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Control of Moving Domains, Shape Stabilization and Variational Tube Formulations

Jean-Paul Zolésio

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1. The use of BV perimeter in shape optimization

In order to derive existence results in classical shape optimization (see for example [9]) I introduced after 1984 ([38], [39]) the concept of a functional regularized with the perimeter:

$$J_\sigma(\Omega) = J(\Omega) + \sigma P_D(\Omega) \tag{1.1}$$

This has been in the context of large water wave modelling (non shallow water free boundary) in which the "small" parameter σ turns to be the surface tension. That result was emphasized in [36] after having been presented to a large audience, then in [37] (that paper was kept two years before being accepted without changes for publication).

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the tube was introduced in [40],[13],[?],... in the following form:

$$J(\zeta, V) = j(\zeta, V) + \sigma \int_0^\tau P_D(\Omega_t(V))\, dt. \tag{1.2}$$

Following that idea we consider in this paper new control functionals for non cylindrical heat equation, non cylindrical wave equations and shape metrics. Many results are completely new : the optimality conditions and existence results for both heat and wave problems as well as the existence of solution to the level set equation with possible topological change in finite time. The Euler equation for the shape geodesic generalizes in some sense the variational formulation for the incompressible Euler equation. In deriving the new geodesic conditions for the new shape metrics $\bar{\delta}(\Omega_1, \Omega_2)$ and $d_E(\Omega_1, \Omega_2)$ we introduce new technical results such as the expression for the boundary shape derivative v'_{Γ_t} and the new weak form for the transverse vector field evolution equation. Also the structure of the adjoint field Λ is clarified when the right hand side is a shape gradient like measure $\gamma_t^*(g\,\vec{n}_t)$, then $\Lambda = \nabla\lambda\nabla\chi_{\Omega_t}$. The cubic energy expression for the wave equation with homogeneous Dirichlet condition derived in 1984 is also generalized for the first time to the "co-nornal Neumann" condition $\frac{\partial y}{\partial t} y + \frac{\partial y}{\partial \nu_t^*} y = 0$ on Γ_t.

2. Shape Evolution

Let be given a bounded "universe" D in R^N with lipschitzian continuous boundary and consider the set $L^1(D, \{0,\,1\}\,)$ of characterisitc functions $\zeta \in L^1(D)$ such that $\zeta^2 = \zeta$. We shall consider the family of measurable subsets $\Omega \subset D$ such that $\zeta = \chi_\Omega$ (that family is then defined up to subsets with zero measure in D). The time evolution of Ω is described with the help of vector fields. The time interval being denoted by $I = [0,\, \tau]$,

$$p > 1,\ V \in E := \{V \in L^p(I \times D, R^N),\quad s.t.\ divV \in L^p(I \times D),\ < V, n_{\partial D} >= 0\ \}$$

The subspace

$$E^{lip} := E \cap\, L^1(0, \tau, W^{1,\infty}(D, R^N)\,)$$

plays an essential role: for any measurable subset $\Omega_0 \subset D$ and any $V \in E^{lip}$ there exists a unique solution to the convection problem:

$$\mathbf{C} := L^1(0, \tau, L^1(D,\,\{0,\,1\}\,)) \cap C^0([0, \tau], L^1(D))$$

$$\zeta \in \mathbf{C},\ \ \frac{\partial}{\partial t}\zeta\ +\ \nabla_x\zeta.V\ =\ 0,\ \ \zeta(0) = \chi_{\Omega_0} \tag{2.1}$$

Indeed it exists the flow mapping $T_t(V)$ so that $\zeta(t, .) = \chi_{\Omega_0} o(T_t(V))^{-1}$. We denote $\Omega_t(V) := T_t(V)(\Omega_0)$ and shall refer to $Q_V := \cup_{0<t<\tau}\{t\} \times \Omega_t(V)$ as being a classical tube (roughly speaking the regularity of the moving boundary being "controlled" by the smoothness of V and $\partial\Omega_0$). We consider the family of subsets with finite perimeter in D:

$$\mathbf{P}_D = \{\ \omega \subset D,\ \chi_\Omega \in BV(D)\ \},\ \mathbf{H} = \mathbf{C} \cap L^1(I, BV(D)) \tag{2.2}$$

A main result for our topic is the following tubes closures:

Theorem 2.1. *Let $p > 1$, $\Omega_0 \subset D$ and a sequence $(\zeta_n, V_n) \in \mathbf{H} \times E$ verifying (2.1) and such that there exists a positive constant $M > 0$ with*

$$||V_n||_E + \int_0^\tau ||\nabla_x \zeta_n||_{M^1(D,R^N)}\, dt \;\leq\; M \tag{2.3}$$

Then there exists a weakly converging subsequence in $L^\infty(]0,\tau[\times D) \times E$. Any limiting element (ζ, V) belongs to $\mathbf{H} \times E$, verifies (2.1) with the bound (2.3). Moreover the convegence is strong in the $L^1(0,\tau, L^1(D))$ norm and notice that we get the following continuity on the limiting element : $\zeta \in \mathbf{H}$.

Proof As $\zeta \in L^\infty$ and $divV$, $V \in L^p(I \times D)$ we have $\zeta V \in L^p$ so that

$$\frac{\partial}{\partial t}\zeta = -\nabla\zeta.V = -div(\,\zeta V) + \zeta divV \;\in L^p(0,\tau, W^{-1,1}(D))$$

. The conclusion follows from the "parabolic Helly compacity results" which is included in [13],[31],[33], (see also the book [26] for a $p = 2$ version). It states that if a sequence ζ_k remains bounded in $L^1(0,\tau, BV(D))$ with $\frac{\partial}{\partial t}\zeta_k$ bounded in $L^p(0,\tau, W^{-1,1}(D))$, then there exists a subsequence converging strongly in $L^1(0,\tau, L^1(D))$.

As we shall see in section (10) the distribution space $W^{-1,1}(D)$ can be replaced by the Banach space of $M^1(D)$ of bounded measures over the bounded domain D.

We consider the weak closure of the family of classical tubes:

$$\mathbf{T}_{\Omega_0} := \{\;(\zeta, V) \in \mathbf{H} \times E\;\; s.t.\;\exists M > 0,\, (\zeta_n, V_n) \in \mathbf{H} \times E^{lip}$$
$$\text{weakly converges to}\,(\zeta, V)\text{ with : }\int_0^\tau P_D(\;T_t(V_n)(\Omega_0)\;)\, dt \leq M \quad \}$$

Corollary 2.2. *The set $\mathbf{T}_{\Omega_0}$ is weakly closed in $\mathbf{H} \times E$*

In section 10 we shall use $M^1(D)$ and derive a similar closure result with $p = 1$ but, in order to recover the continuity $\zeta \in C^0(0,\tau, L^1(D))$, we shall also need $p > 1$. The weak convection (2.1) is studied in [33]. It is interesting to notice that when the initial condition is a smooth enough function Φ_0 then the solution convects the level sets of Φ_0. Also from [10], [1] we know that the oriented distance function $b_{\Omega_t(V)}$ itself is solution to equation (2.1) with speed vector field $V(t, p_t(x))$, where $p_t = I_d - b_{\Omega_t(V)}\;\nabla b_{\Omega_t(V)}$ is the projection onto the boundary $\partial\Omega_t(V)$. The convection (2.1) generalises to boundary measures as follows:

Let $\Gamma_t(V) = \partial\Omega_t(V)$ with $V \in L^1(0,\tau, W_0^{1,\infty}(D, R^N))$ and let Γ_0 be a smooth manifold. Then we consider the element

$$\gamma_t := -\nabla\chi_{\Omega_t(V)}.\nabla b_{\Omega_t(V)}$$

Which is the usual boundary layer measure:

$$< \gamma_t, \Psi >_{H^{-1}(D)\times H_0^1(D)} = \int_{\Gamma_t(V)} \Psi(x)\, d\Gamma_t(x) \tag{2.4}$$

That measure solves the following evolution

$$\gamma(0) \,=\, \Gamma_0 := \partial\Omega_0$$

$$\frac{\partial}{\partial t}\gamma(t) + \nabla\gamma(t).V(t) + < DV(t).\nabla b_{\Omega_t(V)}, \nabla b_{\Omega_t(V)} > \gamma(t) = 0 \qquad (2.5)$$

That results enables us to develop a *tube variational analysis* for the minimization over $\mathbf{T}_{\Omega_0}$ of functionals in the following form

$$j(\zeta, V) = \int_0^\tau \int_{\Omega_t(V)} F(t, y_{Q_V}, \nabla y_{Q_V}) dx dt + \sigma \int_0^\tau P_D(\Omega_t(V))\ dt$$

Where y_{Q_V} stands for the solution of some boundary value problem associated with the tube Q_V. We shall propose two examples concerning the heat equation and the wave equation in that non cylindrical evolution domain Q_V. We shall choose adequate boundary conditions associated with that moving boundary. The equation (2.5) should permit to handle the convergence of boundary integrals and is under consideration in forecoming papers. This will permit to enlarge the present study to functionals J in the form

$$J(\zeta, V) = j(\zeta, V) \,+\, \int_0^\tau \int_{\partial\Omega_t(V)} f(t, y_{Q_V}, \nabla y_{Q_V}\ ) \, d\Gamma_t \, dt$$

$$= j(\zeta, V) + \int_0^\tau < \gamma(t), f(t, y, \nabla y) > \ dt\, .$$

A main point in that study is that for smooth enough tubes in a minimizing sequence, say $V_n \in E \cap L^1(0, \tau, W_0^{1,\infty}(D, R^N))$ the tubes Q_{V_n} is smooth enough so that some classical analysis will furnish the existence and may be uniquiness for the solution y_{Q_n} to the boundary value problem under concern. The point is that any such analysis fails for non smooth limiting tube in $\mathbf{T}_{\Omega_0}$. We shall propose specific choice of function $F(t, y, \nabla y)$ so that the minimization of (ζ, V) will "create" the existence. We deal here with simple linear equation and in a forcoming work we shall extend to Navier Stokes 3D equation.

Indeed that analysis would be efficient with open tubes. There are several obvious ways for dealing with open tubes. For example introducing mollifier on the considered vector fields.We propose here to replace the BV perimeter by the *density perimeter* that we recall now.

2.1. Boundedness of the Density Perimeter

2.1.1. Density Perimeter. Following [3] ,[4] , we consider for any closed set A in D the density perimeter associated to any $\gamma > 0$ by the following.

$$P_\gamma(A) = sup_{\epsilon \in (0,\gamma)} \ \ [\frac{meas(A^\epsilon)}{2\,\epsilon}\ ] \qquad (2.6)$$

Where A^ϵ is the dilation $A^\epsilon = \cup_{x\in A} B(x, \epsilon)$. We recall some main properties:

The mapping $\Omega \to P_\gamma(\partial\Omega)$ is lower-semi continuous in the H^c-topology,

The property $P_\gamma(\partial\Omega) < \infty$ implies that meas$(\partial\Omega) = 0$ and $\Omega \setminus \partial\Omega$ is open in D.

If $P_\gamma(\partial\Omega_n) \leq m$ and Ω_n converges in the H^c-topology to some open subset

$\Omega \subset D$, then the convergence holds in the $L^2(D)$topology .

The "parabolic" situation:

whenever $V \in C^{\infty}(I \times \bar{D})$, the mapping $t \to P_\gamma(\partial\Omega_t)$ is not continuous. So that mapping cannot be an element of $H^1(0,\tau)$. For any smooth vector field, $V \in C^0([0,\tau], W_0^{1,\infty}(D, R^N))$, we consider,

$$\Theta_\gamma(V,\Omega_0) \;=\; Min\;\{\int_0^\tau (\frac{\partial}{\partial t}\mu)^2\; dt \,|\, \mu \in \mathbf{M}_\gamma(V,\Omega_0)\;\} \tag{2.7}$$

Where

$$\mathbf{M}_\gamma(V,\Omega_0) = \{\;\mu \in H^1(0,\tau),\; P_\gamma(\partial\Omega_t(V)) \le \mu(t)\; a.e.t,$$
$$\mu(0) \le (1+\gamma)P_\gamma(\partial\Omega_0)\,\}$$

In general this set is non empty. When that set is empty we put $\Theta_\gamma(V,\Omega_0) = +\infty$. Notice that even when the mapping $p = (t \longrightarrow P_\gamma(\Omega_t(V))\;)$ is an element of $H^1(0,\tau)$ (then $p \in \mathbf{M}_\gamma(V,\Omega_0)\;\}$), we may have: $\Theta(V,\Omega_0) < ||p'||^2_{L^2(0,\tau)}$ as the minimizer will escape to possible variation of the function p.

Proposition 2.3. *Let $V \in C^0([0,\tau], W_0^{1,\infty}(D, R^N))$, $divV = 0$, we have:*

$$P_\gamma(\partial\Omega_t(V)) \le 2P_\gamma(\partial\Omega_0) + \sqrt{\tau}\;\;\Theta(V,\Omega_0)^{1/2} \tag{2.8}$$

Moreover if $V_n \in C^0([0,\tau], W_0^{1,\infty}(D, R^N))$, verifies $V_n \longrightarrow V$ in $L^2((0,\tau)\times D, R^N)$ and the uniform boundedness : $\exists M > 0,\;\; \Theta(V_n,\Omega_0) \le M$ Then

$$\Theta(V,\Omega_0) \le \; liminf\; \Theta(V_n,\Omega_0).$$

An alternative approach is to consider

$$\tilde{\Theta}_\gamma(V,\Omega_0) := ||p_\gamma||_{BV(0,\tau)},\;\; p_\gamma(t) := P_\gamma(\partial\Omega_t(V))$$

We would derive the same kind of estimates.

3. Heat Equation with Insulated Boundary

We consider the non cylindrical situation: the boundary Σ is insulated or adiabatic. As the domains move it is not the usual Neumann boundary condition but the one described below.

Non cylindrical evolution problems, such as Navier Stokes equation for moving boundaries in a fluid (see [5]) is a challenging optimal control issue. In the case of linear problems we deal with easier situations. Nevertheless a difficult issue is that we need to handle such problem with non smooth geometry. The study of non cylindrical heat equation is an old story. Far from being exhaustive here let us quote the works by P. Acquistapace [21], and recentely in [23]. In these works the boundary of the moving domain should be smooth enough. The obvious technique was based on the transport into a cylindrical problem which, in terms of an abstract setting, leads to a dynamical system with a non autonomous operator with a moving domain. Here we revisit that analysis in the scope of the optimal control of the moving domain Ω_t. As classically in shape analysis, the control parameter will be the speed vector field $V(t,x)$ whose flow mapping $T_t(V)$ builds the non cylindrical evolution domain $Q_V = \cup_{0<t<\tau}\{t\}\times\Omega_t$. Let $V \in C^0([0,\tau], C^1(D,R^N))$ with $V.n = 0$ on ∂D, the moving domain is $\Omega_t := T_t(V)(\Omega_0)$ and its characteristic function is $\zeta = \zeta_0 o T_t(V)^{-1}$. We consider the unique solution u to the parabolic problem:

$$\frac{\partial}{\partial t}u - \Delta u = 0 \text{ in } Q_V, \ \frac{\partial}{\partial n_t}u + < V(t), n_t > \ u = 0 \ \text{ on the moving boundary} \tag{3.1}$$

$\Gamma_t, \quad u(0) = u_0$

This b.c. cannot be written as $\frac{\partial}{\partial \nu}u = 0$ on the lateral time-space boundary Σ.

3.1. The weak formulation

$\forall\, \psi \in C^1([0,\tau]\times R^n)$ with $\psi(\tau) = 0$, ,

$$\int_0^\tau \int_{\Omega_t} (-u\frac{\partial}{\partial t}\psi + \nabla u.\nabla\psi)dtdx \tag{3.2}$$

$$= \int_{\Omega_0} \psi(0)(x)\, dx$$

Introducing $U(t,x) = u(t)oT_t(V)(x)$, the transported solution on the cylindrical domain, we get U as solution to the parabolic boundary value problem:

$$U_t + U(ln\, J)_t - J^{-1}div(U\, J\, DT^{-1}.V(t)oT_t(V)) \tag{3.3}$$

$$- J^{-1}\, div(J\, DT^{-1}.(DT^*)^{-1}.\nabla U) = 0$$

with the boundary condition

$$< DT_t(V)^{-1}.(DT_t(V)^{-1})^*.\nabla U, n > \tag{3.4}$$

$$+ < DT_t(V)^{-1}.V(t)oT_t(V), n > \ U \ = \ 0.$$

Given some element $U_d \in L^2(D)$ and $\sigma > 0$, we introduce the cost functional in the following form

$$j(\zeta, V) = 1/2\int_0^\tau\int_D \zeta((u-U_d)^2 + |\nabla u|^2)dxdt \tag{3.5}$$

$$j_1(\zeta,V) = j(\zeta,V) + \sigma/2\int_0^\tau(|V(t)|_E^2 + |\nabla\zeta|_{M^1(D,R^N)})dt.$$

We consider the minimization problem

$$Inf\ \{\ j_1(\zeta,V)\ \ |\ \ (\zeta,V)\in \mathbf{T}_{\Omega_0}\ \} \tag{3.6}$$

For elements (ζ, V) in $\mathbf{T}^{lip}_{\Omega_0}$ we have $\zeta = \zeta_{\Omega_0} o T_t(V)^{-1}$. The element ζ is uniquely associated to the vector field V. Stability of weak relaxed solution : Given a tube $(\zeta, V) \in \mathbf{T}^{lip}_{\Omega_0}$, we consider the solution u to the weak parabolic problem:

$$\forall\, \psi \in C^1([0,\tau]\times R^n) \text{ with } \psi(\tau) = 0, \ , \ \mathbf{H} = \nabla u$$

$$\int_0^\tau \int_D \zeta(-u\,\frac{\partial}{\partial t}\psi \,+\, \mathbf{H}.\nabla\psi\,)dtdx \,=\, \int_{\Omega_0} \psi(0)(x)\,dx \tag{3.7}$$

For the optimal control purpose we introduce the adjoint problem: the weak relaxed dual problem is the following one:

$$\forall\,\psi \in C^1([0,\tau]\times R^n) \text{ with } \psi(0)=0,$$

$$\int_0^\tau \int_D \zeta(p\,\frac{\partial}{\partial t}\psi \,+\, \mathbf{H}.\nabla\psi\,)dtdx + \int_0^\tau \int_{\Omega_t} p\,\psi\,v\,d\Gamma_t\,dt \,=\, \int_{\Omega_\tau} \psi(\tau)(x)\,dx \tag{3.8}$$

Solution stability : In order to get $\mathbf{H}^0 = (\nabla u)^0$ in (3.7), we need the tube to be an open set or at least such property for a.e. t concerning the set Ω_t such that $\zeta(t) = \chi_{\Omega_t},\ a.e.t.$ A technique to that approch is to "replace" the perimeter by the "density perimeter" $P_\gamma\partial\Omega_t$. As far as the functional j_1 is concerned, we have the following stability result:

Proposition 3.1. *Assume that (ζ_v, V_n) is a sequence of smooth tubes , $\zeta_n = \chi_{Q_n}$, we say that Q_n is builds by smooth speed vector fields V_n. For each n we have a solution u_n . Assume that (V_n,ζ_n) converges to (V,ζ) in $\sigma(L^2,L^2)\times L^1$ topology, with $\zeta_n = \zeta_{\Omega_0} o T_t^{-1}(V_n)$. Moreover assume that u_n^0 (the extension by zero) weakly converges in $L^2(0,\tau,D)$ to a limit element u as well as $(\nabla u_n)^0$ to some element $\mathbf{H}$. Then $(u,\mathbf{H}) \in L^2(Q)^{N+1}$ and is solution to the problem (3.7) in Q.*

In order to prove the existence result for a functional governed by the heat equation we need open sets and Hausdorff complementary convergence, then we modify the optimal control problem as follows:

$$j_2(\zeta,V) = j(\zeta,V) + \sigma/2\,\int_0^\tau (\,||V(t)||_E \,+\, ||P_\gamma(\partial\Omega_t)||_{BV(0,\tau)}\ \,)\,dt \tag{3.9}$$

We consider the minimization problem

$$Inf\ \{\ j_2(\zeta,V)\ \ |\ \ (\zeta,V)\in \mathbf{T}^{lip}\ \}. \tag{3.10}$$

Let (V_n,ζ_n) be a minimizing sequence. The tube Q_n is smooth and the solution u_n of heat equation is classically defined. Obviously the null-extensions to the cylinder $[0,\tau]\times D$ of both u_n and the gradients ∇u_n are bounded in $L^2([0,\tau]\times D)$. We consider a weakly converging subsequence, still denoted u_n, weakly converging to u and ∇u_n weakly converging to some vector field Z. On the other hand, as the $BV(0,\tau)$ norm of $P_\gamma(\partial\Omega_t^n)$ is bounded there exists a subsequence, still denoted Ω^n, such that $P_\gamma(\partial\Omega_t^n)$ converges in $L^1(0,\tau)$ to some integrable function f. Then for almost every t, $P_\gamma(\partial\Omega_t^n)\to f(t)$. As a result $a.e.t,\ \ P_\gamma(\partial\Omega_t^n)\ \le M(t)$ then for almost every time t, the open set Ω_t^n converges to some open set Ω_t both in H^c and L^p topologies, $meas(\partial\Omega_t)=0$ and $P_\gamma(\partial\Omega_t)\le liminf_{n->\infty}\,P_\gamma(\partial\Omega_t^n)=f(t)$

Let $\phi\in\mathbf{D}([0,\tau]\times D)$ such that $a.e.t,\ \phi(t)\in\mathbf{D}(\Omega_t)$. For $n\ge N_t$ we have $\phi(t)\in\mathbf{D}(\Omega_t^n)$ so that , a.e. t, we have :

$$\int_{\Omega_t^n} <\nabla u_n(t),\phi(t)>\,dx = -\int_{\Omega_t^n} u_n(t)\,div\phi(t)\,dx$$

Obviously,

$$\int_D < \nabla u_n(t), \phi(t) > dx = \int_{\Omega_t^n} < \nabla u_n(t), \phi(t) > dx,$$

and the same concerning u, so that in the limit we get $Z = \nabla u$.

We consider the Gateaux derivative of that functional, for $(\zeta, V) \in \mathbf{T}^{lip}_{\Omega_0}$ that is for $V \in E^{lip}$ and $\zeta(t) = \chi_{\Omega_0} o T_t(V)^{-1}$.

3.2. The Dual Problem

$$-\frac{\partial}{\partial t} p - \Delta p = 0 \text{ in } Q_V \tag{3.11}$$

$$p(\tau) = u_\tau \tag{3.12}$$

$$\frac{\partial}{\partial n_t} p = 0 \quad \text{on the moving boundary} \, \Gamma_t \tag{3.13}$$

The adjoint weak formulation is the following one:

$$\forall \psi \in C^1([0, \tau] \times R^n) \text{ with } \psi(0) = 0,$$

$$\int_0^\tau \int_{\Omega_t} (p \frac{\partial}{\partial t} \psi + \nabla u . \nabla \psi) dt dx + \int_0^\tau \int_{\partial \Omega_t} p \, \psi \, v \, d\Gamma_t \, dt = \int_{\Omega_\tau} \psi(\tau)(x) \, dx \tag{3.14}$$

Setting $P = p o T_t(V)$ we get the same equation as (3.3), but with final condition at $t = \tau$, and the following boundary condition:

$$(DT_t(V)^{-1}.(DT_t(V)^{-1})^*.\nabla P, n > + < DT_t(V)^{-1}.V(t) o T_t(V), n > P$$

$$+ < V(t) o T_t(V), (DT_t(V)^{-1})^*.n > P = 0. \tag{3.15}$$

4. Energy stabilization in Wave equation by moving domain

We consider the wave equation in a moving domain and the variation of the "acoustic" energy with respect to the dynamical boundary. We analyse passive and active controls in order to decrease the enegy for given initial conditions under Dirichlet or Neumann boundary conditions.

In 1985 [34] , 1987 [35] we addressed the shape stabilization issue for the wave equation in a moving domain under homogeneous Dirichlet boundary conditions. We revisit that results for Neumann homogeneous boundary conditions in view of modeling of smart actuators acting as a periodical (may be small) moving part of the boundary in the acoustic wave equation. As in the previous work the key point is that the time derivative of the energy turns out to be a cubic term with respect to the normal component of the boundary speed.

4.1. Wave Equation

Let $(a, b) \in H^1(\Omega) \times L^2(\Omega)$ be the initial datum. We consider the wave equation :

$$\frac{\partial^2}{\partial t^2} y - \Delta y \;=\; 0 \text{ in } Q, \tag{4.1}$$

with inital conditions:

$$y(0) \;=\; a, \;\; \frac{\partial}{\partial t} y(0) \;=\; b \text{ in } \Omega, \tag{4.2}$$

and one of the three boundary conditions

$$y \;=\; 0 \text{ on } \partial\Omega_t \tag{4.3}$$

or

$$\frac{\partial}{\partial n_t} y(0) \;=\; 0 \text{ on } \partial\Omega_t \tag{4.4}$$

or

$$\frac{\partial}{\partial n_t} y(0) + < V(t), n_t > y_t \;=\; 0 \text{ on } \partial\Omega_t. \tag{4.5}$$

We refer to problem (4.1), (4.2), (4.3) as the Dirichlet problem, and to (4.1), (4.2), (4.5) as the Neumann problem which is known ([46]) to be well posed in the energy norm when the boundary is smooth enough. Note that the system (4.1), (4.2),(4.4) is not known to be well posed.

4.2. Energy Functional

Let

$$E(t) = 1/2 \int_{\Omega_t} (y_t^2 + |\nabla y|^2) dx, \tag{4.6}$$

and

$$\mathbf{E}_t(V) \;=\; \int_0^t E(s)\, ds \tag{4.7}$$

4.3. Weak formulation of the Neumann problem (4.1), (4.2), (4.5)

Let y be such that $y(0) = a$ in Ω with y and $y_t \in L^2(0, \tau, L^2(\Omega_t(V))$ together with $\nabla y \in L^2(0, \tau, L^2(\Omega_t(V)), R^N))$, and verifying, $\forall \psi \in C^1([0, \tau], H^1(R^N))$ with $\psi(\tau) = 0$:

$$\int_0^\tau \int_{\Omega_t(V)} (< \nabla y, \nabla \psi > - y_t\, \psi_t\,) dt + \int_\Omega b\, \psi(0)\, dx = 0. \tag{4.8}$$

Lemma 4.1. *The following equation holds:*

$$\int_0^\tau \int_{\Omega_t(V)} z\psi_t dx dt = - \int_0^\tau \int_{\Omega_t(V)} z_t \psi dx dt - \int_0^\tau \int_{\partial\Omega_t(V)} z\psi < V(t), n_t > d\Gamma_t dt$$
$$+ \int_{\Omega_\tau(V)} z(\tau, x)\psi(\tau, x) dx - \int_{\Omega_0(V)} z(\tau, 0)\psi(\tau, 0) dx. \tag{4.9}$$

Then from (4.8) we get the boundary condition (4.5) on the moving boundary.

4.3.1. Hyperbolic adjoint problem.

$$\frac{\partial^2}{\partial t^2}p - \Delta p \ = \ 2\Delta y \text{ in } Q, \tag{4.10}$$

with inital conditions:

$$p(t) \ = \ 0, \ \frac{\partial}{\partial t}p(t) \ = \ y(t) \text{ in } \Omega_t(V) \tag{4.11}$$

and one of the two boundary conditions

$$p \ = \ 0 \text{ on } \partial\Omega_t, \tag{4.12}$$

or (respectively)

$$\frac{\partial}{\partial n_t}p(t) + < V(t), \, n_t > p_t \ = \ -2\frac{\partial y}{\partial n} \text{ on } \partial\Omega_t. \tag{4.13}$$

4.4. Derivative with respect to the vector field V

We have

$$2\mathbf{E}_t'(V,W) = \int_0^t \int_{\Omega_t} 2(\nabla y.\nabla y' + y_t y_t')dxdt + \int_0^t \int_{\Gamma_t} ((y_t)^2 + |\nabla y|^2) < Z(t), \, n_t > \, d\Gamma_t dt \tag{4.14}$$

where $Z(t)$ is the transverse field (see section 4.5). Making use of Green's theorem and of lemma 4.1 we get:

$$= \int_0^t \int_{\Omega_t} 2(-\Delta y \, y' - \, y_{tt} y')dxdt + \int_0^t \int_{\Gamma_t} (2\frac{\partial y}{\partial n_t} y' + ((y_t)^2 + |\nabla y|^2) z \,) \, d\Gamma_t dt$$

$$- \int_0^\tau \int_{\partial\Omega_t(V)} y_t \, y \, < V(t), n_t > d\Gamma_t \, dt + \int_{\Omega_\tau(V)} (y_t \, y)(\tau, x)dx - \int_{\Omega_0} (y_t \, y)(0, x)dx$$

As $\Delta y = y_{tt}$ we get :

$$= -4 \int_0^t \int_{\Omega_t} \Delta y y' dxdt + \int_0^t \int_{\Gamma_t} (2\frac{\partial}{\partial n_t} y y' \ + ((y_t)^2 + |\nabla y|^2) z \,) \, d\Gamma_t dt \tag{4.15}$$

$$- \int_0^\tau \int_{\partial\Omega_t(V)} y_t \, y \, < V(t), n_t > d\Gamma_t \, dt + \int_{\Omega_\tau(V)} (y_t \, y)(\tau, x)dx - \int_{\Omega_0} (y_t \, y)(0, x)dx$$

We have now to consider the two different boundaty conditions:

4.4.1. Dirichlet Condition 4.3**.**

$$2\mathbf{E}_t'(V,W) = -4 \int_0^t \int_{\Omega_t} \Delta y y' dxdt + \int_0^t \int_{\Gamma_t} (2\frac{\partial}{\partial n_t} y y' \ + ((y_t)^2 + |\nabla y|^2) z \,) \, d\Gamma_t dt$$

$$\int_{\Omega_\tau(V)} (y_t \, y)(\tau, x)dx - \int_{\Omega_0} (y_t \, y)(0, x)dx$$

From $y(t, x(t)) = 0$ we get $\frac{\partial}{\partial t} y(t, x(t)) = y_t(t, x) + \nabla y(t, x).V(t, x) = 0$ so that $y_t = -\frac{\partial}{\partial n_t} y \ < V(t), n_t >$, then:

$$2\mathbf{E}_t'(V,W) = -4 \int_0^t \int_{\Omega_t} \Delta y y' dxdt$$

$$+\int_0^t\int_{\Gamma_t}(2\frac{\partial}{\partial n_t}yy' + (\frac{\partial}{\partial n_t}y)^2\ (< V(t,n_t>^2+1) < Z(t),n_t>)\,d\Gamma_t dt$$

$$+\int_{\Omega_\tau(V)}(y_t\,y)(\tau,x)dx - \int_{\Omega_0}(y_t\,y)(0,x)dx$$

4.4.2. Neumann condition (4.5)**.**

$$2\mathbf{E}_t'(V,W) = -4\int_0^t\int_{\Omega_t}\Delta y y' dxdt$$

$$+\int_0^t\int_{\partial\Omega_t}(\ (y_t)^2 + |\nabla y|^2\ )\ \ < Z(t),\, n_t>\ \ d\Gamma_t dt$$

$$+\int_0^\tau\int_{\partial\Omega_t(V)}\frac{\partial}{\partial n_t}y\ \ (\ y < V(t),n_t> +2\,y'\ )\ \ d\Gamma_t\,dt$$

$$+\int_{\Omega_\tau(V)}(y_t\,y)(\tau,x)dx - \int_{\Omega_0}(y_t\,y)(0,x)dx$$

4.4.3. Characterisation of y' for the Neumann condition (4.5)**.** From (4.8) we obtain that

$$\int_0^\tau\int_{\Omega_t(V)}(\nabla y'.\nabla\psi - y_t'\ \psi_t)dxdt + \int_0^\tau\int_{\partial\Omega_t(V)}(\nabla y.\nabla\psi - y_t\psi_t)Z(t).n_t\,ds_t\,dt = 0.$$

Then we obtai y' being solution to the homogeneous wave equation :

$$-\Delta y' + \frac{\partial^2 y'}{\partial t^2} = 0$$

Concerning the boundary condition, we make use of the following "boundary version" of lemma 4.1

Lemma 4.2. *It holds :*

$$\int_0^\tau\int_{\partial\Omega_t(V)} f\,\psi_t ds_t =$$

$$-\int_0^\tau\int_{\partial\Omega_t(V)}(\ f_t\,\psi\ + (\ H(t)\,f\psi + \frac{\partial f}{\partial n_t}\psi + \frac{\partial\psi}{\partial n_t}f\ )\ < V(t),n_t>)\,ds_t\,dt \quad (4.16)$$

Making use of Green's theorem and of lemma 4.1 we get:

$$\int_0^\tau\int_{\Omega_t(V)}(\nabla y'.\nabla\psi - y_t'\ \psi_t)dxdt = \int_0^t\int_{\Omega_t}(-\Delta y'\,\psi + y_{tt}'\psi)dxdt$$

$$+\int_0^t\int_{\partial\Omega_t}(\frac{\partial y'}{\partial n_t} + y_t'\,\psi\ < V(t),n_t>)\ \psi\ \ d\Gamma_t\,dt$$

$$-\int_{\Omega_\tau(V)}(y_t\,y)(\tau,x)dx + \int_{\Omega_0}(y_t\,y)(0,x)dx$$

While from lemma 4.2, with $f = y_t < Z(t),\nabla b_{\Omega_t(V)}>$, we have :

$$\int_0^\tau\int_{\partial\Omega_t(V)}(\nabla y.\nabla\psi - y_t\psi_t)Z(t).n_t\,ds_t\,dt = \int_0^\tau\int_{\partial\Omega_t(V)}(\nabla y.\nabla\psi\,)Z(t).n_t$$

$$+ \, f_t \, \psi \;\; + (\, H(t) \, f \psi + \frac{\partial f}{\partial n_t} \psi + \frac{\partial \psi}{\partial n_t} f \;) \; < V(t), n_t > \,) ds_t \, dt$$

then

$$\frac{\partial y'}{\partial n_t} + v \, y'_t \; - \; div_{\partial \Omega_t(V)}(< Z(t), n_t > \; y(t) \,)$$

$$+ (\; f_t \;\; + H(t) \, f \; < V(t), n_t > + \frac{\partial f}{\partial n_t} < V(t), n_t > .$$

Where $f = y_t \; < Z(t), \nabla b_t >$ and

$$f_t = y_{tt} \;\; < Z(t), n_t > \; + y_t \; < \frac{\partial}{\partial t} Z(t), n_t > \; + y_t \;\; < Z(t), \nabla (\frac{\partial}{\partial t} b_{\Omega_t(V)} \,) >$$

But

$$\nabla (\frac{\partial}{\partial t} b_{\Omega_t(V)} \,) = - D^2 b_{\Omega_t(V)} . V(t) o p_t \; - \; D^* (\;\; V(t) o p_t \;\;) . \nabla b_{\Omega_t(V)}$$

So that

$$(\;\; f_t \;\;)|_{\partial \Omega_t(V)} \; = \;\; y_{tt} \;\; < Z(t), n_t > \; + y_t \;\; < Z'(t), n_t >$$

$$- \, y_t \;\; < Z(t), D^2_{\Omega_t(V)} . V(t) + D^* V(t) . n_t \;\; > .$$

Concerning the term $\frac{\partial}{\partial n_t} f$ we have

$$\frac{\partial}{\partial n_t} f = - \frac{\partial}{\partial n_t} y(t) \;\; < Z(t), n_t > - y(t) \;\;\; < DZ(t) . n_t, \; n_t > .$$

That is

$$\frac{\partial y'}{\partial n_t} + < V(t), \; n_t > \; y'_t \; = \; div_{\partial \Omega_t(V)} (\;\; < Z(t), n_t > \; \nabla_{\partial \Omega_t(V)} y(t) \;\;\;) \qquad (4.17)$$

$$+ (\;\; y_{tt} \;\; < Z(t), n_t > \; + y_t \;\; < Z'(t), n_t > \; - y_t \;\; < Z(t), \;\; D^2 b_{\Omega_t(V)} . V(t) + D^* V(t) . n_t \; >$$

$$+ H(t) \, y_t \;\; < Z(t), n_t > < V(t), n_t >$$

$$- \; (\;\; \frac{\partial}{\partial n_t} y(t) \;\; < Z(t), n_t > \; + y(t) \;\;\; < DZ(t) . n_t, \; n_t > \;\;) \;\; < V(t), n_t > .$$

The Gradient $\mathbf{G}(V)$ of the functional $\mathbf{E}(V)$ is such that

$$\mathbf{E}'(V; W) := \int_0^\tau \int_D \;\; < \mathbf{G}(t, x), \; W(t, x) >_{R^N_x} , dx dt,$$

where the space integral over D should be understood as distribution over D with compact support included in the moving boundary. In order to get an explicit expression of that gradient we could introduce an adjoint state p in order to "eliminate" y' from the previous expression of the derivative. That adjoint will permit to use the characterization of y' whose previous problem leads to explicit expressions in terms of normal fields $< V(t), n_t >$ but also $< Z(t), n_t >$ on the moving boundary. On that Neumann boundary condition (4.5) which, in the weak formulation, corresponds to "free test functions" ψ lying in the whole linear space $H := H^1(0, \tau, L^2(D)) \cap L^2(0, \tau, H^1(D))$ we shall make use of the min max Lagrangian approach that we briefly describe here :

4.4.4. Min Max approach. We have

$$\mathbf{E}_\tau(V) = Min_{\phi\in H}\ Max_{\psi\in H}\{\ \mathbf{L}(\phi,\psi)\ \mid\ \ \psi(\tau)=0\,,\ \ \phi(0)=a\ \},$$

where the Lagrangian is

$$\mathbf{L}(\phi,\psi) = \int_0^\tau\int_{\Omega_t(V)}[1/2((\phi_t)^2+|\nabla\phi|^2)+\nabla\phi.\nabla\psi-\phi_t\,\psi_t\ ]dxdt+\int_{\Omega_0}b\psi(0)dx$$

Proposition 4.3. *We have*

$$\mathbf{E}'_\tau(V,W) = \int_0^\tau\int_{\partial\Omega_t(V)}[1/2((y_t)^2+|\nabla y|^2)+\nabla y.\nabla p - y_t\,p_t\ ] < Z(t), n_t > ds_t\,dt, \tag{4.18}$$

where the *adjoint* state solves:

$$\forall\phi\in H,\ \phi(0)=0,\ \int_0^\tau\int_{\Omega_t(V)}[\phi_t\,y_t+\nabla\phi.\nabla y\ +\ \nabla\phi.\nabla p-\phi_t\,p_t\ ]dxdt=0 \tag{4.19}$$

That is

$$\int_0^\tau\int_{\Omega_t(V)}[-\Delta p\,+\,p_{tt}]\,\phi dxdt+\int_0^\tau\int_{\partial\Omega_t(V)}(\frac{\partial}{\partial n_t}p- < V(t),n_t > p_t)\,\phi\,d\Gamma_t\,dt$$
$$= -\int_0^\tau\int_{\Omega_t(V)} -2\Delta y\,\phi\,dxdt$$
$$-\int_0^\tau\int_{\partial\Omega_t(V)}(\frac{\partial}{\partial n_t}y+ < V(t),n_t > y_t)\,\phi\,d\Gamma_t\,dt$$

And as , from (4.5) $< V(t),n_t > y_t = \frac{\partial}{\partial n_t}y$ we get :

$$\int_0^\tau\int_{\Omega_t(V)}[-\Delta p\,+\,p_{tt}]\,\phi dxdt+\int_0^\tau\int_{\partial\Omega_t(V)}(\frac{\partial}{\partial n_t}p- < V(t),n_t > p_t)\,\phi\,d\Gamma_t\,dt$$
$$= -\int_0^\tau\int_{\Omega_t(V)} -2\Delta y\,\phi\,dxdt$$
$$-\int_0^\tau\int_{\partial\Omega_t(V)} 2\frac{\partial}{\partial n_t}y\ \phi\,d\Gamma_t\,dt$$

So that p solves the following *backward* wave equation:

$$p(\tau)=0,\ -\Delta p\,+\,p_{tt}\ =\ 2\,\Delta y$$

With the boundary condition :

$$\frac{\partial}{\partial n_t}p- < V(t),n_t > p_t = -2\frac{\partial}{\partial n_t}y. \tag{4.20}$$

Notice that this backward problem is well posed as it boundary condition (3.14) is similar to (4.5). This is not obvious, but we have to keep in mind that when reversing the time variable (i.e. taking $s=\tau-t$) then we consider the function $\tilde{p}(s)=p(\tau-s)$ which is defined in the reverse tube which is itself built from the "initial domain" $\Omega_\tau(V)$ by the speed vector field $\tilde{V}(s) := -V(\tau-s)$ so that the sign is unchanged in the difference which occurs in the boundary condition (3.14) expressed in terms of V and p or in terms of $\tilde{V}$ and $\tilde{p}$.

4.5. Transverse derivative

For two given vector fields V and W the *transverse* field Z is the solution to the Lie bracket evolution

$$H_V.Z = W, \text{ where } H_V.Z := \frac{\partial}{\partial t}Z + [Z, V], \quad [Z, V] := DZ.V - DV.Z. \quad (4.21)$$

4.5.1. Free divergences vector fields. For the sake of simplicity we assume first that V and W are free divergence vector fields: $divV = divW = 0$. Then the transverse field Z is itself a free divergence field. We consider the operator

$H_V \in \mathbf{L}(\mathbf{D}(D, R^N), \mathbf{D}(D, R^N))$.

Its transposed $H_V^* \in \mathbf{L}(\mathbf{D}'(D, R^N), \mathbf{D}'(D, R^N))$ is given by :

$$H_V^*.\Lambda := -\frac{\partial}{\partial t}\Lambda - D\Lambda.V - D^*V.\Lambda. \quad (4.22)$$

Lemma 4.4. *We have :*

$$H_V.(\zeta\, \vec{e}) = (\frac{\partial}{\partial t}\zeta + \nabla\zeta.V)\, \vec{e} + \zeta\, H_V.\vec{e}, \quad (4.23)$$

$$H_V^*(\zeta\, \vec{f}) = -(\frac{\partial}{\partial t}\zeta + \nabla\zeta.V)\, \vec{f} - \zeta\, H_V^*.\vec{f}. \quad (4.24)$$

We see that if (ζ, V) is a tube then the two previous expressions simplify as the first term vanishes by (2.1). In particular, we have

$$H_V^*.(\zeta\, \nabla\lambda) = -\zeta\, (\frac{\partial}{\partial t}\nabla\lambda + D\nabla\lambda.V + D^*V.\nabla\lambda) = -\zeta\nabla(\frac{\partial}{\partial t}\lambda + \nabla\lambda.V).$$

Then if we set the "scalar" operator h_V by:

$$h_V.\psi := \frac{\partial}{\partial t}\psi + \nabla\psi.V$$

we get, (ζ, V) being a tube,

$$H_V^*(\zeta\, \nabla\lambda) = -\zeta\, \nabla(\, h_V.\lambda). \quad (4.25)$$

Notice that when $divV = 0$ the adjoint operator verifies $h_V^* = -h_V$. By "reversing" the time, that operator h_V is then self adjoint.

4.5.2. Adjoint for the transverse field Z. Assume we have a functional derivative in the form

$$J'(V, W) = \int_0^\tau \int_{\Gamma_t(V)} g(t)\, < Z(t), n_t >\, d\Gamma_t\, dt.$$

Let us consider $\tilde{g}$ as being *any* extension of $g(t)$ to D (the most usefull choice being $\tilde{g}(t) = g(t)op_t$, where $p_t := I_d - b_{\Omega_t}\nabla b_{\Omega_t}$, the projection mapping is well defined in a neighbourhood of the moving boundary when it is smooth enough, see ([24]), ([1]). Then, as the fields are divergence free, we have

$$J'(V,W) = \int_0^\tau \int_{\Omega_t(V)} div(\tilde{g}(t)\,Z(t)\,)dxdt = \int_0^\tau \int_D \chi_{\Omega_t(V)} < \nabla\tilde{g}(t),\, Z(t) > dxdt$$

Then considering the adjoint problem

$$H_V^*\Lambda = \chi_{\Omega_t(V)}\,\nabla\tilde{g}(t), \quad \Lambda(\tau) = 0, \tag{4.26}$$

we get

$$J'(V,W) = \int_0^\tau \int_D < H_V^*.\Lambda, Z >_{R^N}\ dxdt = \int_0^\tau \int_D < \Lambda,\, H_V.Z\ >_{R^N}\ dxdt$$

$$= \int_0^\tau < \Lambda,\, W >_{R^N}\ dxdt.$$

Thus Λ is the linear mapping $W \to J'(V;W)$. Let us consider the adjoint problem (4.26) : we search for a solution in the form

$$\Lambda(t,x) = \chi_{\Omega_t(V)}(x)\,\nabla_x\lambda(t,x).$$

Then, as

$$H_V^*(\chi_{\Omega_t}\,\nabla\lambda) = \chi_{\Omega_t(V)}\,\nabla(\ h_V^*.\lambda\ ),$$

it is sufficient to choose λ as solution to the *scalar adjoint* backward problem:

$$-\frac{\partial}{\partial t}\lambda\, -\, \nabla\lambda.V\, =\, \tilde{g}, \quad \lambda(\tau) = 0. \tag{4.27}$$

Therefore we see that the solution Λ to the problem (3.14) is given by $\Lambda = \chi_{\Omega_t(V)}\,\nabla\lambda$ where λ is the solution to the problem (4.27), and that expression is independent of the choice of the extension $\tilde{g}$ of g outside the lateral boundary of the tube. Let us consider any extension such that $\frac{\partial}{\partial n_t}\tilde{g}(t) = 0$ on $\partial\Omega_t(V)$ (this is the case when we choose $\tilde{g}(t) = gop_t$). Then the term $\nabla\lambda\tilde{g}(t)$ simplifies to $\nabla_{\Gamma_t} g(t)$ which is the *tangential* gradient of $g(t)$ on the surface $\Gamma_t := \partial\Omega_t(V)$. Finally, assuming the vector fields V, W to be divergence free (that is the measure of the moving domains to be preserved), we consider the tangential equation

$$\lambda_t + \nabla_{\Gamma_t(V)}\lambda.V\, =\, g(t) \ \text{ on } \Gamma_t\,, \quad \lambda(\tau) = 0, \ \text{ on } \Gamma_\tau \tag{4.28}$$

and

$$J'(V;W) = \int_0^\tau \int_{\Gamma_t} g(t) < Z(t), n_t > d\Gamma_t dt = \int_0^\tau < \Lambda(t), W(t) > dt$$

$$= \int_0^\tau \int_D < -\chi_{\Omega_t(V)}\,\nabla\lambda, W > dxdt = \int_)^\tau \int_{\Omega_t(V)} div(\lambda W) dxdt$$

$$= \int_0^\tau \int_{\Gamma_t(V)} \lambda(t)\ < W(t), n_t >\ d\Gamma_t\, dt,$$

As a result :

Proposition 4.5. *Let V, W be given in $L^1(0,\tau,W_0^{1,\infty}(D,R^N))$, and $g \in L^1(0,\tau,L^1(\Gamma_t(V))$. Consider the solution λ to the tangential backward problem (4.28). Then we have*

$$\int_0^\tau \int_{\Gamma_t(V)} g(t) < Z(t), n_t > d\Gamma_t\, dt \;=\; \int_0^\tau \int_{\Gamma_t} \lambda(t) < W(t), n_t > d\Gamma_t\, dt$$

4.6. Optimal control problem for moving domain in D with given measure.
We consider divergence free fields V, W, Z and we introduce

$$\lambda_t + \nabla_{\Gamma_t(V)}\lambda.V = 1/2((y_t)^2 + |\nabla y|^2) + \nabla y.\nabla p - y_t\, p_t\,, \quad \lambda(\tau)=0.$$

Then we get:

$$\mathbf{E}'_\tau(V,W) = \int_0^\tau \int_{\partial\Omega_t(V)} \lambda \; < W(t), n_t > ds_t\, dt. \tag{4.29}$$

In order to consider the vector field V as a control parameter we shall now "increase" the cost functional by a "small" term (which would be "the price" of the control V):

$$\sigma > 0, \;\; \mathbf{E}_{\tau,\sigma}(V) = \mathbf{E}_\tau(V) \,+\, \sigma/2\,||\Delta V||^2 \tag{4.30}$$

The gradient of $\mathbf{E}_{\tau,\sigma}(V)$ is given by:

$$\mathbf{E}'_{\tau,\sigma}(V;W) = \int_0^\tau \int_{\partial\Omega_t(V)} \lambda(t) \; < W(t), n_t > d\Gamma_t\, dt \; + \\ \sigma \int_0^\tau \int_D < \Delta^2 V, W > \; dx dt$$

The Optimal Speed Synthesis is the following *equation* :

$$V^*(t) = (\Delta^2)^{-1}.\; \lambda(t) \nabla \chi_{\Omega_t(V^*)} \,+\, \nabla_x \Pi(t), \tag{4.31}$$

$$\lambda_t + \nabla_{\Gamma_t(V^*)}\lambda.V^* = 1/2((y_t)^2 + |\nabla y|^2) + \nabla y.\nabla p - y_t\, p_t\,, \quad \lambda(\tau)=0,$$

$$p(\tau)=0, \;\; \frac{\partial}{\partial n_t} p - < V(t), n_t > p_t \,=\, -2\frac{\partial}{\partial n_t} y,$$

$$y(0)=0, \;\; y-t(0)=b, \;\; , \frac{\partial}{\partial n_t} y - < V(t), n_t > y_t \,=\, 0.$$

4.6.1. General case: V is not a divergence free vector field. We shall just notice the (small) changes in the previous analysis: the adjoint problem is

$$H_V^*\Lambda = -\frac{\partial}{\partial t}\Lambda - D\Lambda.V - D^*V.\Lambda \,-\, (divV)\,\Lambda.$$

$$H_V^*(\zeta\, \vec{f}) = (-\frac{\partial}{\partial t}\zeta - \nabla\zeta.V)\vec{f} \;\; + \zeta\; H_V^*.\vec{f}$$

So that if (ζ, V) is a tube we simply get as before

$$H_V^*(\zeta \vec{f}) = \zeta\, H_V^*.\vec{f}.$$

Now

$$H_V^*(\nabla\lambda) = -\nabla\,\lambda_t \,-\, D(\nabla\lambda).V \,-\, D^*V.\nabla\lambda \,-\, divV\,\nabla\lambda$$

$$= -\nabla\,\lambda_t\ \ - \nabla(\,\nabla\lambda.V\ )\ -\ divV\ \nabla\lambda,$$

and

$$H_V^*(\chi_{\Omega_t(V)}\,\nabla\lambda) = -\chi_{\Omega_t(V)}[\ \nabla(\ \ \frac{\partial}{\partial t}\lambda + \nabla\lambda.V\ \ )\ +\ divV\ \nabla\lambda\ ]$$

But also :

$$H_V^*(-\lambda\,\nabla\chi) = (\frac{\partial}{\partial t}\lambda + \nabla\lambda.V)\nabla\chi + \lambda(\ \nabla(\ \frac{\partial}{\partial t}\chi + \nabla\chi.V\ )\ \ +\ divV\,\nabla\chi$$

$$= (\frac{\partial}{\partial t}\lambda + \nabla\lambda.V\ \ +\ \ \lambda\, divV\ )\ \ \nabla\chi$$

Hence we have :

Proposition 4.6. *Let $\lambda \in L^1(0,\tau, L^1(\Gamma_t)$ solve the backward problem*

$$\frac{\partial}{\partial t}\lambda + \nabla\lambda.V\ \ +\ \ \lambda\, divV = g,\ \ on\ \Gamma_t,\ \ \lambda(\tau) = 0.$$

Then we have:

$$\int_0^\tau \int_{\Gamma_t} g\ < Z(t), n_t > d\Gamma_t dt = \int_0^\tau \int_{\Gamma_t} \lambda\ < W(t), n_t > d\Gamma_t dt.$$

4.6.2. Optimality condition : the general case. We consider the minimization of the previous functional $\mathbf{E}_\sigma(V)$ without any constraint on the divergence of the vector field V. We obtain the following characterisation:

$$V^*(t) = \frac{1}{\sigma}\,(\Delta^2)^{-1}.\ \lambda(t)\nabla\chi_{\Omega_t(V^*)}, \tag{4.32}$$

$$\lambda_t + \nabla_{\Gamma_t(V^*)}\lambda.V^* +\ \lambda\, divV^* = 1/2((y_t)^2 + |\nabla y|^2) + \nabla y.\nabla p - y_t\, p_t\,,\quad \lambda(\tau) = 0,$$

$$p(\tau) = 0,\ \ \frac{\partial}{\partial n_t}p - \ < V(t), n_t >\ p_t = \ -2\frac{\partial}{\partial n_t}y,$$

$$y(0) = 0,\ \ y - t(0) = b,\ \ ,\frac{\partial}{\partial n_t}y - \ < V(t), n_t >\ y_t =\ 0.$$

In order to simplify that system, an idea would be to search for a solution V^* proportional to the normal field $n_{\Gamma_t(V^*)}$.

A possibility is to parametrize the domain as a level set of some function $\Phi(t,x)$ say

$$\Omega_t := \{x \in D\ \ \ |\ \Phi(t,x) > 0\ \ \}$$

Then the vector field can be taken as

$$V(t,x) = -\frac{\partial}{\partial t}\Phi\,\frac{\nabla\Phi}{||\nabla\Phi||}$$

which indeed is proportional to $n_{\Gamma_t} = \frac{\nabla\Phi}{||\nabla\Phi||}$. In this case the equation for the scalar boundary adjoint term λ drastically simplifies to the following:

$$\frac{\partial}{\partial t}\lambda\ +\ \lambda\, divV = g := 1/2((y_t)^2 + |\nabla y|^2) + \nabla y.\nabla p - y_t\, p_t.$$

In the last section we shall investigate the fixed point theory for solving that non-linear problem in terms of level sets.

In order to avoid the analysis of the free boundary value problem (the first order necessary optimality condition) we propose here a different analysis which may be "suboptimal" but permits to handle algorithms for the decay of the energy functional $\mathbf{E}(V)$. It is based on the so called *cubic* derivative, that is the cubic (with respect to the normal componant $v(t) =< V(t), n_t >$) expression for the energy density time derivative $E'(t)$. We recall now these old results and extend to the previous Neumann condition (4.5) for the wave equation in a moving domain.

4.7. Time Derivative of the Energy: a passive shape control approach

The cubic expression for the energy derivative was established for the homogeneous Dirichlet condition. We extend it here, as a new result, to the previous "Neumann condition" $\frac{\partial}{\partial \nu^*} u = 0$ on the moving boundary:

4.7.1. Dirichlet Problem. We recall [34] the cubic expression of the derivative for the wave equation with Dirichlet boundary condition (4.3) on the moving domain.

Proposition 4.7. *Let y be the solution of (4.1),(4.2),(4.3), and $v(t) =< V(t), n_t >$ be the normal component of the speed vector field at the moving boundary $\partial\Omega_t$. Then*

$$E'(t) = 1/2 \int_{\partial\Omega_t} (\frac{\partial}{\partial n_t} y)^2 (v^3(t) - v(t)) \, d\Gamma_t \tag{4.33}$$

Proof: Assuming smooth enough solution y, we get

$$E'(t) = \int_{\Omega_t} \frac{\partial}{\partial t} y \frac{\partial^2}{\partial t^2} y + \nabla y . \nabla (\frac{\partial}{\partial t} y) \, dx + \int_{\Gamma_t} 1/2((\frac{\partial}{\partial t} y)^2 + ||\nabla y||^2) \, v \, d\Gamma_t$$

$$= \int_{\Omega_t} \frac{\partial}{\partial t} y (\frac{\partial^2}{\partial t^2} y - \Delta y) dx + \int_{\Gamma_t} [1/2((\frac{\partial}{\partial t} y)^2 + ||\nabla y||^2) v + \frac{\partial}{\partial n_t} y \frac{\partial}{\partial t} y] d\Gamma_t$$

using the boundary condition we conclude the proof.

Corollary 4.8.

$$E(t) = 1/2 \int_{\Omega} (a^2 + |\nabla b|^2) dx + 1/2 \int_0^t \int_{\partial\Omega_s} (\frac{\partial}{\partial n_s} y(s))^2 (v^3(s) - v(s)) \, d\Gamma_s \, ds \tag{4.34}$$

$$\mathbf{E}_t(V) = tE(0) + 1/2 \int_0^t (t-s) \int_{\partial\Omega_s} (\frac{\partial}{\partial n_s} y)^2 (v^3(s) - v(s)) \, d\Gamma_t \, ds. \tag{4.35}$$

4.8. Neumann condition (4.5)

Proposition 4.7 extends to the Neumann condition with some generalisation in the cubic expression.

Proposition 4.9. *Let y be the solution of (4.1),(4.2),(4.5) $v(t) =< V(t), n_t >$ be the normal component of the speed vector field at the moving boundary $\partial\Omega_t$. Then*

$$E'(t) = 1/2 \int_{\partial\Omega_t} (y_t)^2 (v^3(t) - v(t)) \, d\Gamma_t + 1/2 \int_{\partial\Omega_t} |\nabla_{\Gamma_t} y|^2 \, v(t) d\Gamma_t. \tag{4.36}$$

In view of (4.36) the energy derivative, as in the previous Dirichlet case, is governed by a cubic in $v =< V(t), n_t >$ at the boundary, here it takes a different form:

$$E'(t) = 1/2 \int_{\partial\Omega_t} (\,(y_t)^2\, v^3(t) \;+\; (\,|\nabla_\Gamma y|^2 - (y_t)^2)\, v(t)\,)\, ds_t. \tag{4.37}$$

Corollary 4.10.

$$E(t) = E(0) + 1/2 \int_0^t \int_{\partial\Omega_s} (y_t(s))^2 (v^3(s) \;- v(s))\, d\Gamma_s\, ds$$

$$+1/2 \int_0^t \int_{\partial\Omega_s} |\nabla_{\Gamma_s} y(s)|^2 v(s) d\Gamma_s \tag{4.38}$$

$$\mathbf{E}_t(V) = tE(0) + 1/2 \int_0^t (t-s) \int_{\partial\Omega_s} [(y_s\,)^2\,(v^3(s) - v(s)) + |\nabla_{\Gamma_s} y(s)|^2\, v(s)]\, d\Gamma_s\; ds \tag{4.39}$$

4.9. Energy derivative in term of the normal derivative of y.

Making use of (4.5) we rewrite (4.37) in term of the *normal derivative* $\frac{\partial y}{\partial n_t}$, as it was the case for the expression we got for the Dirichlet case, we obtain a much more suitable expression (4.40). What is to be observed is that in (4.40) “the minus sign of (4.37) has been changed” for a plus (as $(\frac{\partial y}{\partial n})^2 + |\nabla_\Gamma y|^2 = |\nabla y|^2$)) , which will make all the difference in the next result:

$$E'(t) = 1/2 \int_{\partial\Omega_t(V)} \frac{1}{v}\, (\,|\nabla y(t)|^2\, v^2 \;-\; (\frac{\partial y(t)}{\partial n_t})^2\,)\, ds_t. \tag{4.40}$$

4.10. The case for a damped material

Assume that the Neumann boundary condition takes the slightly different following form: let $d_0 > 0$ and

$$\frac{\partial y}{\partial n_t} \;+\; (d_0 \;+\; < V(t), n_t >) y_t \;= 0 \;\text{ on } \partial\Omega_t(V). \tag{4.41}$$

Then we get:

$$E'(t) = 1/2 \int_{\partial\Omega_t} (\,(y_t)^2\, v^3(t) \;+\; (\,|\nabla_\Gamma y|^2 - (y_t)^2)\, v(t)\,)\, ds_t \tag{4.42}$$

$$+ \int_{\partial\Omega_t} d_0\, (y_t)^2\; (v^2 + 1/2\, d_0\, v \;- 1\,).$$

The integrand in the additive term is negative for v^2 ”small enough”, that is

$$v^*_- \;<\; v \;<\; v^*_+$$

With $v^*_{+,-} = -1/4\; d_0 \;+- \sqrt{1 + 1/16\; d_0^2}$

$$E'(t) = 1/2 \int_{\partial\Omega_t(V)} \frac{1}{v}\, (\,|\nabla y(t)|^2\, v^2 \;-\; (\frac{\partial y(t)}{\partial n_t})^2\,)\, ds_t. \tag{4.43}$$

4.11. Passive boundary control

4.11.1. Dirichlet problem. In order to have a decay of $E(t)$ we can, *formally*, choose the normal speed component $v(t)$ on the moving boundary such that $E'(t) \leq 0$. For this it is sufficient to impose the following condition

$$0 \leq v(t) \leq 1 \ \text{ or } v(t) \leq -1. \tag{4.44}$$

4.11.2. Neumann problem. In the same formal approach, following (4.37) the sufficient condition in order to derive $E'(t) \leq 0$ would be to "choose" the normal speed verifying the following conditions (which are in fact an equation on v):

$$v(t) \geq 0 \text{ and } v(t)^2 \geq (\frac{|\nabla_{\Gamma_t} y(t)|^2}{(y_t)^2} - 1), \quad \text{or } v(t) \leq 0 \ \text{ and } \ v(t)^2 \leq (\frac{|\nabla_{\Gamma_t} y(t)|^2}{(y_t)^2} - 1). \tag{4.45}$$

Obviously (4.45) is "often" an empty condition.

In view of (4.40) we shall obtain a negative derivative $E'(t) \leq 0$, for the energy if the integrand is non positive at almost every $x \in \partial\Omega_t(V)$.

Proposition 4.11. *Assume $\nabla y(t,x)$ different from zero for a.e.$(t,),\ x \in \partial\Omega_t(V)$, then let*

$$K(t,x) = (\frac{(\frac{\partial y(t)}{\partial n_t})^2}{|\nabla y|^2})^{1/2} \ \leq \ 1.$$

Assume that the speed field V verifies:

$$\textit{If } v(t,x) > 0, \ \textit{ then } v(t,x) \leq K(t,x); \ \textit{ if } v(t,x) < 0, \ \textit{ then } v(t,x) \leq -K(t,x) \tag{4.46}$$

Then $E'(t) \leq 0$ and that inequality is strict if, on a subset of $\partial\Omega_t(V)$ of non zero measure one of the inequalities (4.46) is strict.

Notice that the conditions (4.46) is not really a feedback loop between the state $y(t)$ and the control v at the boundary, but we cannot say that, as in the Dirichlet case, the synthesis (4.46) is a passive control. It can be written in more compact form as follows:

$$a.e.t, a.e.x \in \partial\Omega_t(V), \ \ v(t,x) \in \mathbf{K} =]-\infty, \ \ -K(t,x)\,] \cup [0, \ K(t,x)\,] = \mathbf{K}^- \cup \mathbf{K}^+. \tag{4.47}$$

4.12. An example

In order to satisfy 4.47 choose

$$v(t,x) = 1/\sqrt{2}\ \ K \text{ or } \ v(t,x) = -\sqrt{2}K. \tag{4.48}$$

then the integrand term in the expression (4.40) of $E'(t)$ takes the following form:

$$-\frac{1}{\sqrt{2}}\,|\frac{\partial y}{\partial n}|\ |\nabla y| \quad (\text{when } \ v = -\sqrt{2}\ K \text{ or } \ v = +1/\sqrt{2}K). \tag{4.49}$$

Proposition 4.12. *Assume v satisfies (4.48), then*

$$E'(t) = -\frac{1}{2\sqrt{2}} \int_{\partial\Omega_t(V)} |\frac{\partial y}{\partial n_t}| \, ||\nabla y|| \, ds_t \leq -\frac{1}{2\sqrt{2}} \int_{\partial\Omega_t(V)} |\frac{\partial y}{\partial n_t}|^2 ds_t. \quad (4.50)$$

4.13. Smart material modeling.

We impose a constraint on the normal componantof the speed field which translates the fact that during the time, the shape evolution is periodical or "locally periodical". Assume that only small parts of the boundary centered on given points $x_1, ..., x_m$ of $\partial\Omega_0$ are moving with the speed of a "rigid body motion" for example.

4.13.1. One actuator. Assume a single "small part" γ_t of the boundary $\partial\Omega_t(V)$ is moving, keeping zero curvatures, and with a constantspace normal speed $v(t)$. That is that $\Omega_t(V)$ is a moving domain those boundary $\partial\Omega_t(V)$ moves under a normal a speed $V(t,x).n_t = v$ that is zero out of $\bar{\gamma}_t$. Now we assume $v(t,.)$ to be constant on γ_t and γ_0 flat (without curvatures) so that in fact $\gamma_t = \gamma_0 + d(t)\vec{n}_0$ (is moved by translations). The boundary $\partial\Omega_t = (\partial\Omega_0 - \bar{\gamma}_0) \bigcup \bar{\gamma}_t \bigcup \mathbf{O}_t$. where $\mathbf{O}_t$ is a cylindrical (variable with t) open set on which V is tangential but $v = 0$. Of course the moving boundary $\partial\Omega_t(V)$ is not of class C^1 but is Lipschitzian smooth (which is enough for justifying the previous considerations.) From (4.40) we obtain:

$$E'(t) = 1/2 \, \frac{1}{v(t)} \; \left(v(t)^2 \int_{\gamma_t} ||\nabla y(t,x)||^2 \, dx \; - \int_{\gamma_t} (\frac{\partial y(t,x)}{\partial n_t})^2 \, dx \right). \quad (4.51)$$

We introduce

$$\mathbf{K}_{\gamma_0}(t) = \frac{\int_{\gamma_t} (\frac{\partial y(t,x)}{\partial n_t})^2 \, dx}{\int_{\gamma_t} ||\nabla y(t,x)||^2 \, dx} \; \leq \; 1. \quad (4.52)$$

Proposition 4.13. *Assume that $\int_{\gamma_t} ||\nabla y(t,x)||^2 \, dx$ is different from zero, and $v(t)$ verifies:*

$$v(t) > 0 \;\; and \;\; v(t) \leq \mathbf{K}_{\gamma_0}(t), \quad or \;\; v(t) < 0 \;\; and \;\; v(t) \leq \; -\mathbf{K}_{\gamma_0}(t). \quad (4.53)$$

That is

$$v(t) \; \in \;]-\infty, \; -\mathbf{K}_{\gamma_0}(t)] \bigcup [0, \; \mathbf{K}_{\gamma_0}(t)]$$

Then $E'(t) \leq 0$.

For example chose

$$v(t) = +1/\sqrt{2}\,\mathbf{K}_{\gamma_0}(t) \;\; \text{or} \;\; v(t) = -\sqrt{2}\;\mathbf{K}_{\gamma_0}(t).$$

Then we get:

$$E'(t) = -\frac{1}{2\sqrt{2}} \left(\int_{\gamma_t} (\frac{\partial y(t,x)}{\partial n_t})^2 \, dx \right)^{1/2} \; \left(\int_{\gamma_t} ||\nabla y(t,x)||^2 \, dx \right)^{1/2}. \quad (4.54)$$

This displacement is periodical by the choice of adequate time intervals.

5. The level set Approach for Moving Domain : Asymptotic Analysis

The boundary of a moving domain $\Omega_t \subset D$ can be parametrized as a *level set* of a one parameter smooth function $\Phi(t,.) \in H^1(D)$ such that, for example, $\Phi(t,.)+1 \in H_0^1(D)$ (that is to say that $\Phi(t,.) = -1, \;\; x \in \partial D$).

In that situation

$$\Omega_t := \{ \; x \in D, \; s.t. \;\; \Phi(t,x) > 0 \; \} \tag{5.1}$$

is a quasi open subset in D (it is an open subset up to *zero capacity* subset), and verifies $\bar{\Omega}_t \subset D$.

Of course for a given open tube $Q = \bigcup_{0<t<\tau} \{t\} \times \Omega_t \quad \subset]0,\tau[\times D$, such function Φ is *not unique*.

5.1. Intrinsic geometry function

It is classical that $\Phi(t) = \Phi(t,.)$ can be decomposed as

$$\Phi(t) = \Phi(t)^* o \beta_{\Phi(t)}, \tag{5.2}$$

where the *monotone rearrangement* $\Phi(t)^*$ is a monotone (increasing) mapping defined from the interval $[0, meas(D)]$ into R, while the *intrinsic geometry mapping* $\bar{\beta}(t) =: \beta_{\Phi(t)}$ is defined from D into the interval $[0, meas(D)]$ as :

$$\bar{\beta}(t)(x) = meas(\{ \; y \in D \; s.t. \;\; \Phi(t)(y) < \; \Phi(t)(x) \;\; \}). \tag{5.3}$$

Assume that

$$\partial\Omega_t = \{ \; x \in \; D \;\; s.t. \; \Phi(t)(x) = 0 \; \} \; = \; \Phi(t)^{-1}(0). \tag{5.4}$$

Then we verify that this property (5.4) just depends on the *intrinsic geometry mapping* $\bar{\beta}(t)$, that is : (5.4) holds true for any monotone increasing continuous mapping $\Phi(t)^*(s)$.

The *intrinsic* view point consists in taking $\Phi(t)^*(s) = s, \; \forall s \in [0, \; meas(D)]$

In that situation we see that $\Phi(t) = \bar{\beta}(t)$, that is to say that (5.3) holds for the function $\Phi(t) = \bar{\beta}(t)$:

$$\forall x \in D, \;\; \bar{\beta}(t)(x) = meas(\{ \; y \in D \; s.t. \;\; \bar{\beta}(t)(y) < \; \bar{\beta}(t)(x) \;\; \}) \tag{5.5}$$

and

$$\partial\Omega_t = \bar{\beta}(t)^{-1}(0)$$

. (5.1) can be rewritten as

$$\Omega_t = \; \{ x \in D \; s.t. \; \Phi(t)(x) > 0 \; \} \;\; = \;\; \{ \; x \; \in D \; s.t. \; \bar{\beta}(t)(x) > 0 \; \} \tag{5.6}$$

Of course there are several functions $\bar{\beta}(t)$ verifying (5.6) and (5.5).

5.2. Speed vector field

Assuming (5.1), it is classical ([8],[24]) that a speed vector (whose flow mapping carries that moving domain) is

$$V^{\phi}(t,x) = -\frac{\partial}{\partial t}\phi(t,x)\,\frac{\nabla_x\phi(t,x)}{||\nabla_x\phi(t,x)||^2}. \tag{5.7}$$

Any other field building this tube is of the form $W = V^{\phi}+Z$ with $< Z(t), n_t >= 0$ on Γ_t.

We understand that the presence in the denominator of the term $||\nabla_x\phi||$ will not help to define correctely the flow mapping of that vector field V^{ϕ}. At least we shall have to control the term $||V^{\phi}(t,x)||_{R^N} = |\frac{\partial\phi}{\partial t}|/||\nabla_x\phi||$ in order to use the shape differential equation technique. The way to bypass that difficulty is as follows. Consider any shape gradient descent method for minimizing a shape functional $J(\Omega)$ whose shape gradient $G(\Omega) \in \mathbf{D}'(R^N, R^N)$ is a vector distribution with compact support included in the boundary of the domain (which, in *smooth* situation, takes the form $G(\Omega) = \gamma^*_{\partial\Omega}.(g\,n)$ where g, the *shape gradient density* is a scalar distribution on the boundary with zero transverse order, n beeing the normal field). The Shape differential equation consists in solving the non linear problem (see [7], [8],...,[24]) :

$$\boxed{\forall t, \quad 0 \le t \ \le \ \tau, \quad V(t,.) = -A^{-1}.G(\Omega_t(V))} \tag{5.8}$$

leading to the decrease of the functional :

$$J(\Omega_t(V)) \le \ J(\Omega_0) - \alpha\int_0^t \ ||V(s,.)||^2_{\mathbf{D}(A^{1/2})}ds \ \ . \tag{5.9}$$

Let V^* be *a solution* to (5.8), the problem is then to find a function ϕ such that the associated vector speed V^{ϕ} builds the same tube $Q^* = Q_{V^*}$. The necessary and sufficient condition (under some smoothness) is that the normal components of the two vector fields are equal *on the lateral boundary* Σ_V, that is

$$-\frac{\partial}{\partial t}\phi/|\nabla_x\phi| \ =< V^*(t), n_t > \quad \text{on} \ \ \partial\Omega_t(V^*). \tag{5.10}$$

Assume that ϕ solves the equation (5.10) on the lateral boundary Σ_{V^*}, by "multiplying" that equation by the non negative term $|\nabla\phi(t)|$, we obtain that ϕ solves the problem :

$$\frac{\partial}{\partial t}\phi(t,x) \ \ + < \nabla_x\phi(t,.),\, V^*(t) \ >= 0 \ \text{ on } \ \partial\Omega_t(V^*). \tag{5.11}$$

An obvious way to solve that pb (5.11) is to consider the *global* convection problem:

$$\frac{\partial}{\partial t}\phi(t,x) \ \ + < \nabla_x\phi(t,.),\, \bar{V}^*(t) \ >= 0 \ \text{ in } \ D, \tag{5.12}$$

where $\bar{V}^*$ is *any admissible extension* to the cylinder $(0,\tau)\times D$ of the vector field $V^*|_{\Sigma_{V^*}}$ (the restriction of V^* to the lateral boundary of the tube Q_{V^*}). A possible choice of such vector field $\bar{V}^*$ is V itself, but there are many other examples, one of them is $\bar{V}^* = V^* o p^*$ where p^* stands for the projection mapping or the local (or *narrow*) $(p^*)^h$ projection onto the boundary $\partial\Omega_t(V^*)$. In this paper we furnish existence and uniquiness results for this convection problem (5.12) when $\bar{V}^*,\ div\bar{V}^*$ are in $L^1(0,\tau,L^2(D))$ while the initial data satisfies $\phi_0 \in L^\infty(D)$.

Let V^* and ϕ be solutions to (5.8) and (5.12), respectively, then we get

$$\frac{\partial}{\partial t}\phi(t,x)/|\nabla\phi| =< \nabla_x\phi(t,.)/|\nabla\phi|, \bar{V}^* >,$$

so that

$$\boxed{\ |\ \ \frac{\partial}{\partial t}\phi(t,x)/|\nabla\phi|\ \ \ |\ \le\ ||\bar{V}^*(t,x)||_{R^N}.\ } \tag{5.13}$$

Assume that $\bar{V}^* = V^*$, then if $V^* \in E = L^1(0,\tau,L^2(D,R^N))$ wuth ϕ being solution to (5.12), we get $V^\phi \in E$. Assume that $divV^\phi \in E$ then for any given $\phi_0 \in L^\infty(D)$ we get the existence of a solution to (5.12). In the classical setting (developed in [7], [8],...) the shape gradient G of the shape functional J is bounded (in some "negative" Sobolev space of distributions over the universe D) and continuous (with respect to the *Courant metric* , see [24]), then , $\forall k,\ k \ge 1$, the shape differential equation posesses smooth solutions $V^* \in \mathbf{C}^{0,k} = C^0([0,\tau], C^k(\bar{D},R^N)) \subset E$. Then the flow mapping $T_t(V^*)$ is classically defined and the unique solution to the convection problem 5.12, if $\bar{V}^*$ is also chosen in $\mathbf{C}^{0,k}$, is given by

$$\phi(t) = \phi_0 o T_t(\bar{V}^*)^{-1}. \tag{5.14}$$

As $t \to T_t(\bar{V}^*)^{-1}(.) \in \mathbf{C}^{1,k}$ (see [5]) assuming the initial data $\phi_0 \in C^k(\bar{D})$, we get $\phi \in \mathbf{C}^{0,k-1}(D \setminus K_\phi)$, with the compact set $K_\phi = \{x \in D\ s.t.\ \nabla\phi(x) = 0\ \}$.

As $\phi(t) = \phi_0 o T_t^{-1}$, we get $\nabla\phi(t) =\ (\ (DT_t)^{-1}.\nabla\phi_0\ )oT_t^{-1}$ so that K_{ϕ_0} is void implies that , $\forall t,\ \ K_{\phi(t)}$ is empty. Then $V^\phi \in \mathbf{C}^{0,k-1}$ (assuming now that $k \ge 2$) and $\Omega_t(V^\phi = \Omega_t(\bar{V}^*)\Omega_t(V^*)$ (In other words, the three vector fields $V^*,\ \bar{V}^*,\ V^\phi$ build the same tube Q_V as they have the same normal speed v on the lateral boundary Σ_V). From (5.8) we get

$$-\frac{\partial}{\partial t}\phi(t,x)\ \ + < \nabla_x\phi(t,.),\ A^{-1}.G(\Omega_t(V^\phi))\ >= 0 \tag{5.15}$$

which is a Hamilton-Jacobi like equation for the function ϕ. From 5.9, we get

$$J(\Omega_t(V)) \le\ J(\Omega_0) - \alpha \int_0^t\ \ ||\frac{\partial\phi(s,.)}{\partial t}\,/\,\nabla_x\phi(s,.)||^2_{\mathbf{D}(A_k^{1/2})}ds. \tag{5.16}$$

Briefly we could say that the shape differential equation 5.8 is solved by the fixed point method (see [7], [26]) in a classical setting which does not permit the change of topology in the moving domain. We introduce here the weak setting which permits to handle that equation with possible topological changes by avoiding any homeomorphism.

5.3. Example of the operator A

Let D be a bounded domain in R^N with smooth boundary. Let $\vec{G}$ be a vector distribution with compact support in D, that is $\vec{G} \in \mathbf{E}'(D, R^N)$, of the form $\vec{G} = \gamma^*_\Gamma.(g\,\vec{n})$, where Γ is the boundary , a manifold of regularity C^1, of the domain Ω, with $\bar{\Omega} \subset D$. The trace (or restriction)operator is $\gamma_\Gamma \in \mathbf{L}(H^1_0(D, R^N), H^{1/2}(\Gamma, R^N))$ and its adjoint operator is $\gamma^*_\Gamma \in \mathbf{L}(H^{-1/2}(\Gamma, R^N), H^{-1}(D, R^N))$; the normal field on Γ outgoing from Ω is denoted by n and g given in $L^1(\Gamma)$, is a scalar function defined on the boundary Γ. We consider the linear operator

$A \in \mathbf{L}(H^1_0(D, R^N), H^{-1}(D, R^N))$ defined by

$A.F = (-\Delta F_1, ..., -\Delta F_N)$ for any $F_i \in H^1_0(D), 1 \le i \le N$.

We consider the element $U \in H^1_0(D, R^N)$, $U = -A^{-1}.\vec{G}$. It solves the problem $-\Delta U_i = 0$ in $D \setminus \Gamma$, $[DU.n] = g\,\vec{n}$ in $H^{-1/2}(\Gamma)$, where $[\vec{E}]_\Gamma$ stands for the *jump* term on Γ.

5.4. Shape Gradient Estimate.

In order to perform the fixed point approach in the non linear shape differential equation (5.8) we require the shape gradient G to verify an estimate as follows:

there exist two positive constants s and M such that :

$$\boxed{\forall \Omega \subset D,\ s.t.\ \partial\Omega \text{ is a } C^1 \text{ manifold}, \quad ||G(\Omega)||_{H^{-s}(D,R^N)} \le M} \tag{5.17}$$

We can immediately give several such examples. Consider the following *distributed* functionals : E being a mesurable subset in Ω, $y = y(\Omega) := (-\Delta)^{-1}.f$, f given in $L^2(D)$,

$$J(\Omega) = \int_E (y - Y_d)^{1/2} dx. \tag{5.18}$$

The Eulerian derivative is given by:

$$dj(\Omega, V) = < G, V(0) >$$

with

$$< G(\Omega), W > = \int_\Omega < A'(W).\nabla y, \nabla p > \ dx$$

$$+ \int_\Omega (< \nabla f, W > p \ - \ \chi_E\,(y - Y_d) \ < \nabla y, W >)dx, \tag{5.19}$$

where the symmetric matrix $A'(W) := 2\epsilon(W) - \ divW\ I_d$

is an element of $C^0([0, \tau], C^0(\bar{D}, R^{N^2}) \subset L^\infty([0, \tau] \times D, R^{N^2})$ (with $2\epsilon(W) := DW + DW^*$). The usual estimates holds:

$$||y(\Omega)||_{H^1_0(D)} \le 1/\sqrt{\lambda_1(D)}\ ||f||_{L^2(D)},$$

so that $||p(\Omega)||_{H^1_0(D)} \le M$ and then there exist a constant $C > 0$ such that for all $W \in W^{1,\infty}_0(D, R^N)$ we have

$$| < G(\Omega), W > | \le C\,||W||_{W^{1,\infty}(D,R^N)}.$$

As for $s > 1 + \frac{1}{N}$ we have

$H_0^s(D, R^N) \subset W^{1,\infty}(D, R^N)$, we get the boundedness of the gradient in $H^{-s}(D, R^N)$: There exists a positive constant

$C(||f||_{L^2(D)}, \lambda_1(D), ||Y_d||_{L^2(E)})$ such that

$$\forall \Omega \subset D,\ ||G(\Omega)||_{H^{-s}(D,R^N)} \leq C.$$

Choose for example $s = 2$ and the operator

$\mathbf{A} := A^2 \in \mathbf{L}(H_0^2(D, R^N), H^{-2}(D, R^N))$, we have

$$< \mathbf{A}.W, W >_{H^{-2}(D,R^N)\times H_0^2(D,R^N)} = \int_D \Sigma_{i=1,..,N} (\Delta W_i)^2\, dx = \int_D ||\Delta W||^2\, dx.$$

5.5. Existence of solution to the Shape Differential Equation (5.8).

The domain is bounded in R^N. For $k \geq 1$ we consider:

$$F^k = \{V \in C^0([0,1], C^k(\bar{D}, R^N) \cap H_0^1(D, R^N)\ )\ \}$$

Given a domain $\Omega_0 \subset D$ we consider the family

$$\mathbf{O}^k_{D,\Omega_0} := \{\Omega \subset D,\ \ \partial\Omega \in C^1,\ \exists V \in F^k,\ \ s.t.\ \Omega = T_1(V)(\Omega_0)\ \ \},$$

The family $\mathbf{O}^k_{D,\Omega_0}$ is equipped with the courant metric d_k , it is a complete metric space. Let $\Omega \in \mathbf{O}^k_{D,\Omega_0}$, we consider $B^k(\Omega) = \{\Omega' \in \mathbf{O}^k_{D,\Omega_0}\ \ s.t.\ \ d_k(\Omega, \Omega') \leq 1\ \ \}$. Then, for $k \geq 2$, $B^K(\Omega)$ is compact in $\mathbf{O}^{k-1}_{D,\Omega_0}$. Let us consider

$$\mathbf{F}^k_{D,\Omega_0} = \{\ (t \to \Omega_t)\ \in\ C^0([0,1], \mathbf{O}^k_{D,\Omega_0}\ )\ \ \}$$

and the *Shape Gradient mapping*

$$G\ \in\ C^0(\ \mathbf{O}^1_{D,\Omega_0}\ \ \mathbf{E}'(D, R^N)\ ).$$

Where $\mathbf{E}'D, R^N)$ is the linear space of vector Distributions on the the open set D with compact support (and then with finite order). In practice that shape gradient mapping will range in some "negative Sobolev " space over D, say:

$$G\ \in\ C^0(\ \mathbf{O}^1_{D,\Omega_0}\ \ H^{-s}(D, R^N) \cap \mathbf{E}'(D, R^N)\ )$$

We assume that $G(.)$ verifies the boundedness assumption (5.17). We apply the Leray fixed point theorem on the closed convex

$$K_M = \{V \in F^1\ \ s.t.\ \ ||V(t)||_{C^1(\bar{D},R^N)} \leq M\ \ \}$$

and the mapping

$$f:\ V \in K_M\ \to \Omega_t(V)\ \in \mathbf{K}^k_{D,\Omega_0}\ \to\ G(\Omega_t(V)) \in \mathbf{F}^k_{D,\Omega_0}\ \to\ -\mathbf{A}^{-1}.G(\Omega_t(V)) \in F^k_{D,\Omega_0}.$$

When M is large enough, the mapping f ranges in K_M: $f(K_m) \subset K_M$.

Theorem 5.1. *The gradient mapping g being continuous and bounded (5.17), there exists a vector field $V \in F^1$ such that $f(v) = V$. In other words, there exists $V \in F^1$ solution to the shape differential equation (5.8).*

The proof follows from the equicontinuity of the family of mapings ($t \rightarrow T_t(V)$), when V describes the convex set K_M and from the complete continuity (compactness) of the linear injection maping $H^s(D) \subset H^{s-\epsilon}(D)$, $\epsilon > 0$ as D is a smooth domain. (see [7]or [6] for a complete proof and several weaker results in that direction).

5.6. Asymptotic Domains

Under the global continuity and boundedness assumption (5.17) we can successively apply the previous existence result on all time interval $[n, n+1]$ so that we derive the existence of a continous solution V to equation (5.8) for any time $t > 0$. The asymptotic problem is then to characterise the situation in the limit as $t \rightarrow \infty$. From the decrease of the shape functional J whose G is the shape gradient under consideration we get, for any solution V^* of equation (5.8) :

$$j(\Omega_t(V) \leq J(\Omega_0) - \alpha \int_0^t ||V(s)||^2\, ds,$$

Thus, assuming $J(.) \geq 0$ (or more generally with a finite lower bound) we have

$$\int_0^\infty ||V^*(s)||^2\, ds \ < \ J(\Omega_0),$$

and hence,

$$V^* \in L^2(R_+,\ C^1(\bar{D}, R^N) \cup H_0^1(D, R^N)\ ).$$

From (5.8) we get

$$G^*(t) := G(\Omega_t(V^*)) \ \in \ L^2(0, \infty,\ H^{-1}(D, R^N)\,).$$

Thus there exist sequences $t_n \rightarrow \infty$ such that $V^*(t_n) \rightarrow 0$ and $G^*(t_n) \rightarrow 0$, in the respective topologies.

The main question is what is arriving to the domains Ω_{t_n}. From classical compactness of family of open subsets in D in the complementary Hausdorff topology, there exists an open set Ω_∞ in D such that $\Omega_{t_n} \rightarrow \Omega_\infty$ in the complementary Hausdorff topology. The main property for our purpose is the well known following "compactivorous" stability :

$$\forall \psi \in \mathbf{D}(\Omega_\infty),\ \exists\, N_{K(\psi)},\ s.t.\ n \geq N_{K(\psi)},\ \text{ implies } \ \psi \in \mathbf{D}(\Omega_{t_n}). \tag{5.20}$$

Here $K(\psi)$ designates the compact support of the function ψ. Let $\zeta_n = \chi_{\Omega_{t_n}}$ be the characteristic function of the set Ω_{t_n}, we have the weak-* convergence of ζ_n to some function $\lambda \in L^\infty(D)$, $0 \leq \lambda \ \leq 1$.

5.6.1. The asymptotic analysis for the Dirichlet Problem. Let us consider that the governing state equation of the shape functional J is the solution $y = y(\Omega) \in H_0^1(\Omega) \subset H_0^1(D)$, solution to $y = (-\Delta)^{-1}.F|_\Omega$, for given data $F \in L^2(D)$ (here we denote by $F|_\Omega$ the restriction of F to the open set Ω).

In that case we do have, with $y_n := y^0(\Omega_{t_n})$ (the extension by zero, element of $H_0^1(D)$):

$$(1 \ - \ \zeta_n\,)\, y_n \ = 0$$

So that in the limit (as $||y_n|||_{H^1_0(D)} \leq 1/\sqrt{\lambda_1(D)}\ ||F||_{L^2(D)}$ we assume , after extraction of a new subsequence, that y_n weakly converges in $H^1_0(D)$, then strongly in $L^2(D)$, to some element $y_\infty \in H^1_0(D)$) :

$$(1 - \lambda(x)\,)\ y_\infty(x) = 0\ a.e.x \in D.$$

Now, as $y_\infty \in H^1_0(D)$, we consider

$$\tilde{\Omega}_\infty := \{x \in D\ s.t.\ y_\infty(x) > 0\ \},$$

is a quasi open subset in D (open up to a zero capacity subset) and then we get

$$\lambda \geq \chi_{\tilde{\Omega}_\infty}. \tag{5.21}$$

from the definition ofy_n we get :

$$\forall \phi \in \mathbf{D}(\Omega_{t_n}),\ \int_D (< \nabla y_n, \nabla \psi > -F\,\psi\,)\ dx = 0.$$

(This integral should be taken over the smooth open set Ω_{t_n}, but as ψ is compactly supported in Ω_{t_n} we can write it over the larger domain D). From (5.20) this equality can be extended to any element $\psi \in \mathbf{D}(\Omega_\infty)$ as soon as $N \geq N_{K(\psi)}$. Then, in the limit we derive :

$$\forall \phi \in \mathbf{D}(\Omega_\infty),\ \int_D (< \nabla y_\infty, \nabla \psi > -F\,\psi\,)\ dx = 0. \tag{5.22}$$

Now we must pay attention to the fact that the open sets Ω_∞ and $\tilde{\Omega}_\infty$ are not known to have smooth boundaries (not even with N dimensional zero measures) so we have to distinguish between the two main definitions for the Sobolev space H^1_0. Let us consider

$$H^1_0(\Omega_\infty) := \{\phi \in H^1_0(D)\ s.t.\ \phi(x) = 0\ q.e.x \in D \setminus \Omega_\infty\ \}$$

And the smaller linear subspace :

$$\mathbf{H}^1_0(\Omega_\infty) = \text{closure of } \mathbf{D}(\Omega_\infty\,)$$

$$\text{in } H^1_0(D) = cl_{\{H^1_0(D)\}}(\ \mathbf{D}(\Omega_\infty)\,) \subset H^1_0(\Omega_\infty) \subset H^1_0(D).$$

A priori we do have $y_\infty \in H^1_0(\Omega_\infty)$ verifying the weak equation (5.22) for any test functions ψ in the *smaller* space $\mathbf{H}^1_0(\Omega_\infty)$. A sufficient condition in order to derive the equality of these two subspaces of $H^1_0(D)$ is a Wiener condition on the local capacity on the complementary of Ω_∞ see [41], [42], [43] . In dimension 2 these conditions are fullfield , see [45] and the original 2D result in [44]. The idea is that the initial domain Ω_0 being smooth is such that its complementary in D has a finite number of connected components. (which we denote by $\#(D \setminus \Omega_0)$). Now, this number is lower semi continuous for the Hausdorff complementary convergence of open subset of D, that is $\#(D \setminus \Omega_\infty) \leq liminf\#(D \setminus \Omega_{t_n}) = \#(D \setminus \Omega_0)$. In 2D the non connectivity of two closed subsets implies that there exist a linear segment reaching these sets and give a lower bound on the local relative capacity which enables us to conclude and we get the shape gradient asymptotic stability : As we have the following convergences in $H^1_0(D)$:

$$y_n \to y(\Omega_\infty)$$

$$p_n \to p(\Omega_\infty)$$

$$< G(\Omega_{t_n}), W >= \int_{\Omega_{t_n}} < A'(W).\nabla y_n, \nabla p_n > \, dx$$

$$+ \int_{\Omega_{t_n}} (< \nabla f, W > p_n - \chi_E \, (y_n - Y_d) < \nabla y_n, W >) dx \tag{5.23}$$

$$= \int_D < A'(W).\nabla y_n, \nabla p_n > \, dx$$

$$+ \int_D (< \nabla f, W > p_n - \chi_E \, (y_n - Y_d) < \nabla y_n, W >) dx \tag{5.24}$$

$$\longrightarrow \int_D < A'(W).\nabla y_\infty, \nabla p_\infty > \, dx$$

$$+ \int_D (< \nabla f, W > p_\infty - \chi_E \, (y_\infty - Y_d) < \nabla y_\infty, W >) dx \tag{5.25}$$

As the elements are zero a.e. in $D \setminus \Omega_\infty$ we get :

Proposition 5.2. *The shape gradient $G(\Omega_{t_n})$ defined by (13.1) (weakly) converges as element of $(F^1)'$ to a Distribution $G(\Omega_\infty) \in W_0^{1,\infty}(D, R^N)'$ characterised by :*

$$\forall W \in W_0^{1,\infty}(D, R^N), \; < G(\Omega_\infty), W >= \int_{\Omega_\infty} < A'(W).\nabla y_\infty, \nabla p_\infty > \, dx$$

$$+ \int_{\Omega_\infty} (< \nabla f, W > p_\infty - \chi_E \, (y_\infty - Y_d) < \nabla y_\infty, W >) dx \tag{5.26}$$

5.7. Topological change in finite time

The previous asymptotic analysis when $t \to \infty$ can be brought back at time $s = 1$ by the following change of ("time" variable) :

Let $s = 1 - \, (t+1)^{-1}$ so that $t = \frac{s}{1-s}$ and $\frac{\partial t}{\partial s} = \frac{1}{(1-s)^2}$. When t describes the line $[0 + \infty[$, the variable s describes the interval $[0, 1]$. We set

$$\tilde{V}(s) := V(\frac{s}{1-s}), \; \tilde{\Omega}_s := \Omega_{\frac{s}{1-s}}, \; \tilde{y}_s := y(\Omega_{\frac{s}{1-s}}), \; ...$$

Let $\Phi(t, x)$ be a solution to

$$\forall t, \;\; t > 0 \; , \; \frac{\partial}{\partial t}\Phi + < \nabla_x \Phi, \, V >= 0, \;\; \Phi(0, .) = \Phi_0(.).$$

We also set $\tilde{\Phi}(s, x) = \Phi(\frac{s}{1-s}, x)$ and this function solves the problem

$$\forall s \in [0, \, 1[, \;\; \frac{\partial}{\partial s}\tilde{\Phi}(s, x) + \, < \, \nabla_x \tilde{\Phi}(s, x), \; \frac{1}{(1-s)^2} \, \tilde{V}(s, x) \; >= \, 0, \;\; \tilde{\Phi}(0, .) = \Phi_0(.).$$

Consider

$$\mathbf{V}(s,x) := \frac{1}{(1-s)^2}\,\tilde{V}(s,x).$$

Then $\tilde{\Phi}$ solves the $\mathbf{V}$-convection problem

$$\forall s \in [0,\,1[,\;\; \frac{\partial}{\partial s}\tilde{\Phi} + <\nabla_x \tilde{\Phi}(s,x), \mathbf{V}> = 0,\;\; \tilde{\Phi}(0,.) = \Phi_0(.). \tag{5.27}$$

If $\mathbf{V}$ was smooth enough on the finite interval $s \in [0,\,1]$ we would have, as previously for V the convected unique solution to (5.27) : $\tilde{\Phi}(s) = \Phi_0 o T_s(\mathbf{V})^{-1}$. But now the field $\mathbf{V}$ is non smooth on the closed interval [0, 1]. Effectively from (5.17) we see that $||V(t)||$ is bounded so that

$$||\mathbf{V}(s,.)||_{H^2(D,R^N)\cap H^1_0(D,R^N)} \;=\; \mathbf{O}(\frac{1}{(1-s)^2}),\;\; s \to 1.$$

Moreover we get:

$$\tilde{V}(s)/(1-s) \;\in\; L^2(0,\,1, H^2(D,R^N) \cap H^1_0(D,R^N)\,),$$

so that we cannot conclude for $\mathbf{V}(s,.)$ to be in $L^p(0,\,1, H^2(D,R^N)\cap H^1_0(D,R^N)\,)$, for some $p \geq 1$, which would be necessary in order to derive existence and also uniquiness to the solution of the Hamilton Jacobi-like equation (5.15) using the results derived in [33] (based on the use by L. Ambrosio [22] of Alberti rank one theorem) . One way to get an existence result is to modify the choice of the speed vector field in the previous fixed point construction as follows :

5.7.1. Existence result for the Hamilton Jacobi eq. with possible topological changes.
Let us consider a solution V^{**} to the following fixed point problem :

$$t \leq 0,\;\; V^{**}(t) := -(t+1)^{-2}\; A^{-1}.G(\Omega_t(V^{**})).$$

We get

$$\int_0^\infty (t+1)^2\; ||V^{**}(t)||^2\, dt \;\leq\; \frac{1}{\alpha}\, J(\Omega_0).$$

Then by change of variable :

$$\int_0^1 \; ||V^{**}(\frac{s}{1-s})||^2\; \frac{1}{(1-s)^4}\; ds \leq \frac{1}{\alpha}\, J(\Omega_0)$$

We set

$$\bar{\mathbf{V}}(s,x) := \;\frac{1}{(1-s)^2}\, V^{**}(\frac{s}{1-s}) \;\in\; L^2(0,\,1,\, L^2(0,\,1, H^2(D,R^N)\cap H^1_0(D,R^N)\,)$$

Again with

$$\Phi(t,.) := \Phi_0 o T_t(V)^{-1}$$

Φ solves the convection problem in strong form over $[0,\,\infty[$

$$\Phi(0,.) = \Phi_0,\;\; \frac{\partial}{\partial t}\Phi \;+\; \nabla\Phi.V(t) = 0.$$

Let

$$\Phi^*(s,x) := \Phi(\frac{s}{1-s}, x).$$

Then it also solves the convection problem in strong form for $0 \le s < 1$:

$$\Phi(0,.) = \Phi_0,\ \frac{\partial}{\partial s}\Phi \,+\, \nabla\Phi.\bar{\mathbf{V}}(s) = 0.$$

With the vector field $\bar{\mathbf{V}} \in L^2(0,\tau,H)$ we get existence and uniquiness of solution on any interval $(0,\,\tau)$ hence this solution can be extended for larger values of s so that we obtain solution through the possible topological change.

6. Tubes associated to BV vector fields

Let $V \in \mathbf{E}^p$. Equipped with the graph norm $\mathbf{E}^p$ is a Banach space.

Proposition 6.1. *Let $1 \le p < \infty$, then $H^1_0(D)$ is a dense subspace in $\mathbf{E}^p(D)$.*

The proof is done in three steps.

Lemma 6.2. *For any $V \in \mathbf{E}^p(D)$ let V^0 designate the extension by zero outside of D. Then we have $divV^0 = (divV)^0$ and $V^0 \in \mathbf{E}^p := \{\, V \in L^p(R^N, R^N),\ divV \in L^p(R^N)\ \}$.*

Let b designate the oriented distance function to the bounded domain D . For given $h > 0$, a small enough parameter, consider the "cut oriented distance function " with support in the narrow band h , $b_h := \rho o b$ where the cutting function ρ is smooth , positive, with support in the interval $[-2h,\, +2h]$ and taking the value $\rho = 1$ on the subinterval $[-h, +h]$.We introduce the flow mapping $T_h := T_h(\nabla b_h)$ and consider the following mapping

$$\mathbf{T}_h\ :\ V \in \mathbf{E}^p(D)\ \longrightarrow V_h := det(DT_h)\, DT_h\, V^0 o T_h.$$

Lemma 6.3. *$div(V_h) = detDT_h\ div(V^0)oT_h$.*

It follows that $divV_h \;=\; detDT_h\,(\,(divV)^0)oT_h \;\in\; L^2(R^N)$. Moreover, as $h > 0$, we have $V_h.n = 0$, so that

$$V_h \in \mathbf{E}^p(D).$$

We consider a mollifier $r_h \to \delta_0$ as $h \to 0$ with support of $r_h \subset B(0, h/2)$ so that the support verifies (for $h > 0$) $supt\{\ r_h * (\ \mathbf{T}_h.V\ )\ \}\ \subset\ D$. Thus

$$V^h := r_h * (\ \mathbf{T}_h.V\ ) \quad \in \mathbf{D}(D, R^N) \subset H^1_0(D, R^N).$$

It is now immediate to verify that $V^h \to V$ strongly in $\mathbf{E}^p(D)$, which establishes the density result without any geometrical assumption on the domain D. Then we may restrict to the boounded domain D the results from [22]:

Theorem 6.4. *Let $V \in \mathbf{E}^{1,1} \cap L^1(0,\tau, BV(D, R^N))$. Assume that $(divV)^+$ (resp. $(divV)^-$)is in $L^1(0,\tau, L^\infty(D, R^N))$, then problem (7.5) (resp. (7.2)) has a unique solution ζ such that $(\zeta, V) \in \mathbf{T}^1_{\Omega_0}$.*

Then we see that some regularity on V implies that , for given V, the characteristic solution is unique. We denote it by ζ_V. The converse is false: for given ζ the set of V such that $(\zeta, V) \in \mathbf{T}_\Omega^{p,q}$ is a closed convex set that we denote by $\mathbf{K}(\zeta)$. For $1 < p, q < \infty$, In that convex set we can find a *minimal* element V_ζ (which minizes the ssociated normes in $\mathbf{K}(\zeta)$.

6.1. Tube Energy - *Variational Problem***.**

For any given positive constants $a > 0$, $\sigma \geq 0$, $\mu \geq 0$ and $\nu \geq 0$ we shall consider the minimization associated to the following functionals:

$$J^a(V) = 1/2 \int_I \int_D (a + \xi_V)\,(\,|V(t,x)|^2 + (divV(t,x))^2\,)\,dxdt, \tag{6.1}$$

$$J^a_{\sigma,\mu,\nu}(V) = J^a(V)$$

$$+\sigma \int_0^\tau ||\nabla(\xi_V(t))||_{M^1(D)}dt + \mu\Theta(V,\Omega_0) + \nu \int_0^\tau \int_D DV..DV\,dxdt. \tag{6.2}$$

We shall consider the three situations associated with $\sigma+\mu+\nu > 0$ and $\sigma\,\mu\,\nu = 0$. When ν is zero the terms σ and μ will play a surface tension role at the dynamical interface while the case $\sigma + \mu = 0$ should be considered as a mathematical regularisation, as in the non usual variational interpretation developed in the previous section $\nu > 0$ does not lead to the usual viscosity term (i.e. does not lead to the Navier Stoke equations)

Theorem 6.5. *Assuming $V \in \mathbf{E}^2$, $a > 0$, $\sigma.\eta.\nu > 0$, there exists $V \in \mathbf{E}^2$ such that, $\forall W \in \mathbf{E}^2$:*

$$J^a_{\sigma,\eta,\nu}(V) \leq J^a_{\sigma,\eta,\nu}(W).$$

7. Saddle points

Let us consider the following more general situation :

$$\phi(0) = \phi_0,\ \frac{\partial}{\partial t}\phi + \nabla\phi.V = f, \tag{7.1}$$

$$\psi(0) = \psi_0,\ \frac{\partial}{\partial t}\psi + div(\psi\, V) = g \tag{7.2}$$

Let two real numbers (p, q) be given, with $1 < p \leq \infty$, $1 \leq q < \infty$, and the linear space for speed vector fields:

$$E^{p,q} = \{\ V \in L^p(0,\tau, L^q(D, R^N))\ s.t.\ divV \in L^p(0,\tau, L^q(D))\ \},$$

$$\mathbf{E}^{p,q} = \{\,V \in E^{p,q},\ s.t.\ \ V.n = 0 \text{ in } W^{-1,1}(\partial D)\ \}. \tag{7.3}$$

The null condition on the normal component of the vector filed at the boundary can be weakly written as

$$\forall\phi \in L^2(I, C^1(\bar{D})),\ \int_I \int_D (\,divV\ \phi + \nabla\phi.V\,)\,dtdx = 0. \tag{7.4}$$

Proposition 7.1. *([13]) Assume $V \in \mathbf{E}^{2,2}$. If $(divV)^+ \in L^1(0,\tau,L^\infty(D))$, problem (7.1) has solutions*

$$\phi \in L^\infty(0,\tau,L^2(D)) \cap H^1(0,\tau,H^{-1}(D)) \subset C^0([0,\tau],H^{-1/2}(D)).$$

If $(divV)^- \in L^1(0,\tau,L^\infty(D))$ problem (7.2) has solutions

$$\psi \in L^\infty(0,\tau,L^2(D)) \cap H^1(0,\tau,H^{-1}(D)) \subset C^0([0,\tau],H^{-1/2}(D)).$$

The first idea would be to consider $divV \in L^1(0,\tau,L^\infty(D))$. Then both problems have solutions. They are, formally, adjoints problems of each other, and we could be tempted to conclude uniqueness to both problems. That argument does not apply as one of the two solutions ϕ or ψ should be smooth in order to be "put in duality". Then under previous poor regularity on V we will not get existence nor uniquiness for shape convection problem (7.5):

$$\zeta(0) = \zeta_{\Omega_0},\ \frac{\partial}{\partial t}\zeta + \nabla\zeta.V = 0,\ \ \zeta = \zeta^2 \tag{7.5}$$

Functional setting: To give sense to the product $\nabla\zeta.V$ in (7.5), we write it as

$$\nabla\zeta.V = div(\zeta\, V) - \zeta\, divV.$$

Then, as soon as $\zeta \in L^\infty((0,\tau)\times D)$, that term makes sense in $L^1(0,\tau,W^{-1,1}(D))$ when V and its divergence $divV$ are in $L^1(0,\tau,L^1(D))$. we consider a vector fields V in $\mathbf{E}^{2,2}$ and any function $G \in L^\infty(D)$ verifying $G \geq \alpha > 0$. We consider the Lagrangian expression for the functional

$$g(V) = Inf_{\zeta\in\mathbf{U}_V}\ Sup_{\phi\in\mathbf{H}_V}\ \ \mathbf{L}_V(\zeta,\phi),$$

with

$$\mathbf{L}_V(\zeta,\phi) = \int_0^\tau \int_D \{\ 1/2\ \zeta^2\ G\ \ + \zeta\,(\frac{\partial}{\partial t}\phi + div(\phi\, V)\,)\}dxdt - \int_{\Omega_0} \phi(0)dx\,,$$

$$\mathbf{H}_V\ =\{\ \ \phi \in L^2(0,\tau,L^2(D))\ \ s.t.\ \frac{\partial}{\partial t}\phi + div(\phi\, V)\ \in L^2(I\times D)\,,\ \ \phi(\tau)=0\}\,,$$

$$\mathbf{U}_V = \{\,\zeta \in L^2(I\times D),\ s.t.\ \frac{\partial}{\partial t}\zeta + \nabla\zeta\, V)\ \in L^2(I\times D)\,,\ \ \zeta(0) = \chi_{\Omega_0}\,\}.$$

Notice that the elements of $\mathbf{H}_V$ are continuous ($\phi \in C^0([0,\tau],W^{-1/2,1}(D))$ so that $\phi(\tau)$ makes sense.

The Lagrangian $\mathbf{L}_V$ is concave-convex on $\mathbf{U}_V \times \mathbf{H}_V$. $L^2(I\times D)\times\mathbf{E}^{2,2}$.

Saddle points (ξ,λ) are solution to the system composed of equation (7.1) (with $\Phi_0 = \chi_{\Omega_0},\ \ f=0$) and the following backward "adjoint equation"

$$\frac{\partial}{\partial t}\lambda + div(\lambda V\,)\ =\ \ -\xi_V\, G\,,\quad \lambda(\tau)=0. \tag{7.6}$$

The converse is true when we have an extra density condition on V and $divV$:

$$\text{Assumption on V}:\ \ \{\phi \in C^\infty(I\times D)\cap\mathbf{E}^{2,2}\ \}\text{ is dense in }\mathbf{H}_V. \tag{7.7}$$

The weakly coupled system (7.1), (7.6) posesses solutions when

$(divV)^+ \in L^1(I,L^\infty(D))$. We derive the following uniqueness results for the convection problem (7.1):

Proposition 7.2. *Assume* $V \in \mathbf{E}^{2,2}$*, verifying (7.7) and*
$(divV)^+ \in L^1(I, L^\infty(D))$.

Then, with $f = 0$ *and* $\Phi_0 = \chi_{\Omega_0}$ *(convection problem), or more generally* $\Phi_0 \in L^\infty(D)$*, the problem (7.1) posesses a unique solution* ζ_V *verifying*

$$0 \leq \zeta_V \leq 1 \quad a.e.(t,x) \in I \times D$$

or (in the more general setting)

$$Infess\ \Phi_0 \ \leq \ \zeta_V \ \leq \ Supess\ \Phi_0.$$

We have the monotonicity : $\Omega_0^1 \subset \Omega_0^2$ *(or, in the more general setting* $\Phi_0^1 \leq \Phi_0^2$ *) implies* $\zeta_V^1 \ \leq \ \zeta_V^2$.

Proof : from the strong assumption (7.7) the set of saddle-points is not empty and is completely characterized by the system (7.1)-(7.6). let us denote by $\mathbf{S}_V$ the set of saddle-points. We know that it can be written as $\mathbf{S}_V = A_V \times B_V$, which means that if (ζ^i, λ^i), $i = 1, 2$ are saddle points then ζ^1, λ^2 and ζ^2, λ^1 are also saddle points. We infer that equation (7.6) with right hand side $G\,\zeta^i$ has solutions and we derive uniqueness of ζ_V (from the fact that $G > 0$), being the single element in A_V. From uniqueness we know that $0 \leq \zeta \leq 1$.

7.1. Derivative with respect to the speed field V.

Functionals J in form of min max have a well known Gateaux derivative with respect to V. Now, it is important to notice that in the present saddle point formulation the linear vector spaces depend on the parameter V.

With $div \in L^1(I, L^\infty(D))$ the Ambrosio results apply as we have $H^1(D) \subset W^{1,1}(D) \subset BV(D)$.

Applying the results from [2] and [18] for differentiation under uniqueness of the saddle point, and assuming

$$V \in B := L^2(0, \tau, H_0^1(D, R^N)),\ \mathbf{L}_V^\nu := \mathbf{L}_V + \nu/2 \int_0^\tau \int_D DV..DV\,dxdt,$$

we have

$$J'(V, W) =< (a + \zeta)V - \nabla(\zeta divV) - \nu\Delta V - \lambda\,\nabla\zeta,\ W >_{F' \times F}\ .$$

We shall give now a sense to the term $\lambda\,\nabla\zeta$, using the concept of a *transverse field* Z which has been introduced in [12] and further developed in [11], [13], [25], [5], [17],[26]. Indeed we have

$$\lambda\,\nabla\zeta = \nabla(\zeta\,\lambda) - \zeta\,\nabla\lambda.$$

We give a sense to that last term $\zeta\,\nabla\lambda$ through the *transverse field* problem and its adjoint .

7.2. Transverse derivative

For two given vector fields V and W the *transverse* field Z is the solution to the Lie bracket evolution

$$H_V.Z = W,\ H_V.Z := \frac{\partial}{\partial t}Z + \ [Z, V], \quad [Z, V] := DZ.V \ - \ DV.Z \tag{7.8}$$

For *avoiding some technicalities* we assume now (as it will be the cas in the first example associated with modeling of arteris) that V and W are free divergence vector fields: $divV = divW = 0$. The adjoint oprator is then :

$$H_V^*.\Lambda := -\frac{\partial}{\partial t}\Lambda - D\Lambda.V \ - \ D^*V.\Lambda \tag{7.9}$$

Lemma 7.3.

$$H_V.(\zeta\,\vec{e}) = (\frac{\partial}{\partial t}\zeta + \nabla\zeta.V)\,\vec{e} \,+\, \zeta\,H_V.\vec{e} \tag{7.10}$$

$$H_V^*(\zeta\,\vec{f}) = -(\frac{\partial}{\partial t}\zeta + \nabla\zeta.V)\,\vec{f} \,-\, \zeta\,H_V^*.\vec{f} \tag{7.11}$$

We see that if (ζ, V) is a tube then the two previous expressions simplifies as the first term vanishes from 7.5. Specifiqueley we have

$$H_V^*.(\zeta\,\nabla\lambda) = -\zeta\,(\frac{\partial}{\partial t}\nabla\lambda + D\nabla\lambda.V + D^*V.\nabla\lambda) = -\zeta\nabla(\,\frac{\partial}{\partial t}\lambda + \nabla\lambda.V\,)$$

Then, if we set the "scalar" operator h_V by:

$$h_V.\psi := \frac{\partial}{\partial t}\psi + \nabla\psi.V$$

we get, (ζ, V) being any tube:

$$H_V^*(\zeta\,\nabla\lambda) = -\zeta\,\nabla(\ h_V.\lambda). \tag{7.12}$$

Notice that here, as $divV = 0$, the adjoint operator verifies $h_V^* = -h_V$. By "reversing" the time, that operator h_V is then self adjoint. Nevertheless, it is *not* maximally defined in the admissible setting so that it is not possible to use same classical abstract setting of evolution linear system in order to solve (7.1) and/or (7.2).

8. Shape tube metric for smooth sets

Assuming the domain Ω_0 to be smooth, i.e its boundary being a C^k manifold, for $k \geq 1$, we consider the subfamily

$$\mathbf{O}^k_{\Omega_0} = \{\Omega \in \mathbf{O}_{\Omega_0}, \Omega = \zeta(1),\ \ \zeta \text{ having lateral boundary } \ \Sigma \text{ piecewise } \ C^k\}.$$

Let $\Omega_1,\ \Omega_2$ be given in $\mathbf{O}^k_{\Omega_0}$, we consider

$$T_k(\Omega_1, \Omega_2) = \{\ \ \zeta \in T(\Omega_1, \Omega_2)\ s.t.\,\Sigma \text{ is piecewise } \ C^k\ \ \}$$

and introduce the metric

$$\delta(\Omega_1, \Omega_2) := Inf_{\{\zeta \in T_k(\Omega_1,\Omega_2)\}}\ \ ||v||_{L^1(I, L^1(\Gamma_t))}.$$

In general the infimum is not a minimum. Notice also that, as for any $F \in C^0(\bar{D})$:

$$\int_\Sigma f\, d\Sigma = \int_0^1 dt \int_{\Gamma_t} f(t,x)\, \sqrt{1+v^2}\, d\Gamma_t(x),$$

and we have

$$\delta(\Omega_1,\Omega_2) = Inf_{\{\zeta\in T_k(\Omega_1,\Omega_2)\}} \int_\Sigma \frac{|v|}{\sqrt{1+v^2}} d\Sigma$$

$$\leq Min\ \{\ Inf_{\{\zeta\in T_k(\Omega_1,\Omega_2)\}} \int_\Sigma |v|\, d\Sigma\ ,\quad Inf_{\{\zeta\in T_k(\Omega_1,\Omega_2)\}} \int_\Sigma d\Sigma\quad \}.$$

It can be verified that the two majorant terms to be a metric. The first one does not verify the triangle axiom and the second does not verify the first metric axiom (as it cannot be zero with $\tau = 1$).

Proposition 8.1. *Let $k \geq 1$, then, equipped with δ, $\mathbf{O}^k_{\Omega_0}$ is a metric space.*

Obviously δ is non negative. Assume that $\delta(\Omega_1,\Omega_2) = 0$, then $\forall t \in I,\ v(t,x) = 0$ on Γ_t; then the time space normal vector field satisfies
$\nu(t,x) = \frac{1}{\sqrt{1+v(t,x)^2}}\,(-v(t,x),\vec{n}_t(t,x)\,) = (0,\,\vec{n}_t(x)\,)$.

When the tube is a cylinder and the domain Ω_t does not depend on t , then $\Omega_1 = \Omega_2$. The symmetry of δ is immediate by taking the backward tube : $\zeta'(t,.) := \zeta(1-t,.)$.

Concerning the triangle inequality, assume three domains Ω_i in D and tubes ζ^1 connecting $\Omega_1,\ \ \Omega_2$, ζ^2 connecting $\Omega_2,\ \ \Omega_3$ with both of them realizing the infimum in the δ definition up to some given $\epsilon > 0$.

Let us consider the following new piecewise C^1-tube defined (through its characteristic function ζ) as follows :

$$\zeta(t,x) = \zeta^1(2t,x), \qquad 0 \leq t \leq 1/2$$

$$\zeta(t,x) = \zeta^2(2t-1,x),\ , \qquad 1/2\ \leq\ t\ \leq\ 1$$

It can be easily verified that its normal speed v is given by

$$v(t,x) = 2v^1(2t,x),\ 0 \leq\ t\ \leq\ 1/2$$

$$v(t,x) = 2v^2(2t-1,x), \quad 1/2\ \leq\ t\ \leq\ 1$$

Now by construction ζ connects Ω_1 to Ω_3, $\in \bar{\mathbf{T}}_k(\Omega_1,\,\Omega_3)$, hence we get

$$\delta(\Omega_1,\Omega_3) \leq \int_0^1\ ||\ v(t)\ ||_{L^1(\Gamma_t)}\ dt$$

$$= \int_0^{1/2}\ ||\ v(t)\ ||_{L^1(\Gamma_t)}\ dt + \int_{1/2}^1\ ||\ v(t)\ ||_{L^1(\Gamma_t)}\ dt$$

$$= \int_0^{1/2}\ ||\ 2v^1(2t)\ ||_{L^1(\Gamma^1_{2t})}\ dt + \int_{1/2}^1\ ||\ 2v^2(2t-1)\ ||_{L^1(\Gamma_{2t-1})}\ dt$$

$$= \int_0^1\ 2\,||\ v^1(r)\ ||_{L^1(\Gamma^1_r)}\ 1/2\, dr + \int_0^1\ 2\,||\ v^2(u)\ ||_{L^1(\Gamma_u)}\ 1/2\, du,$$

as v^1 (resp. v^2) realises the infimum (in the definition of $\delta(\Omega_1,\Omega_2)$ (resp. $\delta(\Omega_2,\Omega_3)$) up to $\epsilon > 0$. Then $\forall \epsilon > 0$ we get

$$\delta(\Omega_1,\Omega_3) \leq \delta(\Omega_1,\Omega_2)+\delta(\Omega_2,\Omega_3)+2\,\epsilon$$

9. Geodesic for the metric $\bar{\bar{\delta}}$

9.1. Derivative of the metric terms

9.1.1. Admissibles variations for the godesic: transverse fields Z preserving the extremities of the tube. Let us assume that a tube $(\zeta, V) \in \mathbf{C}_{\Omega_1,\Omega_2}$ is a minimizer for $\delta_2(\Omega_1,\Omega_2)$. Let $\mathbf{Z}(s;\,t,x) \in R^{N+1}$ be a "horizontal vector field" $\mathbf{Z}(s;\,t,x) = (0,\,Z(s;\,t,x))$, where $Z(s;\,t,x) \in R^N$. The variable s will be the (tube) perturbation parameter while the time t is an independant parameter in the horizontal (transverse) flow mapping $T_s(Z(t))$ (here $Z(t) = Z(s;t,x) := Z(t)(s,x)$). The R^{N+1} flow mapping is

$$T_s(\mathbf{Z})(t,x) = (t,x) + \ (0, \int_0^s Z(\sigma;\,t, T_\sigma(Z(t))(x)d\sigma\ )$$

that is $T_s(\mathbf{Z})(t,x) = (t,\,T_s(\,Z(t)\,)(x)\ )$. We are looking for necessary optimality conditions solved by the vector field V (or at least by its normal component $v(t,.) =< V(t,.), n_t(.) >$ on the lateral boundary. Consider any Z such that $Z(s;\,0,x) = Z(s;\,1,x) = 0$. Then if Q is a tube which connect the two sets Ω_1 and Ω_2, as $T_s(,Z(0)\,) = T_s(\,Z(1)\,) = 0$, the tubes

$$Q^s := T_s(\mathbf{Z})(Q) \ \text{ connects the two sets } \ \Omega_1,\ \Omega_2.$$

Let v^s be the normal speed of Σ^s. As usual

$$\nu^s o T_s(\mathbf{Z}) = ||cof(\,D_{t,x}T_s(\mathbf{Z}).\nu||^{-1}_{R^{N+1}}\ cof(\,D_{t,x}T_s(\mathbf{Z}).\nu,$$

where the cofactor is given by $cof A := det A\ (A^*)^{-1}$. From that expression we get the explicit representation for

$$\nu^s o T_s = \frac{1}{\sqrt{1+(v^s o T_s)^2}}\ (-v o T_s,\, \vec{n}^s_t).$$

We will be interested in the expression for $v^s o T_s$ and its (material) derivative

$$\dot{v}(t,s) := \frac{d}{ds}(v^s o T_s(\mathbf{Z})\ )|_{s=0} \quad \text{on } \Gamma_t$$

Then the *shape boundary derivative* is implicitly given through Z as:

$$v(t)'_{\Gamma_t} := \dot{v} \,-\, \nabla_{\Gamma_t} v(t).Z(t)_{\Gamma_t}$$

(see the books [9], [24], [10], [26]). We shall prove the

Theorem 9.1.

$$v(t)\,(v(t)^s)'_{\Gamma_t} \,=\, v(t)\,\frac{\partial}{\partial t}(< Z(0,t,.), n_t(.) >\,) \tag{9.1}$$

9.2. Optimality condition for the vector field V.

Let $p \geq 1$, and consider that the tube Q_V minimizes the term

$$J := \int_0^1 (\int_{\Gamma_t(V)} | < V(t), n_t > | \, d\Gamma_t \ )^p \, dt,$$

Then, considering a "transverse horizontal" field $\mathbf{Z} = (0, Z(s; t, x))$, we get $j(0) \leq j(s)$, with

$$j(s) := \int_0^1 (\int_{T_s(\mathbf{Z})(\Gamma_t(V))} |v^s| \, d\Gamma_t^s \ )^p \ dt, \quad J = j(0),$$

where of course $\Gamma_t^s = T_s(\mathbf{Z})(\Gamma_t)$, $\Sigma^s = T_s(\mathbf{Z})(\Sigma) = \cup_{0<t<1} \{t\} \times \Gamma_t^s$ and ν^s is the normal field on Σ^s in the form $\nu^s = \frac{1}{\sqrt{1+(v^s)^2}}(-v^s, n_t^s)$ where n_t^s is the N-dimensional (horizontal) normal filed to Γ_t^s. With

$$a(t) := p \, (\int_{\Gamma_t(V)} | < V(t), n_t > | \, d\Gamma_t \ )^{p-1},$$

we have:

$$j'_+(0) = \int_0^1 a(t) \, \{ \int_{\Gamma_t} sgn\,(v) \, v(t)'_{\Gamma_t}(Z) \ d\Gamma_t \, \} \, dt$$

$$+ \int_0^1 a(t) \, \{ \int_{\Gamma_t \cap \{v(t)^{-1}(0)\}} |v(t)'_{\Gamma_t}(Z)| \ \ d\Gamma_t \, \} \, dt.$$

Hence from theorem (9.1) we obtain :

Theorem 9.2. *Let ζ be a minimizer of $d(\Omega_1, \Omega_2)$. Then for any smooth vector field Z such that $Z(0, .) = Z(1, .) = 0$, we have*

$$\int_0^1 a(t) \{ \int_{\Gamma_t} sgn \, v(t) \ \frac{\partial}{\partial t} (< Z(t, .), n_t(.) >) \, d\Gamma_t$$

$$+ \int_{\Gamma_t \cap \{v(t)^{-1}(0)\}} |v(t)'_{\Gamma_t}(Z)| \ \ d\Gamma_t \, \} \, dt \ \geq \ 0$$

Formal calculus (fro $p = 1$) would lead to

$$j'(0) = -\int_0^1 \{ \int_{\Gamma_t} [\ \frac{\partial}{\partial t} sgnv(t) \, (< Z(0, t, .), n_t(.) >)$$

$$+ H(t) \, sgn \, v(t) \ \ v(t) \ \ < Z(0, t, .), n_t >] d\Gamma_t \ \} \ \ dt$$

$$+ \int_0^1 \{ \int_{\Gamma_t \cap \{v(t)^{-1}(0)\}} |v(t)'_{\Gamma_t}(Z)| \ \ d\Gamma_t \, \} \, dt$$

(as we already assumed that Z is chosen such that $\frac{\partial}{\partial n_t}(< Z(t), n_t >) = 0$ on Γ_t (as well as the same for $v(t) =< V(t), n_t >$). As $sgnv(t) \ v(t) \ = \ |v(t)|$, the necessary condition leads to the following geodesic condition:

Proposition 9.3. *Let $(\zeta, V) \in \mathbf{T}(\Omega_1, \Omega_2)$ be a smooth minimizer of $d(\Omega_1, \Omega_2)$. Then the time-space normal field ν verifies*

$$\exists\, c(t),\ s.t.\ \forall t,\ \ 0 < t < 1,\ \ -\frac{\partial}{\partial t}(sgnv(t)op_t)$$
$$+\,(H(t)\ |v(t)|\ )op_t = c(t)\ H(t)op_t\ \ on\ \Gamma_t \tag{9.2}$$

Of course the derivative $\frac{\partial}{\partial t}(sgnv(t)op_t)$ is a problem, even assuming the boundaries to be smooth . The idea is to generalise definition of the metric, the easiest case being to replace the L^1 norm by the L^2 norm such that

$$d_2(\Omega_1, \Omega_2) = inf\ \int_0^1 \int_{\Gamma_t} 1/2\ v(t,x)^2\, d\Gamma_t(x)\, dt.$$

Then the same calculus would lead to a nice necessary condition (where $sgnv$ is replaced by $v(t)$). But now the theoritical basement of the metric partially vanishes; d_2 remains a metric but is not a complete one.

9.3. Explicit expression for $v(t)'_{\Gamma_t}$ (proof of theorem 9.1)

The direct calculation of that representation is rather complicate due to the term $cof D_{t,x}T_s(\mathbf{Z})$ (mainely because of the time derivatives in $D_{t,x}$...). We proceed in a more intrinsic way: making the calculus of the derivative

$$\frac{d}{ds}\int_{\Sigma^s} F(s,t,x)\, d\Sigma(t,x)$$

by two different techniques: the $N+1$ boundary integral derivative and the N dimensional one. From the proposition 9.6 and theorem 9.5 below, we have :

$$\frac{d}{ds}(\int_{\Sigma^s} F(s,t,x)\, d\Sigma(t,x))_{s=0} = \int_0^1 \int_{\Gamma_t} \{ \frac{1}{\sqrt{1+v^2}}(-v\partial_t F + \partial_{n_t} F)$$
$$+[-\frac{1}{\sqrt{1+v^2}^3}(< \partial_t V(t), n_t > - < \nabla_{\Gamma_t} v(t), V_{\Gamma_t} >) + \frac{H(t)}{\sqrt{1+v^2}}]\, F\ \ \}$$
$$< Z(t), n_t > d\Gamma_t\ dt \tag{9.3}$$

On the other hand we may consider

$$\frac{d}{ds}(\int_{\Sigma^s} F(s,t,x)\, d\Sigma(t,x))_{s=0}$$
$$= \int_0^1 (\frac{d}{ds}\int_{T_s(Z(t))(\Gamma_t)} F(s,t,x)\sqrt{1+(v^s)^2}\ d\Gamma_t^s\ )_{s=0}\, dt$$
$$= \int_0^1 \int_{\Gamma_t} \{\ (F(s,t,x)\sqrt{1+(v^s)^2}\ )'_{\Gamma_t} + F(0,t,x)\sqrt{1+v(t,x)^2}\ t)\, H(t)$$
$$< Z(t), n_t > \}\, d\Gamma_t(x) dt,$$

where $H(t,x)$ is the mean curvature of the boundary Γ_t, $H(t) = div_{\Gamma_t}(n_t)$. Also we have:

$$(F(s,t,x)\sqrt{1+(v^s)^2}\ )'_{\Gamma_t} = F(s,t,x)'_{\Gamma_t}\sqrt{1+v^2}\ + F(s,t,x)\,(\sqrt{1+(v^s)^2}\ )'_{\Gamma_t}$$

Let us consider specific functions F such that F is independent of the pertubation parameter s and $F(t,x) = F(t, op_{\Gamma_t}(x))$ in a neibourhood of the unperturbed lateral boundary Σ, where p_t is the horizontal N-dimensional projection mapping onto the boundary Γ_t. Then we have $\frac{\partial}{\partial n_t}F = 0$ and

$$F(t)'_{\Gamma_t} = \frac{\partial}{\partial s}F(t) + \frac{\partial}{\partial n_t}F(t) \; < Z(t), n_t >= \frac{\partial}{\partial s}F(t) = 0.$$

In these expressions $V(t,x) \in R^N$ is any vector field bulding the tube Q. We choose

$$V(t,x) = v(t) op_{\Gamma_t} \nabla b_{\Omega_t},$$

so that on Γ_t we have

$$< \frac{\partial}{\partial t}V(t), n_t > \frac{\partial}{\partial t}(v(t) op_{\Gamma_t}) = 0, \text{ and } \; V(t)_{\Gamma_t} = 0.$$

Then $(\frac{d}{ds}\int_{\Sigma^s} F(s) d\Sigma)_{s=0}$

$$= \int_0^1 \int_{\Gamma_t} \{ -\frac{v(t)}{\sqrt{1+v(t)^2}} \frac{\partial}{\partial t} F(t) [-\frac{1}{\sqrt{1+v^2}^3} \partial_t (vop) + \frac{H(t)}{\sqrt{1+v^2}}] F(0,t) \}$$
$$< Z(t), n_t > d\Gamma_t dt$$
$$= \int_0^1 \int_{\Gamma_t} \{ \, F(0,t) (\sqrt{1 + \, (v(t)^s)^2} \,)'_{\Gamma_t} + F(0,t) \sqrt{1+v(t)^2} \,) \, H(t)$$
$$< Z(t), n_t > \} \, d\Gamma_t(x) dt$$

We have

$$(\sqrt{1 + \, (v(t)^s)^2} \,)'_{\Gamma_t} = \frac{v(t)}{\sqrt{1+v(t)^2}} \, v(t)'_{\Gamma_t}$$

For general choices of F we have:

$$\int_0^1 \int_{\Gamma_t} F(0,t) \, (\sqrt{1 + \, (v(t)^s)^2} \,)'_{\Gamma_t} d\Gamma_t$$
$$= \int_0^1 \int_{\Gamma_t} \{ -(\frac{v(t)}{\sqrt{1+v(t)^2}} < z(t), n_t >) \, \frac{\partial}{\partial t} F(t)$$
$$-F(0,t) \sqrt{1+v(t)^2} \,) \, H(t) \; < Z(t), n_t > \} \, d\Gamma_t(x) dt$$
$$+ \int_0^1 \int_{\Gamma_t} [-\frac{1}{\sqrt{1+v^2}^3} \, \partial_t (vop) + \frac{H(t)}{\sqrt{1+v^2}}] F(0,t) < Z(t), n_t > d\Gamma_t dt.$$

Now we have (F is independant on the perturbation variable s but not on t)

$$F'_{\Gamma_t} = \frac{\partial}{\partial t}F + \frac{\partial}{\partial n_t}F \; < V(t), n_t >,$$

so that we get the following formula (integration by parts on Σ):

$$\int_0^1 \int_{\Gamma_t} \partial_t F \, G d\Gamma_t \, dt = -\int_0^1 \int_{\Gamma_t} F \, \partial_t G d\Gamma_t dt$$

$$-\int_0^1\int_{\Gamma_t} H(t)F(t)G(t) < V(t), n_t > \, d\Gamma_t dt$$
$$+\int_{\Gamma_1}(FG)(1)d\Gamma_1 - \int_{\Gamma_0}(FG)(0)d\Gamma_0$$
$$+\int_0^1\int_{\Gamma_t} F(t)\frac{\partial}{\partial n_t}G(t) < V(t), n_t > d\Gamma_t dt$$

So that, with $G = -\frac{v}{\sqrt{1+v^2}} < Z(t), n_t > :$

$$\int_0^1\int_{\Gamma_t} G(t)\,\frac{\partial}{\partial t}F(t)\,d\Gamma_t dt$$
$$= -\int_0^1\int_{\Gamma_t}\frac{\partial}{\partial t}\{\,\sqrt{1+v(t)^2}^{\,-1}\,v(t)\;\; < Z(t), n_t > \}\,F(t)d\Gamma_t dt$$
$$-\int_0^1\int_{\Gamma_t} H(t)\,\sqrt{1+v(t)^2}\,v(t)\,F(t)\;\; < Z(t), n_t > d\Gamma_t dt$$

Moreover

$$\int_0^1\int_{\Gamma_t} F(0,t)\,(\sqrt{1+\,(v(t)^s)^2}\,)'_{\Gamma_t}d\Gamma_t$$
$$= -\int_0^1\int_{\Gamma_t} F(0,t)\sqrt{1+v(t)^2}\,)\,H(t)\;\; < Z(t), n_t > \}\,d\Gamma_t(x)dt$$
$$+\int_0^1\int_{\Gamma_t}\frac{\partial}{\partial t}\{\,\frac{v(t)}{\sqrt{1+v(t)^2}}\;\; < Z(t), n_t > \}\,F(t)d\Gamma_t dt$$
$$+\int_0^1\int_{\Gamma_t} H(t)\,\frac{v(t)^2}{\sqrt{1+v(t)^2}}\;F(t)\;\; < Z(t), n_t > d\Gamma_t dt$$
$$+\int_0^1\int_{\Gamma_t}[-\frac{1}{\sqrt{1+v^2}^3}\,\partial_t(vop) + \frac{H(t)}{\sqrt{1+v^2}}]F(0,t) < Z(t), n_t > d\Gamma_t dt$$
$$+\int_0^1\int_{\Gamma_t}\frac{\partial}{\partial n_t}(\frac{v(t)}{\sqrt{1+v(t)^2}} < Z(t), n_t >)\,v(t)\,F(0,t)d\Gamma_t dt$$

The last term is zero as we have chosen $\frac{\partial}{\partial n_t}v(t) = \frac{\partial}{\partial n_t}z(t) = 0$. That is

$$(\sqrt{1+\,(v(t)^s)^2}\,)'_{\Gamma_t} =$$
$$\{\,-\sqrt{1+v(t)^2}\,)\,H(t)\;+H(t)\,\frac{v(t)^2}{\sqrt{1+v(t)^2}}$$
$$+[-\frac{1}{\sqrt{1+v^2}^3}\,\partial_t(vop) + \frac{H(t)}{\sqrt{1+v^2}}]\,\} < Z(t), n_t >$$
$$+\frac{\partial}{\partial t}\{\,\frac{v(t)}{\sqrt{1+v(t)^2}}\;\; < Z(t), n_t > \}$$
$$= -\frac{1}{\sqrt{1+v^2}^3}\,\partial_t(vop)\;\; < Z(t), n_t > +\frac{\partial}{\partial t}\{\,\frac{v(t)}{\sqrt{1+v(t)^2}}\;\; < Z(t), n_t > \}.$$

We obtain

$$\boxed{(\sqrt{1+(v(t)^s)^2})'_{\Gamma_t} = \frac{v(t)}{\sqrt{1+v(t)^2}}\ \frac{\partial}{\partial t}(< Z(t), n_t >)} \tag{9.4}$$

Hence theorem (9.1) is proved.

We turn now to the proof of proposition 9.6 below that we used at 9.3, in the beginning of that section : we have to compute in theorem 9.5 the time-space mean curvature **H** of the lateral surface Σ in R^{N+1} and then the $N+1$ dimensional boundary derivative concept f'_Σ.

9.4. *Mean curvature* **H** *of the lateral time-space boundary.*

Assuming the moving domain "smooth enough", we consider the normal speed v chosen as $v =< V(t), \nabla b_{\Omega_t(V)} >$ and

$$\frac{\partial}{\partial t}(\frac{v}{\sqrt{1+v^2}}) = \frac{1}{(\sqrt{1+v^2})^3}\ \frac{\partial}{\partial t}v.$$

But

$$\frac{\partial}{\partial t}v =< \frac{\partial}{\partial t}V, \nabla b > + < \frac{\partial}{\partial t}\nabla b,\ V > .$$

Now, we have

$$\frac{\partial}{\partial t}b_{\Omega_t(V)} = - < V(t), n_t > op_t,$$

where p_t is the projection onto the boundary $\Gamma_t(V) = \partial\Omega_t(V)$. Moreover,

$$\frac{\partial}{\partial t}\nabla b_{\Omega_t(V)} = -(\nabla_{\Gamma_t} < V(t), n_t >) op_t$$

and hence,

$$\frac{\partial}{\partial t}v =< \frac{\partial}{\partial t}V(t), n_t > - < (\nabla_{\Gamma_t} < V(t), n_t >) op_t, V_{\Gamma_t} > .$$

we obtain

Proposition 9.4.

$$\frac{\partial}{\partial t}(\frac{v}{\sqrt{1+v^2}}) = \frac{1}{(\sqrt{1+v^2})^3}(< \frac{\partial}{\partial t}V(t), n_t > - < (\nabla_{\Gamma_t} < V(t), n_t >) op_t, V_{\Gamma_t} > . \tag{9.5}$$

On the other hand we have :

$$div_{\Gamma_t}(\frac{1}{(\sqrt{1+v^2})}n) = - < \nabla_{\Gamma_t}(\frac{1}{(\sqrt{1+v^2}}, n > + \frac{1}{(\sqrt{1+v^2})^3}\ div_{\Gamma_t}n.$$

so that we get

$$div_{\Gamma_t}(\frac{1}{\sqrt{1+v^2}}n) = -\frac{1}{(\sqrt{1+v^2})^3}\ < \epsilon(V).n_t, n_t > + \frac{H_t}{\sqrt{1+v^2}},$$

where $\epsilon(V) = 1/2\,(DV + DV^*)$ is the deformation tensor. We consider the situation in which the field V verifies the following property:

$$V(t) = V(t)op_t \;\text{ in a neighbourhood of }\; \Gamma_t, \tag{9.6}$$

where p_t is the R^N projection mapping onto Γ_t ("horizontal" projection). Then we get :

$$p_t = I_d - b_{\Omega_t(V)}\,\nabla b_{\Omega_t(V)},$$

and

$$\frac{\partial}{\partial t}p_t = -\frac{\partial}{\partial t}b_{\Omega_t(V)}\nabla b_{\Omega_t(V)} - b_{\Omega_t(V)}\nabla(\frac{\partial}{\partial t}b_{\Omega_t(V)}).$$

The restriction to the boundary Γ_t leads to the distance $b_{\Omega_t(V)} = 0$ so the expressions simplify as follows (also we shall now denote by b_t that distance function) :

$$\frac{\partial}{\partial t}p_t|_{\Gamma_t} =< V(t), n_t > n_t,$$

and on the boundary $\Gamma_t(V)$ we get $DV(t).n_t = 0.$,

9.4.1. Time-space mean curvature of the lateral boundary Σ:

Theorem 9.5. *Assume that the field V verifies for each t: $V(t) = V(t)op_t$. Then, on the boundary $\Gamma_t(V)$ the mean curvature $\mathbf{H} := \mathbf{D}iv_\Sigma \nu$ is given by:*

$$\mathbf{H} = -\frac{1}{(\sqrt{1+v^2})^3}(< \frac{\partial}{\partial t}V, n_t > - < \nabla_{\Gamma_t}(< V(t), n_t >), V(t)_{\Gamma_t} >)$$
$$+ \frac{1}{\sqrt{1+v^2}}\; H_t.$$

9.5. Lateral Boundary Derivative

9.5.1. Transverse horizontal vector field. The normal component of any horizontal field $\tilde{Z} = (0, Z(t,x))$ is given by :

$$< \tilde{Z}, \nu >= \frac{1}{\sqrt{1+v^2}}\; < Z, n_t > .$$

If $f(\Sigma)$ is the restriction to the lateral boundary Σ of a function $F(t,x)$ defined over R^{N+1}, we get the (lateral) shape $N+1$-dimensional boundary derivative $f'_\Sigma(\tilde{Z})$ in the direction of the horizontal field $\tilde{Z}$ as follows : $f'_\Sigma(\tilde{Z}) = \frac{\partial}{\partial \nu}F$

9.5.2. Lateral shape derivative f'_Σ. We recall that (see [9],[24])

$$f'_\Sigma(\tilde{Z}) = (\frac{d}{ds}(f(\Sigma_s)o\mathbf{T}_s(\tilde{Z}))_{s=0} - \;<\; \nabla_\Sigma f(\Sigma), \tilde{Z}_\Sigma >$$

Notice that the operator ∇_Σ, as a tangential differential operator of the space time surface Σ, is itself a time-space manifold, and we get

$$f'_\Sigma(\tilde{Z}) = \dot{f}(\Sigma, \tilde{Z}) - \frac{vz}{1+v^2}\,\frac{\partial}{\partial t}f - < Z - \frac{z}{1+v^2}n_t, \nabla f > .$$

Consider a given function $F \in C^1([0,\tau] \times \bar{D})$. In a first step we assume that F is zero in the neighbourhood of $t = \tau$ so that the following derivative of the lateral boundary integral could be considered as derivative of the integral on the total boundary of the tube (as it will generate no term on the top $t = \tau$ of the tube). Then the usual derivative expressions apply : we consider the derivative of the lateral integral.

$$\Sigma^s = \{\ (t, T_t(V + sW)(x))\ \mid x \in \partial\Omega_0\ \},$$

$$\frac{\partial}{\partial s}_{s=0}\Big(\int_{\Sigma^s} F\ d\Sigma^s\Big) = \int_{\Sigma}\Big(\frac{\partial}{\partial\nu}F + \mathbf{H}_\Sigma F\Big) < \mathbf{Z}, \nu >_{R^{N+1}} \Big)\, d\Sigma,$$

where $\mathbf{H}_\Sigma$ is the mean curvature of the lateral boundary of the tube. At each point $(t,x) \in \Sigma$ we have :

$$< \mathbf{Z}(t,x), \nu(t,x) >_{R^{N+1}} = \frac{1}{\sqrt{1+ < V(t), n_t >^2}} < Z(t), n_t > .$$

Moreover,

$$\frac{\partial}{\partial\nu}F = \frac{1}{\sqrt{1 + (< V(t), n_t >)^2}}\Big(- < V(t), n_t > \frac{\partial}{\partial t}F + \frac{\partial}{\partial n_t}F\Big).$$

Then,

$$\frac{\partial}{\partial s}_{s=0}\Big(\int_{\Sigma^s} F\ d\Sigma^s\Big) = \int_\Sigma \Big[\frac{1}{\sqrt{1+v^2}}\Big(-v\,\frac{\partial}{\partial t}F + \frac{\partial}{\partial n_t}F\Big) \tag{9.7}$$

$$\Big(-\frac{1}{(\sqrt{1+v^2})^3}\Big(< \frac{\partial}{\partial t}V, n_t > - < \nabla_{\Gamma_t} v,\ V(t)_{\Gamma_t} >\Big)$$

$$+ \frac{1}{\sqrt{1+v^2}}\ H_t\Big) F\Big]\ \frac{1}{\sqrt{1+v^2}} < Z, n >_{R^N}\ \ d\Sigma.$$

Proposition 9.6. *Assume the vector field V in the canonical form $V(t) = V(t) o p_t$ in a neigbourhood of the lateral boundary Σ and let $v =< V(t), n_t >$ on Γ_t, then we have :*

$$\frac{\partial}{\partial s}_{s=0}\Big(\int_{\Sigma^s} F\ d\Sigma^s\Big) = \int_0^\tau \int_{\Gamma_t} \Big[\frac{1}{\sqrt{1+v^2}}\Big(-v\,\frac{\partial}{\partial t}F + \frac{\partial}{\partial n_t}F\Big) \tag{9.8}$$

$$+F\Big(-\frac{1}{(\sqrt{1+v^2})^3}\Big(< \frac{\partial}{\partial t}V, n_t > - < \nabla_{\Gamma_t} v,\ V(t)_{\Gamma_t} >\Big)$$

$$+ \frac{1}{\sqrt{1+v^2}}\ H_t\Big)\Big] < Z, n >_{R^N}\ \ d\Gamma_t\, dt.$$

9.5.3. Tube with minimal lateral boundary. In the specific case where $F = 1$ all the derivatives of F cancel and we have the derivative of the lateral surface of the tube :

$$\frac{\partial}{\partial s}_{s=0}\Big(\int_{\Sigma^s} d\Sigma^s\Big) = \int_\Sigma \Big[-\frac{1}{(1+v^2)^2}\Big(< \frac{\partial}{\partial t}V, n_t > - < \nabla_{\Gamma_t} v,\, V(t)_{\Gamma_t} >\Big) \tag{9.9}$$

$$+\frac{1}{1+v^2}\, H_t)\Big] < Z, n >_{R^N} \, d\Sigma.$$

The optimality condition for a minimal surface tube is easily obtained via the adjoint problem solution λ as

$$\frac{\partial}{\partial s}_{s=0}\Big(\int_{\Sigma^s} d\Sigma^s\Big) = \int_\Sigma \lambda \; < W, n_t > \, d\Sigma, \tag{9.10}$$

where λ solves :

$$\lambda(\tau) = 0, \;\; -\frac{\partial}{\partial t}\lambda - div(\lambda V) \tag{9.11}$$

$$= -\frac{1}{(1+v^2)^2}\Big(< \frac{\partial}{\partial t}V, n_t > - < \nabla_{\Gamma_t} v,\, V(t)_{\Gamma_t} >\Big) + \frac{1}{1+v^2}\, H_t.$$

The optimality condition for a tube with minimal lateral surface is be

$$\boxed{-\frac{1}{(1+v^2)}\Big(< \frac{\partial}{\partial t}V, n_t > - < \nabla_{\Gamma_t} v,\, V(t)_{\Gamma_t} >\Big) + \; H_t = 0.} \tag{9.12}$$

10. Non smooth tubes analysis

We denote by $\mathbf{H}^k$ the family of tubes ζ having a lateral boundary Σ piecewise C^k (in the precise sense of the previous section), with $\zeta = \chi_Q$. Let $\zeta \in \mathbf{H}^k$, we consider the $N+1$ dimensional perimeter

$$P_{I\times D}(Q) = ||\nabla_{t,x}\zeta||_{M^1(I\times D)}$$

$$= \int_0^1 \int_{\Gamma_t} \sqrt{1+v^2}\, d\Gamma_t\, dt \;\leq\; \int_0^1 P_D(\Omega_t)\, dt \;+\; \int_0^1 \int_{\Gamma_t} |v(t)|\, d\Gamma_t\, dt.$$

Consider also the fact that

$$< \frac{\partial}{\partial t}\zeta,\; g >_{\mathbf{M}(I\times D)\times C^0_{comp}(I\times D)} \;=\; \int_0^1 \int_{\Gamma_t} v\, g\, d\Gamma_t\, dt = \int_0^1 \int_{\Omega_t} div(g\, V)\, dx dt.$$

(Where V is any smooth extension to $I \times D$ of v)

As

$$\int_0^1 \int_{\Gamma_t} |v|\, d\Gamma_t\, dt \;=\; ||\frac{\partial}{\partial t}\zeta||_{M^1(I\times D)}, \;\; P_D(\Omega_t) \;=\; ||\nabla_x \zeta(t)||_{M^1(D,R^N)}$$

we have :

$$||\nabla_{t,x}\zeta||_{M^1(I\times D)} \leq \; ||\frac{\partial}{\partial t}\zeta||_{M^1(I\times D)} \;+\; \int_0^1 ||\nabla_x \zeta(t)||_{M^1(D,R^N)}\; dt. \tag{10.1}$$

We shall consider the weak closure of such smooth tubes ζ and verify that the estimate (10.1) still holds true on the closure:

Proposition 10.1. *let $\zeta_n \in \mathbf{H}^k$ be a sequence of tubes such that*

$$||\frac{\partial}{\partial t}\zeta_n||_{M^1(I\times D)} + \int_0^1 ||\nabla_x \zeta_n(t)||_{M^1(D,R^N)}\, dt \leq M \tag{10.2}$$

Then there exists a subsequence (still denoted ζ_n) and ζ such that $\zeta_n \to \zeta$ stongly in $L^1(I\times D)$ (so that $\zeta = \zeta^2$) and :

$$||\nabla_{t,x}\zeta||_{M^1(I\times D)} \leq lim\,inf\, ||\frac{\partial}{\partial t}\zeta_n||_{M^1(I\times D)} + \int_0^1 ||\nabla_x \zeta_n(t)||_{M^1(D,R^N)}\, dt. \tag{10.3}$$

Proof : From (10.1) we get : $\quad ||\zeta_n||_{BV(I\times D)} \leq M + meas(D), \quad$ so that the classical compact embeding of BV in L^1 leads to

$$||\nabla_{t,x}\zeta||_{M^1(I\times D, R^{N+1})} \leq lim\,inf\; ||\nabla_{t,x}\zeta_n||_{M^1(I\times D, R^{N+1})},$$

$$||\frac{\partial}{\partial t}\zeta||_{M^1(I\times D)} \;\leq\; lim\,inf\, ||\frac{\partial}{\partial t}\zeta_n||_{M^1(I\times D)},$$

$$\int_0^1 ||\nabla_x \zeta(t)||_{M^1(D,R^N)}\, dt \;\leq\; lim\,inf \int_0^1 ||\nabla_x \zeta_n(t)||_{M^1(D,R^N)}\, dt.$$

We introduce the weak closure $\mathbf{H}^*$ of $\mathbf{H}^k$ in $BV(I\times D)$:

$$\mathbf{H}^* = \{\zeta = \zeta^2 \in L^1(I\times D),\; s.t.\; \exists \zeta_n \in \mathbf{H}^k,\;\; \zeta_n \to \zeta \text{ in } L^1(I\times D)\;\;,$$

$$\nabla_{t,x}\zeta_n \to \nabla_{t,x}\zeta \text{ (wealky in) } M^1(I\times D) \quad \}.$$

In order to extend the metric δ to that setting we would like to define the families $\mathbf{O}^*_{\Omega_0}$ (resp. $T^*(\Omega_1,\Omega_2)$) similar to $\mathbf{O}_{\Omega_0}$ (resp. $T(\Omega_1,\Omega_2)$). But the difficulty is that elements ζ in the closure $\mathbf{H}^*$ are not continuous so that the connection property cannot be defined. In order to recover that continuity on the closure we propose two directions following the two next propositions 11.1 and 11.2.

11. Compactness result

Proposition 11.1. *Consider ζ_n bounded in $L^1(I, BV(D))$ together with $\frac{\partial}{\partial t}\zeta_n$ bounded in $L^p(I, M^1(D))$ for some $p > 1$.*

Then there exists a subsequence and an element
$\zeta \in L^1(I, BV(D)) \cap W^{1,1}(I, M^1(D)) \subset C^0(I, M^1(D))$
such that ζ_n strongly converges to ζ in $L^1(I, L^1(D))$ with
$\nabla\zeta \in L^p(I, M^1(D, R^N))$
verifying
$||\zeta||_{L^1(I,BV(D))} \leq lim\,inf\; ||\zeta_n||_{L^1(I,BV(D))}$
and
$||\frac{\partial}{\partial t}\zeta||_{L^p(I,M^1(D))} \;\leq\; lim\,inf\;\; ||\frac{\partial}{\partial t}\zeta_n||_{L^p(I,M^1(D))}.$

Proposition 11.2. *Consider ζ_n bounded in $L^1(I, BV(D))$ together with*
a.e.t $||\frac{\partial}{\partial t}\zeta_n(t)||_{M^1(D)} \leq \theta(t)$ *for some* $\theta \in L^1_{loc}(0,1)$.
Then there exists a subsequence and an element
$\zeta \in L^1(I, BV(D)) \cap W^{1,1}(I, M^1(D)) \subset C^0(I, M^1(D))$
such that ζ_n strongly converges to ζ in $L^1(I, L^1(D))$ with
$\nabla\zeta \in L^p(I, M^1(D, R^N))$
verifying
$||\zeta||_{L^1(I,BV(D))} \leq lim\,inf\ ||\zeta_n||_{L^1(I,BV(D))}$
and
$||\frac{\partial}{\partial t}\zeta||_{L^1(I,M^1(D))} \leq lim\,inf\ ||\frac{\partial}{\partial t}\zeta_n||_{L^1(I,M^1(D))}$.

Obviously, if the sequence verifies $(\zeta_n)^2 = \zeta_n$, then in the strong limit, we get $\zeta^2 = \zeta$.

At that point notice that, in both propositions 11.1 and 11.2 ,
$\zeta \in W^{1,1}(I, M^1(D))$ implies $\zeta \in C^0(I, L^1(D))$, more precisely :

Proposition 11.3. *Let $\zeta(t,x) = \zeta^2(t,x)$, $a.e.(t,x) \in I \times D$; $\zeta \in W^{1,1}(I, M^1(D))$ then $\zeta \in C^0(I,\ L^1(D))$ and the mapping:*

$$t \in \bar{I} \to p(t) := ||\nabla_x\zeta(t)||_{M^1(D,R^N)} \ \textit{is lower semi continuous} \tag{11.1}$$

Proof: Let $t_k \to t$ as $k \to \infty$, as $W^{1,1}(I, M^1(D)) \subset C^0(I, M^1(D))$ we get $\zeta(t_k) \to \zeta(t)$ so that:

$$\forall g \in C^0_{comp}(D),\ \int_D (\zeta_k(t,x) - \zeta(t,x))\, g(x)\ dx \to 0.$$

But there exists a subsequence (still denoted t_k) with $\zeta(t_k)$ wealkly $L^2(D)$ convergent to some $\mu, 0 \leq \mu \leq 1$ over D. From the previous convergence it turns out that $\mu = \zeta(t) = \zeta^2 = \mu^2$. Then that convergence is also strong in $L^2(D)$, hence in $L^1(D)$ (as $\zeta^2 = \zeta$). Being defined as supremum of continous terms:
$p(t) = sup_{\{\ g \in C^1_{comp}(D,R^N),\ ||g(x)||_{R^N} \leq 1\,\}} \quad \int_D \zeta(t,x) div_x g\ dx$
then p is l.s.c. We introduce the weak closures $\mathbf{H}^{c,*}_p$ and $\mathbf{H}^{c,*}_\theta$ of $\mathbf{H}^k$:

$$\mathbf{H}^{c,*}_p = \{\zeta = \zeta^2 \in \mathbf{H}^c \cap \mathbf{H}^*,\ s.t.\ \exists \zeta_n \in \mathbf{H}^k,\ \zeta_n \to \zeta \text{ in } L^1(I \times D)\ ,$$

$$\nabla_{t,x}\zeta_n \to \nabla_{t,x}\zeta\ , \sigma M^1(I \times D), \text{ with } \frac{\partial}{\partial t}(\zeta_n - \zeta) \to 0,\ \sigma L^P(I, M^1(D))\ \}$$

$$\mathbf{H}^{c,*}_\theta = \{\zeta = \zeta^2 \in \mathbf{H}^c \cap \mathbf{H}^*,\ s.t.\ \exists \zeta_n \in \mathbf{H}^k,\ \zeta_n \to \zeta \text{ in } L^1(I \times D)\ ,$$

$$\nabla_{t,x}\zeta_n \to \nabla_{t,x}\zeta\,(\text{wealky in })\, M^1(I \times D), \text{ with } a.e.t \in I,\ ||\frac{\partial}{\partial t}\zeta_n(t)||_{M^1(D)} \leq \theta(t)\ \}.$$

12. Fully Eulerian metric spaces

As soon as the speed vector field V verifies the assumption of theorem 6.4, there is a unique tube associated to V. Thus then we have an application $V \to \zeta_V$, and with such regularity on V we can revisit the complete metric d: the non differentiable perimeter and curature terms that we were obliged to introduce in order to apply the compactness theorems are not any more necessary. From the previous tube analysis we consider several interesting choices for the spatial regularity of the speed vector field (together with its divergence field). Let E be a closed subspace in $BV(D) \cap \mathbf{E}^{1,1}$ such that any element $V \in E$ verifies the assumptions of theorem 6.4. A first example is, when working with prescribed volume for the moving domain,

$$E_0 = \{ \; V \in BV(D, R^N) \cap \mathbf{E}^{1,1}, \;\; s.t. \;\; divV = 0 \;a.e.\; (t,x) \in I \times D \;\; \}$$

V be a free divergence vector field with $divV = 0,$, $V \in L^1(I, E_0))$, where $E = BV(D, R^N)$ or any closed subspace

(for example $E = \{ \, V \in \; H^1_0(D, R^N), \;\; s.t. \;\; divV = 0 \; \}$). An *obvious* metric is to consider the set

$$\mathbf{V}(\Omega_1, \Omega_2) = \{ \, V \in \mathbf{E}^{1,1} \; s.t. V, \; divV \; \in L^p(I, E_0), \;\; s.t. \;\; \zeta_0 = \chi_{\Omega_1}, \;\; \zeta(1) = \chi_{\Omega_2} \;\; \}$$

$$\delta_{E_0}(\Omega_1, \Omega_2) = Inf_{V \in \mathbf{V}(\Omega_1, \Omega_2)} \int_0^1 ||V(t)||_{E_0} \, dt. \tag{12.1}$$

As V is divergence free the previous boundedness assumption on the divergence are verified and to each V a tube ζ_V is associated trough the convection. Then following the same proof we get the

Proposition 12.1. *Let E be any subspace of $BV(D, R^N) \cap \mathbf{E}^{1,1}$, such that any element V satisfies to assumptions of theorem* 6.4*. Then equipped with δ_E the family $\mathbf{O}^E_{\Omega_0}$ is a metric space.*

$$d_{E_0}(\Omega_1, \Omega_2) = Inf_{V \in \mathbf{V}(\Omega_1, \Omega_2)} \quad ||V||_{L^1(I, E_0)} \;\; + \;\; ||\frac{\partial}{\partial t} V||_{L^1(I, M^1(D, R^N))} \tag{12.2}$$

Theorem 12.2. *Let E be any subspace of $BV(D, R^N) \cap \mathbf{E}^{1,1}$, such that any element V satisfies to assumptions of theorem* 6.4*. Then equipped with d_E the family $\mathbf{O}^E_{\Omega_0}$ is a complete metric space.*

12.1. Geodesic

The previous transverse tube perturbation will apply. In that setting we are concerned with vector fields $Z(s,t,x) \in R^N$ such that $Z(s,0,x) = Z(s,1,x) = 0$, so that the extrimities of the perturbed tube are preserved. The previous study for the transverse field implies that for given such a vector filed Z, with $div_x Z(s,t,x) = 0$ we get the admissible perturbation of the field V in the following form $V + sW(s,t,x)$ with

$$W(s,t,x) = \frac{\partial}{\partial t} Z(s,t,x) \; + [Z, \, V]$$

more precisely define the Lipschitz-continuous connecting set

$$\mathbf{V}^{1,\infty}(\Omega_1,\Omega_2) = \{\ V \in L^1(I, W^{1,\infty}(D,R^N)) \cap \mathbf{E}^{1,1},\ s.t.\ \ \zeta_V\ \in \bar{\mathbf{T}}(\Omega_1,\Omega_2)\quad \}.$$

And the set of smooth transverse vector fields:

$$\mathbf{Z} = \{\quad Z(t,x) \in C^{\infty}_{comp}(I \times D, R^N)\quad \}$$

(Notice that such Z verifies $Z(0,.) = Z(1,.) = 0$ on D)

Proposition 12.3. *Let* $V \in \mathbf{V}(\Omega_1,\Omega_2)$ *and* $Z(t,x) \in \mathbf{Z}$ *. The Transformation* $\mathbf{T} = T_s(Z)oT_t(V)$ *maps* $\Omega_t(V)$ *onto* $\Omega_t^s := T_s(Z)(\Omega_t(V))$ *so that*

$$\forall s,\ \forall Z,\ V^s(t,x) = \frac{\partial}{\partial t}\mathbf{T}o\,\mathbf{T}^{-1}$$

$$= (\frac{\partial}{\partial t}T_s(Z(t)) + DT_s(Z(t)).V(t)\ )oT_s(Z(t))^{-1} \in\ \mathbf{V}^{1,\infty}(\Omega_1,\Omega_2)$$

We obtain :

Lemma 12.4.

$$\frac{\partial}{\partial s}V^s(t,x)|_{s=0}\ =\ \frac{\partial}{\partial t}Z(t) + [Z(t),\,V(t)]. \tag{12.3}$$

Corollary 12.5. *Consider a functional* $\mathbf{J}(V) = j(\zeta_V)$ *and let* $\bar{V}$ *be a minimizing element of* $\mathbf{J}$ *on* $\mathbf{V}(\Omega_1,\Omega_2)$ *then we have*

$$\forall Z \in \mathbf{Z},\ \frac{\partial}{\partial s}\mathbf{J}(\bar{V}^s)_{s=0} = J'(\bar{V};\,(\frac{\partial}{\partial s}V^s)_{s=0}) = \mathbf{J}'(\bar{V};\ \frac{\partial}{\partial t}Z(t) + [Z(t),\,V(t)]\ )\ \geq\ 0 \tag{12.4}$$

That variational principle extends to vector field $V \in E$ for which the flow mapping $T_t(V)$ is poorly defined. The element $\zeta_V \in \mathbf{H}^c$ is uniquely defined. For any $Z \in \mathbf{Z}$ we have $\zeta_V^s := \zeta_V oT_s(Z)^{-1} \in \bar{\mathbf{T}}(\Omega_1,\Omega_2)$. Moreover we have

Proposition 12.6. $\zeta_V^s = \zeta_{V^s}$ *with*

$$V^s(t,.) := -DT_s^{-1}(-Z(t)).(\,V(t)oT_s(Z(t))^{-1}) - \frac{\partial}{\partial t}T_s(-Z(t))\ )$$

In other words:

$$\frac{\partial}{\partial t}\zeta\ + \nabla\zeta.V = 0\ \ implies\ \ \frac{\partial}{\partial t}(\zeta oT_s(Z(t))^{-1})\ + \nabla(\zeta oT_s(Z(t))^{-1}).V^s\ =\ 0.$$

It can also be verified that the expression (12.3) for the derivative of the field still holds true so that the variational principle (12.4) is valid for any functional $\mathbf{J}$ minimized over the lipschitzian connecting family $\mathbf{V}^{1,\infty}(\Omega_1,\Omega_2)$.

And more generally, without assuming V in E we have :

Proposition 12.7. *Let* $(\zeta, V) \in \mathbf{T}^{p,q}(\Omega_1,\Omega_2)$*, then for all* $s > 0$ *and* $Z \in \mathbf{Z}$ *we have :*

$$(\zeta oT_s(Z)^{-1},\,V^s)\ \in\ \mathbf{T}^{p,q}(\Omega_1,\Omega_2).$$

Notice that, in order to get a differentiable metric we could consider

$$\tilde{d}(\Omega_1,\Omega_2) \;=\; Inf_{V\in\mathbf{V}(\Omega_1,\Omega_2)} \;\; \int_0^1 (\; ||V(t)||_{H_0^1\cap E_0} \;+\; ||\frac{\partial}{\partial t}V||_{L^2(D)} \;)\, dt.$$

Equipped with $\tilde{d}$, $\mathbf{O}_{\Omega_0}$ would be a complete metric space but $\tilde{d}$ fails to be a metric because of the triangle axiom.

The advantage is that now the associated functional is differentiable with repect to V and we can apply the previous variational principle with transverse vector field Z.

Let $\bar{V}$ be a minimizer in $\mathbf{V}(\Omega_1,\Omega_2)$ for $\tilde{d}(\Omega_1,\Omega_2)$. Then $\forall Z\in\mathbf{Z}$ we have

$$\int_0^1 \{\, ||V(t)||^{-1} \;< V(t), Z_t+[Z,V] > + |V'(t)|^{-1}\, ((V'(t)\,(Z_t+Z,V)'\;))\;\; \}dt = 0,$$

where $<,>$ is the $H_0^1(D,R^N)$ inner product while $((,))$ is the $L^2(D,R^N)$. In order to recover a differentiable complete metric we introduce again the constraint on the perimeter as in the begining and set

$$p\geq 1,\;\; \delta_{H^1,p}(\Omega_1,\Omega_2) \;=\; Inf_{V\in\mathbf{V}(\Omega_1,\Omega_2)} \int_0^1 \;\; ||V(t)||^p_{H_0^1\cap E_0}\, dt. \tag{12.5}$$

The optimality condition is :

$$\forall Z\in\mathbf{Z}\;\; s.t.\, divZ=0 \;\; and \quad \int_0^1\int_{\Gamma_t} H(t)\; < Z(t), n_t >\; d\Gamma_t\, dt \;=\; 0,$$

$$p\int_0^1 \; ||V(t)||^{p-2} \;< V(t), Z_t+[Z,V]\; >\; dt = 0.$$

From (7.9) , that last condition can be rewitten as

$$< ||V(t)||^{p-2}\, V(t)\,,\;\; H_V.Z \;\; >= 0.$$

The adjoint operator (for free divergence vector field V) is given by:

$$H_V^*.\Lambda := -\frac{\partial}{\partial t}\Lambda - D\Lambda.V \;-\; D^*V.\Lambda \tag{12.6}$$

Finally, the second condition turns to be:

$$< H_V^*.(||V(t)||^{p-2}\, V(t)\,)\,,\;\; Z \;\; >= \; 0$$

$$\exists c(t),\, P \quad s.t.\;\; \frac{\partial}{\partial t}(||V(t)||^{p-2}\, V(t)) + ||V(t)||^{p-2}\,(\; DV(t).V \;+\; D^*V.V(t)\,)$$
$$= \nabla P + c\;\chi_{\Gamma_t}\, div_{\Gamma_t}(n_t)\;\, n_t.$$

That is,

$$(p-2)||V||^{p-4}((V,\frac{\partial}{\partial t}V))V \;+\; ||V(t)||^{p-2}\,(\frac{\partial}{\partial t}V \;+\; DV(t).V \;+\; D^*V.V(t)\,)$$
$$= c\;\chi_{\Gamma_t}\, div_{\Gamma_t}(n_t)\;\, n_t, \tag{12.7}$$

which can be written as (with the notations $\bar{V}=||V||^{-1}\,V,\;\Pi = P-1/2|V|^2$) :

$$divV = 0,$$

$$\boxed{\frac{\partial}{\partial t}V + (p-2)((\frac{\partial}{\partial t}V, \bar{V}))\,\bar{V} \;+ DV.V \;= \nabla\Pi + c(t)||V||^{2-p}\,\chi_{\Gamma_t}\,div_{\Gamma_t}(n_t)\;n_t} \tag{12.8}$$

13. Level set formulation for the Tube shape metric

13.1. Shape Gradient approximation

From the shape derivative structure theorem, we know that any shape gradient takes the following form

$$\int_{\partial\Omega} g \;< V(0), n > \,d\Gamma, \tag{13.1}$$

where g is the socalled *density gradient*, a measure on the boundary, and $v =< V(0), n >$ is the normal componant of the vector field. In the level set setting, assume that $\Omega = \{x \in D \;\mid\; \Phi(t,x) > 0\;\}$ then $V = -\frac{\partial}{\partial t}\Phi\,\frac{\nabla_x\Phi}{||\nabla_x\Phi||}$, so that obviously, $v = -\frac{\partial}{\partial t}\Phi/||\nabla_x\Phi||$. From Federer measure decomposition theorem we have :

$$\int_{\mathbf{U}_h(\Gamma)} F(x)\,dx = \int_{-h}^{+h} (\int_{\Phi^{-1}(z)} \frac{F}{||\nabla_x\Phi||}\,d\Gamma\;)\,dz,$$

where

$$\mathbf{U}_h^{\Phi}(\Gamma) = \{x \in D \;\mid\; |\Phi(x)| \,<\, h\;\;\}$$

Assuming the mapping $z \in (-h, +h) \to (\int_{\Phi^{-1}(z)} \frac{F}{||\nabla_x\Phi||}\,d\Gamma\;)$ to be continuous we obtain

$$\int_{\Gamma} \frac{F(x)}{||\nabla_x\Phi(x)||}\,d\Gamma(x) = \frac{1}{2h}\int_{\mathbf{U}_h(\Gamma)} F(x)\,dx \;+\; o(1), \quad h \to 0.$$

Applying that approximation in the previous shape derivative we obtain, for any smooth enough extension $\tilde{g}$ of g to the neigboorhoud $\mathbf{U}_h^{\Phi}(\Gamma)$:

$$\int_{\partial\Omega} g \;< V(0), n > \,d\Gamma \;=\; -\frac{1}{2h}\int_{\{x\in D,\;|\Phi(x)|<h\,\}} \tilde{g}(x)\;|\frac{\partial}{\partial t}\Phi(t,x)|\,dx + o(1), \;\; h \to 0.$$

The point being that the denominator $||\nabla_x\Phi(t)||$ has been eliminated.

13.2. h-scale metric d_h

Two open subsets $\Omega_i \subset D$ being given, for $i = 1, 2$, (resp. quasi open subsets) we associate two continuous functions (resp. elements of $H^1(D)$) $\phi_i \in C^o(\bar{D})$ such that $x \in \Omega_i$ iff $\phi_i(x) > 0$, $x \in \partial\Omega_i$ iff $\phi(x) = 0$ (then $x \in D \setminus \bar{\Omega}$ iff $\phi_i(x) < 0$). We consider the following closed convex set

$$\mathbf{K}(\Omega_1, \Omega_2) := \{\;\psi(t,x) \in L^2(0, 1, H^1(D)),\;\; \int_0^1 \psi(t,x)\,dt \;=\; \phi_1(x) \,-\, \phi_2(x) \quad\}. \tag{13.2}$$

Then for any element $\psi \in \mathbf{K}(\Omega_1, \Omega_2)$ we consider the level set function

$$\Phi(t,x) := \phi_1(x) \,+\, \int_0^t \psi(s,x)\,ds,$$

And the moving domain

$$\Omega_t := \{ \ x \in D, \quad \Phi(t,x) \ > \ 0 \ \} \in \bar{\mathbf{T}}(\Omega_1, \Omega_2)$$

Then we set

$$d_h(\Omega_1, \Omega_2) := Inf_{\{\psi \in \mathbf{K}(\Omega_1,\Omega_2)\}} \int_0^1 [\, - \frac{1}{2h} \int_{\{\ |\Phi(t,x)| < h\ \}} |\psi(t,x)| \, dx + ||\psi(t)||_{H^1(D)} \,] \, dt \tag{13.3}$$

That metric turns to be numerically tractable and several experiments are performed at INRIA with J. Picard and L. Blanchard. The choice of h has to be tuned to the pixels density.

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