General preferential entailments as circumscriptions

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Abstract A (general) preferential entailment is defined by a "preference relation" among "states". States can be either interpretations or sets of interpretations, or "copies" of interpretations or of sets of interpretations, although it is known that the second kind and the fourth one produce the same notion. Circumscription is a special case of the simplest kind, where the states are interpretations. It is already known that a large class of preferential entailments where the states are copies of interpretations, namely the "cumulative" ones, can be expressed as circumscriptions in a greater vocabulary. We extend this result to the most general kind of general preferential entailment, the additional property requested here is "loop", a strong kind of "cumulativity". The greater vocabulary needed here is large, but only a very simple and small set of formulas in this large vocabulary is necessary, which should make the method practically useful.

1 Introduction

Preferential entailments are useful in knowledge representation. Four kinds are introduced in Kraus and al. [7], which in fact reduce to three. Till now, no system computing efficiently the most general kinds is known, but systems do compute circumscription, a particular case of the simplest kind of preferential entailment. Costello [4] has shown how, contrarily to an affirmation in [7], an important subclass of an intermediate kind can be translated into circumscription, by extending the vocabulary. We show that an important subclass of the most general kind can also be translated into circumscription by modifying the vocabulary. We begin with notations (§2), definitions (§3) and useful known results (§4). Then, we need two technical definitions: an auxiliary vocabulary in which the theories of the original language correspond to single interpretations in the new one (§5); and a simplified preference relation for a large class of preferential entailments (§6). Finally, we describe the translation (§7) and detail an example (§8).

2 Notations and framework

• We work in a propositional language **L**. As usual, **L** also denotes the set of all the formulas. $V(\mathbf{L})$, the vocabulary of **L**, denotes a set of *propositional symbols*. Letters φ, ψ denote formulas in **L**. A *formula* will generally be *assimilated to its equivalence class*. Letters such as $\mathcal{T}$ or $\mathcal{C}$ denote sets of formulas (i.e. subsets of **L**). Two logical constants $\top$ and $\bot$ denote respectively the true and the false formulas.
• Letters μ, ν denote *interpretations for* **L**, identified with subsets of $V(\mathbf{L})$. $\mu \models \varphi$ and $\mu \models \mathcal{T}$ are defined classically. If $\mathbf{M}_1 \subseteq \mathbf{M}$, $\mathbf{M}_1 \models \mathcal{T}$ means $\mu \models \mathcal{T}$ for any

$\mu \in \mathbf{M}_1$. For a set E, $\mathcal{P}(E)$ denotes the set of the subsets of E. The set $\mathcal{P}(V(\mathbf{L}))$ of the interpretations for $\mathbf{L}$ is denoted by $\mathbf{M}$. A *model* of $\mathcal{T}$ is an interpretation μ such that $\mu \models \mathcal{T}$, $\mathbf{M}(\mathcal{T})$ and $\mathbf{M}(\varphi)$ denote respectively the sets of the models of $\mathcal{T}$ and φ.
• $\mathcal{T} \models \varphi$, $\mathcal{T} \models \mathcal{T}_1$ and $Th(\mathcal{T})$ are defined classically. A *theory* is a subset of $\mathbf{L}$ closed for deduction, $\mathbf{T}$ denotes the set $\{\mathcal{T} \subseteq \mathbf{L} \ / \ \mathcal{T} = Th(\mathcal{T})\}$ of the theories of $\mathbf{L}$. If $\mathcal{T}_1$ is a theory, we get $\mathcal{T} \subseteq \mathcal{T}_1$ iff $\mathcal{T}_1 \models \mathcal{T}$, for any $\mathcal{T} \subseteq \mathbf{L}$.
• A theory $\mathcal{C} \in \mathbf{T}$ is *complete* if $\forall \varphi \in \mathbf{L}$, $\varphi \in \mathcal{C}$ iff $\neg\varphi \notin \mathcal{C}$. We denote by $\mathbf{C}$ the set of all the complete theories of $\mathbf{L}$. $Th(\mu)$ denotes the set of the formulas satisfied by μ. For any subset $\mathbf{M}_1$ of $\mathbf{M}$, $Th(\mathbf{M}_1) = \{\varphi \ / \ \mathbf{M}_1 \models \varphi\} = \bigcap_{\mu \in \mathbf{M}_1} Th(\mu)$. This ambiguous use of Th and of $\models$ (for formulas or interpretations) is usual. For any $\mathcal{T} \in \mathbf{T}$, $\mathcal{T} = \bigcap_{\mathcal{C} \in \mathbf{C},\, \mathcal{C} \models \mathcal{T}} \mathcal{C}$. Th defines a one-to-one mapping between $\mathbf{M}$ and $\mathbf{C}$: $Th(\mu) \in \mathbf{C}$ for any $\mu \in \mathbf{M}$. If $V(\mathbf{L})$ is finite, Θ denotes the canonical one-to-one mapping from $\mathcal{P}(\mathbf{M})$ to $\mathbf{L}$: for any $\mathbf{M}_1 \subseteq \mathbf{M}$, $\Theta(\mathbf{M}_1)$ is the formula such that $\mathbf{M}(\Theta(\mathbf{M}_1)) = \mathbf{M}_1$.
• $\mathbf{T}, \mathbf{C}, \mathbf{M}, Th, \Theta$ and $\models$ should be indexed by $\mathbf{L}$. To keep the notations readable, we will denote two languages by say $\mathbf{L}$ and $\mathbf{L}'$, and all what concerns $\mathbf{L}$ will be denoted as above, while we will use $\mathbf{T}', \mathbf{C}', \mathbf{M}', Th', \Theta'$ and $\models'$ for what concerns $\mathbf{L}'$.

3 The various kinds of preferential entailments

Definition 3.1. A *pre-circumscription* f (in $\mathbf{L}$) is an extensive (i.e., $f(\mathcal{T}) \supseteq \mathcal{T}$ for any $\mathcal{T}$) mapping from $\mathbf{T}$ to $\mathbf{T}$. For any subset $\mathcal{T}$ of $\mathbf{L}$, we use the abbreviation $f(\mathcal{T}) = f(Th(\mathcal{T}))$, assimilating a pre-circumscription to a particular extensive mapping from $\mathcal{P}(\mathbf{L})$ to itself[1]. We write $f(\varphi)$ for $f(\{\varphi\}) = f(Th(\varphi))$. □

Definitions 3.2 1. A set of *states* $\mathbf{S}$ is a set of "copies" of elements of $\mathbf{T}$ (or equivalently [3] a set of "copies" of subsets of $\mathbf{M}$): there exists a mapping l from $\mathbf{S}$ to $\mathbf{T}$ and, for any $\mathcal{T} \in \mathbf{T}$, the subset $l^{-1}(\mathcal{T})$ of $\mathbf{S}$ is the set of the *copies of* $\mathcal{T}$.
2. As usual, we define $l(\mathbf{S}) = \{l(s)\}_{s \in \mathbf{S}} = \{\mathcal{T} \in \mathbf{T} \ / \ l^{-1}(\mathcal{T}) \neq \emptyset\}$. For any $\mathcal{T} \subseteq \mathbf{L}$, $\mathbf{S}(\mathcal{T})$ is the subset of $\mathbf{S}$ defined by $\mathbf{S}(\mathcal{T}) = \{s \in \mathbf{S} \ / \ l(s) \models \mathcal{T}\}$.
3. For any $\mathcal{T} \subseteq \mathbf{L}$ we define the subset of $\mathbf{T}$: $\mathbf{W}(\mathcal{T}) = \{\mathcal{T}_1 \in \mathbf{T} \ / \ \mathcal{T} \subseteq \mathcal{T}_1\}$. We write $\mathbf{W}(\varphi)$ for $\mathbf{W}(\{\varphi\})$. Notice that we get $\mathbf{S}(\mathcal{T}) = l^{-1}(\mathbf{W}(\mathcal{T}))$.

Definitions 3.3 1. A *general preference relation* $\prec_g$ is a binary relation over $\mathbf{S}$. For any $\mathcal{T} \in \mathbf{T}$, we define the subsets $\mathbf{S}_{\prec_g}(\mathcal{T})$ of $\mathbf{S}$ and $\mathbf{W}_{\prec_g}(\mathcal{T})$ of $\mathbf{T}$ as follows: $\mathbf{S}_{\prec_g}(\mathcal{T}) = \{s \in \mathbf{S}(\mathcal{T}) \ / \ s_1 \prec_g s$ for no $s_1 \in \mathbf{S}(\mathcal{T})\}$, and $\mathbf{W}_{\prec_g}(\mathcal{T}) = l(\mathbf{S}_{\prec_g}(\mathcal{T}))$.
2. The *general preferential entailment* $f_{\prec_g}$ is the pre-circumscription defined by $f_{\prec_g}(\mathcal{T}) = \bigcap_{\mathcal{T}_1 \in \mathbf{W}_{\prec_g}(\mathcal{T})} \mathcal{T}_1$ for any $\mathcal{T} \subseteq \mathbf{L}$.

This is the definition of [3, Definitions 3.1, 3.2], originating from [7, Definition 3.11]. Particular cases give the most classical kinds of preferential entailments:

Definitions 3.4 1. If $l(\mathbf{S}) \subseteq \mathbf{C}$ (instead of $l(\mathbf{S}) \subseteq \mathbf{T}$), let us call the general preference relation a *multi preference relation*, which we will denote by $\prec_m$ instead of $\prec_g$ and let us call $f_{\prec_m}$ a *multi preferential entailment*.

[1] For a reader familiar with [7], a pre-circumscription is an *inference operation* satisfying the full (or theory) versions of *reflexivity*, *left logical equivalence*, *right weakening* and *AND*.

2. If $\mathbf{S} = \mathbf{T}$ and $l = identity$, let us call $\prec_g$ a *simplified general preference relation*.
3. If $\mathbf{S} = \mathbf{C}$ and $l = identity$ (i.e. restrictions 1 and 2 apply), then the relation, defined in **C**, is called a *preference relation* $\prec$ and $f_\prec$ is called a *preferential entailment*.

As we work in propositional logic, **C** can be replaced by **M** and **T** by $\mathcal{P}(\mathbf{M})$ (see e.g. [3]). Point 1 originates from [7, Definition 5.6] and point 3 from [18]. The notion of general preferential entailment has been qualified as "cumbersome" in the introducing paper [7]. Then, this notion has been tamed in various texts [1,2,3,6,13,10,14].

The best known kind of preferential entailment is circumscription:

Definition 3.5. $\mathbf{P}, \mathbf{Q}, \mathbf{Z}$ is a partition of $V(\mathbf{L})$. The symbols in $\mathbf{P}, \mathbf{Z}$ and **Q** are respectively *circumscribed*, *varying* and *fixed*. We define the preference relation $\prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$ in **M** by: $\mu \prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})} \nu$ if $\mathbf{P} \cap \mu \subset \mathbf{P} \cap \nu$ and $\mathbf{Q} \cap \mu = \mathbf{Q} \cap \nu$ ($\subset$: strict inclusion).

The *circumscription* $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ is the preferential entailment $f_{\prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}}$.

Definition 3.6. $\Phi \subseteq \mathbf{L}$, $V(\mathbf{L}) = \mathbf{Q} \cup \mathbf{Z}$ (disjoint union), $\mathbf{P}' = \{P'_\varphi\}_{\varphi\in\Phi}$ is a set of distinct propositional symbols not in **L**. The *formula circumscription* of *the set of formulas* Φ, with **Q** *fixed* and **Z** *varying*, is defined as follows, for any $\mathcal{T} \subseteq \mathbf{L}$:

$CIRCF(\Phi, \mathbf{Q}, \mathbf{Z})(\mathcal{T}) = CIRC(\mathbf{P}', \mathbf{Q}, \mathbf{Z})(\mathcal{T} \cup \{\varphi \Leftrightarrow P'_\varphi\}_{\varphi\in\Phi}) \cap \mathbf{L}$.

$CIRC$ is defined in the greater language $\mathbf{L}'$: $V(\mathbf{L}') = V(\mathbf{L}) \cup \mathbf{P}'$.

Remark 3.1. $CIRCF(\Phi, \mathbf{Q}, \mathbf{Z})$ is the preferential entailment $f_\prec$ in **L** associated with the preference relation $\prec_{(\Phi;\mathbf{Q},\mathbf{Z})}$ defined in **M** by:

$\mu \prec_{(\Phi;\mathbf{Q},\mathbf{Z})} \nu$ if $Th(\mu) \cap \Phi \subset Th(\nu) \cap \Phi$ and $\mathbf{Q} \cap \mu = \mathbf{Q} \cap \nu$. □

These are the usual propositional adaptations [17,12,4] of the original predicate calculus versions [8,9,16]. Circumscription is a preferential entailment (Definition 3.4-3) and various systems make useful automatic computation for propositional circumscription[2]. Thus, it is interesting to express more complex formalisms in terms of circumscription. This has already been done for multi preferential entailments [4] (see also [13,11]), what we do now is to extend this technique to general preferential entailments.

4 A reminder: characterization results

Here are known results from [7,17] and other texts (see [13,14] for precise references).

We consider now that $V(\mathbf{L})$ *is finite*.

(Notice that in this case we can restrict our attention to finite sets **S** [7].)

Definition 4.1. A general preference relation $\prec_g$ is *safely founded (sf)*, if for any $s \in \mathbf{S}(\mathcal{T}) - \mathbf{S}_{\prec_g}(\mathcal{T})$, there exists $s_1 \in \mathbf{S}_{\prec_g}(\mathcal{T})$ such that $s_1 \prec_g s$.

Definitions 4.2 Here are various properties a pre-circumscription may possess. $\mathcal{T}_1, \mathcal{T}_2$ are in **T** (remind that intersecting theories corresponds to a disjunction $\vee$ of formulas):

Case reasoning: $\quad f(\mathcal{T}_1 \cap \mathcal{T}_2) \models f(\mathcal{T}_1) \cap f(\mathcal{T}_2)$. **(CR)**

[2] Here are three examples: LWB (http://lwbwww.unibe.ch:8080/LWBtheory.html), SMODELS (http://www.tcs.hut.fi/Software/smodels/), and DLV (http://www.dbai.tuwien.ac.at/proj/dlv/).

Disjunctive coherence: $f(\mathcal{T}_1) \cup f(\mathcal{T}_2) \models f(\mathcal{T}_1 \cap \mathcal{T}_2)$. **(DC)**

Cumulative transitivity: If $\mathcal{T}'' \subseteq f(\mathcal{T}),\ \ f(\mathcal{T} \cup \mathcal{T}'') \subseteq f(\mathcal{T})$. **(CT)**

Cumulative monotony: If $\mathcal{T}'' \subseteq f(\mathcal{T}),\ f(\mathcal{T}) \subseteq f(\mathcal{T} \cup \mathcal{T}'')$. **(CM)**

Cumulativity: If $\mathcal{T}'' \subseteq f(\mathcal{T})$, then $f(\mathcal{T}) = f(\mathcal{T} \cup \mathcal{T}'')$. **(CUMU)**

If $\mathcal{T}_2 \subseteq f(\mathcal{T}_1), \cdots, \mathcal{T}_n \subseteq f(\mathcal{T}_{n-1}), \mathcal{T}_1 \subseteq f(\mathcal{T}_n)$, then $f(\mathcal{T}_1) = f(\mathcal{T}_n)$. **(LOOP$_n$)**

(Loop): For any integer $n \geq 2$, f satisfies (LOOP$_n$). **(LOOP)**

Preservation of consistency: If $f(\mathcal{T}_1) = Th(\bot) = \mathbf{L}$, then $\mathcal{T}_1 = \mathbf{L}$. **(PC)**

Proposition 4.1. *For pre-circumscriptions: 1. (CR) implies (CT).*
2. As (CUMU) is (CM) + (CT), in case of (CR), (CUMU) and (CM) are equivalent.
3. (LOOP$_2$) is equivalent to (CUMU), (LOOP$_{n+1}$) is stronger than (LOOP$_n$).
4. (CR) and (CUMU) imply (LOOP). □

Theorem 4.1. *1. For any general preferential entailment $f_{\prec_g}$, there exists a simplified general preference relation $\prec_{sg}$ such that $f_{\prec_g} = f_{\prec_{sg}}$.*
2. *A pre-circumscription f satisfies (CT) iff it is a general preferential entailment.*
3. *A pre-circumscription f satisfies (CUMU) – respectively (LOOP) – iff it is a general preferential entailment defined by a relation $\prec_g$ satisfying (sf) – respectively a transitive and irreflexive relation (i.e. a strict order) $\prec_g$ satisfying (sf) (cf point 5).*
4. *A pre-circumscription satisfies (CR) iff it is a multi preferential entailment.*
5. *A pre-circumscription satisfies (CR) and (CUMU) iff it is a multi preferential entailment defined in a finite set* **S** *by a relation $\prec_m$ which is a strict order (on a finite set this implies (sf) and, contrarily to 3 for (LOOP), (sf) alone suffices here).*
6. *A pre-circumscription satisfies (CR) and (DC) iff it is a preferential entailment.*
7. *A preferential entailment satisfies (CUMU) and (PC) iff it is defined by a preference relation $\prec$ which is transitive and irreflexive, iff it is a formula circumscription.* □

5 Modifying the vocabulary

Definitions 5.1 $\mathbf{L}$ and $\mathbf{L}'$ are two languages, f is a mapping from $\mathbf{T}$ to $\mathbf{T}$ and f' is a pre-circumscription defined in $\mathbf{L}'$. We say that f is obtained from f' by (Def$\rightleftharpoons$) – respectively by (Def$\rightleftharpoons$4) – if there exist two mappings b_1 from $\mathbf{T}$ to $\mathbf{T}'$ and b_2 from $\mathbf{T}'$ to $\mathbf{T}$ such that the three conditions ($\rightleftharpoons$1–3) – respectively the four conditions ($\rightleftharpoons$1–4) – below are satisfied and such that we have, for any $\mathcal{T} \in \mathbf{T}$: $f(\mathcal{T}) = b_2(f'(b_1(\mathcal{T})))$.

1. b_1 preserves inclusion:
   for any $\mathcal{T}_1, \mathcal{T}_2$ in $\mathbf{T}$, if $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $b_1(\mathcal{T}_1) \subseteq b_1(\mathcal{T}_2)$, ($\rightleftharpoons$1)
2. $b_1 \circ b_2$ is contractive on the set $f'(b_1(\mathbf{T}))$:
   $b_1(b_2(f'(b_1(\mathcal{T})))) \subseteq f'(b_1(\mathcal{T}))$ for any $\mathcal{T} \in \mathbf{T}$, ($\rightleftharpoons$2)
3. $b_2 \circ f' \circ b_1$ is extensive: for any $\mathcal{T} \in \mathbf{T}$, $\mathcal{T} \subseteq b_2(f'(b_1(\mathcal{T})))$. ($\rightleftharpoons$3)
4. b_2 preserves inclusion on the set $f'(b_1(\mathbf{T}))$: For any $\mathcal{T}_1, \mathcal{T}_2$ in $\mathbf{T}$,
   if $f'(b_1(\mathcal{T}_1)) \subseteq f'(b_1(\mathcal{T}_2))$, then $b_2(f'(b_1(\mathcal{T}_1))) \subseteq b_2(f'(b_1(\mathcal{T}_2)))$. ($\rightleftharpoons$4)

($\rightleftharpoons$3) means that $f = b_2 \circ f' \circ b_1$ is a pre-circumscription. Notice that we need only to know the value of b_2 on the subset $f'(b_1(\mathbf{T})) = \{f'(b_1(\mathcal{T})) \ / \ \mathcal{T} \in \mathbf{T}\}$ of $\mathbf{T}'$.

The following preservation results are immediate:

Proposition 5.1. *1. If f' is a pre-circumscription defined in a language $\mathbf{L}'$ which satisfies (CUMU) – resp. (LOOP) – and if f is defined from f' by (Def$\rightleftharpoons$), then f is a pre-circumscription defined in $\mathbf{L}$ which satisfies (CUMU) – resp. (LOOP).*

2. If f' is a pre-circumscription defined in a language $\mathbf{L}'$ which satisfies (CT) – respectively (CM) – and if f is defined from f' by (Def$\rightleftharpoons$4), then f is a pre-circumscription defined in $\mathbf{L}$ which satisfies (CT) – respectively (CM). □

6 A useful simplified general preference relation

Definition 6.1. [7] Let f be a pre-circumscription. We define the following general preference relation $\prec_f^{klm}$: 1. $\mathbf{S} = f(\mathbf{T}) = \{f(\mathcal{T}) \ / \ \mathcal{T} \in \mathbf{T}\}$,

2. l is the mapping from $\mathbf{S}$ to $\mathbf{T}$ defined by $l(f(\mathcal{T})) = f(\mathcal{T})$ for any $\mathcal{T} \in \mathbf{T}$.

3. $f(\mathcal{T}_1) \prec_f^{klm} f(\mathcal{T}_2)$ if $f(\mathcal{T}_1) \neq f(\mathcal{T}_2)$ and there exists $\mathcal{T}_3 \in \mathbf{T}$ such that $f(\mathcal{T}_1) = f(\mathcal{T}_3)$ and $\mathcal{T}_3 \subseteq f(\mathcal{T}_2)$.

The set $f(\mathbf{T})$ is then the set denoted by $l(\mathbf{S})$ in Definition 3.3 for the general preference relation defined here. The relation $\prec_f^{klm}$ is introduced in [7, Theorem 3.25] in order to prove "the hard part" of Theorem 4.1-3 for (CUMU). The relation $\prec_f^{klm}$ can be replaced by a simplified general preference relation (see also [1,2]):

Definition 6.2. Let $\prec_g$ be a general preference relation (defining thus a set $\mathbf{S}$ and a mapping l). We define the following simplified general preference relation $\prec_s$: for any $\mathcal{T}_1, \mathcal{T}_2 \in \mathbf{T}$, $\mathcal{T}_1 \prec_s \mathcal{T}_2$ if 1. $\mathcal{T}_1 = Th(\perp)$ and $\mathcal{T}_2 \notin l(\mathbf{S}) \cup \{Th(\perp)\}$, or 2. $\mathcal{T}_1 = l(s_1)$, $\mathcal{T}_2 = l(s_2) \neq Th(\perp)$, and $s_1 \prec_g s_2$, for some s_1, s_2 in $\mathbf{S}$.

Proposition 6.1. *If a general preference relation $\prec_g$ is such that the mapping l is injective, we have, for any $\mathcal{T} \in \mathbf{T}$, $\mathbf{W}_{\prec_g}(\mathcal{T}) \cup \{Th(\perp)\} = \mathbf{W}_{\prec_s}(\mathcal{T}) \cup \{Th(\perp)\}$. Thus we have $f_{\prec_g} = f_{\prec_s}$ where $\prec_s$ is the simplified general preference relation defined from $\prec_g$ as in Definition 6.2.*

Proof: As l is injective, for any s_1, s_2 in $\mathbf{S}$, $s_1 \prec_g s_2$ iff there exist $\mathcal{T}_1$ and $\mathcal{T}_2$ in $l(\mathbf{S}) = f(\mathbf{S})$ such that $s_1 = l(\mathcal{T}_1)$, $s_2 = l(\mathcal{T}_2)$ and $\mathcal{T}_1 \prec_s \mathcal{T}_2$. Moreover $Th(\perp) \prec_s \mathcal{T}$ for any $\mathcal{T} \notin l(\mathbf{S})$, and $Th(\perp) \in \mathbf{W}(\mathcal{T})$ for any $\mathcal{T} \in \mathbf{T}$. Thus, for any $\mathcal{T} \in \mathbf{T}$, we have $\mathbf{W}_{\prec_g}(\mathcal{T}) \cup \{Th(\perp)\} = \mathbf{W}_{\prec_s}(\mathcal{T}) \cup \{Th(\perp)\}$. As $Th(\perp) \in \mathbf{W}(\varphi)$ for any $\varphi \in \mathbf{L}$, we get that if $\prec_1$ and $\prec_2$ are two general preference relations such that $\mathbf{W}_{\prec_1}(\mathcal{T}) = \mathbf{W}_{\prec_2}(\mathcal{T}) \cup \{Th(\perp)\}$, then $f_{\prec_1}(\mathcal{T}) = f_{\prec_2}(\mathcal{T})$. Thus we get here $f_{\prec_g} = f_{\prec_s}$. □

Definition 6.3. The mapping l of the relation $\prec_f^{klm}$ is injective. We can thus consider the simplified general preference relation, that we call $\prec_{nf}$, defined from $\prec_f^{klm}$ as in Definition 6.2. We call $\prec_{nf}$ the *normal general preference relation associated to* f. □

We get $f_{\prec_f^{klm}} = f_{\prec_{nf}}$ from Proposition 6.1.

As $V(\mathbf{L})$ is finite, we will now generally replace $\mathbf{T}$ by $\mathbf{L}$. $\mathbf{W}(\varphi)$ will be a set of formulas, any simplified general preference relation will be a binary relation in $\mathbf{L}$ and, if f is a pre-circumscription, $f(\varphi) = \psi$ will replace $f(\varphi) = Th(\psi)$.

Proposition 6.2. *If f satisfies (CT), the normal general preference relation $\prec_{nf}$ associated to f is the binary relation described as follows: for any φ_1, φ_2 in $\mathbf{L}$,*
$\varphi_1 \prec_{nf} \varphi_2$ *iff* *1. $\varphi_1 = \bot$ and $\varphi_2 \neq \varphi$ for any $\varphi \in \mathbf{L}$,* *or*
2. $\varphi_2 \neq \bot$, $\varphi_1 \neq \varphi_2$ and there exist φ_3, φ_4 such that $f(\varphi_3) = \varphi_1$, $f(\varphi_4) = \varphi_2$, $\varphi_2 \models \varphi_3$.

Proof: This is a consequence of Definitions 6.1 and 6.3, taking into account two peculiarities of $\prec_f^{klm}$. Firstly, the set $l(\mathbf{S}) = f(\mathbf{L})$ associated to the general preference relation $\prec_f^{klm}$ contains $\bot$: as f is a pre-circumscription, we have $f(\bot) = \bot$. Secondly, we have never $\bot \prec_f^{klm} \varphi$. Indeed, $\bot \prec_f^{klm} \varphi$ iff $f(\bot) \neq f(\varphi)$ and there exists $\varphi_1 \in \mathbf{L}$ such that $f(\varphi_1) = \bot$ and $f(\varphi) \models \varphi_1$. From (CT) we get then $f(f(\varphi)) = f(\varphi) \models f(\varphi_1)$, i.e. $f(\varphi) = \bot = f(\bot)$: a contradiction. □

These results show that all the general preference relations considered in [7] could have been replaced directly by a simplified general preference relation.

Proposition 6.3. *If f is a pre-circumscription satisfying (CUMU), then it is a general preferential entailment which can be defined by $\prec_{nf}$: $f = f_{\prec_{nf}}$.*
More precisely we have, for any $\varphi \in \mathbf{L}$: $\mathbf{W}_{\prec_{nf}}(\varphi) = \{f(\varphi), \bot\}$. □

We omit the proof, as it is an adaptation of a proof given in [7, proof of Theorem 3.25], establishing that we have in this case $\mathbf{W}_{\prec_f^{klm}}(\varphi) = \{f(\varphi)\}$. The fact that we use a simplified general preference relation simplifies even the matter. Notice also that, as in [7, proof of Theorem 3.25] for $\prec_f^{klm}$, we get that in this case $\prec_{nf}$ is (sf).

Here is another result extrapolated from [7], which will be useful in our translation of some general preferential entailments in terms of circumscription (cf the proof of [7, Theorem 4.9], which gives the result for what concerns $\prec_f^{klm}$ and its transitive closure):

Proposition 6.4. *A pre-circumscription f satisfying (CUMU) satisfies (LOOP) iff the transitive closure $\overline{\prec_{nf}}$ of the normal general preference relation $\prec_{nf}$ associated to f is irreflexive. In this case, i.e. if f satisfies (LOOP), we have $\mathbf{W}_{\prec_{nf}}(\varphi) = \mathbf{W}_{\overline{\prec_{nf}}}(\varphi) = \{f(\varphi), \bot\}$, thus $f = f_{\prec_{nf}} = f_{\overline{\prec_{nf}}}$.* □

7 Finite general preferential entailments as circumscriptions

Theorem 7.1. *A pre-circumscription f in $\mathbf{L}$ satisfies (LOOP) iff it can be expressed by (Def$\rightleftharpoons$) — or by (Def$\rightleftharpoons$4) — from a formula circumscription $f' = CIRCF(\Phi', \emptyset, V(\mathbf{L}'))$ defined in a language $\mathbf{L}'$.*

By Proposition 6.4, "*A pre-circumscription f*" could be replaced by "*A general preferential entailment f*". Remind a similar result for multi preferential entailments satisfying (CM) ([11, Theorem 31], extrapolated from [4, Theorem 15]). The reason why we need (LOOP) here instead of just (CUMU) is that we must get a strict order relation in order to get a formula circumscription (see Theorem 4.1, points 3, 5 and 7).

Constructive proof: (if): Any formula circumscription f' satisfies (CUMU) and (LOOP) from Prop. 4.1-4 and Th. 4.1 (-6,7). Then f satisfies (LOOP) from Prop. 5.1-1.

(only if): $f = f_{\overline{\prec_{nf}}}$ from Proposition 6.4, $\prec_{nf}$ being described in Proposition 6.2. $\overline{\prec_{nf}}$ is a strict order from Proposition 6.4 and in fact this proof works for any simplified general preference relation $\prec_s$ such that $f = f_{\prec_s}$ and which is a strict order. We define (1) a language $\mathbf{L}'$ such that there exists a one-to one mapping p from $\mathbf{M}$ to $V(\mathbf{L}')$ and (2) a one-to one mapping b from $\mathcal{P}(\mathbf{M})$ to $\mathbf{M}' = \mathcal{P}(V(\mathbf{L}'))$:

$$\text{For any } \mu \subseteq V(\mathbf{L}), \quad p(\mu) = P'_\mu \in V(\mathbf{L}'). \tag{1}$$

$$\text{For any } \mathbf{M}_1 \subseteq \mathbf{M}, \quad b(\mathbf{M}_1) = p(\mathbf{M} - \mathbf{M}_1) = \{P'_\mu \in V(\mathbf{L}') \;/\; \mu \in \mathbf{M} - \mathbf{M}_1\}. \tag{2}$$

Then, we define (3) a one-to-one mapping $\overline{b}$ from $\mathbf{L}$ to $\mathbf{L}'_C = \{\varphi' \in \mathbf{L}'/Th'(\varphi') \in \mathbf{C}'\} = \{\bigwedge_{P' \in \mathbf{P}'} P' \wedge \bigwedge_{P' \in V(\mathbf{L}') - \mathbf{P}'} \neg P / \mathbf{P}' \subseteq V(\mathbf{L}')\}$ ($\mathbf{L}'_C$ is the subset of $\mathbf{L}'$ corresponding to $\mathbf{C}'$, in the same way than $\mathbf{L}'$ corresponds to $\mathbf{T}'$) and (4) a mapping b_1 from $\mathbf{L}$ to $\mathbf{L}'$. For any $\varphi \in \mathbf{L}$:

$$\overline{b}(\varphi) = \left(\bigwedge_{P'_\mu \in V(\mathbf{L}') \,/\, \mu \in \mathbf{M} - \mathbf{M}(\varphi)} P'_\mu \right) \wedge \left(\bigwedge_{P'_\mu \in V(\mathbf{L}') \,/\, \mu \in \mathbf{M}(\varphi)} \neg P'_\mu \right). \tag{3}$$

$$b_1(\varphi) = \bigwedge_{\mu \in \mathbf{M} - \mathbf{M}(\varphi)} P'_\mu. \tag{4}$$

Thus, $\mathbf{M}'(\overline{b}(\varphi))$ is the singleton $\{b(\mathbf{M}(\varphi))\}$, where $b(\mathbf{M}(\varphi)) = \{P'_\mu \,/\, \mu \in \mathbf{M} - \mathbf{M}(\varphi)\} = \{P'_\mu \,/\, \mu \in \mathbf{M}(\neg\varphi)\}$. Here is a feature of these mappings, which greatly simplifies the translation: for any $\mathbf{M}_1, \mathbf{M}_2 \subseteq \mathbf{M}$: $\mathbf{M}_1 \subseteq \mathbf{M}_2$ iff $b(\mathbf{M}_2) \subseteq b(\mathbf{M}_1)$, i.e.,

$$\text{for any } \varphi, \psi \text{ in } \mathbf{L}, \quad \varphi \models \psi \text{ iff } b(\mathbf{M}(\psi)) \subseteq b(\mathbf{M}(\varphi)). \tag{5}$$

From (4), $b_1(\varphi)$ is the formula which has the set $\{\mu' \;/\; b(\mathbf{M}(\varphi)) \subseteq \mu' \subseteq V(\mathbf{L}')\}$ for set of models. Thanks to (5), we get that $b_1(\varphi)$ is the formula such that $\mathbf{M}'(b_1(\varphi)) = \{b(\mathbf{M}(\psi)) \;/\; \psi \in \mathbf{W}(\varphi)\}$: $b_1(\varphi)$ is an image of the set $\mathbf{W}(\varphi)$ in $\mathbf{L}'$. As b_1 is injective, it defines a one-to-one mapping between $\mathbf{L}$ and the set $b_1(\mathbf{L}) = \{b_1(\varphi) \;/\; \varphi \in \mathbf{L}\} = \{\bigwedge_{P' \in \mathbf{P}'} P' \;/\; \mathbf{P}' \subseteq V(\mathbf{L}')\}$ of all the *conjunctions of atoms of* $\mathbf{L}'$.

We must now come back from $\mathbf{L}'$ to the original language $\mathbf{L}$.

The one-to-one mapping b^{-1} from $\mathbf{M}'$ to $\mathcal{P}(\mathbf{M})$ can be described as follows (cf (2)):

$$b^{-1}(\mu') = \{b^{-1}(P'_\mu) \;/\; P'_\mu \in V(\mathbf{L}') - \mu'\} = \{\mu \;/\; P'_\mu \in V(\mathbf{L}') - \mu'\}.$$

We define the mapping b_2 from $\mathbf{L}'$ to $\mathbf{L}$ by the following two equivalent equations:

$$\text{for any } \varphi' \in \mathbf{L}', \quad b_2(\varphi') = b_1^{-1}(\bigwedge_{P' \in V(\mathbf{L}'),\, \varphi' \models' P'} P'). \tag{6}$$

$$\mathbf{M}(b_2(\varphi')) = b^{-1}(\{P'_\mu \;/\; \varphi' \models' P'_\mu\} = \{\mu \in \mathbf{M} \;/\; \varphi' \not\models' P'_\mu\}). \tag{7}$$

$$\text{From (4) and (6) we get, for any } \varphi \in \mathbf{L} : \; b_2(b_1(\varphi)) = \varphi. \tag{8}$$

The restriction of b_2 to the subset $\mathbf{L}'_C$ of $\mathbf{L}'$ is $(\overline{b})^{-1}$, a one-to one mapping from $\mathbf{L}'_C$ onto $\mathbf{L}$. Indeed we get, P' ranging over $V(\mathbf{L}')$:

$$\text{if } \varphi' \in \mathbf{L}'_C, \quad \text{then } \varphi' = \bigwedge_{\varphi' \models' P'} P' \wedge \bigwedge_{\varphi' \not\models' P'} \neg P'. \tag{9}$$

If $\varphi' \in \mathbf{L}'_C$, then $\mathbf{M}'(\varphi')$ is the singleton $\mathbf{M}'(\varphi') = \{\mu'\}$ for $\mu' = \{P' \in V(\mathbf{L}')/\varphi' \models' P'\}$, and we get: $\mathbf{M}(b_2(\varphi')) = b^{-1}(\mu')$.

It is convenient to introduce the "*exhaustive conjunction*" $\psi' = \bigwedge_{P' \in V(\mathbf{L}')} \; P'$.

We suppose here that $\varphi' = \varphi'_1 \vee \psi'$, with $\varphi'_1 \in \mathbf{L}'_C$. This means that there exists a subset $\mathbf{P}'$ of $V(\mathbf{L}')$ such that $\varphi' = \bigwedge_{P' \in V(\mathbf{L}')-\mathbf{P}'} P' \wedge (\bigwedge_{P' \in \mathbf{P}'} \; \neg P' \vee \bigwedge_{P' \in \mathbf{P}'} \; P')$.

$$\text{We get: if} \quad \varphi' = \varphi'_1 \vee \psi' \quad \text{with} \quad \varphi'_1 \in \mathbf{L}'_C, \quad \text{then} \quad b_2(\varphi') = (\overline{b})^{-1}(\varphi'_1) \tag{10}$$

$$\text{From (7), we get that } b_2 \text{ preserves } \vee \; : b_2(\varphi'_1 \vee \varphi'_2) = b_2(\varphi'_1) \vee b_2(\varphi'_2). \tag{11}$$

We get then, reminding $b_2 = (\overline{b})^{-1}$ on $\mathbf{L}'_C$: $b_2(\varphi') = \bigvee_{\varphi'_c \in \mathbf{L}'_C,\, \varphi'_c \models' \varphi'} (\overline{b})^{-1}(\varphi'_c)$.

We define the following preference relation $\prec'$ on $\mathbf{M}'$ (remember section 2 for Θ):

$$\text{for any } \mu', \nu' \text{ in } \mathbf{M}', \; \mu' \prec' \nu' \; \text{ iff } \; \Theta(b^{-1}(\mu')) \prec_s \Theta(b^{-1}(\nu')). \tag{12}$$

Thus $\prec'$ is the image in $\mathbf{M}'$ of the relation $\prec_s$ on $\mathbf{L}$. It is a strict order and there exists a set Φ' of formulas in $\mathbf{L}'$ such that $f_{\prec'} = CIRCF(\Phi', \emptyset, V(\mathbf{L}'))$ (cf Theorem 4.1-7).

We know from (5) and (4) that $b_1(\varphi)$ has for set of models the set associated to the set $\mathbf{W}(\varphi)$ by b (or $\overline{b}$ if we consider $\mathbf{L}_C$ instead of $\mathbf{M}$). Thus, $\mathbf{W}_{\prec_s}(\varphi)$ is the reverse image of the set $\mathbf{M}'_{\prec'}(b_1(\varphi))$: $\mathbf{W}_{\prec_s}(\varphi) = (\overline{b})^{-1}(\Theta'(\mathbf{M}'_{\prec'}(b_1(\varphi)))) = \{(\overline{b})^{-1}(\varphi'_c)/ \; \varphi'_c \in \mathbf{L}'_C, \mathbf{M}'(\varphi'_c) = \{\mu'\}$ with $\mu' \in \mathbf{M}'_{\prec'}(b_1(\varphi))\}$. From Definition 3.3 we have, for any $\varphi \in \mathbf{L}$, $f_{\prec_s}(\varphi) = \bigvee_{\varphi_1 \in \mathbf{W}_{\prec_s}(\varphi)} \varphi_1$. We get thus, from the definition of $\prec'$: for any $\varphi'_c \in \mathbf{L}'_C$, the only model μ' of φ'_c is in $\mathbf{M}_{\prec'}(b_1(\varphi))$ iff the formula $(\overline{b})^{-1}(\varphi'_c)$ is in $\mathbf{W}_{\prec_s}(\varphi)$. As φ'_c is in $\mathbf{L}'_c$, we get (see (9)): $(\overline{b})^{-1}(\varphi'_c) = b_2(\varphi'_c)$. We get then
$f_{\prec_s}(\varphi) = \bigvee_{\varphi_1 \in \mathbf{W}_{\prec_s}(\varphi)} \; \varphi_1 = \bigvee_{\varphi'_c \in \mathbf{L}'_C \text{ with } \mathbf{M}'(\varphi'_c) = \{\mu'\} \text{ and } \mu' \in \mathbf{M}'_{\prec'}(b_1(\varphi))} \; b_2(\varphi'_c)$.
From (11) we get $f_{\prec_s}(\varphi) = b_2(\bigvee_{\varphi'_c \in \mathbf{L}'_C \text{ with } \mathbf{M}'(\varphi'_c) = \{\mu'\} \text{ and } \mu' \in \mathbf{M}'_{\prec'}(b_1(\varphi))} \; \varphi'_c)$.

$$\text{We get then} \qquad f_{\prec_s}(\varphi) = b_2(f'_{\prec'}(b_1(\varphi))).$$

If we choose $\overline{\prec_{nf}}$ as our $\prec_s$, we get $\mathbf{W}_{\overline{\prec_{nf}}}(\varphi) = \{\bot, f(\varphi)\}$ from Proposition 6.4. Thus, $\mathbf{M}_{\prec'}(b_1(\varphi))$ as at most two elements, $V(\mathbf{L}')$ and the subset of μ' of $V(\mathbf{L}')$ which is the only other model of $f_{\prec'}(b_1(\varphi))$, if there is another model. As moreover we can apply (10) in this case, these peculiarities greatly simplify the effective computation.

It remains to check the conditions. ($\rightleftharpoons$1): $\varphi \models \psi$ iff $\mathbf{M}(\varphi) \subseteq \mathbf{M}(\psi)$ iff $\mathbf{M} - \mathbf{M}(\psi) \subseteq \mathbf{M} - \mathbf{M}(\varphi)$ and, from (4) we get that if $\mathbf{M} - \mathbf{M}(\psi) \subseteq \mathbf{M} - \mathbf{M}(\varphi)$, then $b_1(\varphi) \models' b_1(\psi)$.

($\rightleftharpoons$2): We prove that $b_1 \circ b_2$ is weakening on $\mathbf{L}'$. For any $\varphi' \in \mathbf{L}'$, we get $b_1(b_2(\varphi')) = \bigwedge_{P' \in V(\mathbf{L}'),\, \varphi' \models' P'} P'$ from (4) and (6), thus $\varphi' \models' b_1(b_2(\varphi'))$.

($\rightleftharpoons$3): For any $\varphi \in \mathbf{L}$, we get $\mathbf{M}(b_2(f'(b_1(\varphi)))) = \{\mu \in \mathbf{M} \; / \; f'(b_1(\varphi)) \not\models' P'_\mu\}$ from (7). As f' is a pre-circumscription, we have $f'(b_1(\varphi)) \models' b_1(\varphi)$, thus we get $\mathbf{M}(b_2(f'(b_1(\varphi)))) \subseteq \{\mu \in \mathbf{M} \; / \; b_1(\varphi) \not\models' P'_\mu\}$. Now we have $\mathbf{M}(b_2(b_1(\varphi))) = \{\mu \in \mathbf{M} \; / \; b_1(\varphi) \not\models' P'_\mu\}$. Thus we get $\mathbf{M}(b_2(f'(b_1(\varphi)))) \subseteq \mathbf{M}(b_2(b_1(\varphi)))$, i.e. $b_2(f'(b_1(\varphi))) \models b_2(b_1(\varphi))$, i.e., from (8): $b_2(f'(b_1(\varphi))) \models \varphi$.

($\rightleftharpoons$4): From $\mathbf{M}(b_2(\varphi')) = \{\mu \in \mathbf{M} \; / \; \varphi' \not\models' P'_\mu\}$ (7) we get that, if $\varphi' \models' \psi'$, then we have $b_2(\varphi') \models' b_2(\psi')$.

As the four conditions are satisfied, the translation preserves (LOOP) and also (CM) and (CT) (Proposition 5.1). The preservation of (CT) is interesting: from Theorem 4.1-2, if f' is a general preferential entailment, and if f is defined from f' as here, then f is a general preferential entailment. Thus this translation preserves the main properties which can be preserved in this case. Notice finally that, as the proof of the "if side" does not require condition ($\rightleftharpoons$4), we can formulate the theorem with or without ($\rightleftharpoons$4). $\Box$

A consequence of this proof is the following result:

Corollary 7.1. *A pre-circumscription satisfies (LOOP) iff it is a general preferential entailment defined by a simplified general preference relation which is a strict order.* $\Box$

We do not know what happens if $V(\mathbf{L})$ is infinite. A characterization of formula circumscription is known, but (sf) alone is not enough [12]. Moreover, b_1, b_2 should be defined for each $\mathcal{T} \in \mathbf{T}$ and not only for $\varphi \in \mathbf{L}$, which would complicate the matter.

Notice that we could use a slightly smaller vocabulary $\mathbf{L}'$, starting from the set $f(\mathbf{L})$ instead of the set $\mathbf{L}$, and from $\overline{\prec_f^{klm}}$ instead of $\overline{\prec_{nf}}$. However, this would complicate very seriously the definitions of b_1 and b_2 and we would loose the main advantage of our translation, the easy and natural definitions of b_1 and b_2.

The characterization result extends as follows to general preferential entailments:

Theorem 7.2. *A pre-circumscription f in $\mathbf{L}$ satisfies (CT) iff it can be expressed by (Def$\rightleftharpoons$4) from a preferential entailment $f' = f_{\prec'}$ defined in a language $\mathbf{L}'$.*

Proof: Notice that $V(\mathbf{L})$ must be finite, as for Theorem 7.1 and its corollary.

if: Preferential entailments satisfy (CT), thus f satisfies (CT) from Proposition 5.1, notice however that (Def$\rightleftharpoons$) would not suffice here.

only if: If f satisfies (CT), it is a general preferential entailment defined e.g. by the simplified relation $\prec_f$ introduced in [10, Definition 5.7]. We define $\mathbf{L}'$, b, b_1, b_2 and the preference relation $\prec'$ in $\mathbf{L}'$ as in the proof of Theorem 7.1, $\prec'$ being defined from $\prec_f$ exactly as in (12) from $\prec_s$. From the properties of b_1 and b_2 (mainly from (11)), we get then, as in the proof of Theorem 7.1: $f(\varphi) = b_2(f_{\prec'}(b_1(\varphi)))$ for any $\varphi \in \mathbf{L}$. $\Box$

This result adapts to finite general preferential entailment the characterization result [13, Theorem 4.8 and Preservation result 6.21] showing how to express any finite multi preferential entailment as a preferential entailment in a greater language.

8 A detailed example

Example 8.1. $V(\mathbf{L}) = \{P\}$, $f = f_{\prec_g}$ where $\prec_g$ is defined by $\bot \prec_g P$ and $\bot \prec_g \neg P$.

We get $f(\varphi) = \bot$ if $\varphi \in \{\bot, P, \neg P\}$ and $f(\top) = \top$ and also $\prec_g = \overline{\prec_g} = \prec_{nf} = \overline{\prec_{nf}}$. f falsifies (CR): $f(P \vee \neg P) \not\models f(P) \vee f(\neg P)$. Thus f is one of the simplest examples of a general preferential entailment which is not a multi preferential entailment. It is easy to check that f satisfies (LOOP) here, thus also (CUMU) (cf Theorem 4.1-3).

As f satisfies (LOOP), we apply Theorem 7.1, defining $\prec'$ from $\prec_s = \prec_g = \overline{\prec_{nf}}$. We define p and $V(\mathbf{L}')$ as follows: $p(\emptyset) = P'_0$, $p(\{P\}) = P'_1$, $V(\mathbf{L}') = \{P'_0, P'_1\}$,

$\mathbf{L}$	$\mathcal{P}(\mathbf{M})$	$\mathbf{M}'$	$\mathbf{L}'_C$	$b_1(\mathbf{L}) \subseteq \mathbf{L}'$	$[\in \mathcal{P}(\mathbf{L})]$	$[\in \mathcal{P}(\mathbf{L})]$
φ	$\mathbf{M}(\varphi)$	$b(\mathbf{M}(\varphi))$	$\overline{b}(\varphi)$	$b_1(\varphi)$	$\mathbf{W}(\varphi)$	$\mathbf{W}_{\prec_g}(\varphi)$
$\top$	$\{\emptyset, \{P\}\}$	$\emptyset$	$\neg P'_0 \wedge \neg P'_1$	$\top$	$\{\top, P, \neg P, \bot\}$	$\{\top, \bot\}$
P	$\{\{P\}\}$	$\{P'_0\}$	$P'_0 \wedge \neg P'_1$	P'_0	$\{P, \bot\}$	$\{\bot\}$
$\neg P$	$\{\emptyset\}$	$\{P'_1\}$	$\neg P'_0 \wedge P'_1$	P'_1	$\{\neg P, \bot\}$	$\{\bot\}$
$\bot$	$\emptyset$	$\{P'_0, P'_1\}$	$P'_0 \wedge P'_1$	$P'_0 \wedge P'_1$	$\{\bot\}$	$\{\bot\}$

Table1. Computation of b_1 [and of $\mathbf{W}(\varphi)$ and $\mathbf{W}_{\prec_g}(\varphi)$] for each $\varphi \in \mathbf{L}$

.

getting $\mathbf{M}' = \{\emptyset, \{P'_0\}, \{P'_1\}, \{P'_0, P'_1\}\}$. Table 1 describes $\overline{b}$ and b_1. We get then $\prec'$ described as follows in $\mathbf{M}'$: $\{P'_0, P'_1\} \prec' \{P'_0\}$, $\{P'_0, P'_1\} \prec' \{P'_1\}$.

We get $\mathbf{W}_{\prec'}(\top) = \{\top, \bot\}$ and $\mathbf{W}_{\prec'}(\varphi) = \{\bot\}$ for $\varphi \in \{P, \neg P, \bot\}$.

Using the method given in [15], we get a set Φ' of formulas to circumscribe: We define the greatest pre-order (reflexive and transitive relation) $\preceq'$ on $\mathbf{L}'$, satisfying ($\mu' \prec' \nu'$ iff $\mu' \preceq' \nu'$ and not $\nu' \preceq' \mu'$): $\{P'_0, P'_1\} \preceq' \{P'_0\}$, $\{P'_0, P'_1\} \preceq' \{P'_1\}$, $\{P'_0\} \preceq' \{P'_1\}$, $\{P'_1\} \preceq' \{P'_0\}$ and $\mu' \preceq' \mu'$. Then for each $\mu' \in \mathbf{M}'$, we define the formula $\varphi'(\mu') \in \mathbf{L}'$ having for set of models μ' and its successors for $\preceq'$, getting a set $\Phi' = \{\varphi'(\emptyset), \varphi'(\{P'_0\}), \varphi'(\{P'_0, P'_1\})\}$ such that $f' = f_{\prec'} = CIRCF(\Phi', \emptyset, V(\mathbf{L}'))$ (Φ' is optimal in cardinality for describing f' as a formula circumscription):

$\varphi'(\mu')$	$\mathbf{M}'(\varphi'(\mu'))$
$\varphi'(\emptyset) = \neg P'_0 \wedge \neg P'_1$	$\{\emptyset\}$
$\varphi'(\{P'_0\}) = \varphi'(\{P'_1\}) = \neg(P'_0 \Leftrightarrow P'_1)$	$\{\{P'_0\}, \{P'_1\}\}$
$\varphi'(\{P'_0, P'_1\}) = P'_0 \vee P'_1$	$\{\{P'_0\}, \{P'_1\}, \{P'_0, P'_1\}\}$

As we get $\varphi'(\emptyset) = \neg\varphi'(\{P'_0, P'_1\})$, the formula $\varphi'(\emptyset)$ (or equivalently $\neg\varphi'(\emptyset)$) is "*fixed*" in the circumscription [5], which can help the computation. It is easy to check that this is always true (adding disjunctions of formulas to a set does not modify the circumscription of the set of formulas [15]): the formula associated to the set of models $\mathbf{M}' - \mathbf{M}'(b(f(\mathbf{L}))$ is always obtained by the construction, while the formula associated to the complementary set $\mathbf{M}'(b(\mathbf{L}))$ is the disjunction of the other formulas obtained.

Table 2 describes f' and b_2. Only the framed values are used by the method. The first column gives the formulas $\varphi' \in \mathbf{L}'$ (shortly framed when φ' is in the set $b_1(\mathbf{L})$, i.e. is a conjunction of atoms). The second column describes $f' = CIRCF(\Phi', \emptyset, V(\mathbf{L}'))$: in fact, we only need the (framed) values of $f'(\varphi')$ for the four values in $b_1(\mathbf{L})$. The next three columns give respectively $\mathbf{M}'(\varphi')$, $\mathbf{M}(b_2(\varphi'))$ and the formula $b_2(\varphi') \in \mathbf{L}$ (we need only to consider the two formulas φ' in the set $f'(b_1(\mathbf{L}))$, framed in the f' column, we have made this apparent by long frames in the φ' and b_2 columns).

From the values of $b_1(\varphi)$ for the four $\varphi \in \mathbf{L}$ (Table 1), we compute $b_2(f'(b_1(\varphi)))$, and check that we get indeed $b_2(f'(b_1(\varphi))) = f(\varphi)$ $[f(\varphi) = \bigvee_{\psi \in \mathbf{W}_{\prec_g}(\varphi)} \psi]$.

9 Conclusion and Perspective

We have extended the "expressive power of circumscription", by showing that not only cumulative multi preferential entailments as shown by Costello [4], but also general

$\mathbf{L}'$ φ'	$[\in \mathbf{L}']$ $f'(\varphi')$	$\mathcal{P}(\mathbf{M}')$ $\mathbf{M}'(\varphi')$	$[\in \mathcal{P}(\mathbf{M})]$ $\mathbf{M}(b_2(\varphi'))$	$[\in \mathbf{L}]$ $b_2(\varphi')$
$\boxed{\top}$	$\boxed{P'_0 \Leftrightarrow P'_1}$	$\{\emptyset, \{P'_0\}, \{P'_1\}, \{P'_0, P'_1\}\}$	$\{\emptyset, \{P\}\}$	$\top$
$P'_0 \vee P'_1$	$P'_0 \wedge P'_1$	$\{\{P'_0\}, \{P'_1\}, \{P'_0, P'_1\}\}$	$\{\emptyset, \{P\}\}$	$\top$
$P'_0 \vee \neg P'_1$	$P'_0 \Leftrightarrow P'_1$	$\{\emptyset, \{P'_0\}, \{P'_0, P'_1\}\}$	$\{\emptyset, \{P\}\}$	$\top$
$\neg P'_0 \vee P'_1$	$P'_0 \Leftrightarrow P'_1$	$\{\emptyset, \{P'_1\}, \{P'_0, P'_1\}\}$	$\{\emptyset, \{P\}\}$	$\top$
$\neg P'_0 \vee \neg P'_1$	$\neg P'_0 \vee \neg P'_1$	$\{\emptyset, \{P'_0\}, \{P'_1\}\}$	$\{\emptyset, \{P\}\}$	$\top$
$\boxed{P'_0}$	$\boxed{P'_0 \wedge P'_1}$	$\{\{P'_0\}, \{P'_0, P'_1\}\}$	$\{\{P\}\}$	P
$\boxed{P'_1}$	$\boxed{P'_0 \wedge P'_1}$	$\{\{P'_1\}, \{P'_0, P'_1\}\}$	$\{\emptyset\}$	$\neg P$
$\boxed{P'_0 \Leftrightarrow P'_1}$	$P'_0 \Leftrightarrow P'_1$	$\{\emptyset, \{P'_0, P'_1\}\}$	$\{\emptyset, \{P\}\}$	$\boxed{\top}$
$P'_0 \not\Leftrightarrow P'_1$	$P'_0 \not\Leftrightarrow P'_1$	$\{\{P'_0\}, \{P'_1\}\}$	$\{\emptyset, \{P\}\}$	$\top$
$\neg P'_0$	$\neg P'_0$	$\{\emptyset, \{P'_1\}\}$	$\{\emptyset, \{P\}\}$	$\top$
$\neg P'_1$	$\neg P'_1$	$\{\emptyset, \{P'_0\}\}$	$\{\emptyset, \{P\}\}$	$\top$
$\boxed{\boxed{P'_0 \wedge P'_1}}$	$\boxed{P'_0 \wedge P'_1}$	$\{\{P'_0, P'_1\}\}$	$\{\emptyset\}$	$\boxed{\bot}$
$P'_0 \wedge \neg P'_1$	$P'_0 \wedge \neg P'_1$	$\{\{P'_0\}\}$	$\{\{P\}\}$	P
$\neg P'_0 \wedge P'_1$	$\neg P'_0 \wedge P'_1$	$\{\{P'_1\}\}$	$\{\emptyset\}$	$\neg P$
$\neg P'_0 \wedge \neg P'_1$	$\neg P'_0 \wedge \neg P'_1$	$\{\emptyset\}$	$\{\emptyset, \{P\}\}$	$\top$
$\bot$	$\bot$	$\emptyset$	$\emptyset$	$\bot$

Table2. Theorem 7.1 applied to example 8.1 (only the six framed computations are used)

preferential entailments satisfying (LOOP), can be translated into circumscriptions in another vocabulary. These various kinds of preferential entailment are introduced in Kraus and al. [7]. In order to achieve this translation, we needed two results. Firstly, the notion of general preferential entailment, as introduced in [7], is overly general [10]: we do not need copies of theories (or equivalently, of sets of interpretations). We can define the relation in the simpler set of the theories. Doing this, we have simplified some results in [7]: cumulative inferences correspond to general preferential entailment defined by a simplified relation satisfying (sf), also known as “smooth” (a result already given in [1,2] in much more complex ways). Secondly, we have described a modification of the vocabulary which allows to transpose any general preference relation among theories into a preference relation among complete theories (or among interpretations). This method needs a huge auxiliary vocabulary, however, only a very simple, and small, subclass of formulas in the new vocabulary (the conjunctions of atoms) needs to be considered. Moreover, the translation formulas from the old vocabulary to the new one and back are easy to compute. Thus, the method should be really applicable.

These results should have applications in helping the automatic computations of non monotonic formalisms. The modification of vocabulary introduced here could have other applications, as it is rather general, and relatively simple. Also, the simplification of the originally overly complex notion of general preferential entailment should help future studies on the subject: it is much easier to work with relations among theories that with relations among arbitrary sets of copies of theories. Finally, the translation results given here should also have real applications. This is obvious for the result allowing to translate any finite general preferential entailment satisfying (LOOP) into a circumscription. Indeed, the work on automatic computation of circumscription is still very active, and our work shows that any progress could be applied, not only to cumulative preferential entailments, as already known, but also to the strictly more general notion of general preferential entailment satisfying (LOOP). Also, the result showing how to express any finite cumulative general preferential entailment (a yet strictly more general notion) in terms of preferential entailment (where the relation is directly among interpretations) should have applications, since the notion of ordinary preferential entailment is simpler and more studied than the notion of general preferential entailment.

More studies are needed in order to apply these computations. Moreover, we are still waiting for efficient ways of computing ordinary preferential entailments, or even formula circumscriptions. At least we know now that not only multi, but also general, preferential entailments, would benefit from these demonstrators.

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