
\|g_n\|_{H^s_t(\omega_T)} \leq C \delta^2 \Delta_n \left( \theta_n^{L(s)-1} + \theta_n^{s-\alpha-2} \right),
\]

(144b)

and for all integer $s \in [3, \tilde{\alpha}]$, one has

\[
\|h_n\|_{H^s_t(\Omega_T)} \leq C \delta^2 \Delta_n \left( \theta_n^{L(s)-1} + \theta_n^{s-\alpha-2} \right).
\]

(145)

**Proof.** From (100) we have

\[
f_n = (S_{\theta_n} - S_{\theta_{n-1}}) f^a - (S_{\theta_n} - S_{\theta_{n-1}}) E_{n-1} - S_{\theta_n} e_{n-1}.
\]

Using (92c), (143), we prove

\[
\|(S_{\theta_n} - S_{\theta_{n-1}}) f^a\|_{H^s_t(\Omega_T)} \leq C \theta_n^{s-\alpha-2} \|f^a\|_{H^{s+1}_x(\Omega_T)} \Delta_n - 1,
\]

and

\[
\|(S_{\theta_n} - S_{\theta_{n-1}}) E_{n-1}\|_{H^s_t(\Omega_T)} \leq C \theta_n^{s-\alpha-2} \|E_{n-1}\|_{H^{s+1}_x(\Omega_T)} \Delta_n - 1
\]

\[
\leq C \delta^2 \theta_n^{L(s)-1} \Delta_n - 1 = C \delta^2 \theta_n^{s-\alpha-2} \Delta_n - 1.
\]

From (92), and (142) we have

\[
\|S_{\theta_n} e_{n-1}\|_{H^s_t(\Omega_T)} \leq C \delta^2 \theta_n^{L(s)-1} \Delta_n - 1.
\]

Collecting the above estimates gives (144a), with $\Delta_n - 1, \theta_n - 1$ instead of $\Delta_n, \theta_n$. Using $\theta_n - 1 \leq \theta_n \leq \sqrt{2}\theta_n - 1$, and $\Delta_n - 1 \leq 3 \Delta_n$, yields (144a). Estimate (144b) follows in the same way.

To prove (145), we use (107) to derive:

\[
h_n^+ = (S_{\theta_n} - S_{\theta_{n-1}}) (\mathcal{R}_\Omega \tilde{E}_{n-1,2} - \tilde{E}_{n-1}^+) + S_{\theta_n} (\mathcal{R}_\Omega \tilde{e}_{n-1,2} - \tilde{e}_{n-1}^+).
\]

Then we use (142), and (143) as above. The estimate of $h_n^-$ is the same. 

We now consider problem (101), that gives the solution $(\delta\tilde{V}_n, \delta\psi_n)$. Then we find $\Psi_{n+1}^\pm$, and consequently $(\delta V_n, \delta\Psi_n)$:

**Lemma 16.** Assume $\alpha \geq 7$. If $\delta > 0$ and $\|f^a\|_{H^{s+1}_x(\Omega_T)}/\delta$ are sufficiently small, $\theta_0 \geq 1$ is sufficiently large, then for all $3 \leq s \leq \tilde{\alpha}$, one has

\[
\|(\delta V_n, \delta\Psi_n)\|_{H^s_t(\Omega_T)} + \|\delta\psi_n\|_{H^{s+1}_x(\omega_T)} \leq \delta \theta_n^{s-\alpha-1} \Delta_n.
\]

(146)

**Proof.** Let us consider problem (101), which may be solved because $U + V_{n+1/2}, \Phi^a + \Psi_{n+1/2}$ satisfy the required constraints, in particular the eikonal equations (17), and the Rankine-Hugoniot jump conditions (16). Note that the constraint (15) can be obtained by truncating the coefficients $(V_{n+1/2}, \Psi_{n+1/2}, \psi_{n+1/2})$ by a suitable cut-off function. This truncation does not affect the coefficients on the set $\{ t \in [0, T], x_2 \geq 0, \sqrt{x_1^2 + x_2^2} \leq 1 + 3 \lambda_{\text{max}} T/2 \}$. We can thus consider coefficients with a fixed compact support. In order to apply Proposition 6, let us first
verify (114). By the classical trace estimate, (117), (119a), and (129), we have (note that we
use \( \alpha \geq 7 \)):
\[
\begin{align*}
\| (\dot{U}^a + V_{n+1/2}, \nabla (\dot{\Phi}^a + \Psi_{n+1/2})) \|_{H^\gamma_3(\Omega_T)} + \\
\| (\dot{U}^a + V_{n+1/2}, \partial_{x_2}(\dot{U}^a + V_{n+1/2}), \nabla (\dot{\Phi}^a + \Psi_{n+1/2})) \|_{L^2(0,T)}
\end{align*}
\]
\[
\leq C \| (\dot{U}^a + V_{n+1/2}, \dot{\Phi}^a + \Psi_{n+1/2}) \|_{H^\gamma_3(\Omega_T)} \leq C \delta .
\]
Taking \( \delta \) sufficiently small yields (114). Thus we may apply estimate (115) in order to obtain
\[
\| \delta \dot{V}_n \|_{H^\gamma_3(\Omega_T)} + \| \delta \dot{\psi}_n \|_{H^{s+1}_3(\omega_T)} \leq C \left\{ \| f_n \|_{H^{s+1}_3(\Omega_T)} + \| g_n \|_{H^{s+1}_3(\omega_T)} \right. \\
+ \left( \| f_n \|_{H^2_3(\Omega_T)} + \| g_n \|_{H^{s}_3(\omega_T)} \right) \| (\dot{U}^a + V_{n+1/2}, \dot{\Phi}^a + \Psi_{n+1/2}) \|_{H^{s+3}_3(\Omega_T)} \right\} .
\]
(147)
The particular case \( s = 3 \) yields
\[
\| \delta \dot{V}_n \|_{H^3_3(\Omega_T)} \leq C (\| f_n \|_{H^3_2(\Omega_T)} + \| g_n \|_{H^3_3(\omega_T)}) .
\]
(148)
Given \( \delta \psi_n \), we determine \( \delta \Psi_n \) from the equations (104), and (105). We can perform energy
estimates for \( \delta \Psi_n \). These estimates are obtained by following what was done earlier for the
vorticity equation. For simplicity, we drop the \( \pm \) subscripts. For \( s \in [3, \alpha] \), the energy estimate reads:
\[
\gamma \| \delta \Psi_n \|_{H^s_3(\Omega_T)} \leq C \left\{ \| g_n \|_{H^2_3(\omega_T)} + \| h_n \|_{H^s_3(\Omega_T)} + \| \delta \tilde{u}_n \|_{H^s_3(\Omega_T)} + \left\| \partial_{x_1}(\Phi^a + S_{\theta_n} \Psi_n) \delta \tilde{v}_n \right\|_{H^s_3(\Omega_T)} \\
+ \left\| \dot{U}^a + S_{\theta_n} \dot{u}_n \right\|_{W^{1,\infty}(\Omega_T)} \| \delta \Psi_n \|_{H^s_3(\Omega_T)} + \| \delta \Psi_n \|_{W^{1,\infty}(\Omega_T)} \| \dot{U}^a + S_{\theta_n} \dot{v}_n \|_{H^s_3(\Omega_T)} \\
+ \| \text{coeff} \|_{L^\infty(\Omega_T)} \| \delta \Psi_n \|_{H^s_3(\Omega_T)} + \| \delta \Psi_n \|_{L^\infty(\Omega_T)} \| \text{coeff} \|_{H^s_4(\Omega_T)} \right\} ,
\]
where “coeff” denotes the zero order coefficient in (104), or (105). Choosing \( \delta \) small, and using
Lemma 7, Proposition 7, together with Sobolev’s imbedding Theorem, we end up with
\[
\gamma \| \delta \Psi_n \|_{H^s_3(\Omega_T)} \leq C \left\{ \| g_n \|_{H^2_3(\omega_T)} + \| h_n \|_{H^s_3(\Omega_T)} + \| \delta \dot{V}_n \|_{H^s_3(\Omega_T)} \\
+ \| \delta \dot{V}_n \|_{H^s_3(\Omega_T)} \| \Phi^a + S_{\theta_n} \Psi_n \|_{H^{s+1}_3(\Omega_T)} + \delta \theta_n^{(s+2)\alpha} \| \delta \Psi_n \|_{H^s_3(\Omega_T)} \right\} ,
\]
(149)
for all integer \( s \in [3, \alpha] \). Choosing \( s = 3 \), and using (148), we obtain
\[
\| \delta \Psi_n \|_{H^3_3(\Omega_T)} \leq C \left( \| f_n \|_{H^3_2(\Omega_T)} + \| h_n \|_{H^3_3(\Omega_T)} + \| g_n \|_{H^3_3(\omega_T)} \right) ,
\]
(150)
provided that \( \delta \) is small enough. Therefore, we can combine (149), (150), and (147):
\[
\begin{align*}
\| (\delta \dot{V}_n, \delta \Psi_n) \|_{H^s_3(\Omega_T)} + \| \delta \psi_n \|_{H^{s+1}_3(\omega_T)} & \leq C \left\{ \| f_n \|_{H^{s+1}_3(\Omega_T)} + \| h_n \|_{H^s_3(\Omega_T)} + \| g_n \|_{H^{s+1}_3(\omega_T)} \right. \\
+ \left( \| f_n \|_{H^3_3(\Omega_T)} + \| h_n \|_{H^3_3(\Omega_T)} + \| g_n \|_{H^3_3(\omega_T)} \right) \left( \delta \theta_n^{(s+2)\alpha} + \| \dot{U}^a + V_{n+1/2}, \dot{\Phi}^a + \Psi_{n+1/2} \|_{H^{s+3}_3(\Omega_T)} \right) \right\} .
\end{align*}
\]
(151)
The increment \( \delta \dot{V}_n \) is obtained by the formula (95), which yields for \( s \in [3, \alpha] \):
\[
\| \delta \dot{V}_n \|_{H^3_3(\Omega_T)} \leq C \| (\delta \dot{V}_n, \delta \Psi_n) \|_{H^3_3(\Omega_T)} + \| \delta \Psi_n \|_{H^3_3(\Omega_T)} \| (\dot{U}^a + V_{n+1/2}, \dot{\Phi}^a + \Psi_{n+1/2}) \|_{H^{s+1}_3(\Omega_T)} .
\]
Together with (151), this enables us to get

$$\|(\delta V_n, \delta \Psi_n)\|_{H^s_\gamma(\Omega_T)} + \|\delta \psi_n \|_{H^{s+1}_\gamma(\omega_T)} \leq C \left\{ \|f_n\|_{H^{s+1}_\gamma(\Omega_T)} + \|h_n\|_{H^{s+1}_\gamma(\Omega_T)} + \|g_n\|_{H^{s+1}_\gamma(\omega_T)} \right\} + (\|f_n\|_{H^1_\gamma(\Omega_T)} + \|h_n\|_{H^2_\gamma(\Omega_T)} + \|g_n\|_{H^3_\gamma(\omega_T)}) \left( \delta \theta_n^{(s+2-\alpha)+} + \|\bar{U} + V_{n+1/2}, \Phi^\alpha + \Psi_{n+1/2}\|_{H^{s+3}_\gamma(\Omega_T)} \right),$$

(152)

for all integer $s \in [3, \bar{\alpha}]$. The remaining part of the work is to estimate the right-hand side of (152). Using Lemma 15, (119a), and Proposition 7, (152) becomes

$$\|(\delta V_n, \delta \Psi_n)\|_{H^s_\gamma(\Omega_T)} + \|\delta \psi_n \|_{H^{s+1}_\gamma(\omega_T)} \leq C \left\{ \theta_n^{\gamma-\alpha-1} \left( \|f_n\|_{H^{s+1}_\gamma(\Omega_T)} + \|\phi_n\|_{H^{s+1}_\gamma(\Omega_T)} + \delta^2 \right) + \delta^2 \theta_n^{L(s+1)-1} \right\} \Delta_n$$

$$+ C \delta \Delta_n \left( \theta_n^{2-\alpha} \left( \|f_n\|_{H^{s+1}_\gamma(\Omega_T)} + \delta^2 \right) + \delta^2 \theta_n^{s-2\alpha} \right) \left( \theta_n^{s+3-\alpha} + \theta_n^{s+4-\alpha} \right).$$

(153)

One checks that, for $\alpha \geq 7$, and $s \in [3, \bar{\alpha}]$, the following inequalities hold true:

\[
\begin{align*}
L(s+1) & \leq s - \alpha, \\
(s+3-\alpha)_+ + 2 - \alpha & \leq s - \alpha - 1, \\
(s+3-\alpha)_+ + 9 - 2\alpha & \leq s - \alpha - 1, \\
s + 13 - 3\alpha & \leq s - \alpha - 1.
\end{align*}
\]

From (153), we thus obtain

$$\|(\delta V_n, \delta \Psi_n)\|_{H^s_\gamma(\Omega_T)} + \|\delta \psi_n \|_{H^{s+1}_\gamma(\omega_T)} \leq C \left\{ \|f_n\|_{H^{s+1}_\gamma(\Omega_T)} + \delta^2 \right\} \theta_n^{s-\alpha-1} \Delta_n,$$

and (146) follows.

We now check the three remaining inequalities in (Hn).

**Lemma 17.** Assume $\alpha \geq 7$. If $\delta > 0$ is sufficiently small, $\|f_n\|_{H^{s+1}_\gamma(\Omega_T)} / \delta$ is sufficiently small, and $\theta_0 \geq 1$ is sufficiently large, then for all $3 \leq s \leq \bar{\alpha} - 2$, one has

$$\|L(V_n, \Psi_n) - f_n\|_{H^s_\gamma(\Omega_T)} \leq 2 \delta \theta_n^{s-\alpha-1}.$$  

(154)

**Proof.** Recall that, by summing the relations (109), we have

$$L(V_n, \Psi_n) - f_n = (S_{\theta_{n-1}} - I)f_n + (1 - S_{\theta_{n-1}})E_{n-1} + e_{n-1}.$$

Therefore we get

$$\|L(V_n, \Psi_n) - f_n\|_{H^s_\gamma(\Omega_T)} \leq \|(I - S_{\theta_{n-1}})f_n\|_{H^s_\gamma(\Omega_T)} + \|(I - S_{\theta_{n-1}})E_{n-1}\|_{H^s_\gamma(\Omega_T)} + \|e_{n-1}\|_{H^s_\gamma(\Omega_T)}.$$

If $\alpha + 1 \leq s \leq \bar{\alpha} - 2$, we prove from (92a):

$$\|(I - S_{\theta_{n-1}})f_n\|_{H^s_\gamma(\Omega_T)} \leq C \theta_n^{s-\alpha-1} \|f_n\|_{H^{s+1}_\gamma(\Omega_T)} + \|f_n\|_{H^{s+2}_\gamma(\Omega_T)} \leq \theta_n^{s-\alpha-1} \left( C \|f_n\|_{H^{s+1}_\gamma(\Omega_T)} + \delta \right).$$

In case $s \leq \alpha + 1$, (92b) yields

$$\|(I - S_{\theta_{n-1}})f_n\|_{H^s_\gamma(\Omega_T)} \leq C \theta_n^{s-\alpha-1} \|f_n\|_{H^{s+1}_\gamma(\Omega_T)} \leq C \theta_n^{s-\alpha-1} \|f_n\|_{H^{s+1}_\gamma(\Omega_T)}.$$

Recall that our choice of the integer $r$ is such that $r+1 = \bar{\alpha} - 2$, see Lemma 14. Consequently, for $s \leq \bar{\alpha} - 2$, we prove from (92b), and (143):

$$\|(I - S_{\theta_{n-1}})E_{n-1}\|_{H^s_\gamma(\Omega_T)} \leq C \theta_n^{s-r-1} \|E_{n-1}\|_{H^{s+1}_\gamma(\Omega_T)} \leq C \delta^2 \theta_n^{s-\alpha-1} \leq C \theta_n^{s-\alpha-1} \leq C \delta^2 \theta_n^{s-\alpha-1},$$

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where we have used \( r = \alpha + 1 \), see Lemma 14.

From (142) we finally have
\[
\|e_{n-1}\|_{H^{s}(\Omega_{T})} \leq C \delta^{2} \theta_{n-1}^{L(s)-1} \Delta_{n-1} \leq C \delta^{2} \theta_{n-1}^{L(s)-2} \leq C \delta^{2} \theta_{n}^{\alpha-1}.
\]

From the previous estimates, choosing \( \delta > 0 \), and \( \|f^{a}\|_{H^{s+1}(\Omega_{T})}/\delta \) sufficiently small, we obtain (154).

The following Lemma follows exactly with the same arguments:

**Lemma 18.** Let \( \alpha \geq 7 \). If \( \delta > 0 \) is sufficiently small, and \( \theta_{0} \geq 1 \) is sufficiently large, then for all \( 4 \leq s \leq \alpha \), one has
\[
\|B((V_{n})|_{|z| = 0}, \psi_{n})\|_{H^{s}(\omega_{T})} \leq \delta \theta_{n}^{\alpha-1}.
\]

Moreover, one has
\[
\|\partial_{t}\Psi_{n} + (u^{a} + v_{n})\partial_{x_{1}}\Psi_{n} + v_{n}\partial_{x_{2}}\Phi_{n} - u_{n}\|_{H^{s}(\Omega_{T})} \leq \delta \theta_{n}^{2-\alpha}.
\]

**Lemma 19.** If \( \|f^{a}\|_{H^{s+1}(\Omega_{T})}/\delta \) is sufficiently small, then property \((H_{0})\) holds.

**Proof.** Recall that \( V_{0} = \Psi_{0} = \psi_{0} = 0 \). Thanks to the property of the approximate solution (see Lemma 3), we see that \( U^{a} + V_{0}, \Phi^{a} + \Psi_{0}, \varphi^{a} + \psi_{0} \) satisfy the eikonal equations, and the Rankine-Hugoniot jump conditions. Consequently, the contraction of Proposition 7 yields \( V_{1/2} = \Psi_{1/2} = \psi_{1/2} = 0 \).

Consider the problem
\[
\mathbb{L}_{c}'(U^{a}, \Phi^{a})\delta V_{0} = S_{\theta_{0}}f^{a} \quad \text{in } \Omega_{T},
\]
\[
\mathbb{L}_{c}'(U^{a}|_{|x| = 0}, \varphi^{a})(\delta V_{0}|_{|x| = 0}, \delta \psi_{0}) = 0 \quad \text{on } \omega_{T},
\]
\[
\delta V_{0} = 0, \quad \delta \psi_{0} = 0 \quad \text{for } t < 0.
\]

Since \( \|(U^{a}, \Phi^{a})\|_{H^{2}(\Omega_{T})} \leq \delta \) for all \( s \in [3, \bar{\alpha} + 3] \), we may apply (115) and obtain
\[
||\delta V_{0}\|_{H^{s}(\omega_{T})} + ||\delta \psi_{0}\|_{H^{s+1}(\omega_{T})} \leq C ||S_{\theta_{0}}f^{a}\|_{H^{s+1}(\Omega_{T})}.
\]

Then we find \( \delta \Psi_{0}^{\pm} \) from the equations (104), and (105), that read in this case:
\[
\partial_{t}\delta \Psi_{0}^{\pm} + v_{n}^{a} \partial_{x_{1}}\delta \Psi_{0}^{\pm} + \left( \partial_{x_{2}}\Phi_{0}^{a} \frac{\partial x_{2}}{\Phi_{0}^{a}} - \partial_{x_{2}}\frac{\partial x_{2}}{\Phi_{0}^{a}} \right) \delta \Psi_{0}^{\pm} + \partial_{x_{1}}\Phi_{0}^{a} \delta \tilde{u}_{0} - \partial_{x_{1}}\Phi_{0}^{a} \delta \tilde{u}_{0} = 0.
\]

Standard energy estimates yield, as in the proof of Lemma 16:
\[
\forall s \in [3, \bar{\alpha}], \quad ||\delta \psi_{0}\|_{H^{s}(\Omega_{T})} \leq C ||\delta V_{0}\|_{H^{s}(\omega_{T})}.
\]

We finally obtain from (157), and (158):
\[
||\delta V_{0}, \delta \psi_{0}\|_{H^{s}(\omega_{T})} + ||\delta \psi_{0}\|_{H^{s+1}(\omega_{T})} \leq C ||S_{\theta_{0}}f^{a}\|_{H^{s+1}(\Omega_{T})} \leq C \theta_{0}^{(s-\alpha)+} ||f^{a}\|_{H^{s+1}(\Omega_{T})}.
\]

If \( ||f^{a}\|_{H^{s+1}(\Omega_{T})}/\delta \) is sufficiently small, then
\[
||\delta V_{0}, \delta \psi_{0}\|_{H^{s}(\omega_{T})} + ||\delta \psi_{0}\|_{H^{s+1}(\omega_{T})} \leq \delta \theta_{0}^{s-\alpha-1} \Delta_{0},
\]

for all \( 3 \leq s \leq \bar{\alpha} \).

The other inequalities in \((H_{0})\) are readily satisfied by taking \( ||f^{a}\|_{H^{s+1}(\Omega_{T})} \) small enough. The proof is complete.
From Lemmas 16 - 19, we get that \((H_n)\) holds for every \(n \geq 0\), provided that the parameters are well-chosen and that the source term \(f^a\) is small enough.

Conclusion of the proof of Theorem 1.

Given an integer \(\alpha \geq 7\), in agreement with the requirements of Lemma 14, we set \(\tilde{\alpha} = \alpha + 4\). Let \(\mu = \alpha - 1 \geq 6\). Let us consider initial data \(U_0^+ := (\rho_0^+, v_0^+, u_0^+)\) such that \(U_0^+ = \bar{U}^+ + U_0^+\), where \(\bar{U}_0^+ \in H^{\mu+15/2}(\mathbb{R}_+^2)\), \(\varphi_0 \in H^{\mu+8}(\mathbb{R})\), and that satisfy the compatibility conditions up to order \(\mu + 7\). Thanks to Lemmas 2 and 3, we may find an approximate solution \((U^a, \Phi^a)\) such that \(U^a = \bar{U} + U^a\) with \(\bar{U}^a \in H^{\mu+8}(\Omega)\), \(\Phi^a = \pm x_2 + \Phi^a\), with \(\Phi^a \in H^{\mu+3}(\Omega)\), \(\varphi^a \in H^{\mu+17/2}(\omega)\), and \(f^a \in H^{\mu+7}(\Omega)\).

If \(\|U^1_0\|_{H^{\mu+15/2}(\mathbb{R}_+^2)} + \|\varphi_0\|_{H^{\mu+8}(\mathbb{R})}\) is sufficiently small, from (83), (88) we obtain (117), and the requirements of Lemmas 16, 17, 18, 19. Hence for small, compatible initial data, the property \((H_n)\) holds true for all \(n\). In particular, we have:

\[
\sum_{n \geq 0} \| (\delta V_n, \delta \Psi_n) \|_{H^\mu(\Omega_T)} + \| \delta \psi_n \|_{H^{\mu+1}(\omega_T)} < +\infty,
\]

so the sequences \((V_n)\) and \((\Psi_n)\), converge in \(H^\mu(\Omega_T)\) towards some limits \(V\) and \(\Psi\), and the sequence \((\psi_n)\) converges in \(H^{\mu+1}(\omega_T)\) towards some limit \(\psi\). Passing to the limit in (154), (155) for \(s = \mu = \alpha - 1\), and in (156), we obtain (89). Therefore \(\bar{U} = U^a + V, \Phi = \Phi^a + \Psi\) is a solution on \(\Omega_T^\gamma\) of (3), (4), (5), (6), (7), so the proof of Theorem 1 is complete.

**Remark 10.** The smallness conditions on \(\delta\) and \(\|f^a\|_{H^{\mu+1}(\Omega_T)} / \delta\) are satisfied for sufficiently small initial perturbations \((U_0^+, \varphi_0)\) of the piecewise constant vortex sheet (11), see (83) and (88). Notice also that \((U_0^+, \varphi_0)\) should be sufficiently small also for preserving the linearized well-posedness, obtained under the supersonic condition (12), see Theorems 2 and 4.

A The existence of weakly stable shock waves

This first appendix is devoted to another application of the Nash-Moser type iteration that we have developed in this paper. More precisely, we still consider the Euler equations (1), but here the space dimension equals 2 or 3. We still make the assumption \(p'(\rho) > 0\) for all \(\rho > 0\). Let us denote \(d\) the space dimension, and \(x = (y, x_d)\) a generic point of the space \(\mathbb{R}^d\), \(y \in \mathbb{R}^{d-1}\) and \(x_d \in \mathbb{R}\). We also decompose the velocity \(u\) in (1) as \(u = (v, u)\), \(v \in \mathbb{R}^{d-1}\), and \(u \in \mathbb{R}\).

We are interested in shock waves solutions to (1). These are smooth solutions on either side of a hypersurface \(\Gamma = \{x_d = \varphi(t, y), t \in [0, T], y \in \mathbb{R}^{d-1}\}\), and such that at each time \(t \in [0, T]\) and at each point \((y, x_d)\) of the (curve or) surface \(\Gamma(t)\), the following conditions are satisfied:

\[
\begin{align*}
\rho^+ (u^+ - v^+ \cdot \nabla y \varphi - \partial t \varphi) &= \rho^- (u^- - v^- \cdot \nabla y \varphi - \partial t \varphi) =: j, \tag{159a} \\
j (u^+ - u^-) + (p(\rho^+) - p(\rho^-)) \left( -\nabla \varphi \right) = 0, \tag{159b} \\
j > 0, \quad 0 < \frac{u^+ - v^+ \cdot \nabla y \varphi - \partial t \varphi}{c(\rho^+)} \sqrt{1 + |\nabla \varphi|^2} < 1 < \frac{u^- - v^- \cdot \nabla y \varphi - \partial t \varphi}{c(\rho^-)} \sqrt{1 + |\nabla \varphi|^2}. \tag{159c}
\end{align*}
\]

Observe that the conditions (159a), and (159b) are the Rankine-Hugoniot jump conditions for (1). The condition \(j \neq 0\) means that there is a mass transfer from one side of \(\Gamma(t)\) to the other, and (159c) are Lax’ shock inequalities for a 1-shock wave\(^6\). The existence of shock waves is

\(^6\)The study of the 3-shock waves, for which \(j < 0\), can be carried out in exactly the same way, so we shall not deal with it. However, the results of this appendix extend to the case of 3-shock waves.
again a nonlinear free boundary hyperbolic problem: we wish to solve (1) on either side of $\Gamma(t)$, together with the transmission conditions (159a), (159b), and the constraints (159c). In this case, the free boundary is noncharacteristic, and one can straighten it as in [24, 23, 29] with the change of variables:

$$\Phi^\pm(t, y, x_d) := (t, y, \pm \kappa x_d + \chi(\pm x_d) \varphi(t, y)),$$

where $\chi \in C_0^\infty(\mathbb{R})$, equal to 1 on $[-1, 1]$, and $\kappa$ is a constant that satisfies $\kappa > 2\|\chi\|_{L^\infty(\mathbb{R})}$. The cut-off function is introduced in order to work globally on $\{x_d > 0\}$, and we shall consider solutions for which $\|\varphi\|_{L^\infty((0,T)\times\mathbb{R})} \leq 1$. The problem can thus be rewritten as the following nonlinear hyperbolic system:

$$\partial_t U^\pm + \sum_{j=1}^{d-1} A_j(U^\pm) \partial_x U^\pm + \frac{1}{\partial_{x_d} \Phi^\pm} \left(A_d(U^\pm) - \partial_t \Phi^\pm - \sum_{j=1}^{d-1} \partial_x \Phi^\pm A_j(U^\pm)\right) \partial_{x_d} U^\pm = 0,$$

(160)

in $\{x_d > 0\}$, together with the boundary conditions (159) on $\{x_d = 0\}$. The matrices $A_1, \ldots, A_d$ correspond to the quasilinear form of the Euler equations (1) in space dimension $d$, and $U^\pm = (\rho^\pm, u^\pm)$.

Up to Galilean transformations, the planar shock waves, that is, the piecewise constant solutions of (1), (159) have the form

$$(\rho, v, u) = \begin{cases} U_r := (\rho_r, 0, u_r), & \text{if } x_d > 0, \\
U_l := (\rho_l, 0, u_l), & \text{if } x_d < 0, \end{cases}$$

(161)

where

$$\rho_r u_r = \rho_l u_l =: j, \quad j = \sqrt{\rho_r \rho_l \frac{p(\rho_r) - p(\rho_l)}{\rho_r - \rho_l}}, \quad 0 < \frac{u_r}{c(\rho_r)} < 1 < \frac{u_l}{c(\rho_l)}.$$  

The (linear) stability properties of the planar shock wave (161) are encoded in the following result:

**Proposition 8** (Majda [24]). *The shock wave (161) is uniformly stable if, and only if*

$$\frac{u_r^2}{c(\rho_r)^2} \left(\frac{\rho_r}{\rho_l} - 1\right) < 1.$$  

*In particular, when $p$ is a convex function of $\rho$, this inequality always holds.*

In [24, 23], Majda constructs shock waves that are close to a uniformly stable planar shock (see also [31, 29] for a refined version of Majda's result). In other words, Majda solves the equations (160), (159) for initial data that are perturbations of the planar shock (161), and that satisfy the appropriate compatibility conditions. The initial condition for $\varphi$ is a perturbation of 0 since (161) is a stationary shock wave. Here, we prove the local in time existence of weakly stable shock waves, which answers, at least for the isentropic Euler equations, the question asked in the introduction of [29].

When the shock wave (161) satisfies

$$\frac{u_r^2}{c(\rho_r)^2} \left(\frac{\rho_r}{\rho_l} - 1\right) > 1,$$

(162)

the planar shock wave (161) is only weakly stable. However, it is proved in [9] that the linearized problem around a variable coefficients small perturbation of the planar shock (161) satisfies an a priori estimate with a loss of one tangential derivative. (This weak stability result is the analogue of Theorem 2 for contact discontinuities.) Note that for shock waves, the boundary is
noncharacteristic, so one can control the whole trace of the solution to the linearized problem on
the boundary (compare with Theorem 2). Under the assumption (162), the symbol associated
with the linearized front is elliptic (as for uniformly stable shock waves), so one can also control
the $H^1$ norm of the linearized front in the a priori estimate. Moreover, if we consider the dual
problem defined in [29, page 60], then the (backward) weak Lopatinskii condition is satisfied,
the zeroes of the Lopatinskii determinant are simple, and they are located in the hyperbolic
region of the cotangent of the boundary. This enables us to obtain an energy estimate with a
loss of one tangential derivative for the dual problem, which yields a well-posedness result for
the linearized equations (this is the analogue of Theorem 3). At this stage, we can follow the
analysis of section 3 and prove a tame estimate in the Sobolev spaces $H^m(\Omega_T)$. The analysis is
even simpler than what we have done in section 3 because the boundary is noncharacteristic, so
all the normal derivatives can be estimated directly by the tangential derivatives.

To solve the nonlinear problem, it is convenient to proceed as in [29], and to make the
boundary conditions linear. This is possible because in (159), the symbol associated with the
front $\varphi$ is elliptic. Changing unknowns, we can therefore transform (159a), and (159b) into
boundary conditions of the form

$$\begin{pmatrix}
I_d \\
0
\end{pmatrix} \nabla_{t,y} \varphi + M U_{|x_d=0} = 0,$$

where $M$ is a constant matrix of maximal rank. The analysis of the compatibility conditions
proceeds as in [29, section 4.2]. For the iteration scheme, we use the same Nash-Moser scheme
as in the proof of Theorem 1, with the same chain of spaces, and the same family of smoothing
operators. In brief, to prove the existence of weakly stable shock waves, it is sufficient to force
the Rankine-Hugoniot conditions (in their new linear form) during the iteration scheme. This
is particularly simple since $M$ is of maximal rank, so, up to permuting some columns of $M$, we
can write

$$M = \begin{pmatrix} M_1 & M_2 \end{pmatrix},$$

where $M_2$ is a square invertible matrix. Decomposing the vector $U$ accordingly, we can rewrite
the boundary conditions as

$$-M_2^{-1} \begin{pmatrix} I_d \\
0
\end{pmatrix} \nabla_{t,y} \varphi - M_2^{-1} M_1 V_{|x_d=0} = W_{|x_d=0}.$$

Following the analysis of section 7, we can prove that our modified Nash-Moser iteration con-
verges towards a solution to the nonlinear equations (160), (159), and we thus obtain the fol-
lowing result:

**Theorem 5.** Consider a planar shock wave (161) that satisfies the weak stability condition
(162). Let $T > 0$, and let $\mu \in \mathbb{N}$ be sufficiently large. Then there exists an integer $\tilde{\mu} \geq \mu$, such
that if the initial data $(U_{0,0}^{\pm}, \varphi_0)$ have the form

$$U_0^{\pm} = U_{r,l} + \dot{U}_0^{\pm},$$

with $\dot{U}_0^{\pm} \in H^{\tilde{\mu}+1/2}(\mathbb{R}_+^2)$, $\varphi_0 \in H^{\tilde{\mu}+3/2}(\mathbb{R})$, if they are compatible up to order $\tilde{\mu} - 1$, have a
compact support, and are sufficiently small, then there exists a solution $U^{\pm} = U_{r,l} + \dot{U}^{\pm}, \Phi^{\pm}, \varphi$
to (160), (159), on the time interval $[0, T]$. This solution satisfies $U^{\pm} \in H^\mu([0, T]\times \mathbb{R}^{d-1} \times \mathbb{R}^n)$,
$\varphi \in H^{\mu+1}([0, T]\times \mathbb{R}^{d-1})$, and $(\dot{U}^{\pm}, \varphi)_{|t=0} = (\dot{U}_0^{\pm}, \varphi_0)$.
B The existence of subsonic phase transitions in a Van der Waals fluid

In this second appendix, we are interested in a model of isothermal liquid/vapor phase transitions in a van der Waals fluid. We consider the Euler equations (1) in two, or three space dimensions, and we assume that the fluid obeys an isothermal van der Waals pressure law:

\[ p(\rho) = \pi(v) := \frac{RT}{v - b} - \frac{a}{v^2}, \quad v := 1/\rho, \quad (163) \]

where \( R, a, \) and \( b \) are numerical constants, and \( T \) is the fixed temperature of the fluid. When \( T \) is below the critical temperature \( T_c := 8a/(27bR) \), the pressure law \( p \) is nonmonotone: it is increasing on \( [0, \rho_1] \cup [\rho_2, +\infty[ \), and it is decreasing on the interval \( ]\rho_1, \rho_2[ \). The choice of such a pressure law models the coexistence of liquid (\( \rho > \rho_2 \)), and vapor phases (\( \rho < \rho_1 \)), and we are interested in the existence of propagating phase boundaries that connect a liquid, and a vapor state. Recall that the Maxwell points \( (v_M, v_m) \) are uniquely defined by the relations

\[ v_M < v_m, \quad \pi(v_m) = \pi(v_M) = \pi, \quad \int_{v_m}^{v_M} (\pi - \pi(v)) \, dv = 0, \]

and we denote \( \rho_{m,M} = 1/v_{m,M} \), so that \( \rho_m < \rho_1 < \rho_2 < \rho_M \). We also denote \( \epsilon(\rho) \) the free energy per unit volume, that is defined by

\[ \rho \epsilon'(\rho) - \epsilon(\rho) = p(\rho). \]

From now on, we are interested in piecewise smooth solutions of the Euler equations (1), with the pressure law (163), and such that the states on either side of the boundary belong to distinct phases. When the densities on either side of the boundary are close enough to the Maxwell points, it is shown in [4] that Lax’ shock inequalities are not satisfied. More precisely, in the terminology of Freistühler, see [15], such propagating phase boundaries are undercompressive shock waves of type 0. The Rankine-Hugoniot conditions are not sufficient to determine a phase boundary, because there are not enough characteristics impinging the discontinuity. An additional scalar jump condition is required to determine the “admissible” propagating phase boundaries. We refer to [4, 5] and to the references therein for possible admissibility criteria, and we focus here on the so-called capillary criterion that was considered in [4]. We thus want to construct a solution \( U \) to the Euler equations (1), that is smooth on either side of a hypersurface \( \Gamma = \{ x_d = \varphi(t, y) \} \), that satisfies the Rankine-Hugoniot jump conditions at each point of \( \Gamma \):

\[ \rho^+ (u^+ - v^+ \cdot \nabla y \varphi - \partial_t \varphi) = \rho^- (u^- - v^- \cdot \nabla y \varphi - \partial_t \varphi) =: j, \quad (164a) \]
\[ j (u^+ - u^-) + (p(\rho^+) - p(\rho^-)) \left( -\nabla y \varphi \right)_1 = 0, \quad (164b) \]
\[ j > 0, \quad 0 < \frac{u^+ - v^+ \cdot \nabla y \varphi - \partial_t \varphi}{c(\rho^+) \sqrt{1 + |\nabla y \varphi|^2}} < 1, \quad (164c) \]

together with the generalized equal area rule:

\[ \int_{v^-}^{v^+} \pi(v) \, dv = \frac{\pi(v^+) + \pi(v^-)}{2} (v^+ - v^-). \quad (165) \]

Together with (164), the additional jump condition (165) is equivalent to

\[ \left[ \frac{1}{2} (u - v \cdot \nabla y \varphi - \partial_t \varphi)^2 + (1 + |\nabla y \varphi|^2) \epsilon' \right] = 0. \quad (166) \]

In (166), we have used the classical notation \([q] := q^+ - q^-\) to denote the jump of a quantity \( q \) across the discontinuity.
Straightening the unknown interface as in Appendix A, we thus want to construct a smooth solution to the system

$$\partial_t U^\pm + \sum_{j=1}^{d-1} A_j(U^\pm) \partial_{x_j} U^\pm + \frac{1}{\partial_{x_d} \Phi^\pm} \left( A_d(U^\pm) - \partial_t \Phi^\pm - \sum_{j=1}^{d-1} \partial_{x_j} \Phi^\pm A_j(U^\pm) \right) \partial_{x_d} U^\pm = 0, \quad (167)$$

in \( \{x_d > 0\} \), together with the boundary conditions (164), (166) on \( \{x_d = 0\} \). Once again, the matrices \( A_1, \ldots, A_d \) correspond to the quasilinear form of the Euler equations (1) in space dimension \( d \), and \( U^\pm = (\rho^\pm, \mathbf{u}^\pm) \).

For the sake of completeness, we recall the stability result that was obtained in [4]. Consider a planar phase transition

\[
(\rho, v, u) = \begin{cases} 
U_r := (\rho_r, 0, u_r), & \text{if } x_d > 0, \\
U_l := (\rho_l, 0, u_l), & \text{if } x_d < 0,
\end{cases} \quad (168)
\]

that satisfies \( \rho_r > \rho_M, \rho_l < \rho_m, \) and the jump conditions

\[
\rho_r u_r = \rho_l u_l =: j, \quad j = \sqrt{\frac{\rho_r \rho_l p(\rho_r) - p(\rho_l)}{\rho_r - \rho_l}}, \quad 0 < \frac{u_r - u_l}{c(\rho_r)} < 1,
\]

\[
\int_{v_r}^{v_l} \pi(v) \, dv = \frac{p(\rho_r) + p(\rho_l)}{2} (v_r - v_l).
\]

We have the following:

**Theorem 6** (Benzoni-Gavage [4]). There exist planar phase transitions (168), with \( \rho_r, \rho_l \) close enough to \( \rho_M, \rho_m \), and these planar phase transitions are weakly stable. In any case, the uniform Lopatinski condition is not satisfied.

The weak stability phenomenon in Theorem 6 is somehow less critical than the weak stability of shock waves in isentropic gas dynamics, since it is due to surface waves (that decay exponentially fast in the normal direction). These surface waves are similar to the well-known Rayleigh waves in elastodynamics. Moreover, the symbol associated with the unknown front in (164), (166) is elliptic, and as was done in the preceding section, it is possible to reduce the nonlinear boundary conditions to constant linear boundary conditions.

The basic \( L^2 \) estimate for the linearized equations can be obtained by following exactly the method of [9], and one can also prove the well-posedness of the linearized equations by showing that a dual problem satisfies the backward weak Lopatinski condition. Consequently, all the analysis is similar to the analysis of weakly stable shock waves in isentropic gas dynamics, and in short, we can positively answer the question asked in [4] and show the existence of isothermal phase transitions:

**Theorem 7.** Consider a planar phase transition (168), as given in Theorem 6. Let \( T > 0 \), and let \( \mu \in \mathbb{N} \) be sufficiently large. Then there exists an integer \( \bar{\mu} \geq \mu \), such that if the initial data \((U_0^\pm, \varphi_0)\) have the form

\[
U_0^\pm = U_{r,l} + \dot{U}_0^\pm,
\]

with \( \dot{U}_0^\pm \in H^{\bar{\mu}+1/2}(\mathbb{R}_0^2), \varphi_0 \in H^{\bar{\mu}+3/2}(\mathbb{R}) \), if they are compatible up to order \( \bar{\mu} - 1 \), have a compact support, and are sufficiently small, then there exists a solution \( U^\pm = U_{r,l} + \dot{U}^\pm, \varphi \) to (167), (164), (166) on the time interval \([0, T]\). This solution satisfies \( \dot{U}^\pm \in H^\mu([0, T] \times \mathbb{R}^{d-1} \times \mathbb{R}^+), \varphi \in H^{\mu+1}([0, T] \times \mathbb{R}^{d-1} \setminus \{0\}), \) and \( (\dot{U}^\pm, \varphi)|_{t=0} = (U_0^\pm, \varphi_0) \).
C Nonlinear estimates

In this short appendix, we recall some classical nonlinear tame estimates. The reader is referred to [2, 25, 29] for the details. The introduction of the weight $\exp(-\gamma t)$ in the norms is discussed in [29, page 80].

C.1 Nonlinear estimates in weighted spaces

We recall the notations

$$\Omega = \mathbb{R}^2 \times [0, +\infty[, \quad \omega = \partial \Omega = \mathbb{R}^2,$$

$$\Omega_T = [-\infty, T] \times \mathbb{R} \times [0, +\infty[, \quad \omega_T = [-\infty, T] \times \mathbb{R}.$$ 

We also recall that for all integer $m$ and all $\gamma \geq 1$, the space $H^m_\gamma(\Omega_T) = \exp(\gamma t)H^m(\Omega_T)$ is equipped with the norm

$$\|u\|_{H^m_\gamma(\Omega_T)} := \|e^{-\gamma t}u\|_{H^m(\Omega_T)} \approx \sum_{|\alpha| \leq m} \gamma^{m-|\alpha|} \|e^{-\gamma t}\partial^\alpha u\|_{L^2(\Omega_T)},$$

where the $\approx$ sign denotes equivalent norms, and the constant in the equivalence is independent of $\gamma \geq 1$ and $T > 0$. The definition of the space $H^m_\gamma(\omega_T)$ is similar.

For all real number $p \geq 1$, the space $L^p(\Omega_T)$ denotes the set of measurable functions such that $e^{-2\gamma t/p}u \in L^p(\Omega_T)$. The norm is defined in an obvious way.

With these notations, the Gagliardo-Nirenberg estimates in $H^m_\gamma(\Omega_T)$ read as follows:

**Theorem 8** (Gagliardo-Nirenberg). Let $m \geq 1$ be an integer, let $\gamma \geq 1$, and let $T \in \mathbb{R}$. There exists a constant $C$ (that is independent of $\gamma$ and $T$) such that for all $u \in H^m_\gamma(\Omega_T) \cap L^\infty(\Omega_T)$, for all multi-integer $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq m$, one has

$$\|\partial^\alpha u\|_{L^{2p}_\gamma(\Omega_T)} \leq C \|u\|_{L^\infty(\Omega_T)}^{1-1/p} \|u\|_{H^m_\gamma(\Omega_T)}^{1/p}, \quad \frac{1}{p} = \frac{|\alpha|}{m}.$$

There is a similar result with $\omega_T$ instead of $\Omega_T$. (The constant is still independent of $\gamma$ and $T$).

This result can be used to prove the following tame estimates for products of functions in $H^m_\gamma(\Omega_T)$:

**Theorem 9.** Let $m \geq 1$ be an integer, let $\gamma \geq 1$, and let $T \in \mathbb{R}$. For all functions $u, v \in H^m_\gamma(\Omega_T) \cap L^\infty(\Omega_T)$, the product $uv$ belongs to $H^m_\gamma(\Omega_T)$ and satisfies the estimate

$$\|uv\|_{H^m_\gamma(\Omega_T)} \leq C\left(\|u\|_{L^\infty(\Omega_T)} \|v\|_{H^m_\gamma(\Omega_T)} + \|u\|_{H^m_\gamma(\Omega_T)} \|v\|_{L^\infty(\Omega_T)}\right). \quad (169)$$

Moreover, for all multi-integer $\alpha, \beta$, such that $|\alpha| + |\beta| \leq m$, one has

$$\|\partial^\alpha u \partial^\beta v\|_{L^2_\gamma(\Omega_T)} \leq C\left(\|u\|_{L^\infty(\Omega_T)} \|v\|_{H^m_\gamma(\Omega_T)} + \|u\|_{H^m_\gamma(\Omega_T)} \|v\|_{L^\infty(\Omega_T)}\right). \quad (170)$$

The constant $C$ only depends on $m$, and is independent of $\gamma$ and $T$. The same result holds with $\omega_T$ instead of $\Omega_T$.

There is also a tame estimate for composed functions:

**Theorem 10.** Let $m \geq 1$ be an integer, let $\gamma \geq 1$, and let $T \in \mathbb{R}$. Let $F$ denote a $C^\infty$ function, that is defined on $\mathbb{R}^3$ (or on a neighborhood of the origin in $\mathbb{R}^3$), and that satisfies $F(0) = 0$. Then for all $u \in H^m_\gamma(\Omega_T) \cap L^\infty(\Omega_T)$, the composed function $F(u)$ belongs to $H^m_\gamma(\Omega_T)$ and satisfies the estimate

$$\|F(u)\|_{H^m_\gamma(\Omega_T)} \leq C\left(\|u\|_{L^\infty(\Omega_T)} \|u\|_{H^m_\gamma(\Omega_T)}\right),$$

where $C$ is an increasing function that does not depend on $\gamma$, and $T$. 

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In the paper we also use the following inequalities in $H^m_T$:

**Theorem 11.** The following inequalities hold, with a constant $C$ that is independent of $\gamma \geq 1$:

$$
\| e^{-\gamma t} V \|_{L^\infty(\Omega_T)} \leq \frac{C}{\sqrt{\gamma}} \| V \|_{H^2(\Omega_T)} \quad \forall V \in H^2_\gamma(\Omega_T),
$$

$$
\| e^{-\gamma t} V \|_{W^{1,\infty}(\Omega_T)} \leq C \sqrt{\gamma} \| V \|_{H^3_\gamma(\Omega_T)} \quad \forall V \in H^3_\gamma(\Omega_T),
$$

$$
\| e^{-\gamma t} \psi \|_{L^\infty(\omega_T)} \leq \frac{C}{\gamma} \| \psi \|_{H^2_\gamma(\omega_T)} \quad \forall \psi \in H^2_\gamma(\omega_T),
$$

$$
\| e^{-\gamma t} \psi \|_{W^{1,\infty}(\omega_T)} \leq C \| \psi \|_{H^3_\gamma(\omega_T)} \quad \forall \psi \in H^3_\gamma(\omega_T).
$$

(171)

**C.2 Nonlinear functions**

Following [29, page 81], we introduce the following definition:

**Definition 2.** Let $k \in \mathbb{N}$. A nonlinear function of order $\leq k$ is a finite sum of the form:

$$
\mathcal{F}(u) = F_0(u) + \sum_{\ell=1}^{k} \sum_{\alpha_1 + \cdots + \alpha_\ell \leq k} F_{\ell, \alpha_1, \ldots, \alpha_\ell}(u) \{ \nabla^\alpha_1 u, \ldots, \nabla^\alpha_\ell u \},
$$

where $F_0 \in C^\infty$, and $F_0(0) = 0$, and where the $\ell$-multilinear mappings $F_{\ell, \alpha_1, \ldots, \alpha_\ell}(u)$ are $C^\infty$ functions of $u$.

We have the following generalization of Theorem 10:

**Theorem 12.** Let $k \in \mathbb{N}$, $s \in \mathbb{R}$ with $k \leq s$, and let $\mathcal{F}$ be a nonlinear function of order $\leq k$. Then for all $u \in H^s(\mathbb{R}^2_+) \cap L^\infty(\mathbb{R}^2_+)$, the composed function $\mathcal{F}(u)$ belongs to $H^{s-k}(\mathbb{R}^2_+)$ and satisfies the estimate

$$
\| \mathcal{F}(u) \|_{H^{s-k}(\mathbb{R}^2_+)} \leq C \left( \| u \|_{L^\infty(\mathbb{R}^2_+)} \right) \| u \|_{H^s(\mathbb{R}^2_+)},
$$

where $C$ is an increasing function.

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