SPARSE MODELS AND PURSUIT ALGORITHMS FOR PIV TOMOGRAPHY

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Abstract

The goal of this article is to study the tomographic problem of particle reconstruction within a Sparse Representation framework. By adopting this class of models for the tridimensional (3D) image reconstruction, we account for an additional prior on the signal to reconstruct. In numerical tests involving a recovery problem over a moderate-to-high wide range of seeding densities, the pursuit algorithms are shown to have comparable performance to various state-of-the-art algorithms, for a complexity reduced by a factor of 10 to 100.

1 Introduction

The *Tomographic Particle Image Velocimetry (TomoPIV)* is an experimental technique for the retrieval of Eulerian velocity measurements of turbulent fluids introduced by [**?**]. The technique aims at reconstructing, with very high update rates, 3D motion fields of lightly seeded particles from the images on a finite number of cameras.

A crucial step in solving the velocity fields is estimating the volume distribution of the seeded particles. However, the number of observations available for the reconstruction is very limited, which implies solving an underdetermined linear system. Distinct features of the problem are to be reminded. The 3D signal representing the particles position in space can be considered to be **sparse** in the spatial domain (i.e., there is typically much more empty space than particles in the volume) and **non-negative** (i.e., the energy recovered from each particle is always a positive quantity).

This prior knowledge is henceforth exploited in the current literature. The classical methods commonly adopted in the TomoPIV community are the so-called algebraic methods for reconstruction ([**?**]), amongst which we mention the most popular as being the so-called ART and MART algorithms. The latter looks for a solution satisfying the observation model under an entropy-based optimization criterion and it naturally integrates a **non-negativity constraint** on the sought solution. More recently, Petra et al. set-up the theoretical context for the study of TomoPIV as a sparse representation matter ([**?**]) and have empirically shown that application of reconstruction algorithms for finding **sparse solutions** outperforms state-of-the-art algebraic techniques [**?**].

While these algorithms lead to acceptable reconstruction with respect to accuracy, they each present specific drawbacks: the algebraic techniques induce too dense positive solutions, as opposed to the reconstructed sparse vectors issued by the convex optimization procedure, which suffers from high-complexity without guaranteeing non-negative solutions. Our current investigation focuses on reconstructing a sparse volumetric signal from few projections with respect to accuracy and complexity. In order to achieve a precise sparse signal with a reasonable computational time, we took an interest in a family of algorithms for sparse representations extensively known as **pursuit algorithms**. By applying them to our reconstruction problem, we point out a faster alternative to the state-of-the-art techniques, with comparable performance in terms of probability of correct reconstruction.

2 Related Models

In this section, we describe the mathematical model relating the image data to the sought particle density. The discretized model derived hereafter and commonly adopted in the TomoPIV analysis is inspired from the so-called algebraic image reconstruction model [**?**].

We denote by $\mathcal{V} \subset \mathbb{R}^3$ the cuboid of interest in the considered three-dimensional space. Our goal is to recover the volumetric intensity distribution, say $I(\mathbf{z})$, $\forall\, \mathbf{z} \in \mathcal{V}$ from a vector of measures $\mathbf{y}$. In the context of TomoPIV, $\mathbf{y}$ is made up of a set of pixels collected from a synchronized multi-camera system. Following the classical projection paradigm, the i^{th} element of $\mathbf{y}$ then writes:

$$y_i = \int_{\Omega_i} I(\mathbf{z})\, d\mathbf{z}, \tag{2.1}$$

where Ω_i is the cone of light originated from the camera optical center O_c and passing through the boundaries of the i^{th} pixel (see Figure 1).

Note that any physically-consistent intensity distribution $I(\mathbf{z})$ must correspond to a finite-energy signal, *i.e.*, $I(\mathbf{z}) \in \mathcal{L}^2(\mathcal{V})$. In the sequel, we will moreover assume that $I(\mathbf{z})$ belongs to a finite-dimensional subspace of $\mathcal{L}^2(\mathcal{V})$. Let $\{b_j(\mathbf{z})\}_{j=1}^m$ be a orthogonal basis of this subspace. $I(\mathbf{z})$ can then be rewritten as

$$I(\mathbf{z}) = \sum_{j=1}^{m} x_j\, b_j(\mathbf{z}). \tag{2.2}$$

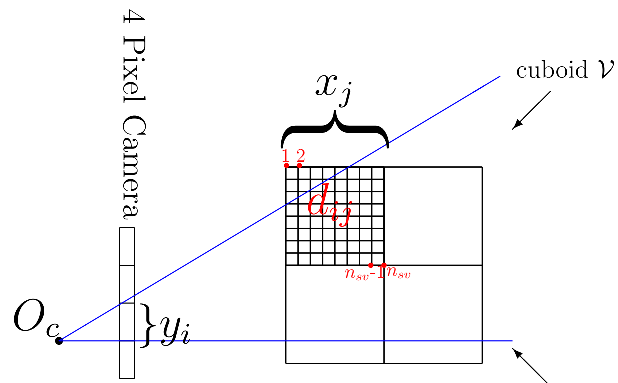
where $x_j = \int_{\mathcal{V}} I(\mathbf{z})\, b_j(\mathbf{z}) d\mathbf{z}$.

There are various choices for the family of basis functions in the literature (ref ?). We assume hereafter that the $b_j(\mathbf{z})$'s are defined as rectangular-pulse functions. In order to properly formalize this choice, let us partition the cuboid into a cartesian grid of cubic **vo**lume **el**ements (voxels) $\zeta_j \subset \mathbb{R}^3$ such that

$$\bigcup_{j=1}^{m} \zeta_j = \mathcal{V}, \qquad \bigcap_{j=1}^{m} \zeta_j = \varnothing. \tag{2.3}$$

The definition of $b_j(\mathbf{z})$ as a rectangular pulse can therefore be expressed as:

$$b_j(\mathbf{z}) \triangleq \begin{cases} 1/\mathrm{vol}(\zeta_j), & \text{if } \mathbf{z} \in \zeta_j, \\ 0, & \text{otherwise,} \end{cases} \tag{2.4}$$



in blue: 2D representation of the cone of light Ω_i passing through the i^{th} pixel

Figure 1: 2D Scheme of a 4-voxels cuboid in focus of a 4-Pixel Camera; j^{th} voxel in $\mathcal{V}$ is divised into 9×9 subvoxels; the cone-of-sight Ω_i passing through the i^{th} pixel intersects the filled subvoxels of the j^{th} voxel.

where $\text{vol}(\zeta_j) \triangleq \int_{\zeta_j} 1\, d\mathbf{z}$ denotes the volume of the voxel.
Coming back to (2.2) and integrating the j^{th} basis function along the i^{th} cone of sight gives us the coefficients:

$$d_{ij} \triangleq \int_{\Omega_i} b_j(\mathbf{z}) d\mathbf{z} = \frac{1}{\text{vol}(\zeta_j)} \int_{\Omega_i \cap \zeta_j} 1\, d\mathbf{z}. \tag{2.5}$$

The integral in the right-hand side of (2.5) represents the volume of the intersection of the i^{th} cone of sight Ω_i with the j^{th} voxels ζ_j. Then, using (2.2) and (2.5), our model can be described by a system of linear equations:

$$y_i = \sum_{j=1}^{m} x_j d_{ij}. \tag{2.6}$$

Expressed in matrix formulation, (2.6) writes:

$$\mathbf{y} = \mathbf{D}\mathbf{x}, \tag{2.7}$$

where $\mathbf{y}$ is the measurement vector, $\mathbf{D} \in \mathbb{R}^{n \times m}$ is a dictionary collecting the elements defined in (2.5) and $\mathbf{x} \in \mathbb{R}^m$ is a vector made up of the unknown projection coefficients x_j. We refer the reader to the appendix for a discussion on the implementation of (2.5).

3 Criterions and Reconstruction Paradigms

In this section, we briefly discuss the main approaches available in the literature to estimate the value of the unknown projection coefficients $\mathbf{x}$ from the vector of pixels $\mathbf{y}$. As mentioned in section 1, the system in (2.7) has specific features which have influenced the choice of estimation paradigms employed within the TomoPIV community. In a nutshell, we most often deal with an underdetermined linear system ($n < m$) and a *non-negative sparse* vector $\mathbf{x}$ to be reconstructed. The non-negativity and the sparsity of the sought signal can therefore be exploited to resolve the ambiguity inherent to undetermined systems.

Regarding algorithms exploiting the non-negativity of $\mathbf{x}$, a classical solution in the TomoPIV literature is the so-called *MART (Multiplicative Algebraic Reconstruction Technique)*. This algorithm belongs to the general class of methods called *row-action algorithms* [?]. Formally, MART addresses the following optimization problem:

$$\mathbf{x}^\star = \arg\min_{\mathbf{x}} \left\{ \sum_{j=1}^{m} x_j \log x_j \right\} \quad \text{subject to} \quad \left\{ \begin{array}{l} \mathbf{x} \geq 0 \\ \mathbf{D}\mathbf{x} = \mathbf{y}. \end{array} \right. \tag{3.1}$$

The first constraint forces the non-negativity of the solution whereas the second one ensures its compatibility with the observed vector $\mathbf{y}$. MART looks for a solution of this problem by means of an iterative scheme based on a closed-form formula; the algorithm converges to the desired solution provided that: *i)* there exists at least one feasible point (*i.e.*, a point satisfying the constraints); *ii)* $d_{ij} \geq 0$ and $y_i > 0 \; \forall i, j$.

A more recent approach to address the TomoPIV reconstruction problem revolves around sparse representations (SR). The idea consists in exploiting the sparsity of the sought vector $\mathbf{x}$ to remove the ambiguity of the solution. More formally, the considered optimization problem writes

$$\mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_0 \text{ subject to } \mathbf{D}\mathbf{x} = \mathbf{y}, \tag{3.2}$$

where $\|\mathbf{x}\|_0$ denotes the ℓ_0 pseudo-norm which counts the number of nonzero elements in $\mathbf{x}$. Solving (3.2) is therefore tantamount to finding the vector with the minimum number of nonzero elements satisfying the observation model (2.7).

Unfortunately, problem (3.2) is NP-hard. In practice, we need therefore to resort to heuristic algorithms to access to its solutions. The design of algorithms addressing the sparse representation problem has recently sparked a surge of interest in the literature. A popular approach consists in approximating the non-convex and non-smooth ℓ_0 norm by another function, easier to handle. In the context of TomoPIV, several authors have explored this option. In [?] the authors considered the ℓ_1 norm as an approximation of the ℓ_0 norm. They proposed an efficient implementation of the corresponding convex problem by means of the *Bregman Iterative Algorithm* [?]. Moreover, in [?] the same authors studied the following approximation: $\|\mathbf{x}\|_0 \simeq \sum_i (1 - e^{-\alpha x_i})$, $\alpha > 0$. This surrogate clearly leads to a good approximation of the ℓ_0 norm as α tends to infinity. The authors then suggested an iterative algorithm looking for a solution of the corresponding problem by solving a linear program at each step.

The emerging TomoPIV documentation outlines an obvious research direction. By resorting to regularization alternative, then to appropriate procedures within the sparse representation context, we are guaranteed to respect this prior information we have on our signal. However, limits on the density of the particles have been empirically reached, with a relatively slow convergence. We further investigate how the use of tractable algorithms for SR allow us to obtain comparable performances in terms of probability of correct reconstruction for a lower complexity.

4 Low-complexity Pusuit for TomoPIV

We pose the problem of TomoPIV as a sparse representation matter (*i.e.*, the sought 3D signal is described as the combination of a small number of atoms chosen from an overcomplete dictionnary) and investigate on which of the families of algorithms addressing the SR issue fits best our expectations in terms of accuracy of the reconstructed signal and complexity.

Finding the exact solution of (**??**) is an NP hard. Therefore, numerous suboptimal (but tractable) algorithms have been devised in the literature to cope with the SR problem. We can distinguish between 3 main families:

- the algorithms based on a problem relaxation, like basis pursuit (BP) [**?**], FOCUSS [**?**] or SL0 [**?**]; these algorithms approximate (**??**), where $p = 1$, by relaxed problems such as (**??**), that can be solved efficiently by standard optimization procedures and have already exploited in TomoPIV litterature [**?**].
- he greedy algorithms which build up the sparse vector x by making a succession of locally-optimal decisions. This family includes MP [1], OMP [2], gradient pursuit (GP) [3] , CoSamp [4] and subspace pursuit (SP) [5] algorithms
- the Bayesian algorithms which express the sparse representation problem as the solution of a Bayesian inference problem and apply statistical tools to solve it. Examples of such algorithms include the relevant vec- tor machine (RVM) algorithm [9], the sum-product [10] and the expectation-maximization [11] SR algorithms.

Nous avons focalisé notre étude sur le problème de la reconstruction des positions des voxels à partir d'un faible nombre de vues sous contraintes de parcimonie à l'aide d'algorithmes de faible complexité. Nous avons basé la reconstruction sur les algorithmes communément appelés "algorithmes de poursuite" dans la littérature ([**?**], [**?**]), qui cherchent à résoudre le critère de minimisation basée sur la mesure de parcimonie correspondante à la pseudo-norme l_0 et supportent une complexité moindre.

La décomposition parcimonieuse dans le dictionnaire D de l'ensemble des observations y que représentent les pixels captés sur les plans image est approchée par la combinaison d'un petit nombre de vecteurs du dictionnaire tel que : $y \approx Dx$, avec $D \in R^{m \times n}$. Nous formalisons la recherche du vecteur de coefficients x le plus parcimonieux qui mène à la reconstruction exacte du vecteur y dans le dictionnaire D de la façon suivante:

$$\min_x \|x\|_0 \text{ soumis à } \mathbf{Dx} = \mathbf{y} \tag{4.1}$$

Les algorithmes de poursuites étudiés nous permettent d'intégrer l'information a priori de parcimonie sur le signal physique à reconstruire pour une complexité réduite d'un facteur de 10 jusqu'à 100. Par ailleurs, nous reprenons les travaux de [**?**] permettant d'exprimer le problème de décomposition parcimonieuse dans un contexte probabilistique Bayésien. Les auteurs proposent un problème d'estimation au sens du maximum a posteriori (MAP) qui prend en compte des probabilités d'occurrence des atomes différents.

Nous rappelons qu'une limitation importante de l'état de l'art ([**?**]) est engendrée par l'absence d'une contrainte de non-négativité sur le signal à reconstruire. Dans le futur immédiat et afin de valider la reconstruction volumique pour la TomoPIV tenant compte des informations physiques propres au système, notre investigation portera sur l'obtention d'une solution parcimonieuse **positive** à partir des algorithmes gloutons.

5 Simulations Results

We assume that $\mathcal{V}$ satisfies the *OVC (Optimal Visibility Constraint)* [?]. For an augmented measurement quality, we set up our system as a response to requirements satisfying imaged features detectability([?])..

A Implementation of (2.5)

The practical implementation of (2.6) requires the numerical evaluation of (2.5). We adopt a "subvoxel" approach inspired by [?]. The latter considers the subdivision of a voxel ζ_j in a set of cubic subvoxels $\chi_{k,j} \subset \zeta_j$ with $k = 1, \dots, n_{sv}$ such that

$$\bigcup_{k=1}^{n_{sv}} \chi_{k,j} = \zeta_j, \qquad \bigcap_{k=1}^{n_{sv}} \chi_{k,j} = \varnothing. \tag{A.1}$$

Using this definition, (2.5) can be rewritten as

$$d_{ij} = \frac{1}{\mathrm{vol}(\zeta_j)} \sum_{k=1}^{n_{sv}} \int_{\Omega_i \cap \chi_{k,j}} 1 d\mathbf{z}. \tag{A.2}$$

We then use the following approximation

$$\int_{\Omega_i \cap \chi_{k,j}} 1 d\mathbf{z} \simeq \frac{\mathrm{vol}(\zeta_j)}{n_{sv}}, \tag{A.3}$$

if the line of sight joining the center of $\chi_{k,j}$ to the optical center Oc crosses the ith pixel and set $\int_{\Omega_i \cap \chi_{k,j}} 1 d\mathbf{z} = 0$ otherwise. Note that this approximation is equivalent to assuming that subvoxel $\chi_{k,j}$ is either is totally included or excluded from cone of sight Ω_i. Letting n_{ij} be the number of subvoxels satisfying (A.3), we finally obtain

$$d_{ij} = \frac{n_{ij}}{n_{sv}}. \tag{A.4}$$