Improving robust stability by increasing the number of controlled degrees of freedom

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In this article we demonstrate the necessity of increasing the number of controlled degrees of freedom when controlling an uncertain system. We show this on an infinite dimensional system representative of reaction-diffusion problems. However, our final goal is concerned with fluid flow control. In both cases, fluid flow control or reaction-diffusion problems, a common approach to control is based on a spatial discretization followed by some linear control design. Unfortunately, this approach is sensitive to uncertainties introduced during linearization of the nonlinear system. In this paper, we show that robust stability is dependent on the number of controlled degrees of freedom and performance of a closed loop system can be improved by considering this aspect.

Notation

$H^l(0,1)$ with $l \in \mathbb{N}^*$ denotes the Sobolev space $W^{l,2}(0,1)$. $H^0(0,1)$ is also denoted as $L^2(0,1)$. We use the subscript notation for partial derivative, for example $\xi_x(x,y)$ represents the derivative of the dependent variable $\xi(x,y)$ with respect to the independent variable x, $\partial\xi/\partial x$, and similarly $\xi_{xy}(x,y) = \partial^2\xi/\partial x\partial y$. $\xi_{i,j}$ denotes the value of the dependent variable in a point $(i\Delta x, j\Delta y)$ on a grid of discretized position, where Δx and Δy represent the discretization steps in each of the two independent variables.

I. Introduction

The objective of this article is to point out some of the difficulties related to the control of systems described by partial differential equations (PDEs) as it is the case in flow control since a flow is governed by the Navier-Stokes equations. These represent systems where the state evolves continuously not only in time but also in space, in contrast to ordinary differential equations (ODE) which describe the evolution only in time.

In the early years of automatic control, finite dimensional systems described by ODE where studied due to the fact that only a finite number of actuators, sensors and states representing different variables of the system (velocity, pressure, temperature, humidity, etc.) at predefined locations where considered. While this is still appropriate for a large number of situations encountered in practice, there are still many physical, mechanical or chemical systems where it is not the case and the evolution in space has to be taken into account too.

In this paper, the position is denoted by x, where $x \in \mathbb{R}^n$ with $n = 1$ (but generally $n \in \{1, 2, 3\}$ depending on the size of the space) and the state of the system by $\xi(x,t)$ which depends on both position x and the time t. Thus $\xi(x,t)$ can be considered as an element of a functional space of infinite dimension. The term distributed parameters system, sometimes used in reference with systems described by PDEs, comes from the fact that the state is considered through the entire space, thus in an infinite number of positions. This is in contrast to ODE systems (called also lumped parameter systems) where the state evolution is considered only in a finite number of points from the domain.

Various discretization methods have been proposed, such as the integral, finite-difference, finite-volume, finite-element, or spectral methods.[1,2,3,4] Control using discretized linearized models of infinite dimensional flow problems has also been proposed[5,6,7,8] (see also[9] where a finite-difference approach is used). However, in this paper we want to cope with the problem of robust stability. In fact, linearization and discretization introduces uncertainties and as soon as the distance between the state of the system and the equilibrium used for liniarization exceeds some robustness margin, the closed loop system becomes unstable. In this paper we demonstrate that robust stability can be greatly improved by increasing the number of controlled degrees of freedom, *i.e.* the number of independent actuators. This will be very helpful in the more complicated case of flow control.[10,11]

In the next sections of this article, we describe the classical approach to controlling infinite dimensional systems (see[9]), which is spatial discretization using a carefully chosen set of grid points. A basic method of approximation is presented in section II and applied in section III to the heat equation with constant diffusivity to point out some of the difficulties that come from using a finite number of actuators in infinite dimensional systems. Then, in section V, by adding actuators it is shown the the robustness of a linear controller is improved.

II. Finite-difference approximation method

Most of the systems encountered in nature can be described by partial differential equations (PDEs). Therefore, in an attempt to control such systems, the need to better understand them arises. As such, numerical analysis has seen a lot of development in the last decades as the computational power of personal computers increased. Numerical analysis is based upon discretisation of PDEs and various techniques have been described.

In this section, a brief review of the finite-difference method (FDM) is presented (for a more in depth description see[12]). This method is based on approximating derivatives by differences, resulting in an algebraic representation of the PDEs. The first step is to replace the continuous domain by a finite-difference mesh. For a two-dimensional (2-D) problem, $\xi(x, y)$ is replaced by $\xi(i\Delta x, j\Delta y)$ (written also $\xi_{i,j}$) for given distance increments Δx and Δy.

Recall next the definition of the derivative of a function $\xi(x, y)$ at $x = x_0,\ y = y_0$:

$$\frac{\partial \xi}{\partial x} = \lim_{\Delta x \to 0} \frac{\xi(x_0 + \Delta x, y_0) - \xi(x_0, y_0)}{\Delta x}. \tag{1}$$

Another way to find this equation is by considering first a Taylor-series expansion for $\xi(x_0 + \Delta x, y_0)$ about $\xi(x_0, y_0)$ which is given by

$$\xi(x_0 + \Delta x, y_0) = \xi(x_0, y_0) + \left.\frac{\partial \xi}{\partial x}\right|_0 \Delta x + \left.\frac{\partial^2 \xi}{\partial x^2}\right|_0 \frac{(\Delta x)^2}{2!} + \ldots \tag{2}$$

or in the (i, j) notation

$$\xi_{i+1,j} = \xi_{i,j} + \left.\frac{\partial \xi}{\partial x}\right|_{i,j} \Delta x + \left.\frac{\partial^2 \xi}{\partial x^2}\right|_{i,j} \frac{(\Delta x)^2}{2!} + \ldots \tag{3}$$

which gives

$$\left.\frac{\partial \xi}{\partial x}\right|_{i,j} = \frac{\xi_{i+1,j} - \xi_{i,j}}{\Delta x} + O(\Delta x), \tag{4}$$

where $(\xi_{i+1,j} - \xi_{i,j})/\Delta x$ represents the "forward" finite-difference approximation of the derivative and $O(\Delta x)$ is the truncation error between the derivative and the proposed representation.

In a similar manner, it is also possible to find a "backward" formula

$$\left.\frac{\partial \xi}{\partial x}\right|_{i,j} = \frac{\xi_{i,j} - \xi_{i-1,j}}{\Delta x} + O(\Delta x) \tag{5}$$

and a "central" formula

$$\left.\frac{\partial \xi}{\partial x}\right|_{i,j} = \frac{\xi_{i+1,j} - \xi_{i-1,j}}{2\Delta x} + O(\Delta x)^2. \tag{6}$$

As for the second derivative, this can be evaluated using the central difference formula

$$\left.\frac{\partial^2 \xi}{\partial x^2}\right|_{i,j} = \frac{\xi_{i+1,j} - 2\xi_{i,j} + \xi_{i-1,j}}{(\Delta x)^2} + O(\Delta x)^2. \tag{7}$$

III. Heat equation

Consider the following diffusion equation, described by the initial boundary value problem (IBVP):

$$\xi_t(x,t) = \theta \xi_{xx}(x,t) + b^T(x)u(t),\ x \in (0,1)\,,\ t > 0, \tag{8}$$

with boundary conditions (BCs)

$$\xi_x(0,t) = \xi_x(1,t) = 0, \tag{9}$$

and initial condition (IC)

$$\xi(x,0) = \xi_0(x) \tag{10}$$

where $\xi_0 \in H^1(0,1)$ represents the initial system state (temperature), $u(t)$ is a control signal, and θ represents the constant diffusivity. This is also known as the heat equation with constant thermal diffusivity.

The input influences the PDE (8) only at certain locations specified by the input shaping function

$$b^T(x) = [b_1(x)\ b_2(x)\ b_3(x)\ b_4(x)] \tag{11}$$

with

$$b_{i+1}(x) = \begin{cases} 1, & 0.175 + 0.2i \leq x \leq 0.225 + 0.2i \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

$\forall i \in \{0,\ 1,\ 2,\ 3\}$.

The output is described by the equation

$$y(t) = \int_0^1 \xi(x,t)c(x)dx,\ t > 0. \tag{13}$$

where $c(x)$ is given by

$$c^T(x) = [c_1(x)\ c_2(x)\ \dots\ c_{400}(x)] \tag{14}$$

with

$$c_{i+1}(x) = \begin{cases} 1, & \frac{i}{400} < x < \frac{i+1}{400} \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

The eigenvalues of the PDE (8) can be computed as

$$\lambda_i = -\theta(i-1)^2\pi^2,\ i \geq 1. \tag{16}$$

The system has one zero eigenvalue and an infinity of negative ones. The numerical simulation of the IBVP can be done by using the method of lines. The semi-discretization of the PDE is done on the spatial domain by dividing the interval on which x is defined into 399 equal subintervals, obtaining thus $N = 400$ grid points. The method of lines is used in initial boundary value problems and it consists in discretizing the system along all minus one of the variables. This results in a system of ordinary differential equations (ODE) which can be solved using various existing methods (see for example the Euler method and the various alternatives of the Runge-Kutta method described in[13]) efficiently implemented in simulation software such as MATLAB, Maple, Mathematica, or Scilab.

After discretization and taking into account the boundary conditions, a lumped parameter system described by a state space representation of the form

$$\dot{\chi}(t) = \frac{\theta}{\Delta x^2} A_L \chi(t) + B_L u(t), \tag{17a}$$

$$y_L(t) = C_L \chi(t), \tag{17b}$$

is obtained, where $\chi(t) \in \mathbb{R}^N$ represents the discretized system state, $A_L \in \mathbb{R}^{N,N}$

$$A_L = \begin{pmatrix} -2 & 2 & 0 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & -2 & 1 \\ 0 & \dots & 0 & 0 & 0 & 2 & -2 \end{pmatrix} \tag{18}$$

is the state matrix, $B_L \in \mathbb{R}^{N,4}$, and $C_L \in \mathbb{R}^{N,N}$ have zero values everywhere exception of a few elements of value 1 at grid positions corresponding to eqs. (12) and (15). Note that with the given output shaping function, $C_L = I_N$ and therefore $y_L(t) = \chi(t)$. The diffusivity constant has been chosen as $\theta = 0.1$ and the spatial discretization step is $\Delta x = \frac{1}{N-1}$.

Using the continuous time system described by eq. (17), it is possible to design an LQR controller[14] that minimizes a quadratic cost function of the form

$$J = \int_0^\infty (\chi^T Q \chi + u^T R u) dt. \tag{19}$$

In the last equation, Q and R are state and input weighting matrices chosen in order to satisfy certain design restrictions. The solution of the state feedback gain matrix is given by

$$K = R^{-1} B_L^T P, \tag{20}$$

where P is found by solving a continuous time algebraic Riccati equation

$$A_L^T P + P A_L - P B_L R^{-1} B_L^T P + Q = 0. \tag{21}$$

Figure 1 shows the evolution of the temperature in the 1-D heat conductor over a 2 *sec* interval. This varies from an initial state $\xi_0(x)$ to a final one. The reference profile is marked as desired temperature profile in figure 2. In the same graphic, the obtained temperature profile is shown shown for comparison. One can observe that very good results are obtained at the point where collocated control and measurement exist while in the rest of the domain the results are not satisfactory. The main reason for this is the small number of point where measurement and control is done. Of course one can increase this number, but there will always be intermediate points where it is difficult to know exactly what the evolution of the system is.

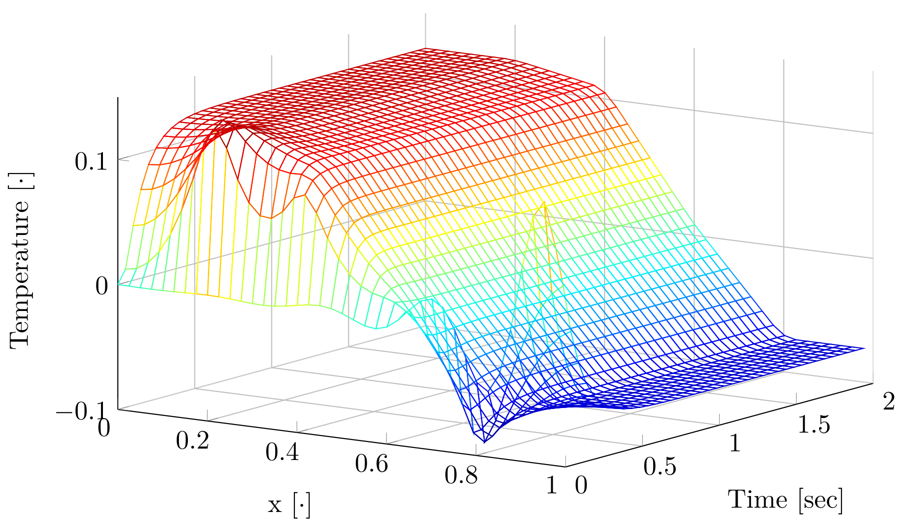

Figura 1. Temperature evolution of the controlled heat conductor.

IV. Nonlinear reaction-diffusion system

Consider now a reaction-diffusion system with nonlinear reaction given by

$$F : L^2(0,1) \to L^2(0,1), \; F(\xi) = 10\xi(1 - 10\xi^2). \tag{22}$$

The IBVP for this system is obtained by modifying eq. (8) of the IBVP described in Section III with

$$\xi_t(x,t) = \theta \xi_{xx}(x,t) + F(\xi(x,t)) + b^T(x) u(t), \tag{23}$$

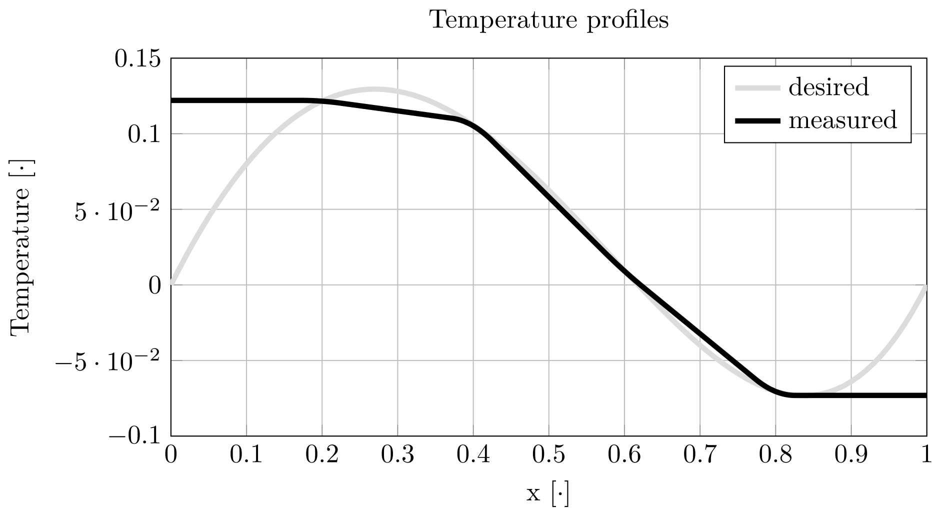

Figura 2. Comparison of desired and obtained temperature profiles (after 2 seconds).

for $x \in (0,1)\,,\ t > 0$. The lumped parameters systems obtained after discretisation in the spatial coordinate x is given by

$$\dot{\chi}(t) = f(\chi(t)) + B_L u \tag{24a}$$

$$y_L(t) = C_L \chi(t), \tag{24b}$$

where

$$f(\chi(t)) = \frac{\theta}{\Delta x^2} A_L \chi(t) + F_L(\chi(t)), \tag{25}$$

with $\chi(t)$ representing the discretized system's state, A_L, B_L, and C_L given previously, and F_L defined as

$$F_L : \mathbb{R}^N \to \mathbb{R}^N,\ F_L(\chi(t)) = 10\chi(t)(1 - 10\chi(t)^2). \tag{26}$$

One approach to control system (24) is to linearize it and then build a controller based on the linear model. Linearizing around the zero temperature yields the state space system

$$\dot{\chi}(t) = \left(\frac{\theta}{\Delta x^2} A_L + 10 I_N\right) \chi(t) + B_L u(t) \tag{27a}$$

$$y_L(t) = C_L \chi(t) \tag{27b}$$

which is unstable having four positive eigenvalues. Nevertheless, an LQR controller can be designed to stabilize the linearized system. Furthermore, a simulation run on the nonlinear system (24), shows that this is also stable, even though the set point value is not the one used for linearization. In figure 3, the temperature profiles obtained in stationary regime with the nonlinear system in closed loop with the LQR controller is compared to the desired profile. It can be observed that the tracking error is quite important in both cases and the controller doesn't manage to follow the desired profile anymore.

Remark: if one would consider a nonlinear term given by $+100\xi^3$ instead of $-100\xi^3$ the closed loop system with the LQR controller is no longer stable and a different design technique has to be considered.

V. Increasing the number of controlled degrees of freedom

In this section, we will show using simulation that it is possible to control even the unstable case with nonlinear term given by $+100\xi^3$ using a linear LQR controller. The only change is in the number of controlled degrees of freedom. In other words, instead of having a length 4 input vector, we will use a length 10 input vector implying thus a constructive change on the physical actuators combined with an increase in the number of actuators. To be more precise, the input shaping function (11) and (12) is now transformed into

$$b^T(x) = [b_1(x)\ b_2(x)\ \ldots\ b_{10}(x)] \tag{28}$$

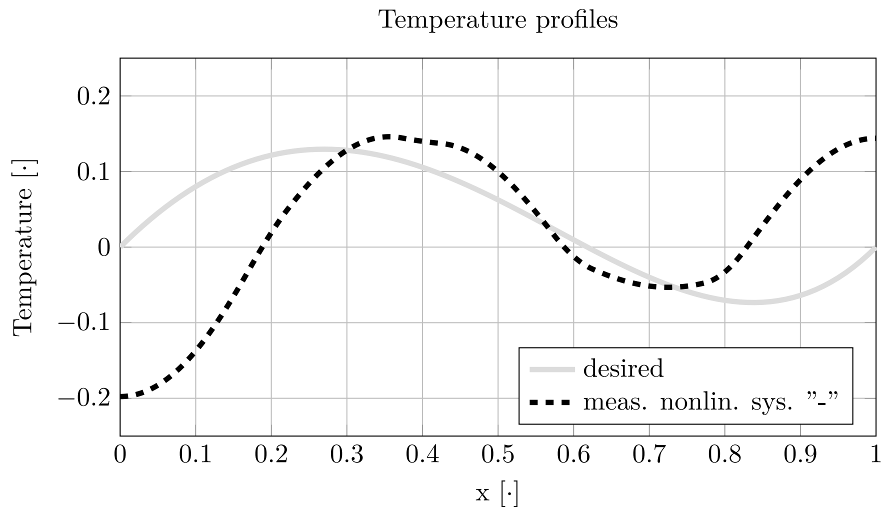

Figura 3. Comparison of desired and obtained temperature profiles (after 2 seconds).

with

$$b_{i+1}(x) = \begin{cases} 1, & 0.1i \leq x \leq 0.05 + 0.1i \\ 0, & \text{otherwise} \end{cases} \tag{29}$$

for $i \in \{0, 1, \ldots, 9\}$. The output shaping function remains unchanged.

With the increased number of actuators, the robustness of the closed loop system is improved. In figure 4, the desired profile and the final temperature profiles obtained in closed loop after the transitory period are shown. For the nonlinear system with negative nonlinearity ”-”, the stationary profile is very close to the desired one. Also for the nonlinear system with positive nonlinearity ”+” the closed loop remains stable and the final profile is close to the desired one. In figure 5, the evolution of the head along the bar is shown.

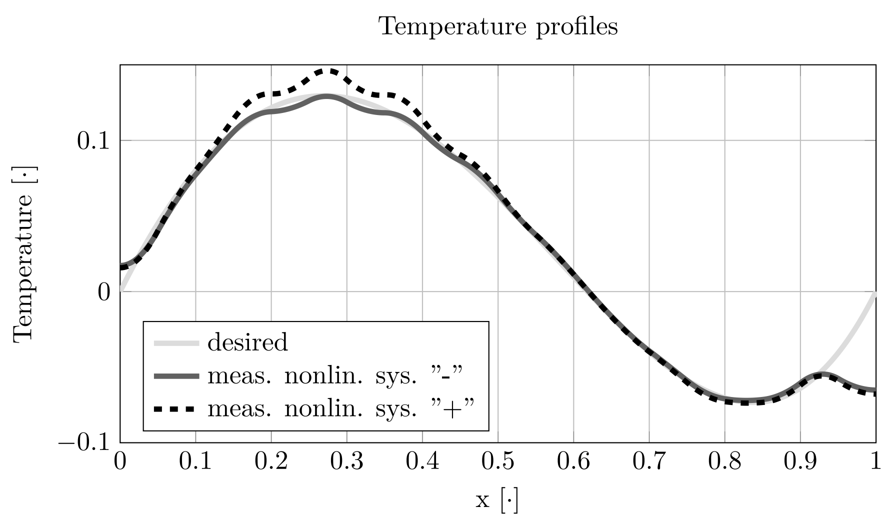

Figura 4. Comparison of desired and obtained temperature profiles for the nonlinear system (after 2 seconds).

VI. Concluding remarks

In this article, we have shown the weaknesses of conventional control, which is appropriate for lumped parameters systems, when applied to distributed parameters systems, as it is the case in flow control. We have

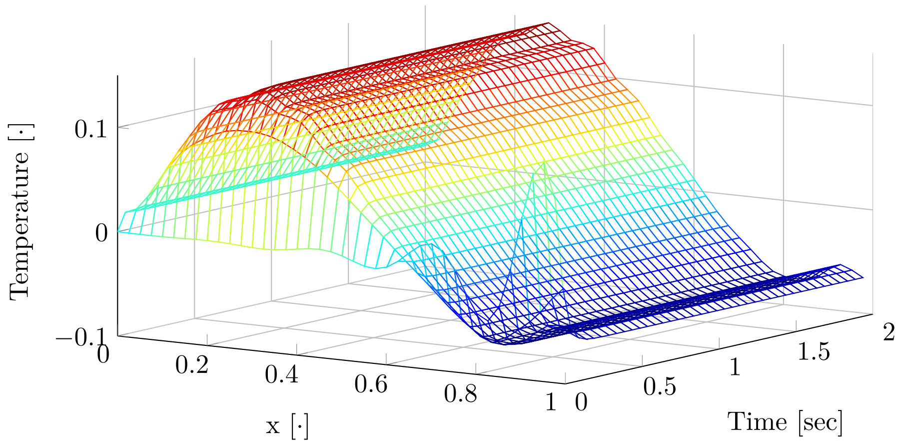

Figura 5. Temperature evolution of the controlled nonlinear heat conductor with positive nonlinear term "+".

shown the basic ideas used when trying to apply conventional control to infinite dimensional systems. The difficulties of applying linear control theory to distributed systems are highlighted. However, we demonstrate by a simulated example the practical advantages of increasing the number of controlled degrees of freedom in such systems, showing that increasing the number of actuators, when possible, can have a beneficial effect on closed loop stability. As already mentioned, this result will be of a great interest in closed-loop flow control. Recall that few degrees of freedom are used in practice, typically one or two in the case of the plane Poiseuille flow.[5,6,7]

Future research will deal with quantifying the robustness gain obtained by the increase in the number of controlled degrees of freedom and propose judicious actuator placement solutions.

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