PURE DISCRETE SPECTRUM DYNAMICAL SYSTEM AND PERIODIC TILING ASSOCIATED WITH A SUBSTITUTION

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Abstract. We give a computable sufficient condition for the symbolic dynamical system associated with a substitution of Pisot type to have a pure discrete spectrum. In the unimodular case, this condition is necessary when the substitution has no nontrivial coboundary; it is satisfied if and only if the Rauzy fractal associated with the substitution generates a self-similar periodic tiling.

The techniques provide a computable sufficient condition for connectivity of the Rauzy fractal of a unimodular substitution.

Introduction

A complete study of the dynamical systems associated with the Morse substitution ($1 \mapsto 12$, $2 \mapsto 21$), the Fibonacci substitution ($1 \mapsto 12$, $2 \mapsto 1$), and the Tribonacci substitution ($1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$) states that the first system has a continuous spectral component whereas the others have pure discrete spectrum and are explicitly semi-conjugate to toral translations, linked with self-similar periodic tilings. Hence, a natural question is to determine which substitutive systems are measure-theoretically isomorphic to a translation on a compact abelian group. An additional motivation comes from the physics of quasicrystals, where sequences generated by substitutions are used as models of atomic configurations, and pure discrete spectrum corresponds to the configurations being pure point diffractive [BT86, Sen95].

The case of substitutions of constant length is well understood: the maximal equicontinuous factor is explicit and systems with pure discrete spectra are characterized [Dek78]. The question of substitutions of non-constant length was first tackled with a complete study of the system associated with the Tribonacci substitution [Rau82]. Then, a significant advance was made by proving that the two main dynamical classifications, up to measure-theoretic isomorphism and topological conjugacy, are equivalent for primitive substitutive systems [Hos86], and also for linearly recurrent maps on Cantor sets [Dur00]. As a consequence, the spectrum of a substitutive system can be divided into two parts: the first part has an arithmetic origin, and depends only on the incidence matrix of the substitution; the second part has a combinatorial origin, and is related to the return words of the fixed point of the substitution [FMN96]. The arithmetic spectrum of substitutive systems of Pisot type is explicit and non-empty, so that these systems are never weakly mixing.

Many papers deal with conditions for a substitutive dynamical system to have pure discrete spectrum (see the bibliography in [PF02], Chap. 7); in particular, the *condition of coincidences* implies the pure discrete spectrum property for substitutions that have constant length or that are of Pisot type over a two-letter alphabet [Dek78, HS03]. About substitutions on more that three letters, the condition of coincidences implies self-similarity of the associated Rauzy fractal, that is, an explicit compact subset of non-zero measure in the product of an Euclidean space and p-adic spaces (depending on the unimodularity of the substitution) [AI01, Sie03]. As a consequence, the associated substitutive system is semi-topologically conjugate to an exchange of domains on the Rauzy fractal. An explicit topological factor is given by a translation on a compact abelian group

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[CS01b, Sie03]; the techniques do not provide results for a semi-topological conjugacy as it is the case for two-letter substitutions.

In this paper, we use a formal and algebraic point of view instead of the usual measurable point of view, to address both the questions of self-similarity and pure discrete spectrum. Our point of view is based on two points. The first one is the combinatorial structure of the substitutive system, deduced from its prefix-suffix automaton. The second one is the algebraic structure contained in the substitutive system and described by the incidence matrix of the substitution. More precisely, we introduce a formal representation of substitutive dynamical systems expressed with a formal power series in α, where α denotes the dominant eigenvalue of the incidence matrix. We obtain a representation map by gathering the set of finite values which can be taken for any topology (archimedean or not) by the formal power series, that is, by taking the completion of $\mathbb{Q}(\alpha)$, with respect to all the absolute values on this field which take a value strictly less than 1 on α.

In that way, the self-similarity of a Rauzy fractal is reduced to the study of sequences of digits such that the associated formal power series tends to zero for all the metrics for which the power series has a limit. This implies that the sequences of digits are paths in a finite graph. The understanding of the structure of the graph is fundamentally connected with pure discrete spectrum. Hence we prove and give many illustrations for the following:

Theorem. *There exists a computable sufficient condition for a substitutive dynamical system to have pure discrete spectrum. This condition is a necessary condition when the substitution is unimodular and has no non-trivial coboundary.*

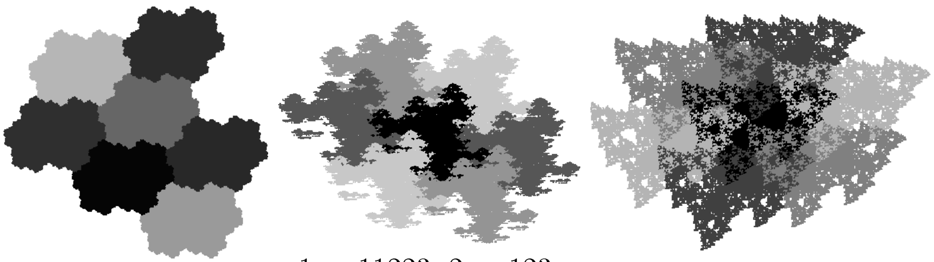
With this condition, we check whether a given translation on a compact abelian group is isomorphic to the substitutive system. When the substitution is unimodular and satisfies an extra combinatorial condition (no non-trivial coboundary), this translation is known to be the maximal equicontinuous factor of the substitutive system. As a consequence, the sufficient condition is also a necessary condition in this specific case.

The reason why we are unable to state that the sufficient condition of the theorem is also a necessary condition is the following. In the general case (non-unimodular or unimodular with non-trivial coboundary), the discrete spectrum of the substitutive system is not explicitly known. Therefore, we cannot prove that the translation on the compact abelian group that is deduced from the construction of the Rauzy fractal has the same discrete spectrum as the substitutive system. When this is wrong, the translation is not the maximal equicontinuous factor of the substitutive system; these systems cannot be isomorphic. However, this does not imply that the substitutive system does not have a pure discrete spectrum, since the good candidate has not been exhibited. Such a phenomenon could occur especially when the spectrum of the substitutive system contains a combinatorial part. However, one should notice that we are unable for the moment to exhibit a substitution of Pisot type that does not satisfy the sufficient condition of the theorem.

As we explained before, the sufficient condition in the theorem above is satisfied if and only if an explicit translation on a compact group is semi-topologically conjugate to the substitutive system. If the substitution is unimodular, the compact group is a torus. In this case, we know that the Rauzy fractal generates a periodic tiling if and only if the toral-translation is semi-topologically conjugate to the substitutive system. Hence we deduce the following, illustrated in Fig. 0.1.

Theorem. *Let σ be a unimodular substitution of Pisot type over a d-letter alphabet. There exists a computable necessary and sufficient condition for the two following equivalent properties:*

- *the Rauzy fractal of the substitution is a fundamental domain for a given lattice $\mathcal{H}_0$, so that it generates a periodic tiling of $\mathbb{R}^{d-1}$.*
- *The substitutive dynamical system is semi-topologically conjugate to the translation on $\mathbb{T}^{d-1}$ given by the vector of frequencies of letters in any infinite word of the substitutive system.*



Tribonacci substitution | $1 \mapsto 11223$, $2 \mapsto 123$, $3 \mapsto 2$ | Flipped Tribonacci substitution

FIGURE 0.1. Periodic tilings generated by Rauzy fractals.

As an intermediate result, we also give an effective sufficient condition for unimodular Rauzy fractals to be connected.

Organization of the paper. The first section is devoted to the definition of substitutions and a combinatorial expansion system associated to it. In the second section is deduced a numeration system associated with a substitution. The non-proper expansions in this system are characterized. The third section introduces the Rauzy fractal associated with a substitution of Pisot type. Its geometrical properties (connectivity and self-similarity) are linked with graphs. In the final fourth section, the dynamics of substitutive dynamical systems of Pisot type is studied. The theorem stated above are deduced.

1. SUBSTITUTIONS : COMBINATORIAL AND FORMAL EXPANSIONS

1.1. **Definitions.** Let $\mathcal{A}$ be a finite alphabet and $\mathcal{A}^*$ the set of finite words in $\mathcal{A}$. The empty word is denoted ε. A doubly-infinite *word* in $\mathcal{A}$ is denoted $w = \dots w_{-2}w_{-1}.w_0w_1\dots$. The metrizable topology of the set of doubly-infinite words $\mathcal{A}^{\mathbb{Z}}$ is the product of the discrete topology on $\mathcal{A}$. A *cylinder* of $\mathcal{A}^{\mathbb{Z}}$ is a clopen set of the form: $[W_1.W_2] = \{(w_i)_i \in \mathcal{A}^{\mathbb{Z}} | w_{-|W_1|}\dots w_{-1}w_0\dots w_{|W_2|-1} = W_1W_2\}$ for $W_1, W_2 \in \mathcal{A}^*$ (if W_1 is empty, the cylinder is denoted $[W_2]$).

Denote by S the *shift map* on $\mathcal{A}^{\mathbb{Z}}$: $S(\,(w_i)_{i\in\mathbb{Z}}\,) = (w_{i+1})_{i\in\mathbb{Z}}$. A word $w \in \mathcal{A}^{\mathbb{Z}}$ such that $S^\nu(w) = w$ with $\nu > 1$ is called *shift-periodic*. The *symbolic dynamical system* generated by a word u is the pair (X_u, S), where X_u denotes the closure in $\mathcal{A}^{\mathbb{Z}}$ of the shift orbit of u. The shift map S is an homeomorphism on this compact subset of $\mathcal{A}^{\mathbb{Z}}$.

A *substitution* σ is an endomorphism of the free-monoid $\mathcal{A}^*$, such that the image of each letter of $\mathcal{A}$ is non-empty, and that for at least one letter a, we have $|\sigma^n(a)| \to +\infty$. A substitution naturally extends to the set of doubly-infinite words $\mathcal{A}^{\mathbb{Z}}$: $\sigma(\dots w_{-2}w_{-1}.w_0w_1\dots) = \dots\sigma(w_{-2})\sigma(w_{-1}).\sigma(w_0)\sigma(w_1)\dots$.

A *periodic point* of σ is a doubly-infinite word $u = (u_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ that satisfies $\sigma^\nu(u) = u$ for $\nu > 0$ and for which there exists a letter a such that every factor W of u is also a factor of the iteration $\sigma^{\nu(W)}(a)$. If $\sigma(u) = u$, then u is a *fixed point* of σ. Every substitution has at least one periodic point [Que87]. The substitution is *shift-periodic* when there exists a point that is periodic for both the shift map and the substitution σ.

We call a substitution σ *primitive* if there exists an integer ν (independent of the letters) such that, for each pair $(a, b) \in \mathcal{A}^2$, the word $\sigma^\nu(a)$ contains at least an occurrence of b. In this case, if u is a periodic point for σ, then X_u does not depend on u and we denote by (X_σ, S) the *symbolic dynamical system generated by* σ. The system (X_σ, S) is *minimal* and *uniquely ergodic*: there exists a unique shift-invariant probability measure μ_{X_σ} on X_σ [Que87].

1.2. **Substitutions of Pisot type.** Let $\mathbf{l}: \mathcal{A}^* \mapsto \mathbb{N}^d$ be the natural homomorphism obtained by abelianization of the free monoid. To each substitution σ on $\mathcal{A}$ is canonically associated its *abelianization linear map* whose matrix $\mathbf{M} = (m_{i,j})_{1\leq i,j\leq d}$ (called *incidence matrix of* σ) is defined by $m_{i,j} = |\sigma(j)|_i$, so that we have $\mathbf{l}(\sigma(W)) = \mathbf{M}\mathbf{l}(W)$ on $\mathcal{A}^*$. If σ is primitive, $\mathbf{M}$ has a simple real positive dominant eigenvalue α (Perron-Frobenius theorem).

A substitution σ is *of Pisot type* if every non-dominant eigenvalue λ of $\mathbf{M}$ satisfies $0 < |\lambda| < 1$. We deduce that the characteristic polynomial of the incidence matrix of such a substitution is irreducible over $\mathbb{Q}$ (see [CS01b]). Consequently, the dominant eigenvalue α is a Pisot number and the other eigenvalues λ are its algebraic conjugates; substitutions of Pisot type are primitive and not shift-periodic (see the proofs in [PF02]). A substitution σ is *unimodular* if $\det \mathbf{M} = \pm 1$.

Note that there exist substitutions whose largest eigenvalue is Pisot but which are not of Pisot type. Examples are $1 \to 12$, $2 \to 3$, $3 \to 4$, $4 \to 5$, $5 \to 1$ and the Morse substitution $1 \to 12$, $2 \to 21$ (the characteristic polynomial is not irreducible).

1.3. **Basis of eigenvectors.** Let $\alpha_2, \ldots, \alpha_r$ denote the non-dominant real eigenvalues of $\mathbf{M}$; let $\alpha_{r+1}, \overline{\alpha_{r+1}}, \ldots, \alpha_{r+s}, \overline{\alpha_{r+s}}$ be the complex eigenvalues of $\mathbf{M}$. Let $\mathbf{u}_\alpha \in \mathbb{Q}(\alpha)^d$ (respectively $\mathbf{v}_\alpha$) be the unique dominant left (respectively right) eigenvector of $\mathbf{M}$ normalized so that the sum of its coordinates is one (respectively, normalized so that $\langle \mathbf{u}_\alpha, \mathbf{v}_\alpha \rangle = 1$).

An eigenvector $\mathbf{u}_{\alpha_k}$ for each eigenvalue α_k is obtained by replacing α with α_k in $\mathbf{u}_\alpha$. From the irreducibility of the characteristic polynomial, these vectors generate $\mathbb{R}^d$; their dual basis is given by the vectors $\mathbf{v}_{\alpha_k}$. More precisely, one has $\mathbf{x} = \sum_{k=1}^d \langle \mathbf{x}, \mathbf{v}_{\alpha_k} \rangle \mathbf{u}_{\alpha_k}$ for every $\mathbf{x} \in \mathbb{R}^d$ [CS01b]. Then, a rational vector is characterized by its first coordinate $\langle \mathbf{x}, \mathbf{v}_\alpha \rangle$ (the other coordinates are deduced by replacing α with its conjugates). This remark underlies the whole contents of the present paper.

Lemma 1.1. *A vector* $\mathbf{x} \in \mathbb{Q}^d$ *with rational coordinates is completely determined by the polynomial* $\langle \mathbf{x}, \mathbf{v}_\alpha \rangle \in \mathbb{Q}[\alpha]$.

1.4. **Combinatorial expansion system associated with a substitution.** Let $w \in X_\sigma$ be a doubly-infinite word generated by a primitive non-shift-periodic substitution σ. In this section a greedy algorithm is defined that allows one to decompose w as a combinatorial power-series. Hence, a combinatorial expansion is defined on X_σ; this combinatorial expansion will play the role of a numeration system.

Desubstitution : a combinatorial division by σ. Every doubly-infinite word $w \in X_\sigma$ has a unique decomposition $w = S^\nu(\sigma(v))$, with $v \in X_\sigma$ and $0 \leq \nu < |\sigma(v_0)|$, where v_0 is the 0^{th} coordinate of v [Mos92]. This means that any word of the dynamical system can be uniquely written of the following form, with $\ldots v_{-n} \ldots v_{-1}.v_0 v_1 \ldots v_n \cdots \in X_\sigma$:

$$w = \ldots \mid \underbrace{\ldots}_{\sigma(v_{-1})} \mid \underbrace{w_{-\nu} \ldots w_{-1}.w_0 \ldots w_{\nu'}}_{\sigma(v_0)} \mid \underbrace{\ldots}_{\sigma(v_1)} \mid \underbrace{\ldots}_{\sigma(v_2)} \mid \ldots$$

Here, the doubly-infinite word v appears to be the quotient of w by the division by σ. The rest of the division lies in the the three-tuple (p, w_0, s), that is the decomposition of $\sigma(v_0)$ of the form pw_0s, where $p = w_{-\nu} \ldots w_{-1}$ (prefix) and $s = w_1 \ldots w_{\nu'}$ (suffix). The word w is completely determined by the quotient v and the rest (p, w_0, s).

Let $\mathcal{P}$ be the finite set of all rests or*digits*, called *prefix-suffix set* associated with σ:

$$\mathcal{P} = \{(p, a, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*;\ \exists\, b \in \mathcal{A},\ \sigma(b) = pas\}.$$

The *desubstitution map* $\theta : X_\sigma \to X_\sigma$ maps a doubly-infinite word w to its quotient v. The decomposition of $\sigma(v_0)$ of the form pw_0s is denoted $\gamma : X_\sigma \to \mathcal{P}$ (mapping w to (p, w_0, s)).

If σ denotes the substitution $1 \mapsto 1112$, $2 \mapsto 12$, one computes $\mathcal{P} = \{(\varepsilon, 1, 112), (1, 1, 12), (11, 1, 2), (111, 2, \varepsilon), (\varepsilon, 1, 2), (1, 2, \varepsilon)\}$.

Prefix-suffix expansion. The *prefix-suffix expansion* is the map $E_{\mathcal{P}} : X_\sigma \to \mathcal{P}^{\mathbb{N}}$ which maps a word $w \in X_\sigma$ to the sequence $(\gamma\,(\theta^i w)\,)_{i\geq 0} \in \mathcal{P}^{\mathbb{N}}$, that is, the itineraries of w through the desubstitution according to the partition defined by γ. For example, the prefix-suffix expansion of periodic points for σ has only empty prefixes.

Let $w \in X_\sigma$ and $E_{\mathcal{P}}(w) = (p_i, a_i, s_i)_{i\geq 0}$ be its prefix-suffix expansion. If an infinite number of prefixes and suffixes are nonempty, then w and $E_{\mathcal{P}}(w)$ satisfy:

$$w = \lim_{n\to\infty} \sigma^n(p_n)\ldots\sigma(p_1)p_0.a_0s_0\sigma(s_1)\ldots\sigma^n(s_n).$$

Hence, the prefix-suffix expansion can be considered as an expansion of the points of X_σ in a "combinatorial" power series. The tuples (p_i, a_i, s_i) play the role of digits in this combinatorial expansion.

Degree of precision of the combinatorial expansion. Any prefix-suffix expansion is the label of an infinite path in the so-called *prefix-suffix automaton of* σ, whose set of nodes is the alphabet $\mathcal{A}$ and such that there is an edge labeled by $(p, a, s) \in \mathcal{P}$ from a toward b if $pas = \sigma(b)$. Examples are shown in Figure 1.1.

Denote by $X_{\mathcal{P}}$ the set of labels of path in the prefix-suffix automaton; it is the support of a subshift of finite type. Any such path is the expansion of a doubly-infinite word, since the map $E_{\mathcal{P}}$ is continuous and onto $X_{\mathcal{P}}$. A countable number of doubly-infinite words are not characterized by their prefix-suffix expansion: $E_{\mathcal{P}}$ is one-to-one except on the orbit denoted X_σ^{per} of periodic points of σ, where it is finite-to-one (see the proofs in [CS01a, HZ01]).

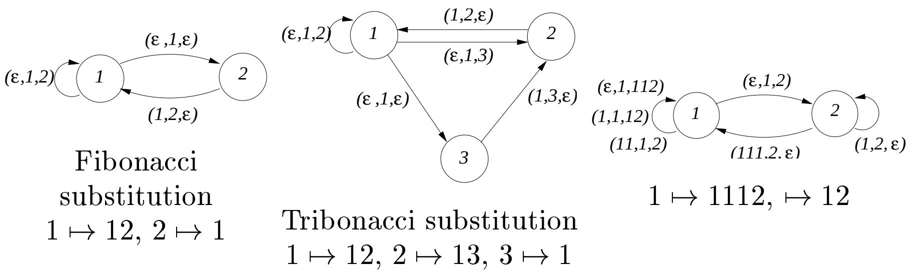


FIGURE 1.1. Prefix-suffix automatons.

1.5. **Formal expansion associated with a substitution.** The desubstitution expands a doubly-infinite word $w \in \mathcal{A}^{\mathbb{Z}}$ as the limit of the words $\sigma^n(p_n)\ldots\sigma(p_1)p_0.a_0s_0\sigma(s_1)\ldots\sigma^i(s_i)$. Since a substitutive set has entropy zero, quite no information is lost by considering only the right side of this decomposition $\sigma^n(p_n)\ldots\sigma(p_1)p_0$. Such a word embeds in $\mathbb{R}^d$ as the vector $\mathbf{l}(\sigma^n(p_n)\ldots\sigma(p_1)p_0)$. The underlying idea in this paper is to represent the word w by the limit of these vectors. Since the limit does not exist in $\mathbb{R}^d$, the vectors $\mathbf{l}(\sigma^n(p_n)\ldots\sigma(p_1)p_0)$ are considered formally, thanks to Lemma 1.1. Indeed, such a rational vector is completely determined by the algebraic number $\langle \mathbf{l}(\sigma^n(p_n)\ldots\sigma(p_1)p_0), \mathbf{v}_\alpha\rangle = \sum_0^n \langle \mathbf{M}^i\mathbf{l}(p_i), \mathbf{v}_\alpha\rangle = \sum_0^n \langle \mathbf{l}(p_i), \mathbf{v}_\alpha\rangle\alpha^i$. A representation of w is given by the formal limit of these polynomials in α. Let $\mathbb{Q}[[\alpha]]$ denote the set of formal power series in α.

Definition 1.2. The *formal expansion* of a doubly-infinite word $w \in X_\sigma$ is the formal power series

$$\varphi_{\mathbb{Q}[[\alpha]]}(w) = \sum_{i\geq 0} \langle \mathbf{l}(p_i), \mathbf{v}_\alpha\rangle\alpha^i \in \mathbb{Q}[[\alpha]].$$

2. Numeration system associated with a substitution of Pisot type

The aim of this section is to represent as precisely as possible any formal expansion associated with a substitution of Pisot type. We first define a representation space and an embedding of $\mathbb{Q}[[\alpha]]$ into this set. In a second part, the lack of injectivity of the embedding is studied : formal power series that map to 0 are characterized. In a third part, examples are developed.

2.1. Numeration system. Let $\mathcal{D}$ be the finite set of digits that appear either in the formal expansion of doubly-infinite words of X_σ, or in the difference between such two formal expansions.

$$\mathcal{D} = \left\{ \langle \mathbf{l}(p) - \mathbf{l}(p'), \mathbf{v}_\alpha \rangle ;\ (p,a,s),(p',a',s') \in \mathcal{P}^2 \right\}.$$

The formal expansion map $\varphi_{\mathbb{Q}[[\alpha]]}$ maps X_σ to the set of formal power series with coefficients in $\mathcal{D}$, say $\mathcal{D}[[\alpha]]$.

A power series in $\mathcal{D}[[\alpha]]$ is simply a formal object. There is no difficulties with giving a value to this formal object, as soon as $\mathbb{Q}[\alpha]$ is provided with a complete metrics such that α^n tends to zero. The representation of a formal power series we want to define will gather the complete set of such values.

Topologies on $\mathbb{Q}(\alpha)$. The metrizable topologies on $\mathbb{Q}(\alpha)$ are of two types [Ami75]:

- Either the topology is archimedean: its restriction to $\mathbb{Q}$ is the usual topology on $\mathbb{Q}$. Then there exists a conjugate $\lambda < 1$ of α such that the absolute value of any $P(\alpha) \in \mathbb{Q}(\alpha)$, where $P \in \mathbb{Q}[X]$, is $|P(\lambda)|$.
- Or the topology is non-archimedean: there exists a prime p such that the restriction of the topology to $\mathbb{Q}$ is the p-adic topology. Then there exists a prime ideal $\mathcal{I}$ of the ring of integers of $\mathbb{Q}(\alpha)$ such that $\alpha, p \in \mathcal{I}$ and the topology is the completion of $\mathbb{Q}(\alpha)$ for the $\mathcal{I}$-adic topology on this last field. Notice that this completion is a $\mathbb{Q}_p$ space with dimension the ramification index of $\mathcal{I}$.

Representation space of the formal power series set $\mathcal{D}[[\alpha]]$. Let us fix the following notations:

- For $2 \leq k \leq r+s$, $\mathbb{K}_{\alpha_k}$ is equal to $\mathbb{R}$ if α_k is a real number; either, it is equal to $\mathbb{C}$.
- $\mathcal{I}_1, \ldots, \mathcal{I}_\nu$ are the prime ideals that contain α, in the ring of integers of $\mathbb{Q}(\alpha)$.
- For $k \leq \nu$, $\mathbb{K}_{\mathcal{I}_k}$ is the completion of $\mathbb{Q}(\alpha)$ for the $\mathcal{I}_k$-adic topology.
- $p_1, \ldots, p_\gamma$ are the prime divisors of the norm of α, that is, $\det \mathbf{M}$.
- For $k \leq \gamma$, let $D_{p_k} = \sum_{1 \leq j \leq \nu, p_k \in \mathcal{I}_j} e(\mathcal{I}_j)$, where $e(\mathcal{I}_j)$ denotes the ramification index of $\mathcal{I}_j$.

The *representation space of* $\mathcal{D}[[\alpha]]$ is the direct product $\mathbb{K}$ of all these fields:

$$\mathbb{K} = \mathbb{K}_{\alpha_2} \times \ldots \mathbb{K}_{\alpha_{r+s}} \times \mathbb{K}_{\mathcal{I}_1} \times \ldots \mathbb{K}_{\mathcal{I}_\nu} \simeq \mathbb{R}^{d-1} \times {\mathbb{Q}_{p_1}}^{D_{p_1}} \times \cdots \times {\mathbb{Q}_{p_\gamma}}^{D_{p_\gamma}}.$$

Endowed with the product of the distances of each of its elements, $\mathbb{K}$ is a metric abelian group. Notice that $\mathbb{K}$ is an Euclidean space if and only if α is a unit. The sequence α^n tends to zero in this space.

The *canonical embedding* of $\mathcal{D}[[\alpha]]$ into $\mathbb{K}$ is defined by:

(2.1)
$$\delta:\ \sum_{i\geq 0} d_i \alpha^i \in \mathcal{D}[[\alpha]] \ \mapsto \Big(\underbrace{\sum_{i\geq 0} d_i {\alpha_2}^i}_{\in \mathbb{K}_{\alpha_2}}, \ldots, \underbrace{\sum_{i\geq 0} d_i {\alpha_{r+s}}^i}_{\in \mathbb{K}_{\alpha_{r+s}}}, \underbrace{\sum_{i\geq 0} d_i \alpha^i}_{\in \mathbb{K}_{\mathcal{I}_1}}, \ldots, \underbrace{\sum_{i\geq 0} d_i \alpha^i}_{\in \mathbb{K}_{\mathcal{I}_\nu}} \Big) \in \mathbb{K}.$$

Lemma 1.1 allows one to embed finite words into $\mathbb{K}$, so that a relation exists between doubly-infinite words and the representation of $\mathcal{D}[[\alpha]]$:

$$\eta : W \in \mathcal{A}^* \mapsto \eta(W) = \delta(\langle \mathbf{l}(W), \mathbf{v}_\alpha \rangle) \in \mathbb{K}.$$

2.2. **Degree of injectivity of the canonical embedding.** The canonical embedding δ provides a representation for the formal set $\mathcal{D}[[\alpha]]$. In this section, we determine how precise is this embedding: its "kernel" is described thanks to a finite graph. The following result was settled and partially proved in [Rau88].

Proposition 2.1. *The representation of a formal power expansion $\sum_{i\geq 0} d_i\alpha^i \in \mathcal{D}[[\alpha]]$ is null if and only if the sequence of digits $(d_i)_i$ labels an infinite path in a finite graph called the* arithmetic graph associated with *σ.*

Arithmetic graph. Let us gather in a subset $\mathcal{W}_a$ of $\mathbb{Q}(\alpha)$ the renormalized finite sums formal power series having a null representation $\mathcal{W}_a = \Big\{\xi = \sum_{i<N} d_i\alpha^{i-N} \in \mathbb{Q}(\alpha), \quad \delta(\sum_{i\geq 0} d_i\alpha^i) = 0.\Big\}$.

The *arithmetic graph of σ* is the connected component of the node zero in a big graph. Its set of nodes is denoted $\mathcal{V}_a \subset \mathcal{W}_a$ and is the biggest set such that:

- An edge from $\xi \in \mathcal{V}_a$ toward $\nu \in \mathcal{V}_a$ is labeled by $d \in \mathcal{D}$ if $\xi = \sum_{i<N} d_i\alpha^{i-N}$, $\nu = \sum_{i<N+1} d_i\alpha^{i-N-1}$, $d = d_N$ and $\delta(\sum_{i\geq 0} d_i\alpha^i) = 0$.
- Each node $\xi \in \mathcal{V}_a$ belongs to an infinite path that starts in $\xi_0 = 0$.

Proof of Proposition 2.1. There are four steps in the proof.

1. *For any absolute value $|\cdot|_v$ on $\mathbb{Q}(\alpha)$, there exists a constant M_v such that $|\cdot|_v$ is bounded by M_v on $\mathcal{W}_a$.*
   Let $\xi = \sum_{i<N} d_i\alpha^{i-N} \in \mathcal{W}_a$, with $\delta(\sum_{i\geq 0} d_i\alpha^i) = 0$. If $|\alpha|_v > 1$ then $|\xi|_v \leq M_v = \max\{|d|_v,\, d \in \mathcal{D}\}/(|\alpha|_v - 1)$. If $|\alpha|_v < 1$, then v is the metric associated with one of the fields that appear in the definition of $\mathbb{K}$ as a product, so that $\xi = -\sum_{i\geq 0} d_{i+N}\alpha^i$ for this topology. Hence $|\xi|_v \leq M_v = \max\{|d|_v,\, d \in \mathcal{D}\}/(1 - |\alpha|_v)$. If $|\alpha|_v = 1$, then the absolute value $|\cdot|_v$ is non-archimedean so that $|\xi|_v \leq M_v = \max\{|d|_v,\, d \in \mathcal{D}\}$.
2. *A sequence $(d_i) \in \mathcal{D}^\infty$ labels an infinite path in the graph if and only if $\delta(\sum_{i\geq 0} d_i\alpha^i) = 0$.*
   Let $(d_i) \in \mathcal{D}^\infty$ label a path in the arithmetic graph. There exists a sequence of nodes $\xi_i \in \mathcal{V}_a$ such that $\xi_0 = 0$ is the initial node of the path and $\alpha\xi_{i+1} = d_i + \xi_i$ for every $i \geq 0$. For every absolute value such that $|\alpha|_v < 1$, since $|\xi_N|_v \leq M_v$, we know that $|\sum_{i<N} d_i\alpha^i|_v = |\alpha^N\xi_N|_n \leq M_v{|\alpha|_v}^N$ tends to zero. This means that $\delta(\sum_{i\geq 0} d_i\alpha^i) = 0$. The converse is a direct consequence of the definition of the arithmetic graph.
3. *There exists $A \in \mathbb{N}$ such that for every $\xi \in \mathcal{V}_a$, $A\xi$ is an algebraic integer.*
   Since $\mathcal{D}$ is a finite set of algebraic numbers, there exists $A \in \mathbb{N}$ such that Ad is an algebraic integer for every $d \in \mathcal{D}$. Let $|\cdot|_v$ be a non-archimedean absolute value on $\mathbb{Q}(\alpha)$. Let $\xi = \sum_{i<N} d_i\alpha^{i-N} \in \mathcal{V}_a$ with $\delta(\sum_{i\geq 0} d_i\alpha^i) = 0$. Since α is an algebraic integer, either $|\alpha|_v = 1$, hence $|A\xi|_v \leq \max\{|Ad|_v,\, d \in \mathcal{D}\} \leq 1$, or $|\alpha|_v < 1$, hence $|A\xi|_v = |-\sum_{i\geq 0} d_{i+N}\alpha^i|_v \leq \max\{|Ad\alpha^i|_v,\, d \in \mathcal{D}, i \geq 0\} \leq 1$. Finally, $|A\xi|_v \leq 1$ for every non-archimedean absolute value on $\mathbb{Q}(\alpha)$, which means that $A\xi$ is an algebraic integer.
4. *$\mathcal{V}_a$ is a finite set.*
   The set $A\mathcal{V}_a$ consists of algebraic integers whose algebraic conjugates are all uniformly bounded, so that the set of minimal polynomials of $A\mathcal{V}_a$ consists of polynomials in $\mathbb{Z}[X]$ with bounded coefficients. Hence, it is finite and so is $A\mathcal{V}_a$. ∎

Effective construction of the arithmetic graph. The smallest integer A such that $A\mathcal{D}$ contains only algebraic integers is easily computable since an algebraic number is an algebraic integer if all the symmetric functions of its algebraic conjugates belong to $\mathbb{N}$. The other steps of the proof are algorithmic, so that the arithmetic graph is computable.

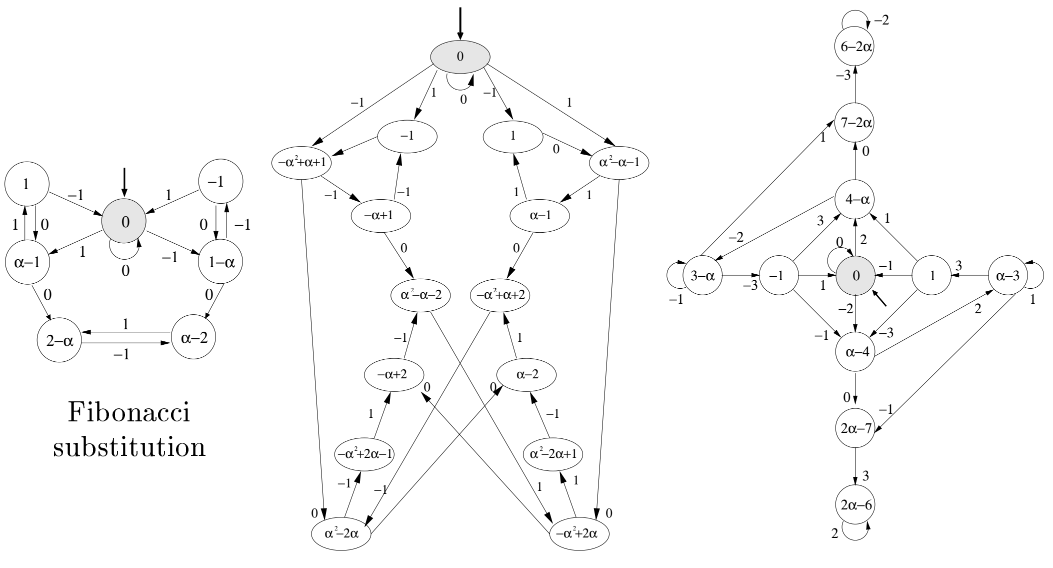
2.3. **Examples.**

The Fibonacci and the Tribonacci substitutions. $1 \mapsto 12$, $2 \mapsto 1$ The eigenvalues of the matrix of $1 \mapsto 12$, $2 \mapsto 1$ are the Fibonacci number α and $\theta = (1-\sqrt{5})/2$. The prefixes appearing in the prefix-suffix automaton are all empty, so that $\mathcal{D} = \{0, 1, -1\}$. The arithmetic graph has 7 nodes. It is shown in Fig. 2.1. By construction, paths starting from 0 are labeled by the sequences $(d_i)_i \in \{0, 1, -1\}^{\mathbb{N}}$ such that $\lim_{N\to\infty} \sum_{i\leq N} \delta(d_i\alpha^i) = 0$. Since σ is unimodular over a two-letter alphabet, δ is the morphism from $\mathcal{D}[[\alpha]]$ to $\mathbb{R}$ that preserves $\mathbb{Q}$ and maps α to θ. Hence paths are labeled by the sequences $(d_i)_i \in \{0, 1, -1\}^{\mathbb{N}}$ such that $\sum_{i\geq 0} d_i\theta^i = 0$.

Similarly, the prefixes that appear in the prefix-suffix automaton of the Tribonacci substitution $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$ (see Fig. 1.1) are all empty, so that $\mathcal{D} = \{0, 1, -1\}$. The arithmetic graph is shown in Fig. 2.1; there are 15 nodes. Paths are labeled by the sequences $(d_i)_i \in \{0, 1, -1\}^{\mathbb{N}}$ such that $\sum_{i\geq 0} d_i\theta^i = 0$ where θ is the *Tribonacci number* (root of the characteristic polynomial $X^3 - X^2 - X - 1$ of modulus less than 1).

The flipped Tribonacci substitution. The characteristic polynomial of $1 \mapsto 12$, $2 \mapsto 31$, $3 \mapsto 1$ is again $X^3 - X^2 - X - 1$. Let α be its dominant root. Prefixes that appear in the prefix-suffix automaton are the empty prefix, 1 and 3, so that $\mathcal{D} = \{0, 1, -1, \alpha^2 - \alpha - 1, -\alpha^2 + \alpha + 1, \alpha^2 - \alpha - 2, -\alpha^2 + \alpha + 2\}$. The arithmetic graph has 65 nodes and does not fit to be seen. Infinite paths are labeled by the sequences $(d_i)_i \in \mathcal{D}^{\mathbb{N}}$ such that $\sum_{i\geq 0} \overline{d_i}\theta^i = 0$, where θ is the Tribonacci number and $\overline{d_i}$ denotes the algebraic conjugate of d_i (obtained by replacing each α with θ in the expression of d_i).

A non-unimodular substitution of Pisot type $1 \mapsto 1112$, $\mapsto 12$. The characteristic polynomial is $X^2 - 4X + 2$. The eigenvalues are $\alpha = 2 + \sqrt{2}$ and $\rho = 2 - \sqrt{2}$. Prefixes that appear in the prefix-suffix automaton are the empty one, 1, 11 and 111, so that $\mathcal{D} = \{0, 1, -1, 2, -2, 3, -3\}$. The arithmetic graph has 11 nodes (see Fig. 2.1). Paths are labeled by the sequences $(d_i)_i \in \mathcal{D}^{\mathbb{N}}$ such that, first, $\sum_{i\geq 0} d_i\rho^i = 0$ for the real topology; second, the power series $\sum_{0\leq i\leq N} d_i\alpha^i$ tends to zero with respect to the unique topology on $\mathbb{Q}(\sqrt{2})$ where 2^i tends to zero.



Fibonacci substitution

Tribonacci substitution

$1 \mapsto 1112$, $\mapsto 12$

FIGURE 2.1. Arithmetic graphs.

3. Rauzy Fractal associated with a substitution of Pisot type

3.1. **Definition and main properties.** The **Rauzy representation** φ_σ of a doubly-infinite word $w \in X_\sigma$ is the representation in $\mathbb{K}$ of the formal power series associated with w: if $(p_i, a_i, s_i)_i$ is the prefix-suffix expansion of w, then

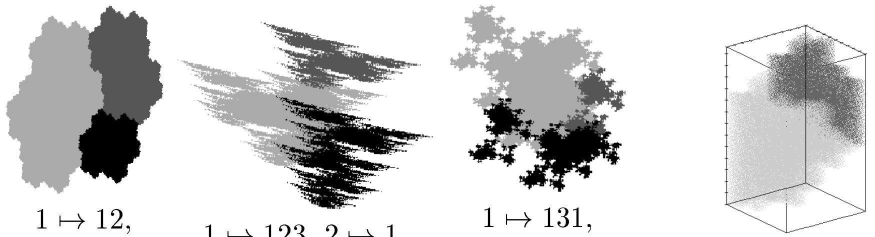
$$\varphi_\sigma(w) = \delta\varphi_{\mathbb{Q}[[\alpha]]}(w) = \delta\left(\sum_{i\geq 0}\langle \mathbf{l}(p_i), \mathbf{v}_\alpha\rangle \alpha^i\right).$$

The representation of the whole substitutive set is called the *Rauzy fractal of the substitution* and denoted $\mathcal{R}$. The images of cylinders are denoted $\mathcal{R}_a$.

$$\mathcal{R} = \varphi_\sigma(X_\sigma) \subset \mathbb{K}, \qquad \mathcal{R}_a = \varphi_\sigma([a]),\ a \in \mathcal{A}. \tag{3.1}$$

Rauzy fractals first appeared in [Rau82], defined and studied in the case of the Tribonacci substitution $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$. Then the definition was generalized to unimodular substitutions of Pisot type in [Rau88, AI01, CS01b]; the dynamics of Rauzy fractals is studied in these papers, as in [Sir00]. Finally, the definition and properties were extended to non-unimodular substitutions of Pisot type in [Sie03].

Examples of Rauzy fractals are given in Fig. 3.1. The different shades hold for the cylinders.



$1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$ — $1 \mapsto 123$, $2 \mapsto 1$, $3 \mapsto 31$ — $1 \mapsto 131$, $2 \mapsto 1$, $3 \mapsto 1132$ — $1 \mapsto 1112$, $2 \mapsto 12$

Figure 3.1. Rauzy fractal (in $\mathbb{R}^2$ and in $\mathbb{R} \times {\mathbb{Z}_2}^2$) of substitutions of Pisot type.

Properties of Rauzy fractals. The Rauzy fractal of a substitution of Pisot type satisfies the following properties in $\mathbb{K}$. See [PF02] for details and references.

- It is compact, with a non-empty interior, and it is the closure of its interior.
- It is fixed by the action of a pseudo-exchange of domains, that is, an piecewise translation defined on domains that may not be disjoint:

$$\mathcal{R} = \cup_{a\in\mathcal{A}}(\mathcal{R}_a + \eta(a)). \tag{3.2}$$

- It is a subset of the closed subgroup $\mathcal{H}$ of $\mathbb{K}$ generated by the vectors $\eta(a)$ for $ain\mathcal{A}$.
- It has non-zero measure for the Haar measure of $\mathbb{K}$.
- The image $\varphi_\sigma \star \mu_\sigma$ of the shift invariant measure on X_σ is absolutely continuous with respect the Haar measure on $\mathbb{K}$.
- The Rauzy fractal is very close to satisfy a property of self-similarity since there exists a contraction $\mathbf{h}$ on $\mathbb{K}$ such that for every letter $b \in \mathcal{A}$,

$$\mathcal{R}_b \quad = \quad \cup_{\{a\in\mathcal{A},\, \sigma(b)=pas\}}\mathbf{h}(\mathcal{R}_a) + \eta(p). \tag{3.3}$$

3.2. **Geometrical problems on Rauzy fractals.** Geometrical questions arisen on Rauzy fractals are of different kinds : are they self-similar, that is, do their cylinders $\mathcal{R}_a$ intersect ? Do the pieces $(\mathcal{R}_a + \eta(a))$ intersect ? Are the Rauzy fractals connected ? What about simply connectivity ? What is the Hausdorff dimension of their boundary ? See [PF02] for a review on these questions. In this paper, we focus on two geometrical questions : connectivity and self-similarity. Let us now state known partial results on these two points.

Connectivity. In [Can], some equivalence relations are defined, related to the connectivity of Rauzy fractals.

- A first relation $\mathcal{E}$ the describes intersections between the cylinders of the Rauzy fractal: for every $a_1, a_2 \in \mathcal{A}^2$, $a_1 \mathcal{E} a_2$ if $\mathcal{R}_{a_1}$ and $\mathcal{R}_{a_2}$ have a non-empty intersection in $\mathcal{R}$.
- For every letter a_0, the relation $\mathcal{E}_{a_0}$ describes the intersections between the sub-cylinders that appear in the self-similar decomposition of $\mathcal{R}_a$ (Formula 3.3). It is defined on the set $\{(p,a,s,b) \in \mathcal{P} \times \mathcal{A};\, a = a_0,\, pas = \sigma(b)\}$ as follows: $(p_1, a_0, s_1, b_1)\mathcal{E}_{a_0}(p_2, a_0, s_2, b_2)$ if the sets $\varphi_\sigma(S^{|p_1|}\sigma[b_1])$ and $\varphi_\sigma(S^{|p_2|}\sigma[b_2])$ have a non-empty intersection in $\mathcal{R}_{a_0}$.

According to [Can], as soon as a unimodular Rauzy fractal is connected, each pair $(\mathcal{R}_{a_1}, \mathcal{R}_{a_2})$ of cylinders is joined by a finite chain of cylinder that successively intersect. This condition is not sufficient; an information about the geometry of cylinders is needed. More precisely:

- If the Rauzy fractal of a unimodular substitution of Pisot type σ is connected, then the equivalence relation obtained by transitive closure of $\mathcal{E}$ has a unique equivalence class [Can].
- If the transitive closures of $\mathcal{E}$ and every $\mathcal{E}_a$, for $a \in \mathcal{A}$, each have a unique equivalence class, then the Rauzy fractal $\mathcal{R}$ is connected as well as each cylinder $\mathcal{R}_a$ [Can].

Notice that non-unimodular Rauzy fractals cannot satisfy such a property, since p-adic spaces are not connected.

The problem raised by such a characterization is that the intersection points between the cylinders have to be explicitly exhibited to check the relation. In Sec. 3.3, the arithmetic graph is used so that the intersection points do not more have to be explicit.

Self-similarity. Two properties are satisfied as soon as the cylinders $\mathcal{R}_a$ intersect on a zero measure set: *1.* Self-similarity of the Rauzy fractal. *2.* Formula (3.2) defines almost everywhere an exchange of domains on $\mathcal{R}$, that satisfies a commutation relation with the shift map.

It is proved in [Rau82, AI01, Sie03] (depending on the class of substitutions that is studied) that the combinatorial condition of strong coincidences is a sufficient condition for the sets $\mathcal{R}_a$ to be disjoint up to a set of zero measure. In this case, the dynamics of the exchange of domains defined by $x \mapsto x + \eta(a)$ when $x \in \mathcal{R}_a$, is correctly described by the shift map on the substitutive dynamical system.

The condition was introduced in [Dek78] for substitutions of constant length. It was generalized to non-constant length substitutions by Host and others in unpublished manuscripts. A formal definition appears in [AI01]: a substitution σ satisfies the *condition of strong coincidences* on prefixes (respectively suffixes) if for every pair $(b_1, b_2) \in \mathcal{A}^2$, there exists a constant n such that: $\sigma^n(b_1) = p_1 a s_1$ and $\sigma^n(b_2) = p_2 a s_2$ with $\mathbf{l}(p_1) = \mathbf{l}(p_2)$ (respectively $\mathbf{l}(s_1) = \mathbf{l}(s_2)$), where $\mathbf{l}$ denotes the abelianization map. An example of substitution with no coincidences is the Morse substitution.

It is conjectured that every substitution of Pisot type satisfies the condition of coincidences: no counter-example is known. It was recently proved by Barge and Diamond that this conjecture is true for two-letter substitutions: every substitution of Pisot type over a two-letter alphabet satisfies the condition of strong coincidences [BD02]. They have partial results in the case of more than two letters.

The condition of coincidences implies self-similarity. In Sec. 3.5, a necessary and sufficient condition for self-similarity is defined.

3.3. **The geometrical graph.** The aim of this section is to define a graph, called the *geometrical graph*, that describes some geometrical properties of Rauzy fractals. Applications to connectivity and self-similarity are given in the following sections. Let us state and prove the following theorem:

Theorem 3.1. *Let σ be a substitutive dynamical system of Pisot type. Two doubly-infinite words w_1 and $w_2 \in X_\sigma$ have the same representation in the Rauzy fractal, that is, $\varphi_\sigma(w_1) = \varphi_\sigma(w_2)$, if and only if their prefix-suffix expansions label two symmetric paths in a symmetric finite graph, called the* geometrical graph of σ.

The geometrical graph is an alteration of the arithmetic graph. It has to characterize digit sequences $(d_i)_i$ such that

- the representation of the formal power series $\sum d_i \alpha^i$ is equal to zero, that is, the sequence $(d_i)_i$ labels a path in the arithmetic graph (Sec. 2);
- the power series $\sum d_i \alpha^i$ is the difference between the formal representations of two points in the dynamical system. Equivalently, the digits $(d_i)_i$ correspond to the difference between the formal representations of two paths is the prefix-suffix automaton.

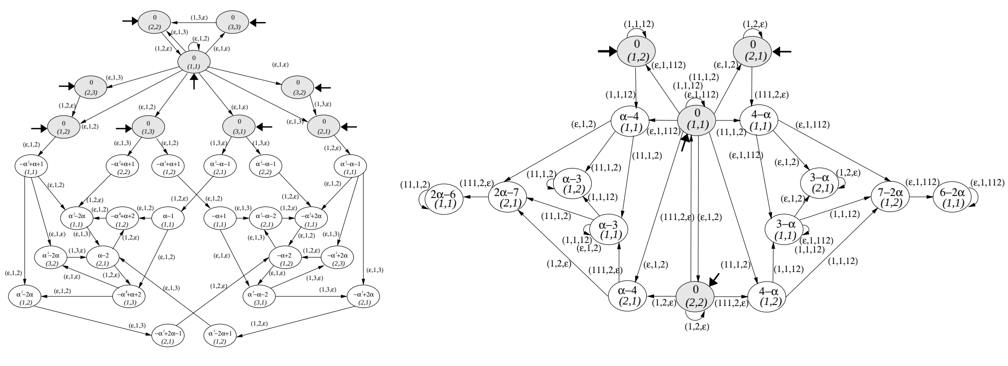
In concrete terms, the geometrical graph is a subgraph of the product of the arithmetical graph with the square of the prefix-suffix automaton: its nodes belong to $\mathcal{V}_a \times \mathcal{A} \times \mathcal{A}$, where $\mathcal{V}_a$ denotes the set of nodes of the arithmetic graph. Its edges are labeled by the prefix-suffix expansion set. Notice that $\mathcal{P}$ projects on the set $\mathcal{D}$ of digits thanks to the mapping $(p, a, s) \to \eta(p) = \delta(\langle \mathbf{l}(W), \mathbf{v}_\alpha \rangle) \in \mathbb{K}$.

Definition. The *geometrical graph of* σ, whose set of nodes is denoted by $\mathcal{V}_g \subset \mathcal{V}_a \times \mathcal{A} \times \mathcal{A}$, and which is labeled by $\mathcal{D}$, is the largest graph that satisfies:

- An edge from $(\xi_1, a_1, a_1{}') \in \mathcal{V}_g$ toward $(\xi_2, a_2, a_2{}') \in \mathcal{V}_g$ is labeled by $(p, a, s) \in \mathcal{P}$ if
  - $a = a_1$ and $\sigma(a_2) = pas$;
  - there exists $(p', a', s') \in \mathcal{P}$ such that $a' = a_1{}'$ and $\sigma(a_2{}') = p'a's'$;
  - $\alpha\, \xi_2 = \xi_1 + (\eta(p) - \eta(p')) = \xi_1 + \langle \mathbf{l}(p) - \mathbf{l}(p'), \mathbf{v}_\alpha \rangle$.
- Each node belongs to an infinite path that starts in the initial set $\mathcal{V}_g \cap \{(0, a_1, a_1{}'),\ a_1, a_1{}' \in \mathcal{A}\}$.

Symmetries in the geometrical graph. The edge from $(\xi_1, a_1, a_1{}')$ toward $(\xi_2, a_2, a_2{}')$ labeled by (p, a, s) is said to be *symmetric* to the edge from $(-\xi_1, a_1{}', a_1)$ toward $(-\xi_2, a_2{}', a_2)$ labeled by (p', a', s').

Examples of geometrical graphs are shown in Fig. 3.2 for the unimodular Tribonacci substitution and the non-unimodular substitution $1 \mapsto 1112$, $2 \mapsto 12$.



Tribonacci substitution $\qquad$ $1 \mapsto 1112$, $2 \mapsto 12$

FIGURE 3.2. Geometrical graphs.

Proof of Theorem 3.1. Let $w_1, w_2 \in X_\sigma$ be two doubly-infinite words such that $\varphi_\sigma(w_1) = \varphi_\sigma(w_2)$. Let us expand w_1 and w_2 into combinatorial power series : $E_\mathcal{P}(w_1) = (p_i, a_i, s_i)_i$ and $E_\mathcal{P}(w_2) = ({p_i}', {a_i}', {s_i}')_i$. The formal expansion of w_1 and w_2 are the power series $\sum_i \eta(p_i)\alpha^i$ and $\sum_i \eta({p_i}')\alpha^i$. Hence, the representations of w_1 and w_2 are equal if and only $\delta(\sum_i \eta(p_i)\alpha^i) = \delta(\sum_i \eta({p_i}')\alpha^i)$, that is, $\delta(\sum_i(\eta(p_i) - \eta({p_i}'))\alpha^i) = 0$. From Proposition 2.1, the sequence of digits $d_i = \eta(p_i) - \eta({p_i}')$ has to label a path (ξ_i) in the arithmetic graph, with $\xi_0 = 0$. One easily checks that $(p_i, a_i, s_i)_i$ then labels the path $(\xi_i, a_i, {a_i}')$ in the geometrical graph, whereas $({p_i}', {a_i}', {s_i}')_i$ labels the symmetric path $(-\xi_i, {a_i}', a_i)$.

Conversely, suppose that $E_\mathcal{P}(w_1) = (p_i, a_i, s_i)_i$ and $E_\mathcal{P}(w_2) = ({p_i}', {a_i}', {s_i}')_i$ label two symmetric paths of the geometrical graph. By construction, the digits $d_i = \eta(p_i) - \eta({p_i}')$ label a path in the arithmetic graph, so that $\delta(\sum_i \eta(p_i)\alpha^i) = \delta(\sum_i \eta({p_i}')\alpha^i)$ and w_1 has the same representation as w_2 in the Rauzy fractal. ■

3.4. **Application to connectivity.** The following lemma means that the first nodes of the geometrical graph provide information about the intersections between the cylinders of the Rauzy fractal.

Lemma 3.2.
- *$(0, a_1, a_2)$ is a node of the geometrical graph if and only if the cylinders $\mathcal{R}_{a_1}$ and $\mathcal{R}_{a_2}$ have a non-empty intersection in the Rauzy fractal.*
- *Let $a \in \mathcal{A}$ and $((p_1, a, s_1), b_1), ((p_2, a, s_2), b_2) \in [\mathcal{P} \times \mathcal{A}]^2$, such that $p_1 a s_1 = \sigma(b_1)$ and $p_2 a s_2 = \sigma(b_2)$. The relation $(p_1, a, s_1, b_1)\, \mathcal{E}_a\, (p_2, a, s_2, b_2)$ is satisfied if and only if the geometrical graph contains an edge from $(0, a, a)$ toward $(\frac{\langle \mathbf{l}(p_1) - \mathbf{l}(p_2), \mathbf{v}_\alpha \rangle}{\alpha}, b_1, b_2)$.*

Proof This is a direct consequence of Theorem 3.1: if $E_\mathcal{P}(w) = (p_i, a_i, s_i)_i$ denotes the combinatorial expansion of $w \in X_\sigma$, then a_0 is the zero-indexed letter of w. Moreover, $w = S^{|p|}\sigma(w_1)$ if and only if $p = p_0$ and a_1 is the zero-index letter of w_1. ■

Corollary 3.3. *The geometrical graph provides a computable sufficient condition for the connectivity of the Rauzy fractal associated with a unimodular substitution of Pisot type.*

Proof The geometrical graph is easily computable from the arithmetic graph by doing the product of the arithmetic graph and the square of the prefix-suffix automaton; then by extracting the connected components of initial nodes that belong to an infinite path. Lemma 3.2 provides an effective algorithm to check the connectivity criterion from [Can]. ■

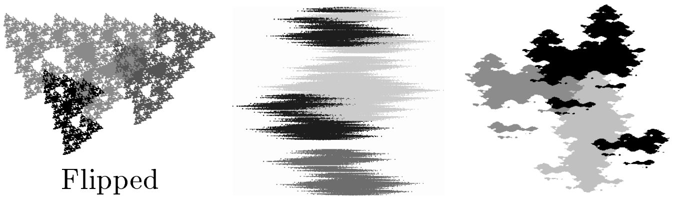
Connected Examples. The geometrical graph of the Tribonacci substitution is shown in Fig. 3.2. Since $(0,1,2)$, $(0,1,3)$ and $(0,2,3)$ are nodes of the graph, the letters 1, 2 and 3 are in relation through $\mathcal{E}$. The transitive closure of $\mathcal{E}_1$ has a unique class, that is $\{((\varepsilon,1,2),1), ((\varepsilon,1,3),2), ((\varepsilon,1,\varepsilon), 3)\}$, as well as the trivial relations $\mathcal{E}_2$ and $\mathcal{E}_3$. This provides an "algorithmic" proof for the connectivity of the Rauzy fractal associated with the Tribonacci substitution (see Fig. 3.1) [Rau82].

The geometrical graph associated with the flipped Tribonacci substitution has 69 nodes and 117 edges. Since $(0,1,2)$ and $(0,1,3)$ appear to be nodes of the graph, the letters 1, 2 and 3 are in relation through the transitive closure of $\mathcal{E}$. However, $(0,2,3)$ is not a node, meaning that the subsets $\mathcal{R}_2$ and $\mathcal{R}_3$ have no intersection in the Rauzy fractal of the substitution (see Fig. 3.3). The graph contains edges that go from $(0,1,1)$ toward $(0,1,3)$ and $(2\alpha - \alpha^2, 1, 2)$. Hence, the Rauzy fractal of the substitution is connected as well as the three pieces $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$.

A non-connected Rauzy fractal $1 \mapsto 3321$, $2 \mapsto 3$, $3 \mapsto 23321$. This example was proposed by E. Harriss as a candidate for non-connectivity. The fractal is shown in Fig. 3.3. The set of nodes of the geometrical graph contains $(0,2,3)$ but not $(0,1,2)$ and $(0,1,3)$. Hence the relation $\mathcal{E}$ has two equivalence classes $\{1\}$ and $\{2,3\}$. This means that the piece corresponding to 1 is not linked with the pieces corresponding to 2 and 3. Hence, the necessary condition of [Can] is not satisfied so that the Rauzy fractal is not connected.

This example is of full interest since it appears to be invertible. However, a substitution of Pisot type over a two-letter alphabet is invertible if and only if its Rauzy fractal is connected. The given example proves that this property cannot be generalized to the three-letter case [Har02].

A non-connected Rauzy fractal $1 \mapsto 11223$, $2 \mapsto 123$, $3 \mapsto 2$ that does not satisfy Canterini's conditions. As shown in Fig. 3.3, the Rauzy fractal seems to be non-connected. However, the connectivity graph contains the nodes $(0,1,2)$, $(0,1,3)$ and $(0,2,3)$. This is not surprising since each cylinder seems to get in contact with the others in Fig. 3.3. In concrete terms, the sufficient condition of Canterini is not satisfied. However, a refinement of the connectivity conditions allows one to prove that this Rauzy fractal is not connected [Can].



Flipped Tribonacci substitution — $1 \mapsto 3321$, $2 \mapsto 3$, $3 \mapsto 23321$ — $1 \mapsto 11223$, $2 \mapsto 123$, $3 \mapsto 2$

FIGURE 3.3. Rauzy fractals.

3.5. **Application to self-similarity.** By construction, the geometrical graph describes points in the Rauzy fractal that represent at least two points of the substitutive system. In the following theorem, we prove that this allows one to compute the measure of the intersections between the cylinders of the Rauzy fractal. Hence, we get a characterization for self-similarity.

Theorem 3.4. *Let σ be a substitution of Pisot type. The Rauzy fractal is self-similar if and only if the geometrical graph associated with σ satisfies the following two conditions:*

1. *For every $a \in \mathcal{A}$, every edge toward $(0,a,a)$ starts in a node $(0,b,b)$;*
2. *$X_{\mathcal{P}}^{strict} \neq X_{\mathcal{P}}$, where $X_{\mathcal{P}}$ is the prefix-suffix expansion set and $X_{\mathcal{P}}^{strict}$ denotes the set of labels of paths $(\xi_i, a_i, a'_i)_i$ of the geometrical graph such that $(\xi_i, a_i, a'_i) \notin \{(0,a,a),\ a \in \mathcal{A}\}$ for all i.*

Proof of Theorem 3.4. As explained in Sec. 3.1, self-similarity is equivalent to the cylinders of the Rauzy fractal to intersect on a zero measure set. We know that the Haar measure on the Rauzy fractal is the image by the representation map φ of the shift-invariant measure on the substitutive set X_σ. Hence, we need to identify which doubly-infinite word w generate an intersection in the Rauzy fractal, that is, have the same representation of a word $w' \neq w$. Then we will have to prove that the set of such doubly-infinite words has a zero measure in X_σ.

Notice that w always has the same representation as itself. In the geometrical graph, this corresponds to the path $(0, a_i, a_i)_i$ where $(p_i, a_i, s_i)_i = E_{\mathcal{P}}(w)$ denotes the prefix-suffix expansion of w. Notice that this path is characterized as being its own symmetric path.

Consequently, pairs of words (w, w') that share the same representation are concerned with two situations. Either w and w' have the same representation simply because they share the same prefix-suffix expansion. [CS01a] states that this happens only for the countable orbit X_σ^{per} of periodic points. Or w has the same representation as w' and their prefix-suffix expansion are distinct. These expansions then label symmetric and distinct paths in the geometrical graph. Let ${X_{\mathcal{P}}}^{nosym} \subset X_{\mathcal{P}}$ denote the set of path in the geometrical graph that are not their own symmetric.

Finally, the cylinders of the Rauzy fractal overlap on the countable set $\varphi({X_\sigma}^{per})$, hence negligible, and on the set $\varphi({E_{\mathcal{P}}}^{-1}{X_{\mathcal{P}}}^{nosym})$. As a consequence, overlaps in the Rauzy fractal have zero measure if and only if $\mu_{X_\sigma}({E_{\mathcal{P}}}^{-1}{X_{\mathcal{P}}}^{nosym}) = 0$.

However, the prefix-suffix expansion set $X_{\mathcal{P}}$ can be endowed with two distinct measures: first, the measure of maximal entropy for the shift map $S_{\mathcal{P}}$ on $\mathcal{P}^{\mathbb{N}}$, invariant for the shift-map on $X_{\mathcal{P}}$; second, the image $E_{\mathcal{P}}^{-1} * \mu_{X_\sigma}$ of the shift-invariant measure on X_σ. As S, this measure is uniquely ergodic. These measures are proved to be equivalent: $E_{\mathcal{P}}^{-1} * \mu_{X_\sigma}$ has a density with respect to μ_{max} [Ver81, Sie00]. Hence, $E_{\mathcal{P}}{}^{-1} X_{\mathcal{P}}{}^{nosym}$ has zero measure in the substitutive set X_σ if and only if $X_{\mathcal{P}}{}^{nosym}$ has zero measure in $X_{\mathcal{P}}$ for the shift-invariant measure with maximal entropy μ_{max}.

We now have to prove that the two conditions of Theorem 3.4 imply $\mu_{max}(X_{\mathcal{P}}{}^{nosym}) = 0$. We need the following properties of shifts of finite type: any non-empty cylinder of $X_{\mathcal{P}}$ has a non-zero measure; any shift-invariant subset of $X_{\mathcal{P}}$ has measure zero or one.

- Suppose that Condition 1. is not satisfied: the geometrical graph contains a non-symmetric edge e from a node (ξ, a, a') toward a node $(0, b, b)$. Let $V_0 = (0, a_0, a_0')$, $\ldots$, $V_{\eta-1}$, $V_\eta = (\xi, a, a')$, $V_{\eta+1} = (0, b, b)$ be a finite path that starts in an initial node and ends in $(0, b, b)$. Let $e_0, e_1, \ldots, e_\nu$ be the labeling this path. Then every digit sequence $(e_i)_i \in [e_0, e_1, \ldots, e_\nu] \cap X_{\mathcal{P}}$, with $e_i = (p_i, a_i, s_i)$, labels the non-symmetric path $(V_i)_i$ in the geometrical graph, with $V_i = (0, a_i, a_i)$ for $i > \eta$. Hence, the cylinder $[e_0, e_1, \ldots, e_\nu] \cap X_{\mathcal{P}}$ has a non-zero measure for μ_{max} and satisfies $[e_0, e_1, \ldots, e_\nu] \subset X_{\mathcal{P}}^{nosym}$. Therefore, the set $X_{\mathcal{P}}^{nosym}$ has a non-zero measure for μ_{max} and the pieces in the Rauzy fractal are not disjoint in measure.
- Suppose now that Condition 1. is satisfied: every non-symmetric path do not go toward $\{(0, a, a),\ a \in \mathcal{A}\}$ from some rank. Hence, $X_{\mathcal{P}}{}^{nosym} = \cup_{W \in \mathcal{P}^\star} W.X_{\mathcal{P}}^{strict}$.
  - When Condition 2. is not satisfied, one has $X_{\mathcal{P}}^{strict} = X_{\mathcal{P}}$. Let W label any finite path in the prefix-suffix automaton. then $X_{\mathcal{P}}{}^{nosym} \supset W X_{\mathcal{P}}^{strict} = W.X_{\mathcal{P}} = [W]$ has a non-zero measure for μ_{max}, so that the cylinders in the Rauzy fractal are not measurably disjoint.
  - When Condition 2. is satisfied, the language of $X_{\mathcal{P}}^{strict}$ is strictly contained in the language of $X_{\mathcal{P}}$. Hence, the language of $X_{\mathcal{P}}$ contains the label W_0 of a finite path in the prefix-suffix automaton that is the factor of no element in $X_{\mathcal{P}}^{strict}$. Consequently, the shift orbit of $X_{\mathcal{P}}^{strict}$ is contained in $X_{\mathcal{P}} \setminus [W_0]$, whose measure is strictly less than 1. Therefore, the orbit of $X_{\mathcal{P}}^{strict}$ has zero measure. Similarly, every set $W.X_{\mathcal{P}}^{strict}$ has zero measure for μ_{max} so that the overlaps in the Rauzy fractal are negligible. ∎

Corollary 3.5. *The geometrical graph provides a computable necessary and sufficient condition for the self-similarity of the Rauzy fractal associated with a substitution of Pisot type.*

Proof There is no difficulty to check in practice whether every edge toward $(0, a, a)$ starts in a node $(0, b, b)$. Condition 2 is checked by constructing the *deterministic graph* of the geometrical graph, recursively defined as follows. A node of the deterministic graph is defined as any subset $\mathcal{V} \subset \mathcal{V}_g$ that satisfies : a). there exists $a \in \mathcal{A}$ such that $\mathcal{V} = \mathcal{V}_g \cap \{(\xi, a, a'),\ a' \neq a\}$; b). there exists a node $\mathcal{V}'$ in the deterministic graph and $(p, a, s) \in \mathcal{P}$ such that $\mathcal{V} = \{(\eta, a_2, a_2'),\ \exists\, (\xi, a_1, a_1') \in \mathcal{V}',\ (p, a, s)$ labels an edge from (ξ, a_1, a_1') toward $(\eta, a_2, a_2')\}$. One checks that $X_{\mathcal{P}}^{strict}$ is the set of labels of paths of the deterministic graph. Hence, as soon as Condition 1 is satisfied, $X_{\mathcal{P}}^{strict} = X_{\mathcal{P}}$ if and only if the deterministic graph satisfies the following property: for every node $\mathcal{V}$, let $(\xi_0, a_0, a_0') \in \mathcal{V}$. Then for every $(p, a, s) \in \mathcal{P}$ such that $a = a_0$, there exists an edge labeled with (p, a, s) that begins in $\mathcal{V}$. This condition is computable. ∎

The Tribonacci substitution. Application to non-proper expansions in the Tribonacci system of numeration. The geometrical graph is shown in Fig. 3.2. Conditions of Theorem 3.4 are satisfied, providing a new proof for the self-similarity of the Rauzy fractal. From a number theoretic point of view, a new graph is obtained by replacing each node (ξ, a_1, a_2) by ξ, and each label (p, a, s) by the digit $\langle \mathbf{l}(p), \mathbf{v}_\alpha \rangle$ (see Fig. 3.4). This graph recognizes the set of points in the Rauzy fractal that have at least two distinct expansions in the numeration system associated with the Tribonacci number α. More precisely, infinite paths in this graph are labeled by sequences

$d_i \in \{0,1\}^{\mathbb{N}}$ such that $d_i d_{i+1} d_{i+2} = 0$ for every i and such that there exists another sequence $d_i' \in \{0,1\}^{\mathbb{N}}$ that satisfies $d_i' d_{i+1}' d_{i+2}' = 0$ for every i and $\sum_{i \geq 0} d_i \alpha^{-i} = \sum_{i \geq 0} d_i' \alpha^{-i}$. Such a graph was previously obtained in [Mes00].

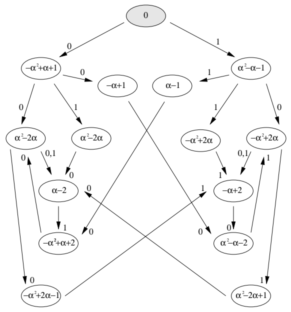


FIGURE 3.4. Expansions of double points for the Tribonacci system of numeration.

Non-unimodular example. The geometrical graph associated with the non-unimodular substitution $1 \mapsto 1112$, $2 \mapsto 12$ was shown in Fig. 3.2. The conditions are again satisfied. The geometry of the associated Rauzy fractal is not so interesting, since it is not an Euclidean object. However, self-similarity allows one to defines a dynamics on the Rauzy fractal. Hence, the theorem provides a new proof that the substitutive system is semi-topologically conjugate to an exchange of two domains in the Rauzy fractal.

More generally, every example that has been previously cited in this paper satisfies the conditions of Theorem 3.4. The associated geometrical graphs are too big to be shown here.

4. Abelian dynamics associated with a substitution of Pisot type

The Rauzy fractal is naturally endowed with a contracting dynamics deduced form self-similarity. In this section, we are concerned with another dynamics that exists on the Rauzy fractal, that is, a linear dynamics. Geometrically, the existence of such a dynamics means that the Rauzy fractal generates a regular tiling. A more dynamical interpretation leads to pure discrete spectrum. Hence, the aim of this section is to make use of the geometrical graph to: first, determine a computable necessary and sufficient condition for the Rauzy fractal of a unimodular substitution of Pisot type to generate a periodic tiling; second, deduce a computable sufficient condition for a substitutive dynamical system to have a pure discrete spectrum.

4.1. **Abelian dynamics.**

Abelian representation. Let a_0 denote a fixed letter in $\mathcal{A}$, for instance $a_0 = d$. Let $\mathcal{H}_0$ be the closed subgroup of $\mathbb{K}$ generated by the vectors $\eta(a) - \eta(a_0)$, for $a \neq a_0 \in \mathcal{A}$. It is a discrete subgroup of $\mathbb{K}$ since its projection in $\mathbb{R}^{d-1}$ is discrete [CS01b]. If the substitution is unimodular, $\mathcal{H}_0$ is a lattice in $\mathbb{R}^{d-1}$.

$$\mathcal{H}_0 = \sum_{a \neq a_0 \in \mathcal{A}} \mathbb{Z}(\eta(a) - \eta(a_0)).$$

Let $\mathcal{H}$ be the smallest group that contains the Rauzy fractal $\mathcal{R}$. Let $\pi_{\mathcal{G}}$ denote the projection of $\mathcal{H}$ onto the quotient group $\mathcal{G} = \mathcal{H}/\mathcal{H}_0$. We call *abelian representation of the substitution* the map

$$\pi_{\mathcal{G}} \varphi_\sigma : X_\sigma \to \mathcal{G}.$$

By construction, it is continuous, onto, and realizes a commutation relation between the shift map S on X_σ and the translation by $\eta(a_0)$ on $\mathcal{G}$, that is $\varphi_\sigma \pi_{\mathcal{G}} \circ S = \varphi_\sigma \pi_{\mathcal{G}} + \eta(a_0)$.

Hence, the translation by $\eta(a_0)$ on $\mathcal{G}$ appears to be an equicontinuous factor of the shift map on X_σ, since all the maps are continuous in the following diagram. Consequently, a sufficient condition for X_σ to have a pure discrete spectrum is that $\varphi_\sigma \pi_{\mathcal{G}}$ is almost everywhere one-to-one.

$$
\begin{tikzcd}
X_\sigma \arrow[r, "prefix-suffix", "expansion"'] \arrow[d, "S", "\substack{shift\\map}"'] & X_{\mathcal{P}} \arrow[r, "Rauzy", "representation"'] \arrow[d, "\substack{adic\\transfor-\\mation}"'] & \mathcal{R} \subset \mathcal{H} \arrow[r, "quotient", "map"'] \arrow[d, "\substack{\mathcal{R}_a \to\\ \mathcal{R}_a + \eta(a)}", "\substack{exchange\\of\,domains}"'] & \mathcal{G} \arrow[d, "\substack{translation\\by\,\eta(a_0)}"] \\
X_\sigma \arrow[r, "E_{\mathcal{P}}"'] & X_{\mathcal{P}} \arrow[r, "E_{\mathcal{P}}^{-1}\varphi_\sigma"'] & \mathcal{R} \arrow[r, "\pi_{\mathcal{G}}"'] & \mathcal{G}
\end{tikzcd}
$$

Isomorphism to the factor? A natural question is whether the substitutive system is measure-theoretically isomorphic to this factor. Since the invariant measures for the systems appearing in the diagram are equivalent, the following conditions are equivalent when the substitution satisfies the condition of coincidences:

1. the quotient map $\pi_{\mathcal{G}}$ is almost everywhere one-to-one on $\mathcal{R}$;
2. the Haar representation map $\pi_{\mathcal{G}}\varphi_\sigma$ is almost everywhere one-to-one on the substitutive dynamical system (X_σ, S), so that the system has a pure discrete spectrum;
3. the sets $\mathcal{R} + \mathbf{v}$, for $\mathbf{v} \in \mathcal{H}_0$, are disjoint up to a zero measure set.

In the unimodular case, it has been proved that the group $\mathcal{H}$ generated by the Rauzy fractal is equal to the full space $\mathbb{R}^{d-1}$ [CS01b]. Hence, the conditions above ensure that the Rauzy fractal generates a periodic tiling of $\mathbb{R}^{d-1}$.

The aim of this section is to determine an explicit condition for these conditions to be satisfied. let us first state partial results on that subject.

Coincidences versus pure discrete spectrum. Coincidences where introduced since they have a deep relation with pure discrete spectrum in some specific cases. Dekking proved that a substitution of constant length generates a pure discrete spectrum dynamical system if and only if the substitution satisfies the condition of coincidences [Dek78]. In this case, the spectrum of the substitutive system is explicit: the substitutive system is measure theoretically isomorphic to a translation on the direct product of a finite group and the group of d-adic integers $\mathbb{Z}_d$, where d is the cardinality of the alphabet. Later, Hollander and Solomyak proved that a substitution of Pisot type over a two-letter alphabet generates a pure discrete spectrum dynamical system if and only if the substitution satisfies the condition of strong coincidences [HS03]. Hence, two-letter substitutive dynamical systems of Pisot type all have a pure discrete spectrum. The techniques used cannot be generalized to substitutions over an alphabet which contains strictly more than two letters.

Partial description of the spectrum. The notion of *coboundaries* introduced by B. Host allows one to understand more the structure of the spectrum of a substitutive system. A *coboundary* of a substitution σ is defined as a map $h : \mathcal{A} \to \mathbb{U}$ (where $\mathbb{U}$ denotes the unit circle) such that there exists a map $f : \mathcal{A} \to \mathbb{U}$ with $f(b) = f(a)h(a)$ for every word ab of length 2 that appears in a periodic point for σ. The coboundary defined by $h(a) = 1$ for every letter a (that is, $f(a) = f(b)$ for every ab in the language) is called the *trivial coboundary.* Roughly speaking, a substitution with coincidences does not have a nontrivial coboundary. For substitutions of constant length, the finite group contained in the maximal equicontinuous factor plays the role of nontrivial coboundaries. Details can be found in [PF02], Chap. 7.

The *spectrum* of a primitive substitutive dynamical system (X_σ, S) is the spectral type of the unitary operator $f \to f \circ S$, well defined on $L^2(X_\sigma, S)$. The structure of the spectrum is described by Host: a complex number $\lambda \in \mathbb{U}$ is an eigenvalue of a primitive non-shift-periodic substitutive dynamical system (X_σ, S) if and only if there exists $p > 0$ such that for every $a \in \mathcal{A}$, the limit $h(a) = \lim_{n\to\infty} \lambda^{|\sigma^{pn}(a)|}$ is well defined, and h is a coboundary of σ [Hos86].

Eigenvalues associated with the trivial coboundary are explicitly determined for unimodular substitutions of Pisot type: in this case, the group of eigenvalues of X_σ associated with the trivial coboundary is generated by the frequencies of letters in any infinite word of the system, that is, by the coordinates of a right normalized dominant eigenvector of the incidence matrix of the substitution. We say that such eigenvalues are "commutative", since they depend only on the incidence matrix of the substitution. On the contrary, eigenvalues associated only with a nontrivial coboundary are "non-commutative": they depend heavily on the combinatorics of the substitution. Notice that we do not know any example of substitution of Pisot type with irrational non-commutative eigenvalues.

Consequence on Rauzy fractals. It is known that the spectrum of the addition of $\eta(a_0)$ on $\mathcal{G}$ is equal to the "commutative" spectrum of the substitutive system [PF02]. Hence, having no nontrivial coboundary is a necessary condition for a unimodular substitution to satisfy the conditions 1, 2 or 3.

4.2. **Tiling graph.** In this section, we expand the definition of the geometrical graph to a so-called *tiling graph*, and prove the following result.

Theorem 4.1. *Let σ be a substitution of Pisot type. The following three properties are equivalent:*

- *The Rauzy fractal $\mathcal{R}$ is self-similar and disjoint almost everywhere from its copies $\mathcal{R} + \mathbf{v}$, for $\mathbf{v} \in \mathcal{H}_0$;*
- *The Abelian representation $\varphi_\sigma \pi_{\mathcal{G}} : X_\sigma \to \mathcal{G}$ is onto and almost everywhere one-to-one;*
- *the following three computable conditions are satisfied by the geometrical graph and its expansion, called the* tiling graph*:*
  1. *for any $a \in \mathcal{A}$, every edge toward $(0, a, a)$ in the geometrical graph starts in a node $(0, b, b)$;*
  2. *$X_{\mathcal{P}}^{strict} \neq X_{\mathcal{P}}$, where $X_{\mathcal{P}}$ is the prefix-suffix shift of finite type and $X_{\mathcal{P}}^{strict}$ denotes the set of labels of paths $(\xi_i, a_i, a_i')_i$ of the geometrical graph such that $(\xi_i, a_i, a_i') \notin \{(0, a, a),\ a \in \mathcal{A}\}$ for every i.*
  3. *the set of labels of infinite paths in the tiling graph is a strict subset of $X_{\mathcal{P}}$.*

To prove Theorem 4.1, we need to determine pairs $(p_i, a_i, s_i)_i, (q_i, b_i, r_i)_i \in X_{\mathcal{P}}$ that satisfy:

$$\exists\, \mathbf{v} \in \mathcal{H}_0, \quad \delta\left(\textstyle\sum_{i\geq 0} \langle \mathbf{l}(p_i) - \mathbf{l}(q_i), \mathbf{v}_\alpha \rangle \alpha^i\right) = \mathbf{v}. \tag{4.1}$$

Such vectors $\mathbf{v} \in \mathcal{H}_0$ join two points in the Rauzy fractal $\mathcal{R}$. As $\mathcal{R}$ is bounded, only a finite number of such vectors $\mathbf{v} \in \mathcal{H}_0$ can satisfy this. In the sequel, these vectors are characterized by their coordinates in the basis $(\eta(a) - \eta(a_0))_{a\neq a_0 \in \mathcal{A}}$.

Proposition 4.2. *There exist $d-1$ bounds $(M_a)_{a\neq a_0\in\mathcal{A}}$, such that, for all $(n_a)_{a\neq a_0\in\mathcal{A}} \in \mathbb{Z}^{d-1}$, as soon there exists $b \neq a_0 \in \mathcal{A}$ with $n_b > M_b$, then the vector $\mathbf{v} = \sum_{a\neq a_0\in\mathcal{A}} n_a(\eta(a) - \eta(a_0)) \in \mathcal{H}_0$ do not join two points in the Rauzy fractal $\mathcal{R}$, that is, do not satisfy Formula (4.1).*

Proof Let α_j, for $j = 2 \dots r+s$, denote the non-dominant eigenvalues of $\mathbf{M}$ and let $\mathbf{v}_{\alpha_j}$ denote the eigenvector of $\mathbf{M}$ obtained by replacing each α with α_j in the coordinates of $\mathbf{v}_\alpha$. Let us define the following easily computable constants:

$$\forall\, j = 2 \dots r+s, \quad m_j = \max_{\substack{(p_1,a_1,s_1)\\(p_2,a_2,s_2)\in\mathcal{P}}} \left| \frac{\langle \mathbf{l}(p_1) - \mathbf{l}(p_2), \mathbf{v}_{\alpha_j}\rangle}{1-\alpha_j} \right|.$$

From the definition of δ, if the vector $\mathbf{v} \in \mathcal{H}_0$ satisfies Formula (4.1), then $\mathbf{v}$ belongs to the bounded box: $\mathcal{B} = \left\{\mathbf{v} = (v_2, \dots v_{r+s}, v_{\mathcal{I}_1}, \dots, v_{\mathcal{I}_\eta}) \in \mathbb{K},\ \forall\, j = 2 \dots r+s,\ |v_j| < m_j\right\}$.

The space $\mathbb{K}$ canonically projects onto $\mathbb{R}^{r-1} \times \mathbb{C}^s$, that is isomorphic to $\mathbb{R}^{d-1}$. Let $\pi_{\mathbb{R}^{d-1}}$ denote the induced canonical projection from $\mathbb{K}$ onto $\mathbb{R}^{d-1}$. Then $\pi_{\mathbb{R}^{d-1}}(\mathcal{B})$ is contained in the box $\mathcal{B}'$ whose vertices are

$$\mathcal{V}_{\mathcal{B}'} = \{(\pm m_2, \pm m_3 \cdots \pm m_r, \pm m_{r+1}, \pm m_{r+1} \cdots \pm m_{r+s}, \pm m_{r+s})\} \subset \mathbb{R}^{d-1}.$$

According to [CS01b], a basis of $\mathbb{R}^{d-1}$ is given by the vectors $\mathbf{w}_a = \pi_{\mathbb{R}^{d-1}}(\eta(a) - \eta(a_0)) \in \mathbb{R}^{d-1}$, for $a \neq a_0 \in \mathcal{A}$. For every $a \neq a_0 \in \mathcal{A}$, and any vector $\mathbf{w} \in \mathbb{R}^{d-1}$, let $c_a(\mathbf{w})$ denote the a-th coordinates of $\mathbf{w}$ in this basis, and define $M_a = [\max\{c_a(\mathbf{w}), \mathbf{w} \in \mathcal{V}_{\mathcal{B}'}\}] + 1$. Then for every vertex $\mathbf{w} \in \mathcal{V}_{\mathcal{B}'}$, the a-th coordinates of $\mathbf{w}$ in the new basis is less than M_a. This implies the whole box $\mathcal{B}'$ belongs to the half-space defined by $c_a(\mathbf{w}) < M_a$.

Let $(n_a)_{a \neq a_0 \in \mathcal{A}} \in \mathbb{Z}^{d-1}$, with $n_b \geq M_b$ for a given $b \neq a_0$. Suppose that $\mathbf{v} = \sum_{a \neq a_0 \in \mathcal{A}} n_a(\eta(a) - \eta(a_0)) \in \mathcal{B}$. This implies $\mathbf{w} = \pi_{\mathbb{R}^{d-1}}\mathbf{v} = \sum_{a \neq a_0 \in \mathcal{A}} n_a \mathbf{w}_a \in \pi_{\mathbb{R}^{d-1}}\mathcal{B} \subset \mathcal{B}'$. But, simultaneously, $c_b(\mathbf{w}) = n_b < M_b$, which is impossible on $\mathcal{B}'$. ■

Tiling vectors. Proposition 4.2 implies that, for any vector $\mathbf{v} \in \mathcal{H}_0$ that joins two points in the Rauzy fractal, there exist a tuple $(n_a)_{a \neq a_0 \in \mathcal{A}} \in \mathbb{Z}^{d-1}$, with $n_a \leq M_a$ for every a, such that $\mathbf{v} = \sum_{a \neq a_0 \in \mathcal{A}} n_a(\eta(a) - \eta(a_0))$. Notice that the converse is not true. Let $\mathcal{T}$ denote the set of such vectors, called *tiling vectors*:

$$\mathcal{T} = \left\{\mathbf{v} = \textstyle\sum_{a \neq a_0 \in \mathcal{A}} n_a(\eta(a) - \eta(a_0)),\ n_a \leq M_a\right\} \subset \mathcal{H}_0.$$

Then, Formula (4.1) for such a vector $\mathbf{v}$ is satisfied if and only if

$$\delta\left(\textstyle\sum_{i \geq 0} \langle \mathbf{l}(p_i) - \mathbf{l}(q_i), \mathbf{v}_\alpha \rangle \alpha^i - \sum_{a \neq a_0 \in \mathcal{A}} n_a \langle \mathbf{l}(a) - \mathbf{l}(d), \mathbf{v}_\alpha \rangle\right) = 0. \tag{4.2}$$

In Sections 2.2 and 3.5 were defined graphs that allow to identify sequences (p_i) satisfying Formula (4.2) for the null vector, that is, with coordinates $n_a = 0$ for every a. In the sequel, the starting nodes of these graphs are modified, to characterize prefix-suffix expansions that satisfy Formula (4.2) in the general case.

Definition of the tiling graph. Let $\mathcal{S}_\mathcal{T}$ denote the arithmetic set of initial nodes

$$\mathcal{S}_\mathcal{T} = \{-\textstyle\sum_{a \neq a_0 \in \mathcal{A}} n_a \langle \mathbf{l}(a) - \mathbf{l}(a_0), \mathbf{v}_\alpha \rangle, \quad \mathbf{v} = \sum_{a \neq a_0 \in \mathcal{A}} n_a(\eta(a) - \eta(a_0)) \neq 0 \in \mathcal{T}\}.$$

Let $\mathcal{W}_\mathcal{T}$ correspond to the nodes in the arithmetic tiling graph :

$$\mathcal{W}_\mathcal{T} = \{\xi = \xi_0 \alpha^{-N} + \textstyle\sum_{i<N} d_i \alpha^{i-N},\ \xi_0 \in \mathcal{S}_\mathcal{T},\ \delta(\xi_0 + \sum_{i \geq 0} d_i \alpha^i) = 0\} \subset \mathbb{Q}(\alpha).$$

The *tiling graph of* σ, whose set of nodes is denoted by $\mathcal{V}_\mathcal{T} \subset \mathcal{W}_\mathcal{T} \times \mathcal{A} \times \mathcal{A}$, and which is labeled by $\mathcal{P}$, is the largest graph that satisfies:

- An edge from $(\xi_1, a_1, a_1') \in \mathcal{V}_\mathcal{T}$ toward $(\xi_2, a_2, a_2') \in \mathcal{V}_\mathcal{T}$ is labeled by $(p, a, s) \in \mathcal{P}$ if there exist $(p', a', s') \in \mathcal{P}$, $\xi_0 \in \mathcal{S}_\mathcal{T}$, and a sequence of digits $d_i \in \mathcal{D}^\mathbb{N}$ such that :
  - $a = a_1$ and $\sigma(a_2) = pas$,
  - $a' = a_1'$ and $\sigma(a_2') = p'a's'$,
  - $\xi = \xi_0 \alpha^{-N} + \sum_{i<N} d_i \alpha^{i-N}$,
  - $\nu = \xi_0 \alpha^{-N-1} + \sum_{i<N+1} d_i \alpha^{i-N-1}$,
  - $d_N = (\eta(p) - \eta(p_2)) = \langle \mathbf{l}(p) - \mathbf{l}(p'), \mathbf{v}_\alpha \rangle$,
  - $\delta(\xi_0 + \sum_{i \geq 0} d_i \alpha^i) = 0$.
- Each node belongs to an infinite path that starts in the initial set $\mathcal{V}_\mathcal{T} \cap \{(\xi_0, a_1, a_1'),\ \xi_0 \in \mathcal{S}_\mathcal{T},\ a_1, a_1' \in \mathcal{A}\}$.

Proposition 4.3. *The tiling graph associated with a substitution of Pisot type is finite. Let w_1 be a doubly-infinite word in the substitutive system. There exists a word w_2 such that $\varphi_\sigma(w_1) - \varphi_\sigma(w_2) \in \mathcal{H}_0 \setminus \{0\}$ if and only if the prefix-suffix expansion of w_1 labels a walk in the tiling graph of σ.*

Proof The proof of Proposition 2.1 can be transposed here so that the tiling graph is finite.

Suppose that there exists a word w_2 such that $\varphi_\sigma(w_1)-\varphi_\sigma(w_2) = \mathbf{v} \in \mathcal{H}_0\setminus\{0\}$. From Proposition 4.2, there exist coefficients $n_a \le M_a$ such that $\mathbf{v} = \sum_{a\neq a_0\in\mathcal{A}} n_a(\eta(a) - \eta(a_0))$. Hence $\xi_0 = -\sum_{a\neq a_0\in\mathcal{A}} n_a\langle \mathbf{l}(a) - \mathbf{l}(a_0), \mathbf{v}_\alpha\rangle \in \mathcal{S}_\mathcal{T}$. Let $(p_i, a_i, s_i)_i$ and $(p_i', a_i', s_i')_i$ be the prefix-suffix expansions of w_1 and w_2, and $d_i = \langle \mathbf{l}(p_i) - \mathbf{l}(p_i{}'), \mathbf{v}_\alpha\rangle$. One checks immediately that $(p_N, a_N, s_N)_N$ labels the infinite walk $(\xi_0\alpha^{-N} + \sum_{i<N} d_i\alpha^{i-N}, a_N, a_N')_N$.

Conversely, suppose that be the prefix-suffix expansion $(p_N, a_N, s_N)_N$ of the doubly-infinite word $w_1 \in X_\sigma$ labels an infinite walk (ξ_N, a_N, a_N') in the tiling graph. The definition of the edges implies that there exists a sequence $(p_N', a_N', s_N')_N$ that appears to be the prefix-suffix expansion of another doubly-infinite word, say $w_2 \in X_\sigma$. Moreover, one has $\xi_{N+1} = \alpha^{-1}\xi_N + \langle \mathbf{l}(p_N) - \mathbf{l}(p_N'), \mathbf{v}_\alpha\rangle$. We deduce that $\alpha^{N+1}\xi_{N+1} = \xi_0 + \sum_{i<N+1}\langle \mathbf{l}(p_N) - \mathbf{l}(p_N'), \mathbf{v}_\alpha\rangle\alpha^i$, so that $\delta(\xi_0 + \sum_{i<N+1}\langle \mathbf{l}(p_N) - \mathbf{l}(p_N'), \mathbf{v}_\alpha\rangle\alpha^i) = \delta(\alpha^{N+1}\xi_{N+1})$ tends to zero as N tends to infinity (since the vertices ξ_{N+1} are in finite number). Finally, $\xi_0 \in \mathcal{S}_\mathcal{T}$ so that there exists $\mathbf{v} = \sum_{a\neq a_0\in\mathcal{A}} n_a(\eta(a) - \eta(a_0)) \neq 0 \in \mathcal{T}$ such that $\xi_0 = -\sum_{a\neq a_0\in\mathcal{A}} n_a\langle \mathbf{l}(a) - \mathbf{l}(a_0), \mathbf{v}_\alpha\rangle$. Hence, $\varphi_\sigma(w_1) - \varphi_\sigma(w_2) = \mathbf{v} \in \mathcal{H}_0 \setminus \{0\}$. ∎

Proof of Theorem 4.1. The self-similarity of the Rauzy fractal was characterized in Sec. 3.5. The proof of Theorem 3.4 can be transposed here so that the copies of the Rauzy fractal do not overlap if and only if the set of label of paths in the tiling graph is not equal to the prefix-suffix shift of finite type $X_\mathcal{P}$. Again, this condition is computable. ∎

4.3. **Unimodular substitutions: application to tilings.** In the unimodular case, $\mathbb{K}$ is equal to the Euclidean space $\mathbb{R}^{d-1}$ and the subgroup $\mathcal{H}_0$ is a lattice in $\mathbb{R}^{d-1}$. Moreover, we know that the copies of the Rauzy fractal along the lattice $\mathcal{H}_0$ cover the full space $\mathbb{R}^{d-1}$: $\bigcup_{\mathbf{v}\in\mathcal{H}_0}(\mathcal{R} + \mathbf{v}) = \mathbb{R}^{d-1}$.

Hence, Theorem 4.1 can be directly interpreted as a necessary and sufficient condition for the Rauzy fractal to generate a tiling. More precisely, we have the following result:

Corollary 4.4. *Let σ be a unimodular substitution of Pisot type over a d-letter alphabet. There exists a computable necessary and sufficient condition for the two following equivalent properties:*

- *the Rauzy fractal of the substitution is a fundamental domain for the lattice $\mathcal{H}_0$, so that it generates a periodic tiling of $\mathbb{R}^{d-1}$;*
- *the substitutive dynamical system is semi-topologically conjugate to the translation on $\mathbb{T}^{d-1}$ by the vector of frequencies of letters in any doubly-infinite word of the substitutive system.*

Proof The conditions of Theorem 4.1 are satisfied if and only if the Haar representation $\varphi_\sigma\pi_\mathcal{G}$ is almost everywhere one-to-one. We already know that it is onto $\mathbb{T}^{d-1}$ [CS01a]. By construction, this map realizes a commutation relation with the translation by $\eta(a_0)$ on the torus $\mathcal{H}/\mathcal{H}_0$. The group of eigenvalues of this translation on $\mathcal{H}/\mathcal{H}_0$ is generated by the frequencies of letters in the substitutive dynamical system [CS01a]. Therefore, the translation is conjugate to the toral translation on $\mathbb{T}^{d-1}$ by the vector of frequencies. Consequently, the substitutive system and the toral translation on $\mathbb{T}^{d-1}$ are semi-topologically conjugate if and only if the conditions of Theorem 4.1 are satisfied. ∎

Differences with the non-unimodular case.

- In the non-unimodular case, the group $\mathcal{H}$ generated by the Rauzy fractal is strictly contained in $\mathbb{K}$. Therefore, the Abelian representation is onto the quotient of $\mathcal{H}$ by the lattice $\mathcal{H}_0$, and cannot be onto the full quotient of $\mathbb{K}$ by $\mathcal{H}_0$. As a consequence, a substitution of Pisot type satisfies the conditions of Theorem 4.1 if and only if its Rauzy fractal generates a tiling of the closed subgroup $\mathcal{H}$ instead of the full space $\mathbb{K}$.
- In the non-unimodular case, the group of eigenvalues of the translation by $\eta(a_0)$ on $\mathcal{H}/\mathcal{H}_0$ is not explicitly known in the general case; the group generated by the frequencies of letters is strictly included in it. Therefore, the second point of Corollary 4.4 cannot be true when the substitution is not unimodular.

4.4. **Effective condition for pure discrete spectrum.** As explained at the end of Sec. 4.1, the tiling property has a dynamical interpretation in terms of pure discrete spectrum.

Theorem 4.5. *There exists a computable sufficient condition for a substitutive dynamical system to have a pure discrete spectrum. This condition is a necessary condition when the substitution is unimodular and has no non-trivial coboundary.*

Proof When the conditions of Theorem 3.4 are satisfied, the substitutive system is semi-conjugate to a translation on a compact group (more precisely to the translation by $\eta(a_0)$ on the group $\mathcal{H}/\mathcal{H}_0$). In particular, it has a pure discrete spectrum. Hence, Theorem 3.4 provides a sufficient condition for a substitutive system of Pisot type to have a pure discrete spectrum.

Conversely, Theorem 3.4 provides a necessary condition for a pure discrete spectrum substitutive system as soon as the translation by $\eta(a_0)$ on the group $\mathcal{H}/\mathcal{H}_0$ is the maximal equicontinuous factor of the substitutive system. When the substitution is unimodular, it was recalled that the translation by $\eta(a_0)$ on the group $\mathcal{H}/\mathcal{H}_0$ is isomorphic to the $\mathbb{T}^{d-1}$-translation given by the frequencies of letters; if the substitution also has no non-trivial coboundary, then this toral translation is the maximal equicontinuous factor of the substitutive system [Hos86] (see also [PF02], Exercise 7.5.15). ∎

In practical, since all substitutions of Pisot type that have been checked have no nontrivial coboundary, Theorem 3.4 appears to be a suitable criterion to check whether a unimodular substitution has a pure discrete spectrum. A program, implemented in *Mu PAD*, tests whether any given substitution of Pisot type satisfies the conditions of Theorem 4.1 [Sie00]. Practically, all the examples that have been checked up to now satisfy the conditions of Theorem 3.4 so that they have a pure discrete spectrum. Hence, we do not know any substitutive system of Pisot type whose spectrum is not purely discrete. **It is a conjecture that all substitutive systems of Pisot type have pure discrete spectra.**

4.5. **Examples.**

New proofs for known results : The Fibonacci and the Tribonacci substitutions. The lattice associated with the Fibonacci substitution is $\mathcal{H}_0 = \frac{3+\sqrt{5}}{2}\mathbb{Z}$. The maximal equicontinuous factor is the translation by 1 on $\mathbb{R}/(\frac{3+\sqrt{5}}{2}\mathbb{Z})$. Only $\pm\frac{-3+\sqrt{5}}{2}$ provide intersections between the copies of the interval $\mathcal{R} = \varphi_\sigma(X_\sigma)$. The points -1 and $\frac{1+\sqrt{5}}{2}$ are the only points in $\mathcal{R}$ that are equal modulo $\mathcal{H}_0$. Consequently, they are the extremities of this interval and the length of $\mathcal{R}$ is $(3+\sqrt{5})/2$.

The conditions of Theorem 4.1 are also satisfied by the Tribonacci substitution. Hence, the associated system is semi-topologically conjugate to a translation on $\mathbb{T}^2$ [Rau82]. The Rauzy fractal generates a tiling of $\mathbb{C}$ by the lattice $\mathcal{H}_0 = \mathbb{Z}(2+\alpha-\alpha^2)\oplus\mathbb{Z}(2\alpha-\alpha^2)$, where α denotes the Tribonacci number. Vectors that connect two points in the Rauzy fractal are $\pm(\alpha-2)$, $\pm(2\alpha-\alpha^2)$ and $\pm(\alpha^2-\alpha-2)$. The tiling graph have loops, implying that a non-countable set of points in the Rauzy fractal are joined by non-zero vectors in $\mathcal{H}_0$.

Tilings : unimodular examples. The lattice $\mathcal{H}_0$ associated with the flipped Tribonacci substitution is the same as the Tribonacci substitution lattice. Vectors in this lattice that join two points in the Rauzy fractal of the substitution are 0, $\pm(\alpha^2-\alpha-2)$, $\pm(\alpha-2)$, $\pm(2\alpha-\alpha^2)$, $\pm(\alpha^2-4)$, $\pm(2\alpha^2-3\alpha-2)$ and $\pm(\alpha^2-3\alpha+2)$. The conditions of Theorem 4.1 are satisfied.

As mentioned before, all the examples that have been checked also satisfy the conditions.

Corollary 4.6. *The dynamical systems associated with each of the unimodular substitutions* $1\mapsto 12 \quad 2\mapsto 31 \quad 3\mapsto 1$, $1\mapsto 12 \quad 2\mapsto 13 \quad 3\mapsto 132$, $1\mapsto 123 \quad 2\mapsto 1 \quad 3\mapsto 31$ *and* $1\mapsto 131 \quad 2\mapsto 1 \quad 3\mapsto 1132$, *are semi-topologically conjugate to a minimal translation on* $\mathbb{T}^2$*, so that they have pure discrete spectra. All the associated Rauzy fractals generate periodic tilings of the plane (Fig. 0.1).*

An example where 0 is not an interior point. The Rauzy fractal associated with the substitution $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 12$ is quite specific: the three pieces have a lot of contacts, and may overlap (see Fig. 4.1). In spite of this visual observation, the geometrical graph satisfies the self-similar conditions and the connectivity conditions. The tiling conditions are also satisfied. The point that explain why the Rauzy fractal is so intricate is that 0 is the representation of the 3 periodic points of the substitutive systems, so that it is at the intersection of the 3 cylinders, providing a spiral shape. Moreover, 0 it at the intersection of the Rauzy fractal with its translation by $\eta(1) - \eta(3)$. Hence, 0 does not belong to the interior of the fractal, which allows the tiling property.

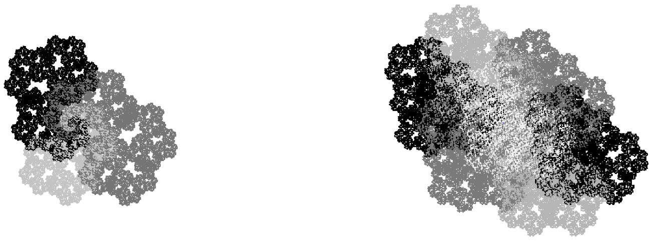


FIGURE 4.1. Rauzy fractal and tiling associated with$1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 12$.

Two-letter non-unimodular substitutions. The conditions of Theorem 4.1 are satisfied by $1 \mapsto 1112 \quad 2 \mapsto 12$, so that this substitution generates a dynamical system with a pure discrete spectrum. This spectrum is explicitly known thanks to [Hos86]: it is generated by the numbers $1/\sqrt{2}^n$ for $n \geq 0$. Hence, the dynamical system associated with the $1 \mapsto 1112$, $2 \mapsto 12$ is semi-topologically conjugate to a minimal translation on the direct product of the torus $\mathbb{T}$ with the 2-solenoid. Notice that this was already known since this substitution satisfies the condition of coincidences over a two-letter alphabet [HS03]. The same holds for $1 \mapsto 11122$, $2 \mapsto 1222$.

The non-unimodular substitution $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 11233$. The characteristic polynomial is $X^3 - 2X^2 - X - 2$, so that this substitution is of Pisot type and non-unimodular. Let α be its dominant root. The constants provided by Proposition 4.2 are $M_1 = M_2 = 17$. The geometrical graph satisfies the self-similar conditions. Non-zero vectors in $\mathcal{H}_0$ that join two points in the Rauzy fractal are given in the Table shown in Fig. 4.2, as well as the size of the tiling graph that correspond to each tiling vectors. The tiling conditions are satisfied, therefore $\mathcal{R}$ intersect its translated copies through $\mathcal{H}_0$ on a zero measure set.

Corollary 4.7. *The dynamical system associated with the non-unimodular substitution* $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 11233$ *has a pure discrete spectrum.*

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Vector $\mathbf{v} \in \mathcal{T}$	vertices	edges	Vector $\mathbf{v} \in \mathcal{T}$	vertices	edges
$\delta(\alpha^2 - 1)$	205	403	$\delta(-3\alpha^2 + 2\alpha + 1)$	72	118
$\delta(2\alpha^2 - 2)$	213	419	$\delta(-2\alpha^2 + 2\alpha)$	221	439
$\delta(3\alpha^2 - 3)$	66	108	$\delta(-\alpha^2 + 2\alpha - 1)$	205	401
$\delta(-3\alpha^2 + \alpha + 2)$	83	141	$\delta(2\alpha - 2)$	205	402
$\delta(-2\alpha^2 + \alpha + 1)$	212	417	$\delta(-3\alpha^2 - 3\alpha)$	71	117
$\delta(-\alpha^2 + \alpha)$	215	427	$\delta(-2\alpha^2 + 3\alpha - 1)$	72	119
$\delta(\alpha - 1)$	203	399	$\delta(-\alpha^2 + 3\alpha - 2)$	69	114
$\delta(\alpha^2 + \alpha - 2)$	218	427	$\delta(3\alpha - 3)$	66	108
$\delta(2\alpha^2 + \alpha - 1)$	65	107	$\delta(-2\alpha^2 + 4\alpha - 2)$	65	107
$\delta(3\alpha^2 + \alpha - 4)$	66	108			

FIGURE 4.2. Vectors between two adjacent copies of the Rauzy fractal for $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 11233$. Size of associated subgraphs.

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