Smallest Equivalent Sets for Finite Propositional Formula Circumscription*

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Abstract Circumscription uses classical logic in order to modelize rules with exceptions and implicit knowledge. Formula circumscription is known to be easier to use in order to modelize a situation. We describe when two sets of formulas give the same result, when circumscribed. Two kinds of such equivalence are interesting: the ordinary one (two sets give the same circumscription) and the strong one (when completed by any arbitrary set, the two sets give the same circumscription) which corresponds to having the same closure for logical "and" and "or". In this paper, we consider only the finite case, focusing on looking for the smallest possible sets equivalent to a given set, for the two kinds of equivalence. We need to revisit a characterization result of formula circumscription. Then, we are able to describe a way to get all the sets equivalent to a given set, and also a way to get the smallest such sets. These results should help the automatic computation, and also the translation in terms of circumscription of complex situations.

1 Introduction

Circumscription uses classical logic for representing rules with exceptions. It is often better to use the formula version. An important aspect of formula circumscription has almost not been studied: what are exactly the sets of formulas which give rise to the same circumscription. Answering this question should have important consequences on the automatization of circumscription, and on the knowledge representation side. A possible explanation for the lack of studies on the subject is the complexity of the predicate versions of circumscriptions. We answer this problem in the finite propositional case, providing a way to obtain all the sets which give the same circumscription as a given set. We describe also a way to get the smallest sets, in terms of cardinality. The method given is only semi-constructive but, to our knowledge, no previous study exist.

Section 2 introduces ordinary and formula propositional circumscriptions. Section 3 gives two kinds of equivalence between sets of formulas, the strongest of these equivalences corresponds simply to have the same closure for logical "and" and "or". Section

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4 gives a few preliminary technical results, including a new study about a known characterization of formula circumscription. Section 5 gives useful indications in order to find the sets of formulas with as few elements as possible, which are "strongly equivalent", as defined in Section 3, to a given set of formulas Φ. Section 6 uses the results of Sections 4 and 5 in order to get the sets of formulas with as few elements as possible which are simply "equivalent" to a given set Φ, meaning which give the same circumscription as Φ. Section 7 provides a few examples. In particular the case where we start from an ordinary circumscription f and where we look for the smallest sets of formulas giving a formula circumscription equal to f, is considered there.

2 Propositional Circumscription

$\mathbf{L}$ being a propositional logic, $V(\mathbf{L})$ is the set of its propositional symbols. We suppose $V(\mathbf{L})$ finite in this text. As usual, $\mathbf{L}$ denotes also the set of all the formulas. We allow empty sets in **partitions** of $V(\mathbf{L})$. $Th(\mathcal{T}) = \{\varphi / \mathcal{T} \models \varphi\}$, the set of **theories** is $\mathbf{T} = \{Th(\mathcal{T}) / \mathcal{T} \subseteq \mathbf{L}\}$. Letters φ, ψ denote formulas in $\mathbf{L}$. Letters $\mathcal{T}$, and also Φ, Ψ, or X, Y denote subsets of $\mathbf{L}$. $\neg\Phi = \{\neg\varphi \;/\; \varphi \in \Phi\}$. Letters μ and ν denote interpretations for $\mathbf{L}$. We denote the interpretations by the subset of $V(\mathbf{L})$ that they satisfy: if $V(\mathbf{L}) = \{P, Q, Z\}$ and $\mu = \{P, Z\}$, then $Th(\mu) = Th(P \wedge \neg Q \wedge Z)$. $\mathbf{M} = \mathcal{P}(V(\mathbf{L}))$ denotes the set of all the interpretations for $\mathbf{L}$. For any subset $\mathbf{M}'$ of $\mathbf{M}$, we note $Th(\mathbf{M}')$ for the set $\{\varphi \in \mathbf{L} \,/\, \mu \models \varphi$ for any $\mu \in \mathbf{M}'\}$. This ambiguous meaning of $\models$ and Th is usual in logic and should not provoke confusion. $\mathbf{M}(\mathcal{T})$ denotes the set of the models of $\mathcal{T}$. The term **formula** will also be used for the quotient of this notion by logical equivalence: $\varphi = \psi$ iff $\mathbf{M}(\varphi) = \mathbf{M}(\psi)$.

Definition 2.1. [15] *A* **preference relation** *in* $\mathbf{L}$ *is a binary relation* $\prec$ *over* $\mathbf{M}$. $\mathbf{M}_{\prec}(\mathcal{T})$ *is the set of the elements* μ *of* $\mathbf{M}(\mathcal{T})$ **minimal for** $\prec$*:* $\mu \in \mathbf{M}(\mathcal{T})$ *and no* $\nu \in \mathbf{M}(\mathcal{T})$ *is such that* $\nu \prec \mu$*. The* **preferential entailment** $f = f_{\prec}$ *is defined by*
$f_{\prec}(\mathcal{T}) = Th(\mathbf{M}_{\prec}(\mathcal{T}))$*, i.e., as* $V(\mathbf{L})$ *is finite,* $\mathbf{M}(f_{\prec}(\mathcal{T})) = \mathbf{M}_{\prec}(\mathcal{T})$. □

Lemma 2.1. [immediate] If $\prec$ is irreflexive, then $f_{\prec} = f_{\prec'}$ iff $\prec = \prec'$. □

Definition 2.2. [9, 13, 14] $(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ *is a partition of* $V(\mathbf{L})$. $\mathbf{P}$ *is the set of the* **circumscribed propositional symbols***,* $\mathbf{Z}$ *of the* **variable** *ones,* $\mathbf{Q}$ *of the* **fixed** *ones.*

A circumscription is a preferential entailment $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = f_{\prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}}$ *where* $\prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$ *is defined by:* $\mu \prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})} \nu$ *if* $\mathbf{P} \cap \mu \subset \mathbf{P} \cap \nu$ *and* $\mathbf{Q} \cap \mu = \mathbf{Q} \cap \nu$.

We define also $\mu \preceq_{(\mathbf{P},\mathbf{Q},\mathbf{Z})} \nu$ *by* $\mathbf{P} \cap \mu \subseteq \mathbf{P} \cap \nu$ *and* $\mathbf{Q} \cap \mu = \mathbf{Q} \cap \nu$. □

Instead of using a propositional circumscription, it is often more natural and easier to use formula circumscription [9]. Here is the propositional version.

Definition 2.3. $\Phi, \mathcal{T}$ *are subsets of* $\mathbf{L}$*, and* $\mathbf{Q}, \mathbf{Z}$ *is a partition of* $V(\mathbf{L})$*. The* **formula circumscription** $CIRCF$ *of the formulas of* Φ*, with* $\mathbf{Q}$ *fixed, is as follows: We introduce the set* $\mathbf{P} = \{P_\varphi\}_{\varphi \in \Phi}$ *of new and distinct propositional symbols.*

$CIRCF(\Phi; \mathbf{Q}, \mathbf{Z})(\mathcal{T}) = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})(\mathcal{T} \cup \{\varphi \Leftrightarrow P_\varphi\}_{\varphi \in \Phi}) \cap \mathbf{L}$.

If $\mathbf{Q}$ *is empty, we write* $CIRCF(\Phi)$ *for* $CIRCF(\Phi; \emptyset, V(\mathbf{L}))$. □

Remark 2.1. Any ordinary circumscription is a formula circumscription: $CIRC(\mathbf{P},\mathbf{Q},\mathbf{Z}) = CIRCF(\mathbf{P};\mathbf{Q},\mathbf{Z}\cup\mathbf{P})$. □

For a formal proof of this immediate result, we can either start directly from Definition 2.3, or use Property 2.2 given below. In $CIRCF(\mathbf{P};\mathbf{Q},\mathbf{Z}\cup\mathbf{P})$, the first occurrence of **P** denotes a set of fomulas (each propositional symbol P denoting also an atomic formula). Notice that, as $CIRCF(\mathbf{P};\mathbf{Q}\cup\mathbf{P},\mathbf{Z})$ is clearly the identity, we must allow the symbols of **P** to vary also.

When considering formula circumscriptions, we may generally consider only the case where $\mathbf{Q}=\emptyset$:

Proposition 2.1. [5] $CIRCF(\Phi;\mathbf{Q},\mathbf{Z}) = CIRCF(\Phi\cup\mathbf{Q}\cup\neg\mathbf{Q})$.
Thus, $CIRC(\mathbf{P},\mathbf{Q},\mathbf{Z}) = CIRCF(\mathbf{P}\cup\mathbf{Q}\cup\neg\mathbf{Q})$.
(In $CIRCF$, **P** *and* **Q** *are considered as sets of formulas.)* □

Definition 2.4. *Let* μ,ν *be in* **M** *and* Φ *be a subset of* **L***.*

(a) We define the set of formulas $\Phi_\mu = \{\varphi\in\Phi \,/\, \mu\models\varphi\} = Th(\mu)\cap\Phi$.
(b) We define two relations in **M***:* $\mu\preceq_\Phi\nu$ *if* $\Phi_\mu\subseteq\Phi_\nu$*, and* $\mu\prec_\Phi\nu$ *if* $\Phi_\mu\subset\Phi_\nu$.
(b′) If $(\mathbf{Q},\mathbf{Z})$ *is a partition of* $V(\mathbf{L})$*, we define two relations in* **M***:* $\mu\preceq_{(\Phi;\mathbf{Q},\mathbf{Z})}\nu$ *if* $\Phi_\mu\subseteq\Phi_\nu$ *and* $\mathbf{Q}_\mu=\mathbf{Q}_\nu$*, and* $\mu\prec_{(\Phi;\mathbf{Q},\mathbf{Z})}\nu$ *if* $\Phi_\mu\subset\Phi_\nu$ *and* $\mathbf{Q}_\mu=\mathbf{Q}_\nu$. □

From these definitions we immediately get:

Lemma 2.2. [immediate]

0. $\preceq_{(\Phi;\mathbf{Q},\mathbf{Z})} = \preceq_{\Phi\cup\mathbf{Q}\cup\neg\mathbf{Q}}$ and $\prec_{(\Phi;\mathbf{Q},\mathbf{Z})} = \prec_{\Phi\cup\mathbf{Q}\cup\neg\mathbf{Q}}$.

1a. $\mu\prec_\Phi\nu$ iff $\mu\preceq_\Phi\nu$ and $\nu\not\preceq_\Phi\mu$.
1b. $\mu\preceq_\Phi\nu$ iff for any $\varphi\in\Phi$, if $\mu\models\varphi$ then $\nu\models\varphi$.

2. For any μ,μ_i,ν in **M** and $\varphi\in\mathbf{L}$, we have $\{\varphi\}_\mu\subseteq\{\varphi\}$. Thus we have ($\mu\preceq_{\{\varphi\}}\nu$ or $\nu\preceq_{\{\varphi\}}\mu$), and we cannot have ($\mu_1\prec_{\{\varphi\}}\mu_2$ and $\mu_2\prec_{\{\varphi\}}\mu_3$).

3a. $\mu\preceq_\Phi\nu$ iff for any $\varphi\in\Phi$, $\mu\preceq_{\{\varphi\}}\nu$.
3b. $\mu\prec_\Phi\nu$ iff for any $\varphi\in\Phi$, $\mu\preceq_{\{\varphi\}}\nu$, and there exists $\varphi\in\Phi$ such that $\mu\prec_{\{\varphi\}}\nu$.

4. $\prec_\Phi$ and $\preceq_\Phi$ are transitive, $\prec_\Phi$ is irreflexive while $\preceq_\Phi$ is reflexive. □

Thus, to know the “useful relation” (see Proposition 2.2 just below) $\prec_\Phi$, we need more than each $\prec_{\{\varphi\}}$, we must know all the $\preceq_{\{\varphi\}}$’s, a more precise information. Here is an alternative definition of formula circumscription (proof easy from Definition 2.3):

Proposition 2.2. [folklore] $CIRCF(\Phi) = f_{\prec_\Phi}$, $CIRCF(\Phi;\mathbf{Q},\mathbf{Z}) = f_{\prec_{(\Phi;\mathbf{Q},\mathbf{Z})}}$. □

3 Equivalences between Sets of Circumscribed Formulas

We examine when two sets Φ and Φ' give the same formula circumscription. From a knowledge representation perspective, and also from a formal perspective, two kinds of such “equivalences” are to be considered.

Definition 3.1. *Φ and Φ' are* **c-equivalent**, *written $\Phi \equiv_c \Phi'$, if $CIRCF(\Phi) = CIRCF(\Phi')$. Φ and Φ' are* **strongly equivalent**, *written $\Phi \equiv_{sc} \Phi'$, if, for any set Φ'' of formulas, $CIRCF(\Phi \cup \Phi'') = CIRCF(\Phi' \cup \Phi'')$.* □

If $\Phi \equiv_{sc} \Phi'$, then $\Phi \equiv_c \Phi'$. The strong version is useful because, when another rule, or even, as we are in the propositional case, another "individual", is added, this corresponds to an addition of formula(s): e.g. a new bird B_k, when we know that birds B_i generally fly F_i, provokes the addition of a new formula $B_k \wedge \neg F_k$ to be circumscribed. If we have only the standard equivalence, we may then loose this equivalence.

We need a few definitions now.

Definition 3.2. *The* $\wedge$**-closure of** *Φ is the set $\Phi^\wedge = \{\bigwedge_{\varphi \in \Psi} \varphi /$ for any finite $\Psi \subseteq \Phi\}$. The* $\vee$**-closure** *$\Phi^\vee$ is defined similarly. The* $\wedge\vee$**-closure of** *Φ is the set $\Phi^{\wedge\vee} = (\Phi^\wedge)^\vee$. $\Phi^\wedge$ (resp. $\Phi^\vee$, or $\Phi^{\wedge\vee}$) is called a set* **closed for** $\wedge$ *(resp.* **for** $\vee$, **for** $\wedge$ **and** $\vee$*).* □

We get always $\top \in \Phi^\wedge, \bot \in \Phi^\vee$ ($\Psi = \emptyset$) and also, from the de Morgan distribution laws, $(\Phi^\wedge)^\vee = (\Phi^\vee)^\wedge$ (remember that a formula is assimilated to its equivalence class).

Definition 3.3. *Let $\prec$ be a preference relation and Φ be a set of formulas.*

1. *φ is* **accessible for** *$f = f_\prec$ if $\varphi \in f(\mathcal{T}) - \mathcal{T}$ for some theory $\mathcal{T}$. The set of the formulas* **inaccessible for** *f is $I_f = I_\prec = \mathbf{L} - \bigcup_{\mathcal{T}\in\mathbf{T}}(f(\mathcal{T}) - \mathcal{T}) = \bigcap_{\mathcal{T}\in\mathbf{T}}(\mathbf{L} - (f(\mathcal{T}) - \mathcal{T}))$.*
2. *The set of the formulas* **positive for** *$\prec$ is the set $Pos(\prec)$ of the formulas φ such that, if $\mu \models \varphi$ and $\mu \prec \nu$, then $\nu \models \varphi$.*
   If $\prec = \prec_\Phi$ of Definition 2.4, we write $Pos_e(\Phi)$ for the set $Pos(\prec_\Phi)$, called the set of the formulas **positive in Φ, in the extended acception**.
   If $\prec = \preceq_\Phi$, we write $Pos_m(\Phi)$ for the set $Pos(\preceq_\Phi)$ of the formulas **positive in Φ, in the minimal acception**. □

One technical role of the inaccessible formulas for circumscriptions is developed in [10]. We will see now that I_f is the greatest (for $\subset$) set Ψ such that $f = CIRCF(\Phi) = CIRCF(\Psi)$ (Theorem 3.3-1 below). $Pos(\prec)$ is closed for $\wedge$ and $\vee$ (next proposition, this gives a justification for the name of this set). The following results are proved elsewhere (see also [11] for details and examples). However, we provide the proofs here again. Notice that these proofs are sometimes simpler than the ones appearing in the referenced text, due to the restriction to the finite case made here.

Proposition 3.1. [12, Proposition 3.1]

1. *If $\prec$ is a binary relation in* **M**, *then $Pos(\prec)$ is closed for $\wedge$ and $\vee$.*
2. *If $\Phi \subseteq \mathbf{L}$, then $\Phi \subseteq Pos_m(\Phi) \subseteq Pos_e(\Phi)$,*
   $\Phi^{\wedge\vee} = Pos_m(\Phi)$ and $Pos_e(\Phi) = I_{\prec_\Phi}$. □

Proof: 1. Obvious.

2. $Pos_e(\Phi) = I_{\prec_\Phi}$: cf [10, Property 4.9].

$\Phi \subseteq Pos_m(\Phi) \subseteq Pos_e(\Phi)$: Obvious. Notice that each $\subseteq$ can be strict. As an example, choose $V(\mathbf{L}) \neq \emptyset$ and $\Phi = \emptyset$. Then, $\preceq_\Phi$ is always true and $\prec_\Phi$ is always false. Thus, we get $Pos_m(\Phi) = Pos(\preceq_\Phi) = \{\top, \bot\}$ and $Pos_e(\Phi) = Pos(\prec_\Phi) = \mathbf{L}$.

$\Phi^{\wedge\vee} = Pos_m(\Phi)$: We already know $\Phi \subseteq Pos_m(\Phi)$, thus, from point 1 we get $\Phi^{\wedge\vee} \subseteq Pos_m(\Phi)$. Conversely, let us suppose $\varphi \in Pos_m(\Phi) = Pos(\preceq_\Phi)$. As $\{\top, \bot\} \subseteq \Phi^{\wedge\vee}$, we may suppose that there exist μ, ν such that $\mu \models \varphi, \nu \models \neg\varphi$. Then, $\Phi_\mu \not\subseteq \Phi_\nu$ from the definitions of $\preceq_\Phi$ and of $Pos(\preceq_\Phi)$. With each such (μ, ν) we associate one formula $\varphi_{(\mu,\nu)} \in \Phi_\mu - \Phi_\nu$. With each ν such that $\nu \models \neg\varphi$, we associate $\varphi_\nu = \bigvee_{\mu\models\varphi} \varphi_{(\mu,\nu)}$. Thus $\varphi_\nu \in \Phi^\vee$, and also $\varphi \models \varphi_\nu$ and $\nu \models \neg\varphi_\nu$. We consider now the formula $\psi = \bigvee_{\nu\models\neg\varphi} \neg\varphi_\nu$. Thus $\psi \in (\neg(\Phi^\vee))^\vee = \neg((\Phi^\vee)^\wedge) = \neg(\Phi^{\wedge\vee})$ and $\mathbf{M}(\neg\varphi) \subseteq \mathbf{M}(\psi)$, i.e., $\neg\varphi \models \psi$. As these $\neg\varphi_\nu$'s satisfy $\neg\varphi_\nu \models \neg\varphi$, we get also $\psi \models \neg\varphi$. Thus $\varphi = \neg\psi \in \Phi^{\wedge\vee}$. $\square$

It is natural to call the formulas in $\Phi^{\wedge\vee}$, "positive in Φ", thus our notation $Pos_m(\Phi)$. There are also good reasons to call the formulas in the generally greater set $Pos_e(\Phi)$, "positive in Φ", in an **extended acception** (see Definition 3.3-2, Proposition 3.1-1, and the example of ordinary circumscription $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ given in [11, 12]).

Here are two easy lemmas:

Lemma 3.1. [12, Lemma 3.6] If $\Phi \subseteq \Psi \subseteq \Phi^{\wedge\vee}$, we have $\preceq_\Phi = \preceq_\Psi$, thus a fortiori $\prec_\Phi = \prec_\Psi$, i.e. $CIRCF(\Phi) = CIRCF(\Psi)$. $\square$

<u>Proof:</u> We get $\mu \preceq_{\Phi^{\wedge\vee}} \nu$ if $\mu \preceq_\Phi \nu$ (Lemma 2.2-1b). From Lemma 2.2-3a, $\mu \preceq_\Phi \nu$ if ($\Phi \subseteq \Psi$ and $\mu \preceq_\Psi \nu$), thus $\mu \preceq_\Phi \nu$ if $\mu \preceq_{\Phi^{\wedge\vee}} \nu$. Thus $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}}$. Thus $\preceq_\Phi = \preceq_\Psi$ iff $\preceq_{\Phi^{\wedge\vee}} = \preceq_{\Psi^{\wedge\vee}}$. Now, $\Phi^{\wedge\vee} = \Psi^{\wedge\vee}$ if $\Phi \subseteq \Psi \subseteq \Phi^{\wedge\vee}$. $\square$

Lemma 3.2. [12, Lemma 3.7] $CIRCF(\Phi) = CIRCF(I_{\prec_\Phi}) = CIRCF(Pos_e(\Phi))$. $\square$

This lemma (false in the infinite case) is contained in [10, Property 5.6].

Theorem 3.3. [12, Theorem 3.8] *Φ and Ψ are sets of formulas.*

1. *$\Phi \equiv_c \Psi$ iff $\prec_\Phi = \prec_\Psi$ iff $Pos_e(\Phi) = Pos_e(\Psi)$.*
   $\prec_\Phi = \prec_{Pos_e(\Phi)} = \prec_{Pos_m(\Phi)}$.
   $Pos_e(\Phi)$ is the greatest (for $\subseteq$) set Ψ satisfying $\Psi \equiv_c \Phi$.
2. *(a) $\Phi \equiv_{sc} \Psi$ iff $\preceq_\Phi = \preceq_\Psi$ iff $\Phi^{\wedge\vee} = \Psi^{\wedge\vee}$. Also $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}}$, thus $\prec_\Phi = \prec_{\Phi^{\wedge\vee}}$.*
   (b) $Pos_m(\Phi) = \Phi^{\wedge\vee}$ is the greatest set Ψ satisfying $\Psi \equiv_{sc} \Phi$ (cf Lemma 3.1).
3. *$\Phi \cup \{\varphi\} \equiv_c \Phi$ iff $\Phi \cup \{\varphi\} \equiv_{sc} \Phi$ iff $\varphi \in \Phi^{\wedge\vee}$.*

Point 1 provides necessary and sufficient (remember that we are in the finite case) conditions for two sets of formulas to give the same circumscription. Point 2 provides necessary and sufficient conditions for two sets of formulas to give the same circumscription, even when they are augmented in a similar way. One of these conditions is simply having the same $\wedge\vee$-closure. Point 3 provides a case where c-equivalence is identical to strong equivalence: informally, from point 2 and Lemma 2.2-3, when there is c-equivalence and not strong equivalence between a set and one of its subsets, the added formulas oppose each other. This mutual cancellation is impossible when the two sets differ by one formula only.

<u>Proof:</u> <u>1.</u> First "iff": Lemma 2.1 and Proposition 2.2.

Second "iff": If $\prec_\Phi = \prec_\Psi$, then $Pos(\prec_\Phi) = Pos(\prec_\Psi)$, i.e. $Pos_e(\Phi) = Pos_e(\Psi)$. If $Pos_e(\Phi) = Pos_e(\Psi)$, then $\prec_\Phi = \prec_\Psi$ (Lemmas 2.1, 2.2-4 and 3.2). Maximality of

$Pos_e(\Phi)$ comes from $\Phi \subseteq Pos_e(\Phi)$, thus, if $\Psi \equiv_c \Phi$, as $Pos_e(\Psi) = Pos_e(\Phi)$ from 1a, we get $\Psi \subseteq Pos_e(\Phi)$. Notice that the result $\prec_\Phi = \prec_{Pos_e(\Phi)}$, and the maximality of $Pos_e(\Phi)$, is also an immediate consequence of Property 5.6 in [10].

2a. First "iff". "if": If $\preceq_\Phi = \preceq_\Psi$, then for any Ψ', $\preceq_{\Phi \cup \Psi'} = \preceq_{\Psi \cup \Psi'}$, thus $\prec_{\Phi \cup \Psi'} = \prec_{\Psi \cup \Psi'}$, i.e. $f_{\prec_{\Phi \cup \Psi'}} = f_{\prec_{\Psi \cup \Psi'}}$ (Lemma 2.2-3a).

"only if": Let us suppose $\preceq_\Phi \neq \preceq_\Psi$. If $\prec_\Phi \neq \prec_\Psi$, then $\Phi \not\equiv_c \Psi$, thus $\Phi \not\equiv_{sc} \Psi$. If $\prec_\Phi = \prec_\Psi$, then there exist μ and ν such that $\mu \preceq_\Phi \nu$, $\nu \preceq_\Phi \mu$, and $\mu \not\preceq_\Psi \nu$, $\nu \not\preceq_\Psi \mu$. As $\mu \neq \nu$, there is a formula φ such that $\mu \models \varphi, \nu \not\models \varphi$. We get, from Lemma 2.2-3b, $\nu \prec_{\Phi \cup \{\varphi\}} \mu$. As $\nu \not\preceq_\Psi \mu$, we get $\nu \not\prec_{\Psi \cup \{\varphi\}} \mu$, thus $\Psi \cup \{\varphi\} \not\equiv_c \Phi \cup \{\varphi\}$. In any case $\Psi \not\equiv_{sc} \Phi$.

Second "iff". Part "only if": If $\preceq_\Phi = \preceq_\Psi$, then $Pos(\preceq_\Phi) = Pos(\preceq_\Psi)$, i.e. $Pos_m(\Phi) = Pos_m(\Psi)$.

Part "if" is a consequence of the second sentence, proved below: If $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}}$ and $Pos(\preceq_\Phi) = Pos(\preceq_\Psi)$ (i.e. $\Phi^{\wedge\vee} = \Psi^{\wedge\vee}$), then $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}} = \preceq_{\Psi^{\wedge\vee}} = \preceq_\Psi$.

Second sentence: From Lemma 3.1 and its proof, we know $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}}$, thus $\prec_\Phi = \prec_{\Phi^{\wedge\vee}}$.

2b. $Pos_m(\Phi) = \Phi^{\wedge\vee}$ (Proposition 3.1-2). Prop. If $\Psi \equiv_{sc} \Phi$, then $\Psi^{\wedge\vee} = \Phi^{\wedge\vee}$ (2a) thus $\Psi \subseteq \Phi^{\wedge\vee}$.

3. First "iff": Let us suppose $\Phi \cup \{\varphi\} \equiv_c \Phi$, i.e., $\prec_{\Phi \cup \{\varphi\}} = \prec_\Phi$, i.e., from Lemma 2.2-3a, ($\mu \preceq_{\{\varphi\}} \nu$ whenever $\mu \prec_\Phi \nu$), and also $\preceq_{\Phi \cup \{\varphi\}} \neq \preceq_\Phi$. From Lemma 2.2-3b there exist μ, ν such that $\mu \preceq_\Phi \nu$, $\nu \preceq_\Phi \mu$, $\mu \not\preceq_{\{\varphi\}} \nu$, $\nu \not\preceq_{\{\varphi\}} \mu$, which contradicts Lemma 2.2-2. Thus, $\Phi \cup \{\varphi\} \equiv_c \Phi$ implies $\preceq_{\Phi \cup \{\varphi\}} = \preceq_\Phi$, i.e., $\Phi \cup \{\varphi\} \equiv_{sc} \Phi$.

Second "iff": As $\varphi \in \Phi^{\wedge\vee}$ iff $(\Phi \cup \{\varphi\})^{\wedge\vee} = \Phi^{\wedge\vee}$, 2a above gives the result. □

4 About the Characterization of Formula Circumscription

From theorem 3.3-2a, we know that the smallest sets strongly equivalent to a given set Ψ are the smallest sets Φ such that $\Phi^{\wedge\vee} = \Psi$. We also know that all the elements of the set of all the sets of formulas strongly equivalent to a given set Ψ correspond to the same pre-order relation $\preceq_\Psi$. We will study this set, focusing on the smallest sets of formulas belonging to this set. From theorem 3.3-1 (or directly from lemmas 2.1 and 2.2-4 and from the definitions of $CIRCF$ and $\prec_\Psi$), we know that the set of all the sets of formulas which are c-equivalent to Ψ correspond to the same strict order relation $\prec_\Psi$. From the definitions of $\preceq_\Psi$ and $\prec_\Psi$, we get that various large relations $\preceq_\Psi$ can be associated to a given strict relation $\prec_\Psi$, which is the relation defining a given formula circumscription $CIRCF(\Psi)$. This leads us, firstly to make precise the connections between $\prec_\Phi$ and $\preceq_\Phi$, then to revisit the characterization of finite formula circumscription.

Definition 4.1. *Let $\prec$ be some binary relation on a set E. We denote by $s_\prec(e)$ and $p_\prec(e)$ respectively the sets $s_\prec(e) = \{e' \in E / e \prec e'\}$ and $p_\prec(e) = \{e' \in E / e' \prec e\}$.*

We suppose now that $\prec$ is a **strict order** *(irreflexive and transitive relation). We call* **associated with** $\prec$ *any* **pre-order** *(reflexive and transitive) relation $\preceq$ on E such that we have, for any e_1, e_2 in E: $e_1 \prec e_2$ iff $e_1 \preceq e_2$ and $e_2 \not\preceq e_1$.* **(Assoc)**

We define the following two relations $\preceq_0$ and $\preceq_1$ on E:

$e_1 \preceq_0 e_2$ if $e_1 \prec e_2$ or $e_1 = e_2$, and

$e_1 \preceq_1 e_2$ if $e_1 \prec e_2$ or ($s_\prec(e_1) = s_\prec(e_2)$ and $p_\prec(e_1) = p_\prec(e_2)$). □

Lemma 4.1. 1. $\preceq_0$ and $\preceq_1$ are associated with $\prec$ and, if $\preceq$ is associated with $\prec$ we have, for any e_1, e_2 in E: (1) if $e_1 \preceq_0 e_2$ then $e_1 \preceq e_2$, and
(2) if $e_1 \preceq e_2$ then $e_1 \preceq_1 e_2$.
2. $\preceq_0$ is the only order relation $\preceq$ satisfying (Assoc).
3. A pre-order $\preceq$ "between $\preceq_0$ and $\preceq_1$" is not necessarily associated with $\prec$: $\preceq$ is associated with $\prec$ iff its graph is the reunion of the graphs of $\preceq_0$ and of some universal relations restricted to subsets of E over which $\preceq_1$ is the universal relation.

Proof: Point 1: It is obvious that $\preceq_0$ is associated with $\prec$, that each pre-order $\preceq$ associated with $\prec$ satisfies implication (1), that $\preceq_1$ is reflexive and satisfies (Assoc).

$\preceq_1$ is transitive: We suppose $e_1 \preceq_1 e_2$ and $e_2 \preceq_1 e_3$. If ($e_1 \prec e_2$ and $e_2 \prec e_3$), or ($e_1 \prec e_2$ and $p_\prec(e_2) = p_\prec(e_3)$), or ($s_\prec(e_1) = s_\prec(e_2)$ and $e_2 \prec e_3$), then $e_1 \prec e_3$. If $s_\prec(e_1) = s_\prec(e_2)$, $p_\prec(e_1) = p_\prec(e_2)$, $s_\prec(e_2) = s_\prec(e_3)$ and $p_\prec(e_2) = p_\prec(e_3)$, then $s_\prec(e_1) = s_\prec(e_3)$ and $p_\prec(e_1) = p_\prec(e_3)$. In any case $e_1 \preceq_1 e_3$.

Implication (2) [proof due to Éric Badouel]: We suppose $e_1 \preceq e_2$. By (Assoc) $e_1 \prec e_2$ or $e_2 \preceq e_1$. If $e_1 \prec e_2$, then $e_1 \preceq_1 e_2$, thus we suppose $e_2 \preceq e_1$. If $e \in p_\prec(e_1)$, then $e \preceq e_1$ by (Assoc) and $e \preceq e_2$ by transitivity. Thus we get $e \prec e_2$ or $e_2 \preceq e$ by (Assoc). If $e_2 \preceq e$, we get $e_1 \preceq e$ by transitivity, thus $e \not\prec e_1$ by (Assoc), a contradiction: we get $e \prec e_2$. Thus $p_\prec(e_1) \subseteq p_\prec(e_2)$. By similar arguments, we get $s_\prec(e_2) \subseteq s_\prec(e_1)$. As our hypothesis is symmetrical in (e_1, e_2), we get $p_\prec(e_1) = p_\prec(e_2)$ and $s_\prec(e_1) = s_\prec(e_2)$: $e_1 \preceq_1 e_2$.

Point 2: Immediate.

Point 3: "only if": $\preceq$ is associated with $\prec$. We suppose $e_1 \preceq e_2$, $e_2 \preceq e_3$, $\cdots$, $e_{n-1} \preceq e_n$, $e_1 \not\preceq_0 e_2$, $e_2 \not\preceq_0 e_3$, $\cdots$, $e_{n-1} \not\preceq e_n$. Then $e_i \not\prec e_{i+i}$ and, by (Assoc), $e_{i+1} \preceq e_i$: $\preceq$ is universal on the set $\{e_1, e_2, \cdots, e_n\}$. By implication (2), $\preceq_1$ is also universal on this set. The result is then immediate.

"if": $\prec$ is a strict order and the graph of $\preceq$ is the union of the graph of $\preceq_0$ and of the graphs of the universal relation on some subsets of E over which the restriction of $\preceq_1$ is universal.

$\preceq$ is reflexive and transitive: Reflexivity is obvious. Let e_1, e_2, e_3 be distinct in E such that $e_1 \preceq e_2$ and $e_2 \preceq e_3$. If $e_1 \preceq_0 e_2$ and $e_2 \preceq_0 e_3$, then $e_1 \preceq_0 e_3$. If $e_1 \preceq_0 e_2$ and $e_2 \not\preceq_0 e_3$ then $e_2 \preceq_1 e_3$ and $e_3 \preceq_1 e_2$ thus $p_\prec(e_2) = p_\prec(e_3)$ and $e_1 \prec e_3$. Similarly, if $e_1 \not\preceq_0 e_2$ and $e_2 \preceq_0 e_3$ then $s_\prec(e_1) = s_\prec(e_2)$ and $e_1 \prec e_3$. In any case $e_1 \preceq e_3$.

$\preceq$ satisfies (Assoc): If $e \preceq e'$ and $e' \not\preceq e$, $\preceq$ is not universal on $\{e, e'\}$, thus $e \preceq_0 e'$ and $e \neq e'$, thus $e \prec e'$. Conversely, we suppose $e \prec e'$. Then $e \preceq e'$ and, if $e' \preceq e$, then $\preceq$, thus $\preceq_1$, is universal on $\{e, e'\}$. As $\prec$ is a strict order, $e' \not\prec e$. Thus, from $e' \preceq_1 e$ we get $s_\prec(e) = s_\prec(e')$, a contradiction. Thus, we get $e \preceq e'$ and $e' \not\preceq e$. □

The number of all the pre-orders $\preceq$ associated with a given strict order $\prec$ is then $\Pi_{C \in \mathbf{C}} B_{card(C)}$, where B_k is the Bell (or exponential) number of index k and $\mathbf{C}$ is the set of all the maximal subsets C of E (with $card(C) > 1$) on which $\preceq_1$ is universal.

Theorem 4.2. [6, 3, 11] *A pre-circumscription f is a formula circumscription iff it is a preferential entailment associated with a strict order $\prec$.*

Proof: Theorem 4.2 has independently appeared at least three times as [6, Theorems 13-14], [3, Theorem 7] and [11, Proposition 5.24-1]. However, in [6, 11] the proof relies on the strict relation (i.e. in fact on the large relation $\preceq_0$), and [3] uses any large relation, but without constructive definition. For our purpose, we need to consider large relations and constructive definitions, which is why we provide a new proof. Also, we need to consider all the possible large relations associated with a strict relation, which is why we have developed this point above.

"only if": Lemma 2.2-4 and Proposition 2.2.

"if": The proof is built on an old result ([8, Lemma 11.8, Theorems 11.6 and 11.9]): any pre-order $\preceq$ on a set E can be put in correspondence with the relation $\supseteq$ on a subset of $\mathcal{P}(E)$: For any e_1, e_2 in E, we have $e_1 \preceq e_2$ iff $s_{\preceq}(e_2) \subseteq s_{\preceq}(e_1)$. **(MN)**

Step 0: Let $\prec$ be a strict order on **M** and $\preceq$ be any pre-order associated with $\prec$.

Step 1, first application of (MN): Let us define the formula $\varphi_{\preceq}(\mu)$, for any $\mu \in \mathbf{M}$, and the set of formulas $\Phi_{\preceq}$ by: $\mathbf{M}(\varphi_{\preceq}(\mu)) = s_{\preceq}(\mu)$ and $\Phi_{\preceq} = \{\varphi_{\preceq}(\mu)\}_{\mu \in \mathbf{M}}$.

From (MN) we get $\mu \preceq \nu$ iff $\varphi_{\preceq}(\nu) \models \varphi_{\preceq}(\mu)$. (MN1)

Step 2, second application of (MN): Starting from the set $\Phi_{\preceq}$ pre-ordered by $\models$, for each $\psi \in \Phi_{\preceq}$, we define the set of formulas $s_{\models}(\psi) = \{\psi' \in \Phi_{\preceq} \ / \ \psi \models \psi'\}$. From (MN) we get, for any μ, ν in **M**: $\varphi_{\preceq}(\mu) \models \varphi_{\preceq}(\nu)$ iff $s_{\models}(\varphi_{\preceq}(\nu)) \subseteq s_{\models}(\varphi_{\preceq}(\mu))$. (MN2)

We get $\varphi_{\preceq}(\nu) \in s_{\models}(\varphi_{\preceq}(\mu))$ iff $\varphi_{\preceq}(\mu) \models \varphi_{\preceq}(\nu)$, $\varphi_{\preceq}(\mu) \models \varphi_{\preceq}(\nu)$ iff $\nu \preceq \mu$ from (MN1), and $\nu \preceq \mu$ iff $\mu \in s_{\preceq}(\nu)$ iff $\mu \models \varphi_{\preceq}(\nu)$ from the definitions of $s_{\preceq}$ and of $\varphi_{\preceq}$. As we have $\mu \models \varphi_{\preceq}(\nu)$ iff $\varphi_{\preceq}(\nu) \in Th(\mu)$, we get $\varphi_{\preceq}(\nu) \in s_{\models}(\varphi_{\preceq}(\mu))$ iff $\varphi_{\preceq}(\nu) \in Th(\mu)$. This means, for any $\mu \in \mathbf{M}$: $s_{\models}(\varphi_{\preceq}(\mu)) = Th(\mu) \cap \Phi_{\preceq}$. From (MN1) and (MN2) we get then $\mu \preceq \nu$ iff $Th(\mu) \cap \Phi_{\preceq} \subseteq Th(\nu) \cap \Phi_{\preceq}$, thus $\mu \prec \nu$ iff $Th(\mu) \cap \Phi_{\preceq} \subset Th(\nu) \cap \Phi_{\preceq}$. Thus, from Proposition 2.2, $f = f_{\prec} = CIRCF(\Phi_{\preceq})$. □

Step 0 allows many possibilities. If we want to reduce the size of the set $\Phi_{\preceq}$ of formulas to circumscribe, obtained in step 1, we will see (in the next paragraph, and more importantly in Proposition 6.1 below) why it is good to choose $\preceq_1$ and why the worst possibility is $\preceq_0$, as done in [6, 11]. In step 1, with each pre-order $\preceq$, we associate naturally a set of formulas $\Phi_{\preceq}$. In step 2, with each set Φ of formulas, we associate naturally a pre-order $\preceq_{\Phi}$ and we get $\preceq = \preceq_{\Phi_{\preceq}}$. Remember that Theorem 3.3-2 allows to reduce the size of the set $\Phi_{\preceq}$ of formulas to circumscribe.

Let us give here a few additional comments about the size of the set of formulas $\Phi_{\preceq}$ obtained after steps 0 and 1. The two relations $\preceq_0$ and $\preceq_1$ introduced in step 0 are respectively the smallest one and the greatest one. Notice also that if we choose $\preceq = \preceq_0$ in step 0, the set $\Phi_{\preceq} = \{\varphi_{\preceq}(\mu)\}_{\mu \in \mathbf{M}}$ that we get in step 1 is always made of $card(\mathbf{M})$ different formulas. Thus, any other choice for $\preceq$ in step 0 will give a set $\Phi_{\preceq}$ with at most $card(\mathbf{M})$ formulas.

5 The Smallest Sets Strongly Equivalent to a Given Set

We need a few definitions, often given in two forms: in terms of subsets of a set E, and in terms of formulas, where $E = \mathbf{M}$ and a formula can be associated with each subset. This is to relate these results to familiar general results about finite sets.

Definition 5.1. *1. (a) The* $\cup$**-closure** *of a set $\mathcal{E}$ of subsets of some given set E is the set $\mathcal{E}^{\cup}$ of the subsets of E which are unions of elements in $\mathcal{E}$: $\mathcal{E}^{\cup} =$*

*$\{\bigcup_{E' \in \mathcal{E}'} E' / \mathcal{E}' \subseteq \mathcal{E}\}$. The $\cap$-**closure** of $\mathcal{E}$ is defined similarly and the $\cup\cap$-**closure** of $\mathcal{E}$ is the set $\mathcal{E}^{\cup\cap} = (\mathcal{E}^{\cup})^{\cap} = (\mathcal{E}^{\cap})^{\cup}$.*

(b) For any set $\mathcal{E}$ of subsets of E, we define the sets $\mathcal{E}_{\cup} = \{E' \in \mathcal{E} / E' \notin (\mathcal{E} - \{E'\})^{\cup}\}$ and $\mathcal{E}_{\cap} = \{E' \in \mathcal{E} / E' \notin (\mathcal{E} - \{E'\})^{\cap}\}$.

*(c) For any set $\mathcal{E}$ of subsets of E, we define an $\cup$-**basis of** $\mathcal{E}$ as any set $\mathcal{E}'$ of subsets of E having the same $\cup$-closure as $\mathcal{E}$ and which has a cardinal minimal with this property. An $\cap$-**basis of** $\mathcal{E}$ and an $\cup\cap$-**basis of** $\mathcal{E}$ are defined similarly.*

2. *Subsets of $E = \mathbf{M}$ correspond to formulas. Let Ψ be a set of formulas in $\mathbf{L}$. Replacing "subset" by "formula", $\cup$ by $\wedge$ and $\cap$ by $\vee$, gives respectively:*
   (a) Definition 3.2.
   (b) The sets $\Phi_{\wedge} = \{\varphi \in \Phi / \varphi \notin (\Phi - \{\varphi\})^{\wedge}\}$ and $\Phi_{\vee} = \{\varphi \in \Phi / \varphi \notin (\Phi - \{\varphi\})^{\vee}\}$.
   *(c) The notions of $\wedge$-**basis**, $\vee$-**basis** of Ψ (unique from Proposition 5.1 below), and $\wedge\vee$-**basis** of a set Φ of formulas (generally not unique).* □

The following result is obvious, and probably well-known in finite set theory (see also [10, Definition 5.2 and Property 5.3]):

Proposition 5.1. *If E is finite, then any set $\mathcal{E}$ of subsets of E has exactly one $\cup$-basis, which is $\mathcal{E}_{\cup}$ and one $\cap$-basis, which is $\mathcal{E}_{\cap}$. Since $\mathbf{M}$ is finite, any set Ψ of formulas in $\mathbf{L}$ has exactly one $\wedge$-basis, which is $\Psi_{\wedge}$ and one $\vee$-basis, which is $\Psi_{\vee}$.* □

Notice that it can exist several $\cup\cap$-basis of $\mathcal{E}$ and that a $\cup\cap$-basis of $\mathcal{E}$ needs not be a subset of $\mathcal{E}$. We also need to complete Definition 2.4:

Definition 5.2. *1. Let $\mathcal{E}$ be a set of subsets of E and $\preceq$ be a pre-order in E. We define:*
   (a) For each $e \in E$, the set $\mathcal{E}_e$ of subsets of $\mathcal{E}$ by $\mathcal{E}_e = \{E_i \in \mathcal{E} \ / \ e \in E_i\}$.
   (b) Three relations on E: $e_1 \prec_{\mathcal{E}} e_2$ *if* $\mathcal{E}_{e_1} \subset \mathcal{E}_{e_2}$, $e_1 \preceq_{\mathcal{E}} e_2$ *if* $\mathcal{E}_{e_1} \subseteq \mathcal{E}_{e_2}$, *and* $e_1 \preceq\succeq_{\mathcal{E}} e_2$ *if* $\mathcal{E}_{e_1} = \mathcal{E}_{e_2}$.
   (c) The subsets $F_{\preceq} = \{s_{\preceq}(e) / e \in E\}$ and $F_{\mathcal{E}} = F_{\preceq_{\mathcal{E}}}$ of $\mathcal{P}(E)$.
   *(d) A subset E_1 of E is a **filter for** $\preceq$ if, for any $e_1 \in E_1, e \in E$ such that $e_1 \preceq e$, we have $e \in E_1$. We denote by $\mathcal{F}_{\preceq}$ the set of all the filters for $\preceq$ and by $\mathcal{F}_{\mathcal{E}}$ the set $\mathcal{F}_{\preceq_{\mathcal{E}}}$ of the **filters for** $\mathcal{E}$.*

2. *Let X be a set of formulas in $\mathbf{L}$ and $\preceq$ be a pre-order in $\mathbf{M}$. We define:*
   (a) (b) Definition 2.4 (a), and (b) respectively.
   (c) The sets of formulas $\Phi_{\preceq}$ (corresponding to $F_{\preceq}$) as in "step 1" in the proof of Theorem 4.2, and $\Phi_X = \Phi_{\preceq_X} = \{\varphi_X(\mu) \ / \ \mu \in \mathbf{M}\}$, where $\varphi_X(\mu) = \varphi_{\preceq_X}(\mu)$.
   (d) The sets of formulas $\mathbf{F}_{\preceq} = \{\varphi \in \mathbf{L} /$ if $\mu \in \mathbf{M}(\varphi)$, $\nu \in \mathbf{M}$ and $\mu \preceq \nu$, then $\nu \in \mathbf{M}(\varphi)\}$, and $\mathbf{F}_X = \mathbf{F}_{\preceq_X}$. □

Remark 5.1. The set $\mathcal{F}_{\preceq}$ of all the filters for $\preceq$ in E may be described as follows:
$\mathcal{F}_{\preceq} = \{s_{\preceq}(E_1) \ / \ E_1 \subseteq E\}$.
$\mathcal{F}_{\preceq}$ is the set of the sets of models of all the formulas in $\mathbf{F}_{\preceq}$.
"F" stands for 'forward", or "elementary filter", and "$\mathcal{F}$" for "filter".
$\preceq\succeq_{\mathcal{E}}$ is an equivalence relation on E.
By definition, $\perp \notin \Phi_{\preceq}$, thus $\perp \notin \Phi_X$.

We get $f_{\prec_X} = CIRCF(X)$ from Proposition 2.2.
Also, it is obvious that $\preceq_{\mathcal{E}}$ is associated with $\prec_{\mathcal{E}}$, and $\preceq_X$ with $\prec_X$. □

Lemma 5.1. 1. Let E be some finite set, and $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2$ be any set of subsets of E.

(a) $\preceq_{\mathcal{E}_1} = \preceq_{\mathcal{E}_2}$ iff $\mathcal{E}_1^{\cup\cap} = \mathcal{E}_2^{\cup\cap}$.
The set of all the filters for $\mathcal{E}$ is the closure of $\mathcal{E}$ for $\cup$ and $\cap$: $\mathcal{F}_{\mathcal{E}} = \mathcal{E}^{\cup\cap}$.

(b) The set $F_{\mathcal{E}}$ is the $\cup$-basis of the set $\mathcal{F}_{\mathcal{E}} = \mathcal{E}^{\cup\cap}$: $F_{\mathcal{E}} = (\mathcal{E}^{\cup\cap})_{\cup}$. This means that we have $\mathcal{F}_{\mathcal{E}} = (F_{\mathcal{E}})^{\cup}$ and that $F_{\mathcal{E}}$ is the smallest (for $\subset$, and a fortiori in terms of cardinality) of all the sets of subsets of E having this property.
Thus, the elements of $F_{\mathcal{E}}$ are $\cup$**-irreducible in** $\mathcal{F}_{\mathcal{E}}$: For any $e \in E$, it does not exist any non trivial union of elements of $\mathcal{F}_{\mathcal{E}}$ which is equal to $s_{\preceq_{\mathcal{E}}}(e)$ (non trivial means without element equal to the union).

(c) For any pre-order $\preceq$ on E, the set $F_{\preceq}$ is such that $\preceq_{F_{\preceq}} = \preceq$.
Thus (see 1a above), we get $\preceq_{\mathcal{F}_{\preceq}} = \preceq_{F_{\preceq}} = \preceq$ and (by definition), $\preceq_{\mathcal{F}_{\mathcal{E}}} = \preceq_{F_{\mathcal{E}}} = \preceq_{\mathcal{E}}$.

2. In terms of formulas, X and Y being sets of formulas in **L**, we get:

(a) $\preceq_X = \preceq_Y$ iff $X^{\wedge\vee} = Y^{\wedge\vee}$.
The set of all the filters for $\preceq_X$ is the closure of X for $\wedge$ and $\vee$: $\mathbf{F}_X = X^{\wedge\vee}$.

(b) The set Φ_X is the $\vee$-basis of the set $\mathbf{F}_X = X^{\wedge\vee}$. This means that we have $\mathbf{F}_X = (\Phi_X)^{\vee}$ and that Φ_X is the smallest (for $\subset$) of all the sets of formulas of **L** having this property.
The elements of $\Phi_{\preceq}$ are $\vee$**-irreducible in** $\mathbf{F}_{\preceq}$: For any $\varphi \in \Phi_{\preceq}$, it does not exist any non trivial (without any term equal to the result) disjunction of elements of $\mathbf{F}_{\preceq}$ which is equal to φ.

(c) For any pre-order $\preceq$ in **M** we get $\preceq_{\Phi_{\preceq}} = \preceq$, thus (see point 2a), we get $\preceq_{\mathbf{F}_{\preceq}} = \preceq_{\Phi_{\preceq}} = \preceq$. Thus (by definition), $\preceq_{\mathbf{F}_X} = \preceq_{\Phi_X} = \preceq_X$. □

These results are known in finite set theory and distributive lattice theory (see e.g. [8, 2, 4]), even if the precise references would need developments, not necessary here. Thus, we omit the proofs, which moreover have no difficulty, or have been given already. Point 2a is Theorem 3.3-2, given here again in order to make precise the correspondence with the "set versions" of point 1a. The left most equalities in point 2c come from point 2a, the right most equalities being step 2 in the proof of Theorem 4.2. Thus, only point 1b (or equivalently 2b) remains, and the proof is straightforward.

Here is an immediate consequence of this lemma:

Proposition 5.2. *The two operations $\preceq_{\mathcal{E}}$ and $\mathcal{F}_{\preceq}$ define two reciprocal one-to-one mappings between the set of all the subsets $\mathcal{E}$ of E which are closed for $\cup$ and $\cap$ and the set of all the pre-orders $\preceq$ defined on E.*

The set of all the subsets $\mathcal{E}$ of E which are closed for $\cup$ and $\cap$ is the set of all the distributive lattices (with $\top$ and $\bot$) which are sublattices of $\mathbf{P}(E)$ (the operations of the lattices being $\cap$ and $\cup$ for meet and join).

The two operations $\preceq_X$ and $\mathbf{F}_{\preceq}$ define two reciprocal one-to-one mappings between the set of all the sets X of formulas of **L** *which are closed for $\wedge$ and $\vee$ and the set of all the pre-orders $\preceq$ defined on* **M**.

The set of all the sets X of formulas of **L** *which are closed for $\wedge$ and $\vee$, is the set of all the distributive lattices (with $\top$ and $\bot$) which are sublattices of* **L**, *where the operations of the lattices are $\wedge$ (meet) and $\vee$ (join).* □

This result can also be extracted from the classical literature about distributive lattices (see e.g. [2, Theorem 3 and Corollary 1, p. 59] or more precisely [4, Theorem 8.19 and Subsection 8.20]). However, this literature uses only orders, and not pre-orders ([8] is a notable exception). The formulation given here makes precise the exact correspondence between having the same pre-order relation on **M** and being "strongly equivalent" with respect to formula circumscription in **L**. Let us give here another useful auxiliary result:

Definition 5.3. *A* **chain of strict entailment** *in a set X of formulas is a sequence $(\varphi_i)_{i\in\{0,1,2,\cdots,n\}}$ of elements of X such that $\varphi_i \models \varphi_{i+1}$ and $\varphi_{i+1} \not\models \varphi_i$ for $i \in \{0,1,2,\cdots,n-1\}$. The* **length of this chain** *is n, and the length $l(X)$ of X is the length of the longest chains of strict entailment in X.* □

Lemma 5.2. For any set of formulas X, $l(\mathbf{F}_X)$ is equal to $card(\Phi_X)$.

Notice that $card(\Phi_X)$ is the number of equivalence classes for $\preceq\succeq_X$ in **M**. Thus, $card(\Phi_X) \leq card(\mathbf{M}) = 2^{card(V(\mathbf{L}))}$. One possible easy proof of this lemma uses the fact that Φ_X constitutes an $\vee$-basis of $\mathbf{F}_X$. Also, it is a consequence of the representation theorem ([2, Theorem 3, p. 59], or e.g., [4, Theorem 8.17]) for finite distributive lattices ($\mathbf{F}_X$ is a distributive lattice). □

Here is a construction of an $\wedge\vee$-basis for any set X of formulas.

Definition 5.4. *1. If $\preceq$ is a pre-order in a finite set E, $Fir_{\preceq} = \{s_{\preceq}(e)/e \in E, s_{\preceq}(e) \notin (F_{\preceq} - \{s_{\preceq}(e)\})^{\cap}\}$ denotes the set of the elements of $F_{\preceq}$ which are $\cap$-***irreducible in** *$F_{\preceq}$. If $\mathcal{E} \subseteq \mathcal{P}(E)$, we define $Fir_{\mathcal{E}} = Fir_{\preceq_{\mathcal{E}}}$.*

*2. If $\preceq$ is a pre-order in **M**, $\Phi ir_{\preceq} = \{\varphi \in \Phi_{\preceq}/\varphi \notin (\Phi_{\preceq} - \{\varphi\})^{\wedge}\}$ denotes the set of the formulas of $\Phi_{\preceq}$ which are $\wedge$-***irreducible in** *$\Phi_{\preceq}$.*
If $X \subseteq \mathbf{L}$, we define $\Phi ir_X = \Phi ir_{\preceq_X}$. □

Notice that $\perp \notin \Phi ir_{\preceq}$ and $\top \notin \Phi ir_{\preceq}$. The following result is immediate:

Proposition 5.3. *1. For any set $\mathcal{E}$ of subsets of a finite set E, we have $\mathcal{E}^{\cup\cap} = Fir_{\mathcal{E}}^{\cup\cap}$. Thus, a set $\mathcal{B}$ of subsets of a finite set E is an $\cup\cap$-basis of $\mathcal{E}$ iff we have: $\mathcal{B} \subseteq \mathcal{F}_{\mathcal{E}}$, $Fir_{\mathcal{E}} \subseteq \mathcal{B}^{\cap}$ and $\mathcal{B}$ has the smallest possible cardinal.*

2. For any set of formulas X, $X^{\wedge\vee} = (\Phi ir_X)^{\wedge\vee}$. Thus, Y is an $\wedge\vee$-basis of X iff we have: $Y \subseteq \mathbf{F}_X$, $\Phi ir_X \subseteq Y^{\wedge}$ and Y has the smallest possible cardinal. □

This result provides a semi-constructive way for getting an $\wedge\vee$-basis of a given set X of formulas (i.e. one of the smallest sets strongly equivalent to X) by splitting this generally not trivial task in two steps. The first step is constructive and easy: getting the set Φir_X. The second step is generally not an immediate task, but it is much easier than the direct search for an $\wedge\vee$-basis because we need only to search for what could be called an **$\wedge$-basis with respect to $\mathbf{F}_X$ of a given set Φir_X**. In some particular cases, this step can be made easy and constructive (see below). This non trivial problem concerns in particular a famous result by Sperner (it is, e.g., "Sperner's theorem" in [1, Part 1.2] and "Sperner's lemma" in [2, p. 98-99]).

Definition 5.5. *Let E be some finite set with $card(E) = n$. An* **antichain** *in E is a set of subsets of E uncomparable for $\subseteq$. A* **weak antichain** *$\mathcal{E}$ in E is a set of subsets of E which is equal to its $\cup$-basis $\mathcal{E}_{\cup}$ (i.e., no set in $\mathcal{E}$ is the union of some other sets in $\mathcal{E}$). $A(n)$ and $WA(n)$ denote respectively the maximal number of elements in an antichain and in a weak antichain in E.* □

As any antichain is a weak antichain, we get $WA(n) \geq A(n)$. Sperner's theorem states that $A(n)$ is the central binomial coefficient $(n, \lfloor n/2 \rfloor)$. This means that the set of all the subsets of E having $\lfloor n/2 \rfloor$ elements is one of the largest possible antichains in E. This result gives, in some cases (case 2 in next proposition), a fully constructive way for getting an $\wedge\vee$-basis Y of a set X of formulas in **L**. It does not seem that the exact value of $WA(n)$, not given in [16], is known. It is easy to realize that we get $WA(n) \geq WA1(n) = \Sigma_{i=0}^{n-1}(i, \lfloor i/2 \rfloor)$. Kleitman [7] has shown that a better lower bound for $WA(n)$ is $WA2(n)$ defined by $(n, n/2) + (n, n/2-1)/n$ for even n and $2(n-1, (n-1)/2) + (n-1, (n-3)/2)/n$ for odd n. $WA2(n)$ is approached by $(1 + 3/(2n))A(n)$ for odd n and by $(1 + 1/(n+2))A(n)$ for even n [7]. For $n \leq 12$, we have $WA2(n) \leq WA1(n)$ (and maybe $WA(n) = WA1(n)$) but for any $n > 12$ we have $WA2(n) > WA1(n)$.

Let us give the proofs of the preceding assertions:

$WA(n) \geq WA1(n)$: Let us choose one element, 1, and all the subsets of $\{2, \cdots, n\}$ having $\lfloor (n-1)/2 \rfloor$ elements, completed by 1, this gives an antichain as great as possible made of elements containing 1. Then, do the same thing, starting from $\{2, \cdots, n\}$ instead of $\{1, \cdots, n\}$: take all the subsets of $\{3, \cdots, n\}$ having $\lfloor (n-2)/2 \rfloor$ elements, completed by 2. And so on till the last element n which gives rise to the single singleton $\{n\}$. We get a weak antichain of $\{1, \cdots, n\}$ having $WA1(n) = \Sigma_{i=0}^{n-1}(i, \lfloor i/2 \rfloor)$ elements.

$WA(n) \geq WA2(n)$ [7]: Let us assign to each subset of $\{1, \cdots, n\}$ the sum of its elements, modulo n. Then for even n, let us choose as our collection: all sets of size $n/2$ along with all sets of size $n/2 - 1$ which have the number i assigned to them, for any one value of i. This gives a bound of $(n, n/2) + (n, n/2-1)/n$ for even n. Obviously each set of cardinality $n/2$ in our collection can contain only one set of cardinality $n/2 - 1$ having value i. For odd n we use this construction for sets not containing the element n, replacing n by $n-1$, and we add all sets containing n which have cardinality $(n+1)/2$. Thus we get an antichain having $WA2(n)$ elements, where $WA2(n) = (n, n/2) + (n, n/2-1)/n$ for even n and $WA2(n) = 2(n-1, (n-1)/2) + (n-1, (n-3)/2)/n$ for odd n.

Comparisons between these numbers [7]: That $WA1(n) \geq WA(n)$ for each n can be seen by calculating these numbers. Now, we get that $WA1(n)$ is like $(n, n/2)(1 + 4/3n)$ for odd n, and $(n, n/2)(1 + 2/3n)$ for even n. Indeed, if we add a new element, we can take $(n-1, (n-1)/2)$ sets containing that element and $WA1(n-1)$ that don't, and we can satisfy the condition here. For even n, $(n-1, (n-1)/2)$ is exactly half of $(n, n/2)$ while for odd n it is $(n, n/2)(1/2 + 1/n)$. We can prove similarly (these results are also from Kleitman) that $WA2$ approaches $(n.[n/2]) * (1 + 3/(2n))$ for odd n, and $(n, [n/2]) * (1 + 1/(n+2))$ for even n. Thus, we get the following two ratios $(WA1 - A(n)) * n/A(n)$ which approaches $4/3$ for odd n and $2/3$ for even n, and $(WA2 - A(n)) * n/A(n)$ which is almost always 1.5 for odd n and which approaches 1 for even n. Thus, $(WA2(n) - A(n))/(WA1(n) - A(n))$ approaches $9/8$ for odd n and $3/2$ for even n, which shows precisely how $WA2(n)$ is better than $WA1(n)$, when compared to $A(n)$ (except for a few small values of n).

For what concerns us here, we get the following results:

Proposition 5.4. *Let X be some finite set of formulas in* **L**. *We are looking for an $\wedge\vee$-basis Y of X (i.e., some set Y with the minimal cardinality k such that $Y \equiv_{sc} X$). Let $\Phi ir_X = \{\varphi_1, \cdots, \varphi_n\}$ be the set introduced in Definition 5.4, with $n = card(\Phi ir_X)$.*

1. *In any case, $k = card(Y)$ must satisfy $WA(k) \geq n$.*
2. *If Φir_X is made of mutually exclusive formulas ($\varphi \models \neg\psi$ for any distinct φ, ψ), $card(Y)$ is the smallest k satisfying $A(k) \geq n$ and it exists a constructive way for finding Y (from any enumeration of the subsets of $\{1, 2, \cdots, k\}$ having $\lfloor k/2 \rfloor$ elements). This case is very particular (it means that $\preceq_X$ is an equivalence relation, thus $\prec_X$ is empty), but it may help to solve more general cases: Example 7.4 below shows that we can sometimes apply this method even when Φir_X is not made of exclusive formulas, and Proposition 7.1 shows how applying this method to subsets of Φir_X may help to find an $\wedge\vee$-basis Y of $\mathbf{F}_X$ (i.e. of X).*
3. *If Φir_X is made of a single chain of strict entailment, then Φir_X is a possible Y (thus, $card(Y) = card(\Phi ir_X)$).*

<u>Proof:</u> <u>1.</u> From Proposition 5.3, we know that we want a subset $Y = \{\varphi'_1, \cdots, \varphi'_k\}$ of $\mathbf{F}_X$ such that each $\varphi_i \in \Phi ir_X$ is a conjunction of elements of Y: there exists an injective mapping l from Φir_X to a set of subsets of $\{1, \cdots, k\}$ such that we have

$$\varphi_i = \bigwedge_{j \in l(\varphi_i)} \varphi'_j, \text{ where } l(\varphi_i) \subseteq \{1, \cdots, k\} \text{ for any } i \in \{1, \cdots, n\}. \tag{1}$$

Since the formulas in Φir_X are $\vee$-irreducible in $\mathbf{F}_X$, thus in Φir_X, $\{l(\varphi_i)\}_{i \in \{1, \cdots, n\}}$ must be a weak antichain in $\{1, \cdots, k\}$ and k satisfies $WA(k) \geq n$. We will see below (condition ($C : \supseteq \rightsquigarrow \models$) in Example 7.1) how to improve this lower bound for k.

<u>2.</u> Let l be an injective mapping from Φir_X to a set of subsets of $\{1, \cdots, k\}$ and $\{\varphi'_j\}_{j \in \{1, \cdots, k\}}$ be some set of formulas satisfying (1). As the formulas in Φir_X are mutually exclusive, we get $\varphi_i \not\models \varphi_j$ for any i, j distinct in $\{1, \cdots, n\}$: indeed, $\bot \notin \Phi ir_X$, thus $\varphi_i \neq \bot$ and $\varphi_i \models \neg\varphi_j$ implies $\varphi_i \not\models \varphi_j$. Thus the set $l(\Phi ir_X) = \{\{l(\varphi_i)\}_{i \in \{1, \cdots, n\}}$ must be an antichain in $\{1, \cdots, k\}$ and k must satisfy $A(k) \geq n$. For any k such that $A(k) \geq n$, it exists an injective mapping l from Φir_X to an antichain in $\{1, \cdots, k\}$. Let us choose one such l, and define the formulas φ'_j as follows:

$$\varphi'_j = \bigvee_{j \in l(\varphi_i)} \varphi_i, \text{ for any } j \in \{1, \cdots, k\}. \tag{2}$$

Clearly, $\varphi'_j \in (\Phi ir_X)^\vee \subseteq \mathbf{F}_X$. As the formulas φ_i are mutually exclusive, it is immediate to see that we get (1). This shows that the smallest k such that $A(k) \geq n$ is the smallest possible k in this case. Moreover, this gives a fully constructive way for finding a set $Y = \{\varphi'_j\}_{j \in \{1, \cdots, k\}}$.

<u>3.</u> Obvious. Notice also that this is a consequence of Proposition 5.5 given below.

□

Example 7.1 below shows that we can sometimes do a slightly better job with weak antichains than with antichains. Case 3 shows that in some cases, we can be much above the best possibility given in 1. Propositions 5.3 and 5.4 provide useful tricks which considerably help the search for an $\wedge\vee$-basis (see also the examples given in Section

7 below). Here is another (generally rather rough, but very precise in some important particular cases) indication about the size of an $\wedge\vee$-basis of X:

Proposition 5.5. *For any set of formulas X, the cardinal of an $\wedge\vee$-basis Y of X is such that we have: $l(\Phi_X) \leq card(Y) \leq card(\Phi ir_X)$.*

<u>Proof</u> From Proposition 5.3 we know that we have $card(Y) \leq card(\Phi ir_X)$: indeed, in the worst case, we can take Φir_X as our $\wedge\vee$-generator of X. $l(\Phi_X) \leq card(Y)$ is straightforward, we get even $l(\Phi_X) < card(Y)$ if $\top \notin \Phi_X$. □

6 The Smallest Sets *c*-Equivalent to a Given Set

We consider now a given $CIRCF(X)$, and we look for the smallest (in terms of cardinality) sets Y such that $CIRCF(Y) = CIRCF(X)$.

Lemma 6.1. Let $\mathcal{E}$ be a set of subsets of some finite set E, closed for $\cup$ and $\cap$ (i.e. such that $\mathcal{E} = \mathcal{F}_{\mathcal{E}}$), and let us consider the relations $\prec = \prec_{\mathcal{E}}$ and $\preceq = \preceq_{\mathcal{E}}$ (Definition 5.2). Let us call $\cup\cap$**-generator of** $\mathcal{E}$ any subset $\mathcal{G}$ of $\mathcal{E}$ such that $\mathcal{G}^{\cup\cap} = \mathcal{E}$. Let e_1 and e_2 be two elements in E not equivalent for $\preceq\succeq$ and satisfying $p_{\prec}(e_1) = p_{\prec}(e_2)$ and $s_{\prec}(e_1) = s_{\prec}(e_2)$ (Definition 4.1). Let us define the relation $\preceq'$ in E as follows: $e \preceq' e'$ if $e \preceq e'$ or ($e \preceq\succeq e_1$ and $e' \preceq\succeq e_2$). Then $\preceq'$ is a pre-order in E associated with $\prec$, and we define the set $\mathcal{E}' = \mathcal{F}_{\preceq'}$ of all the filters for $\preceq'$ (clearly $\mathcal{E}' \subseteq \mathcal{E}$).

1. Let $\mathcal{G}$ be a $\cup\cap$-generator of $\mathcal{E}$. Let us denote by $\mathcal{G}'$ the set of all the subsets G' of E defined as follows: For any $G \in \mathcal{G}$,
$$G' = \begin{cases} G \cup \{e \in E \mid e \preceq\succeq e_2\} & \text{if } \{e_1, e_2\} \cap G = \{e_1\} \\ G - \{e \in E \mid e \preceq\succeq e_2\} & \text{if } \{e_1, e_2\} \cap G = \{e_2\} \\ G & \text{otherwise, i.e. if } \{e_1, e_2\} \cap G \text{ is } \emptyset \text{ or } \{e_1, e_2\}. \end{cases}$$
Then $\mathcal{G}'$ is a $\cup\cap$-generator of $\mathcal{E}'$ and $card(\mathcal{G}') \leq card(\mathcal{G})$.
2. Any $\cup\cap$-basis of $\mathcal{E}'$ has at most as many elements as a $\cup\cap$-basis of $\mathcal{E}$.
Notice that if we start from $\mathcal{G} = F_{\mathcal{E}}$, which is a $\cup\cap$-generator of $\mathcal{E}$, we get as our set $\mathcal{G}'$ the set $\mathcal{G}' = \mathcal{G} - \{s_{\preceq}(e_1),\, s_{\preceq}(e_2)\} \cup \{(s_{\preceq}(e_1))', (s_{\preceq}(e_2))'\}$ where $(s_{\preceq}(e_1))' = s_{\preceq'}(e_1) = s_{\preceq'}(e_2) = s_{\preceq}(e_1) \cup s_{\preceq}(e_2)$ and $(s_{\preceq}(e_2))' = s_{\preceq}(e_1) \cap s_{\preceq}(e_2)$.

<u>Proof:</u> <u>Point 1:</u> If we define $\preceq_0$ and $\preceq_1$ from $\prec$ as in Definition 4.1, we know, from Lemma 4.1 (points 1 and 3) that $\preceq'$ is asoociated with $\prec$ and that we have, assimilating a relation to its graph: $\preceq_0 \subseteq \preceq \subseteq \preceq' \subseteq \preceq_1$. Let $\mathcal{G}$ and $\mathcal{G}'$ be as above, and let H be some element of $\mathcal{E}'$. Clearly $card(\mathcal{G}') \leq card(\mathcal{G})$. Let us define the set H'' as follows:
$$H'' = \begin{cases} H \cup \{e \mid e \preceq\succeq e_2\} & \text{if } s_{\prec}(e_1) \subseteq H \text{ and } \{e_1, e_2\} \cap H = \emptyset \\ H - \{e \mid e \preceq\succeq e_2\} & \text{if } \{e_1, e_2\} \subseteq H \\ H & \text{otherwise, i.e. if } s_{\prec}(e_1) \not\subseteq H. \end{cases}$$
It is immediate that if $e \in H''$, then $s_{\preceq}(e) \subseteq H''$. Thus, $H'' \in \mathcal{E}$. From $\mathcal{G}^{\cup\cap} = \mathcal{E}$, we get $H'' = \bigcup_{i=1}^{I} \bigcap_{j=1}^{J_i} G_{i,j}$ where each $G_{i,j}$ is in $\mathcal{G}$. Now, it is straightforward to check that we get $H = \bigcup_{i=1}^{I} \bigcap_{j=1}^{J_i} G'_{i,j}$, which establishes $H \in \mathcal{G}'^{\cup\cap}$.

<u>Point 2</u> The first sentence is an immediate consequence of point 1. The particular case where $\mathcal{G} = F_{\mathcal{E}}$ is a mere application of the definitions involved. □

Proposition 6.1. *(See Definition 4.1) If two pre-orders $\preceq$ and $\preceq'$ on E are associated with the same strict order $\prec$ and if the graph of $\preceq$ is included in the graph of $\preceq'$, then a $\cup\cap$-basis of $\mathcal{F}_{\preceq'}$ has at most as many elements as a $\cup\cap$-basis of $\mathcal{F}_{\preceq}$. In particular, any $\cup\cap$-basis of $\mathcal{F}_{\preceq_1}$ has at most as many elements as a $\cup\cap$-basis of $\mathcal{F}_{\preceq}$, which has itself at most as many elements as a $\cup\cap$-basis of $\mathcal{F}_{\preceq_0}$.*

Proof: From Lemma 4.1-3, there exists pre-orders $\preceq^i$ with $\preceq = \preceq^0$, $\preceq' = \preceq^p$ and, for any $i \in \{1, \cdots, p\}$, there exists two distinct e_1, e_2 in E such that $e_1 \preceq_1 \succeq_1 e_2$ and the graph of $\preceq^i$ is the graph of $\preceq^{i-1}$ plus $\{(e, e_2) \ / \ e \preceq\succeq e_1\}$. Then, Lemma 6.1 gives the general result, and $(\preceq, \preceq') = (\preceq_0, \preceq)$ or $(\preceq, \preceq_1)$ give the results for $\preceq_0$ and $\preceq_1$. $\square$

Theorem 6.2. *Let Ψ be a set of formulas and $\preceq_1$ and $\preceq_0$ be respectively the greatest and the smallest pre-orders associated with $\prec = \prec_\Psi$ (see Lemma 4.1).*

1. *Any $\wedge\vee$-basis Ψ_1 of the set of formulas $\Phi_{\preceq_1}$ (see Definition 5.2 and Proposition 5.3), is a set of formulas having the smallest possible number of elements such that $CIRCF(\Psi) = CIRCF(\Psi_1)$.*
2. *The set $\mathbf{F}_{\preceq_0}$ (see Definition 5.2) is the greatest (for $\subseteq$) set of formulas such that $CIRCF(\Psi) = CIRCF(\Psi')$.*
   The set of all the sets Ψ' which are c-equivalent to Ψ is the set of the sets Ψ' such that $\preceq_{\Psi'^{\wedge\vee}}$ (i.e. $\preceq_{\Psi'}$) is associated with $\prec$.

Proof: 1: Lemmas 4.1-1, 5.1-2a and Proposition 6.1.
2: Rewritted from Theorem 3.3: indeed, $\mathbf{F}_{\preceq_0} = Pos_e(\Psi) = I_{CIRCF(\Psi)}$. $\square$

Thus, the set of formulas $\mathbf{F}_{\Psi_1} = \mathbf{F}_{\preceq_1}$ is the "simplest" set Ψ' of formulas closed for $\wedge\vee$, such that $\Psi \equiv_c \Psi'$, for at least three respects:

1. It has the minimal cardinal possible such that $CIRCF(\Psi) = CIRCF(\Psi')$ (Lemma 4.1).
2. It has the smallest maximal chain of strict entailment (Lemma 5.2).
3. Any $\wedge\vee$-basis Ψ_1 of $\mathbf{F}_{\preceq_1}$ (i.e. of $\Phi_{\preceq_1}$) is a set with the smallest possible cardinal such that $\Psi \equiv_c \Psi_1$, and Propositions 5.3 and 5.4 give a way to find such an $\wedge\vee$-basis.

Here is a (possibly deceiving?) consequence of these results: Let us consider some circumscription $f = CIRCF(\Psi)$ in a finite language **L**. We define $\prec$ as the strict order relation $\prec_\Psi$, and $\preceq_0$ and $\preceq_1$ respectively as the smallest and the greatest transitive and reflexive relations associated with $\prec$ in the meaning of Definition 4.1. We have not only shown that $\mathbf{F}_{\preceq_1}$ is the smallest set Ψ'_1 closed for $\wedge\vee$ and giving rise to the same circumscription as Ψ (i.e. such that $CIRCF(\Psi) = CIRCF(\Psi'_1)$) but also that $\mathbf{F}_{\preceq_0}$ is the greatest set Ψ'_0 with these two properties. More precisely we have proved that any set Ψ' having these two properties is equal to $\mathbf{F}_{\preceq}$ for some transitive and reflexive relation associated with $\prec$, and that we have: $\Psi'_1 \subseteq \Psi' \subseteq \Psi'_0$. Remind that it is known (see [10, Theorem 3.2 and Property 5.6] or [11, Lemma 5.32 and Proposition 6.39]) that f can be expressed as an X-mapping defined by a set X iff $X^\wedge = \mathbf{F}_0$. Thus, in order to express a given (finite) formula circumscription as an X-mapping, we must use the greatest possible set Ψ'_0 satisfying $CIRCF(\Psi) = CIRCF(\Psi'_0)$. This could be an inconvenient when using the method of X-mappings in order to facilitate the computation of a given circumscription: the sad news is that we must consider the largest possible set (or any of its $\wedge$-basis, but this gives rise to the largest Ψ_0'' set of elements $\vee$-irreducible such that $CIRCF(\Psi) = CIRCF(\Psi_0'',))$. More investigations are needed in order to examine whether this fact has really bad consequences on this method or not.

7 A few examples

When looking for an $\wedge\vee$-basis Y of a set X of formulas as explained in Section 5, we may get a $k = card(Y)$ smaller than the one given by $A(k) \geq n$ (see Proposition 5.4):

Example 7.1. $V(\mathbf{L}) = \{A, B, C\}$, $X = \{A, B, A \wedge C, B \wedge C\}$. We get $\preceq_X$ (Definition 2.4) described as follows: $\mu \preceq_X \nu$ iff $\mu = \nu$ or $\{\mu, \nu\} = \{\emptyset, \{C\}\}$ or $\mu \prec_X \nu$ with $\mu \prec_X \nu$ iff $((\mu \neq \{C\}, \nu \neq \{C\}$ and $\mu \subset \nu)$ or $(\mu = \{C\}$ and $\nu \notin \{\emptyset, \{C\}\}))$.

We get then the following set $\Phi_X = \{\varphi(\mu)\}_{\mu \in \mathcal{P}(V(\mathbf{L}))-\{C\}}$ (Definition 5.2) with: $\varphi(\mu) = \bigwedge_{P \in \mu} P$.

We get then: $\varphi(\emptyset) = \top$ (also, $\varphi(\emptyset) = \varphi(\{C\})$),
$\varphi(\{A\}) = A,\ \ \varphi(\{B\}) = B,\ \ \varphi(\{A, B\}) = A \wedge B,$
$\varphi(\{A, C\}) = A \wedge C,\ \ \varphi(\{B, C\}) = B \wedge C,\ \ \varphi(\{A, B, C\}) = A \wedge B \wedge C.$

Thus, $\Phi ir_X = X$ (Definition 5.4). The smallest k such that $A(k) \geq 4 = card(\Phi ir_X)$ is $k = 4$, but we can choose $k = 3$ if we take a weak antichain and not an antichain: indeed $WA(3) = WA1(3) = (0, 0) + (1, 0) + (2, 1) = 1 + 1 + 2 = 4$, while $A(3) = (3, \lfloor 3/2 \rfloor) = (3, 1) = 3$.

Here is an injective mapping l from Φir_X to a weak antichain in $\{1, 2, 3\}$: $l(A) = \{1\}, l(B) = \{2\}, l(A \wedge C) = \{1, 3\}, l(B \wedge C) = \{2, 3\}$. As we want to get (1) page 13, l must satisfy, for any φ, ψ in Φir_X: $\varphi \models \psi$ if $l(\psi) \subseteq l(\varphi)$. Let us call $(C :\supseteq\rightsquigarrow\models)$ this condition. We define $Y = \{\varphi_i'\}_{i \in \{1,2,3\}}$, as in (2) page 13, getting $\varphi_1' = A, \varphi_2' = B, \varphi_3' = C \wedge (A \vee B)$. Even if l respects $(C :\supseteq\rightsquigarrow\models)$, defining φ_j' by (2) does not always imply that we get (1) page 13, as shown by example 7.2 below. However, here (1) is satisfied, thus $CIRCF(X) = CIRCF(Y)$. As $card(Y) = 3$, Y is one of the sets with the smallest cardinality satisfying $Y \equiv_{sc} X$ from Proposition 5.4. As $\preceq_X = \preceq_Y$ is the greatest relation $\preceq_1$ of Definition 4.1, we get that Y is even one of the smallest sets satisfying $Y \equiv_c X$ from Theorem 6.2-1. $\square$

Let us make here a parenthesis about condition $(C :\supseteq\rightsquigarrow\models)$ on l introduced in example 7.1. Notice that any antichain satisfies trivially condition $(C :\supseteq\rightsquigarrow\models)$. Thus, from Proposition 5.4-3, any example where Φir_X is made of a single chain (and $card(\Phi ir_X) \geq 4$ since $A(i) = i$ for $i \in \{1, 2, 3\}$) provides a counter-example showing that condition $(C :\supseteq\rightsquigarrow\models)$ is not sufficient in order to be sure that (2) page 13 implies (1) page 13. Here is a less trivial counter-example:

Example 7.2. $V(\mathbf{L}) = \{P_i\}_{i \in \{1, \cdots, 6\}}$, $X = \{P_i\}_{i \in \{1, \cdots, 6\}}$. Thus, we circumscribe all the propositional symbols (we need at least 5 symbols for this counter-example to work). We get $\Phi_X = \{\bigwedge_{i \in I} P_i\}_{I \subseteq \{1, \cdots, 6\}}$ and $\Phi ir_X = X$. Thus, $card(\Phi ir_X) = 6$ and $k = card(Y)$ where Y is an $\wedge\vee$-basis of X must be at least equal to 4, as $A(3) = 3 < 6 \leq A(4) = 6$. Notice that here we must consider $A(k)$ and not $WA(k)$

as in Proposition 5.4-1. Indeed, we have seen in Example 7.1 that l must always satisfy $(C\ :\supseteq\!\rightsquigarrow\models)$. As the elements of $\Phi ir_X = X$ are uncomparable for $\models$ (we never have $\varphi \models \psi$ if φ, ψ are two distinct elements in X), the set $l(\Phi ir_X) = \{l(\varphi)\}_{\varphi\in X}$ must be an antichain. Let us define l as follows: $l(P_1) = \{1,2\},\ l(P_2) = \{1,3\},\ l(P_3) = \{1,4\},\ l(P_4) = \{2,3\},\ l(P_5) = \{2,4\},\ l(P_6) = \{3,4\}$. Since $l(\Phi ir_X)$ is an antichain, $(C\ :\supseteq\!\rightsquigarrow\models)$ is trivially satisfied. We define $Y = \{\varphi'_i\}_{i\in\{1,\cdots,4\}}$ by (2) page 13. Then, we do not have (1) page 13: for instance, we get $\bigwedge_{j\in l(P_1)} \varphi'_j = \varphi'_1 \wedge \varphi'_2 = (\bigvee_{1\in l(P_i)} P_i) \wedge (\bigvee_{2\in l(P_i)} P_i) = (P_1 \vee P_2 \vee P_3) \wedge (P_1 \vee P_4 \vee P_5) = P_1 \vee ((P_2 \vee P_3) \wedge (P_4 \vee P_5))$ which is not equal to P_1. In fact, we cannot do better than with X here: X is an $\wedge\vee$-basis of X, by Proposition 5.5 (see Theorem 7.1 below). Thus, we get that $k = 6$ (and not 4) is the lowest possibile choice. Notice that Theorem 7.1 below shows that not only X is one of the smallest (in cardinality) sets Y such that $X \equiv_{sc} Y$, as shown here, but that it is also one of the smallest sets Y such that $X \equiv_c Y$. $\square$

Here ends our parenthesis about condition $(C\ :\supseteq\!\rightsquigarrow\models)$.

The next example shows how the results given in Sections 5 and 6 allow to find the various sets of formulas describing a given formula circumscription:

Example 7.3. $V(\mathbf{L}) = \{A, B, C\}$. We consider the set $X = \{A\vee B, C\vee(A\wedge B), \neg C\vee(\neg A\wedge\neg B), (A\vee B)\wedge\neg C, A\wedge B\wedge\neg C, A\wedge\neg B\wedge C, \neg A\wedge B\wedge C, A\wedge B\wedge C\}$ and $f = CIRCF(X)$. The relations $\preceq_X$ and $\prec_X$ defined on $\mathbf{M} = \mathcal{P}(\{A,B,C\})$ are then described as follows on $\mathbf{M} = \mathcal{P}(\{A,B,C\})$:

$X_\emptyset = \{\neg C\vee(\neg A\wedge\neg B)\}$, $X_{\{A\}} = X_{\{B\}} = \{A\vee B, \neg C\vee(\neg A\wedge\neg B), (A\vee B)\wedge\neg C\}$, $X_{\{C\}} = \{C\vee(A\wedge B), \neg C\vee(\neg A\wedge\neg B)\}$, $X_{\{A,B\}} = \{A\vee B, C\vee(A\wedge B), \neg C\vee(\neg A\wedge\neg B), (A\vee B)\wedge\neg C, A\wedge B\wedge\neg C\}$, $X_{\{A,C\}} = \{A\vee B, C\vee(A\wedge B), A\wedge\neg B\wedge C\}$, $X_{\{B,C\}} = \{A\vee B, C\vee(A\wedge B), \neg A\wedge B\wedge C\}$, $X_{\{A,B,C\}} = \{A\vee B, C\vee(A\wedge B), A\wedge B\wedge C\}$. Thus we get the following descriptions of $\prec_X$ and $\preceq_X$ in $\mathbf{M} = \{\emptyset, \{A\}, \{B\}, \{C\}, \{A,B\}, \{A,C\}, \{B,C\}, \{A,B,C\}\}$:

$\emptyset \prec_X \mu$ if $\mu \in \mathbf{M} - \{\emptyset\}$, $\{A\} \prec_X \{A,B\}$, $\{B\} \prec_X \{A,B\}$, $\{C\} \prec_X \{A,B\}$.

$\mu \preceq_X$ if $\mu \prec_X \nu$ or $\mu = \nu$ or $\{\mu,\nu\} = \{\{A\},\{B\}\}$.

Thus, $\Phi_X = \{\varphi(\mu)\}_{\mu\in\mathcal{P}(V(\mathbf{L}))-\{B\}}$ with

$\varphi_X(\emptyset) = \top,\ \ \varphi_X(\{A\})\ \ [= \varphi_X(\{B\})]\ \ = (A\vee B)\wedge\neg C,$

$\varphi_X(\{C\}) = (A\wedge B\wedge\neg C)\vee(\neg A\wedge\neg B\wedge C),\ \ \varphi_X(\{A,B\}) = A\wedge B\wedge\neg C,$

$\varphi_X(\{A,C\}) = A\wedge\neg B\wedge C,\ \ \varphi_X(\{B,C\}) = \neg A\wedge B\wedge C,\ \ \varphi_X(\{A,B,C\}) = A\wedge B\wedge C.$

As $\top = \bigwedge_{\varphi\in\emptyset}\varphi$ and as $A\wedge B\wedge\neg C = ((A\vee B)\wedge\neg C)\wedge((A\wedge B\wedge\neg C)\vee(\neg A\wedge\neg B\wedge C))$, we get the following set Φir_X (remember that $\top$ can never be in Φir_X):

$\Phi ir_X = \{\varphi_X(\{A\}), \varphi_X(\{C\}), \varphi_X(\{A,C\}), \varphi_X(\{B,C\}), \varphi_X(\{A,B,C\})\}.$

Even if we are not in case 2 of Proposition 5.4, we are "close enough" to this case and we can apply the constructive method described there in order to find an $\wedge\vee$-basis for $\mathbf{F}_X = (\Phi_X)^\vee$. We are looking for a subset Y of $\mathbf{F}_X$, minimal such that $\Phi ir_X \subseteq Y^\wedge$. As Φir_X has five elements, We must use four elements in $\mathbf{F}_X$: $WA(3) = 4 < 5 = card(\Phi ir_X) \leq 7 = WA(4)$. As we have also $A(3) = 3 < 5 \leq 6 = A(4)$, we can start from an antichain without bothering with weak antichains (it is simpler to use antichains because condition $(C\ :\supseteq\!\rightsquigarrow\models)$ is always satisfied). Let us

choose the following injective mapping l from Φir_X into an antichain made of the subsets of $\{1,2,3,4\}$: $l(\varphi_X(\{A\})) = \{1,2\}$, $l(\varphi_X(\{C\})) = \{1,3\}$, $l(\varphi_X(\{A,C\})) = \{1,4\}$, $l(\varphi_X(\{B,C\})) = \{2,3\}$, $l(\varphi_X(\{A,B,C\})) = \{2,4\}$, which gives, by (2) page 13, $Y_a = \{\varphi'_1, \varphi'_2, \varphi'_3, \varphi'_4\}$ with
$\varphi'_1 = \varphi_X(\{A\}) \vee \varphi_X(\{C\}) \vee \varphi_X(\{A,C\}) = (A \vee B \vee C) \wedge (\neg B \vee \neg C)$,
$\varphi'_2 = \varphi_X(\{A\}) \vee \varphi_X(\{B,C\}) \vee \varphi_X(\{A,B,C\}) = (A \wedge \neg C) \vee B$,
$\varphi'_3 = \varphi_X(\{C\}) \vee \varphi_X(\{B,C\}) = (A \wedge B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C) \vee (\neg A \wedge B \wedge C)$,
$\varphi'_4 = \varphi_X(\{A,C\}) \vee \varphi_X(\{A,B,C\}) = A \wedge C$.

We get a set Y_a of four formulas with $Y_a \equiv_{sc} X$ (thus $f = f_{\prec_X} = CIRCF(Y_a)$) and Proposition 5.4 gives that no set Y'_a with fewer elements exist with $Y'_a \equiv_{sc} X$ (which again can be checked directly in this simple example).

Let us consider now the greatest pre-order $\preceq_1$ associated with $\prec_X$ (Definition 4.1). We get $\mu \preceq_1 \nu$ iff $\mu \prec_X \nu$ or $\mu = \nu$ or $\{\mu,\nu\} \subseteq \{\{A\},\{B\},\{C\}\}$ or $\{\mu,\nu\} \subseteq \{\{A,C\},\{B,C\},\{A,B,C\}\}$. We get $\Phi_{\preceq_1} = \{\varphi_{\preceq_1}(\mu)\}_{\mu \in \{(\emptyset),\{A\},\{A,B\},\{A,C\}\}}$ where $\varphi_{\preceq_1}(\emptyset) = \top$, $\varphi_{\preceq_1}(\{A\}) = [= \varphi_{\preceq_1}(\{B\}) = \varphi_{\preceq_1}(\{C\}) =]((A \vee B) \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C)$, $\varphi_{\preceq_1}(\{A,B\}) = A \wedge B \wedge \neg C$, $\varphi_{\preceq_1}(\{A,C\}) = (A \vee B) \wedge C$. The set $\Phi ir_{\preceq_1}$ is then $Y_b = \Phi_{\preceq_1} - \{\varphi_{\preceq_1}(\emptyset)\}$, with three elements in it. Y_b is not an antichain ($\varphi_{\preceq_1}(\{A,B\}) \models \varphi_{\preceq_1}(\{A\})$), but here this does not make a difference because this set has three elements, and, since $WA(2) = 2 < 3 = card(Y_b) \leq WA(3) = 4$, Propositions 5.3 and 5.4 state that we need three elements in any subset X' of $\mathbf{F}_{\preceq_1} = (\Phi_{\preceq_1})^\vee$ such that $Y_b \subseteq (X')^\wedge$. As clearly taking the three elements of Y_b gives a solution, we cannot do better: Y_b is one of the sets of formulas X' with as few elements as possible such that $f = f_{\prec_X} = CIRCF(X')$. This example shows that choosing the relation $\preceq_1$ instead of $\preceq_X$ (thus a fortiori of $\preceq_0$) allows to get a smaller set of formulas. However, we have only $X' \equiv_c X$, and we have $X' \not\equiv_{sc} X$.

Finally, let us use now the smallest pre-order $\preceq_0$ associated with $\prec$, defined by $\mu \preceq_0 \nu$ if $\mu \prec_X \nu$ or $\mu = \nu$. We get the corresponding set $\Phi_{\preceq_0} = \{\varphi_{\preceq_0}(\mu) \;/\; \mu \in \mathbf{M}\}$ where: $\varphi_{\preceq_0}(\emptyset) = \top$, $\varphi_{\preceq_0}(\{A\}) = A \wedge \neg C$, $\varphi_{\preceq_0}(\{B\}) = B \wedge \neg C$,
$\varphi_{\preceq_0}(\{C\}) = (A \Leftrightarrow B) \wedge (A \Leftrightarrow \neg C)$, $\varphi_{\preceq_0}(\{A,B\}) = A \wedge B \wedge \neg C$,
$\varphi_{\preceq_0}(\{A,C\}) = A \wedge \neg B \wedge C$, $\varphi_{\preceq_0}(\{B,C\}) = \neg A \wedge B \wedge C$,
$\varphi_{\preceq_0}(\{A,B,C\}) = A \wedge B \wedge C$.

The set $\Phi ir_{\preceq_0}$ is then $\Phi ir_{\preceq_0} = \Phi_{\preceq_0} - \{\varphi_{\preceq_0}(\emptyset), \varphi_{\preceq_0}(\{A,B\})\}$ (indeed, $A \wedge B \wedge \neg C = (A \wedge \neg C) \wedge (B \wedge \neg C)$) with six elements in it. $\Phi ir_{\preceq_0}$ is an antichain thus we know that we need four elements in any subset Y_c of $\mathbf{F}_{\preceq_0} = (\Phi_{\preceq_0})^\vee$ such that $\Phi ir_{\preceq_0} \subseteq (Y_c)^\vee$. Any enumeration of a the set of the subsets of $\{1,2,3,4\}$ with two elements, indexed by $\Phi ir_{\preceq_0}$ (or equivalently by $\mathbf{M} - \{\emptyset, \{A,B\}\}$) will provide a solution:
$\{A\} \rightsquigarrow \{1,2\}, \{B\} \rightsquigarrow \{1,3\}, \{C\} \rightsquigarrow \{1,4\}, \{A,C\} \rightsquigarrow \{2,3\}, \{B,C\} \rightsquigarrow \{2,4\}$,
$\{A,B,C\} \rightsquigarrow \{3,4\}$, gives rise to the set $Y_c = \{\psi_1, \psi_2, \psi_3, \psi_4\}$ with
$\psi_1 = \varphi_{\preceq_0}(\{A\}) \vee \varphi_{\preceq_0}(\{B\}) \vee \varphi_{\preceq_0}(\{C\}) = ((A \vee B) \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C)$,
$\psi_2 = \varphi_{\preceq_0}(\{A\}) \vee \varphi_{\preceq_0}(\{A,C\}) \vee \varphi_{\preceq_0}(\{B,C\}) = (A \wedge \neg C) \vee ((A \Leftrightarrow \neg B) \wedge C)$,
$\psi_3 = \varphi_{\preceq_0}(\{B\}) \vee \varphi_{\preceq_0}(\{A,C\}) \vee \varphi_{\preceq_0}(\{A,B,C\}) = (B \wedge \neg C) \vee (A \wedge C)$,
$\psi_4 = \varphi_{\preceq_0}(\{C\}) \vee \varphi_{\preceq_0}(\{B,C\}) \vee \varphi_{\preceq_0}(\{A,B,C\}) = (A \wedge B) \vee (\neg A \wedge C)$.

We get another set Y_c of four formulas such that $f = f_{\prec_X} = CIRCF(Y_c)$, which here is not greater than when starting from $\preceq_X$. Remind that the set $\mathbf{F}_{\preceq_0} = (\Phi_{\preceq_0})^\vee =$

$(Y_c)^{\wedge\vee}$ is the set of all the formulas which are inaccessible for f (Definition 3.3), thus that we may describe f as an X-mapping by $f = F_{\mathbf{F}_{\preceq_0}}$ or by $f = F_{\Phi_{\preceq_0}}$ and that the set $\Phi_{\preceq_0}$ is the smallest (for $\subseteq$) set having this property.

From Lemma 4.1-3, we get here $25 = B_3 \times B_3$ pre-orders $\preceq$ associated with $\prec$. □

As a last example, let us apply our results to the case of ordinary circumscription We start from some circumscription $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$. We know that we get $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\mathbf{P}; \mathbf{Q}, \mathbf{Z} \cup \mathbf{P}) = CIRCF(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})$. We want to find one of the smallest possible (in terms of cardinality) sets of formulas Ψ where $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Psi; \mathbf{Q}, \mathbf{Z} \cup \mathbf{P})$ and we ask the same question for $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Psi)$. Remember that a syntactical description of the greatest (for $\subset$, and a fortiori in terms of cardinality) possible sets is already known [11, 12].

For the first problem, we cannot do better than the set **P** (which moreover is made of the simplest possible formulas):

Theorem 7.1. *1.* **P** *is one of the sets* Ψ *with fewer elements such that we have* $CIRC(\mathbf{P}, \emptyset, \mathbf{Z}) = CIRCF(\Psi; \emptyset, \mathbf{Z} \cup \mathbf{P}) = CIRCF(\Psi)$.
2. **P** *is one of the sets* Ψ *with fewer elements such that we have* $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Psi; \mathbf{Q}, \mathbf{Z} \cup \mathbf{P})$ *(and it is the simplest such set).*

Sketch of the proof: 1. It is straightforward (not immediate, but automatic) to see that we have, if $X = \mathbf{P}$ is a set of atomic formulas: $l(\Phi_X) = card(\Phi ir_X) = card(X)$. This is an important case where Proposition 5.5 is very precise. Moreover, if we define $\prec = \prec_{(\mathbf{P}, \emptyset, \mathbf{Z})}$ (Definition 2.2) and $\preceq_1$ as in Definition 4.1, we get $\Phi_{\preceq_1} = \{\bigwedge_{P \in \mathbf{P}'} P\}_{\mathbf{P}' \subseteq \mathbf{P}}$ (and $\mathbf{F}_{\preceq_1} = \mathbf{P}^{\wedge\vee}$). Then use Theorem 6.2.

2. Immediate extension of point 1. □

The second problem is tougher and partially left to any interested reader:

Proposition 7.1. *There exists a set* Ψ *such that we have* $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Psi)$ *satisfying* $card(\Psi) = card(\mathbf{P}) + k_{\mathbf{Q}}$ *where* $k_{\mathbf{Q}}$ *is the smallest integer* k *such that the central binomial coefficient* $A(k) = (k, \lfloor k/2 \rfloor)$ *satisfies* $A(k) \geq 2^{card(\mathbf{Q})}$.

Proof: Let **P**, **Q**, **Z** be a partition of $V(\mathbf{L})$ and Φ be the set of formulas $\Phi = \mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}$. We know that we have $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Phi)$. The set of the formulas Φir_Φ contains a subset composed by formulas involving only the symbols of **Q**. Let us call $\Phi ir(\mathbf{Q})$ this set. This set is the set of the $2^{card(\mathbf{Q})}$ conjunctions of literals involving all the elements of **Q**. Thus, this set is made of mutually exclusive formulas. The subset $\mathbf{F}(\mathbf{Q})$ of the formulas of $\mathbf{F}_\Phi$ composed by formulas involving only the symbols of **Q** is obviously the set $(\mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee}$ of all the formulas made in this vocabulary. We use the method of Proposition 5.4 (case 2) for this subset $\Phi ir(\mathbf{Q})$ of Φir_Φ. We choose a set I with $k_{\mathbf{Q}}$ elements and an injective mapping l from $\Phi ir(\mathbf{Q})$ to the set of subsets of I having $\lfloor k/2 \rfloor$ elements. As with (2) page 13, we define the set $\mathbf{B}(\mathbf{Q}) = \{\varphi'(i)\}_{i \in I}$ where $\varphi'(i) = \bigvee_{i \in l(\varphi), \varphi \in \Phi ir(\mathbf{Q})} \varphi$. Then, we get (1) page 13, thus $\mathbf{B}(\mathbf{Q})^{\wedge\vee} = \mathbf{F}(\mathbf{Q})$. As in the proof of case 2 in Proposition 5.4, we cannot do better: $\mathbf{B}(\mathbf{Q})$ is an $\wedge\vee$-basis of $\mathbf{F}(\mathbf{Q})$. Thus, the set $\Psi = \mathbf{P} \cup \mathbf{B}(\mathbf{Q})$ is such that $\Psi^{\wedge\vee} = (\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee}$ and we get $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Psi)$. □

This proof can easily be extended to prove that this set Ψ is minimal (in cardinality) among the sets Ψ which are unions of subsets of $\mathbf{P}^{\wedge\vee}$ and of subsets of $(\mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee}$ and such that we have $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Psi)$. However, this does not prove that Ψ is minimal without this condition: we could imagine some tricky subset of $(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee}$ with fewer elements. We tend to think that this choice is optimal:

Conjecture 7.2. *Any set* Ψ *such that* $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Psi)$ *has at least* $card(\mathbf{P}) + k_{\mathbf{Q}}$ *elements.* □

Example 7.4. $\mathbf{P} = \{P\}, \{\mathbf{Q}\} = \{Q_1, Q_2, Q_3\}, \mathbf{Z} = \emptyset$.

We get $card(\Phi ir(\mathbf{Q})) = 2^3 = 8$, $card(\mathbf{Q} \cup \neg\mathbf{Q}) = 2 \times 3 = 6$ and $A(5) = (5, 2) = 10 \geq 8$. Thus, the set $\mathbf{B}(\mathbf{Q})$ has 5 elements, which is better than $\mathbf{Q} \cup \neg\mathbf{Q}$. The difference is not negligible, as when $q = card(\mathbf{Q})$ tends to infinity, $\mathbf{Q} \cup \neg\mathbf{Q}$ has $2 \times q$ elements while $k_{\mathbf{Q}} = card(\mathbf{B}(\mathbf{Q}))$ is approximated by $q + \ln(q)$. Here is a possible mapping l, from $\Phi ir(\mathbf{Q})$ to a set of subsets of $\{1, \cdots, 5\}$ with two elements:
$l(Q_1 \wedge Q_2 \wedge Q_3) = \{1, 2\}$, $l(Q_1 \wedge Q_2 \wedge \neg Q_3) = \{1, 3\}$, $l(Q_1 \wedge \neg Q_2 \wedge Q_3) = \{1, 4\}$, $l(Q_1 \wedge \neg Q_2 \wedge \neg Q_3) = \{1, 5\}$, $l(\neg Q_1 \wedge Q_2 \wedge Q_3) = \{2, 3\}$, $l(\neg Q_1 \wedge Q_2 \wedge \neg Q_3) = \{2, 4\}$, $l(\neg Q_1 \wedge \neg Q_2 \wedge Q_3) = \{2, 5\}$, $l(\neg Q_1 \wedge \neg Q_2 \wedge \neg Q_3) = \{3, 4\}$.

We get then $\mathbf{B}(\mathbf{Q}) = \{\varphi'(i)\}_{i \in \{1, \cdots, 5\}}$ with:

$$\begin{aligned}
\varphi'(1) &= l^{-1}(\{1,2\}) \vee l^{-1}(\{1,3\}) \vee l^{-1}(\{1,4\}) \vee l^{-1}(\{1,5\}) \\
&= Q_1, \\
\varphi'(2) &= l^{-1}(\{1,2\}) \vee l^{-1}(\{2,3\}) \vee l^{-1}(\{2,4\}) \vee l^{-1}(\{2,5\}) \\
&= (\neg Q_1 \wedge (Q_2 \vee Q_3)) \vee (Q_2 \wedge Q_3), \\
\varphi'(3) &= l^{-1}(\{1,3\}) \vee l^{-1}(\{2,3\}) \vee l^{-1}(\{3,4\}) \\
&= (Q_1 \wedge Q_2 \wedge \neg Q_3) \vee (\neg Q_1 \wedge Q_2 \wedge Q_3) \vee (\neg Q_1 \wedge \neg Q_2 \wedge \neg Q_3), \\
\varphi'(4) &= l^{-1}(\{1,4\}) \vee l^{-1}(\{2,4\}) \vee l^{-1}(\{3,4\}) \\
&= (Q_1 \wedge \neg Q_2 \wedge Q_3) \vee (\neg Q_1 \wedge \neg Q_3), \\
\varphi'(5) &= l^{-1}(\{1,5\}) \vee l^{-1}(\{2,5\}) \\
&= \neg Q_2 \wedge \neg(Q_1 \Leftrightarrow Q_3).
\end{aligned}$$

Thus, we get as our set Ψ (which in this simple case can be proved to have as few elements as possible) the set $\Psi = \{P\} \cup \mathbf{B}(\mathbf{Q})$ with $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Psi)$. □

Even if we gain a significative number of formulas with respect to the obvious set $\Psi = \mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}$ when $\mathbf{Q}$ is big enough, the formulas involved are more complicated. However, it is interesting to know that the set $\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}$ is not optimal in cardinality, when we allow to replace the set $\mathbf{Q} \cup \neg\mathbf{Q}$ by another subset of $(\mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee}$. Conjecture 7.2 (not absolutely certain...) states that even if we allow to "break" the set $(\mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee}$, i.e. to forget the information that $\mathbf{Q}$ must be fixed, we cannot do better.

8 Conclusion and Perspective

We have described all the sets of formulas Φ which, when circumscribed, give rise to the same result as a given set Ψ. Also, we have described all the sets Φ which, when completed by any arbitrary set Φ', give rise to the same result as a given set Ψ, when completed by Φ'. Our description is syntactical and very simple in the second case ("strong equivalence"). In the first case ("ordinary equivalence"), we have described a method to get all the possible sets. The method is fully constructive if we consider only sets of formulas which are closed for $\wedge$ and $\vee$ (or for $\wedge$ alone, or $\vee$, thanks to the constructive definitions of the $\wedge$-basis and $\vee$-basis). In particular, we have described the unique greatest and the unique smallest (for set inclusion) such sets which are equivalent to a given set. Also, we have described the greatest (unique, it is the same one as the preceding greatest set) set of formulas which is equivalent to a given set, without other condition. The problem of finding the smallest sets (in terms of cardinality, there is no longer uniqueness here) involves the search for one of the smallest sets having the same closure for $\wedge$ and $\vee$ than a given set of formulas. We have described a semi constructive method for finding these sets in all the cases. The method is fully constructive in two particular but instructive cases, which help finding the solution for more general cases. One of these cases is when we start from an ordinary circumscription $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ where the propositional symbols of **P** are circumscribed and those of **Q** fixed, the remaining ones, in **Z**, varying. In this case, we have proved that the natural and well known set of formulas $\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}$ is one of the a smallest possible sets of formulas Ψ which keeps unchanged the set $\mathbf{Q} \cup \neg\mathbf{Q}$ of the fixed literals. More surprisingly, we have shown that we can do better if we allow modifications inside the set of the fixed propositions, and conjectured that our proposition is still the best one, if we “forget the fixed propositions” altogether.

As future work, we should extend these results to the infinite propositional case, then to the predicate case.

Let us add a few words about the importance of such a study. Firstly, this is one of the most fundamental questions to ask: when sets of formulas are equivalent for what we do with them. Secondly, this could help the automatic computation, as we could choose the “easiest” equivalent sets in order to make the computation of a given circumscription. Clearly. a lot of work remains in that direction. Thirdly, this could help the modelization by circumscriptions of complex situations. The idea is to associate with each rule a set of formulas to be circumscribed. Then, in order to combine rules, we would combine the sets. For defining such combinations, it is important to know precisely what are these “sets” and the notions of equivalence give the answers.

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