INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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N° 4007

Octobre 29, 2000

THÈME 1

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4007--FR+ENG

On Cyclic Orders and Synchronisation Graphs

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Thème 1 — Réseaux et systèmes
Action MCR

Rapport de recherche n° 4007 — Octobre 29, 2000 — 26 pages

Abstract: We use a dedicated quotient operation called *winding* to give a new characterization of orientable cyclic orders; the characterization is twofold, in terms of partial orders and of separation properties. For this, we show the application to Petri nets: by winding causal nets, one obtains cyclic orders on synchronization graph that yield live and safe markings and describe the concurrent behavior of the system under these markings. Moreover, we characterize admissible action refinements on such nets by quotients of the corresponding (cyclic or acyclic) order structure.

Key-words: Petri nets, quotients, concurrency, partial orders, cyclic orders.

* Supported by the European TMR project ALAPEDES, Contract Reference ERB-FMRX-CT-96-0074.

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Sur les Ordres Cycliques et les Graphes de Synchronisation

Résumé : Nous utilisons une opération de quotient spécialisée, l'*enroulement*, pour obtenir une nouvelle caractérization des ordres cycliques orientables; la caractérization est double, par rapport aux ordres partiels ainsi qu'aux proprietés de séparation. Nous en démontrons également l'application aux Réseaux de Pétri: par l'enroulement de réseaux causaux, on obtient des ordres cycliques sur des graphes de synchronization, donnant des marquages vivants et sûrs et décrivant le comportement parallèle du système sous ces marquages.

En outre, nous caractérisons, à l'aide des structures d'ordre (cyclique ou acyclique), les opérations de raffinements d'actions admissibles sur de tels réseaux.

Mots-clés : Réseaux de Pétri, quotients, concurrence, ordres partiels, ordres cycliques

1 Introduction

Unlike acyclic (partial) orders, cyclic orders are not binary relations; rather, they are modeled as sets of *triplets* (x, y, z) that satisfy certain closure and consistency axioms to be discussed below. In the simplest case, a cyclic order consists of just one cycle of elements; we will call this a total cyclic order. In the general case, the question arises whether it is possible to find a total order consistent with all ternary cyclic arrangements; one would expect a cyclic anologon of Szpilrajn's Theorem [Szp30], which states that every acyclic partial order can be extended to a total order, to hold. However, as Megiddo [Meg76] and others showed, there exist cyclic orders without a total extension; in addition, Megiddo [Meg76] showed that the problem of *orientability* is NP-hard. Subsequently, characterizations of orientable cyclic orders have been given by by several authors, in particular Alles, Nešetřil, Poljak [ANP91], Chajda and Novák [ChN83], Genrich [Gen71], Jakubík [Jak94], Quilliot [Qui89], [Qui91] and Stehr [Ste98].

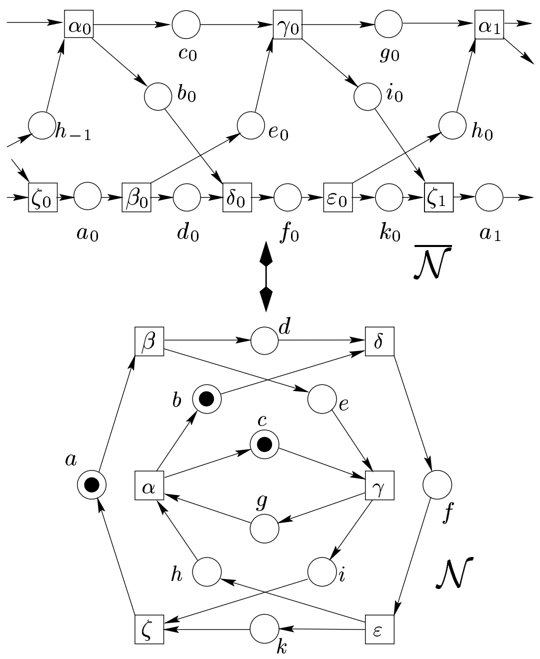

Figure 1: Winding and unwinding

All of the above use combinatorial or geometric arguments; Genrich and Stehr linked cyclic orders to the dynamics of concurrent systems and *Concurrency Theory* (cf. [KS97]). We generalize their results using a quotient approach, **windings**, that allows to generate a cyclic order from a partial order and leads to a twofold characterization of orientability for cyclic orders of arbitrary size. This approach differs from those in the literature; note that generating cyclic orders by rotational closure as in Alles, Nešetřil, Poljak [ANP91], does not reflect cyclic concurrent processes (see discussion below).

To give an idea, consider Figure 1. On the life and safe synchronization graph $\mathcal{N}$, there is a cyclic order describing cyclically repeated sequences of occurrences such as $(\beta, \delta, \varepsilon)$. These relations can be retrieved from the partial order on $\overline{\mathcal{N}}$, which in its turn represents the concurrent behavior of $\mathcal{N}$ as a two-side infinite non-sequential process. The net quotient that generates $\mathcal{N}$ from $\overline{\mathcal{N}}$ is a *winding* (for the definition see below). We will see that the main result on orientability, Theorem 3.14, is a proper generalization of Genrich and Lautenbach's [GL73] Theorem which states that a synchronization graph is live and 1-safe iff (i) all cycles contain at least one token and (ii) the net is covered by cycles with exactly one token. In fact, Theorem 3.14 gives an order theoretic proof of this, allowing at the same time to be applied to any cyclic orders with or without a connection to Petri Nets.

So, the use of windings permits a new characterization of orientability in terms of partial orders and of separation properties. Moreover, windings are coherent with the semantics of acyclic and cyclic concurrent systems modeled by causal nets or synchronization graphs, respectively. The quotient approach allows, at the same time, to characterize admissible action refinements in causal nets and synchronization graphs.

The paper is organized as follows: Section 2 introduces relational structures, cyclic orders and the orientability problem. Section 3 is devoted to the proof of the main result on orientable cyclic orders, Theorem 3.14; the section discusses quotients of relational structures, windings and density properties essential for the result. Section 4 investigates the connections with Petri Nets; we show how Theorem 3.14 specializes to Petri nets, i.e. synchronization Graphs, and that it constitutes an order theoretic generalization of Genrich and Lautenbach's Theorem. Moreover, we investigate action refinements (more precisely, of the dual operation that we call *coarsening*) in causal nets and synchronization graphs and give order theoretic characterizations of the admissible coarsenings. The interaction between windings and coarsenings will turn out to be particularly smooth in the case of *co-faithful* windings, i.e. those that preserve and respect the *cuts*; we prove a sufficient condition for faithfulness and show some possible effects of non-faithfulness.

2 Cyclic Orders

2.1 Relations

A relational structure consists of a set $\mathcal{X}$ and one or more binary or n-ary relations over $\mathcal{X}$; for instance, relational structures can arise as graphs and hyper-graphs, Petri nets, ordered structures, and so forth. Our emphasis is on partial and cyclic orders and their quotients.

Let $\mathcal{X}$ be a non-empty set $n \geqslant 2$. For $\mathcal{M} \subseteq \mathcal{X}$, we write $\complement\mathcal{M} := \mathcal{X} - \mathcal{S}$, and $\mathcal{P}(\mathcal{X}) := \{\mathcal{M} : \mathcal{M} \subseteq \mathcal{X}\}$. A non-empty subset $R \subseteq \Pi_{i=1}^{n}\mathcal{X}_i$ is an **n-ary relation**; the important cases here will be $n = 2$ and $n = 3$. R is **over** $\mathcal{X}$ iff $\mathcal{X}_i = \mathcal{X}$ for all $1 \leqslant i \leqslant n$. Throughout this paper, we will write $R(x_1, \dots, x_n)$ to express that $(x_1, \dots, x_n) \in R$. R is **simple** iff $R(x_1, \dots, x_n)$ implies $x_i \neq x_j$ for $1 \leqslant i < j \leqslant n$. The **simple kernel** $simp(R)$ of R is the union of the simple n-ary relations contained in R (note that simplicity is preserved under unions). Let $r \subseteq \mathcal{X} \times \mathcal{Y}$ and R an n-ary relation over $\mathcal{X}$; then the **lifting** $(R)^r$ of R w.r.t. r is the n-ary relation over $\mathcal{Y}$ given by

$$R^r \quad := \quad \{(y_1, \dots, y_n) : \; \exists\, x_1, \dots, x_n : \; R(x_1, \dots, x_n) \wedge \forall i \in \{1, \dots, m\} \; r(x_i, y_i)\}$$

For a non-empty $\mathcal{Y} \subseteq \mathcal{X}$ and R an n-ary relation over $\mathcal{X}$, denote by $R_{\mathcal{Y}} := R \cap \mathcal{Y}^n$ the **restriction** of R to $\mathcal{Y}$. If $\mathcal{R}$ is a set of relations over $\mathcal{X}$, we call $\Theta = (\mathcal{X}, \mathcal{R})$ a **relational structure (RS)**. If $\Theta_1 = (\mathcal{X}_1, \mathcal{R}_1)$ and $\Theta_2 = (\mathcal{X}_2, \mathcal{R}_2)$ are two RS such that $\mathcal{X}_1 \subseteq \mathcal{X}_2$ and

$\mathcal{R}_1 = \{R_2|_{\mathcal{X}_1} : R_2 \in \mathcal{R}_2\}$, Θ_1 is called a **substructure** of Θ_2 and Θ_2 a **superstructure** of Θ_1. If $\Theta_1 = (\mathcal{X}, \mathcal{R}_1)$ and $\Theta_2 = (\mathcal{X}, \mathcal{R}_2)$ are two RS over the same set $\mathcal{X}$ such that there exists a bijection $f : \mathcal{R}_1 \to \mathcal{R}_2$ satisfying $R_1 \subseteq f(R_1)\ \forall R_1 \in \mathcal{R}_1$, Θ_2 is an **embedding** for Θ_1. Note that embeddings are, in general, **not** superstructures.

Now, let R be a *binary* relation over $\mathcal{X}$. The **image** of x under R is $R[x] := \{y : R(x,y)\}$; denote as $R^T := \{(y,x) : R(x,y)\}$ the **inverse** or **transpose** of R. For $R \subseteq \mathcal{X} \times \mathcal{Y}$ and $R' \subseteq \mathcal{Y} \times \mathcal{Z}$, the **concatenation** of R and R' is $R \circ R' := \{(x,z) : \exists\, y : R(x,y) \wedge R'(y,z)\}$ (note that $R \circ R' \subseteq \mathcal{X} \times \mathcal{Z}$). R is called **(binary) transitive** iff $R \circ R \subseteq R$. With $R \subseteq \mathcal{X} \times \mathcal{X}$ and $R^0 := id_{\mathcal{X}}$, $R^{n+1} := R^n \circ R$, the transitive closure of R is $R^+ := \bigcup_{i \in \mathbb{N}} R^n$; further, $R^* := id_{\mathcal{X}} \cup R^+$. Denote by $id_{\mathcal{X}} := \{(x,x) : x \in \mathcal{X}\}$ the identity relation of $\mathcal{X}$. R is called **reflexive** iff $id_{\mathcal{X}} \subseteq R$ and **symmetric** iff $R^T = R$. If R is transitive, symmetric and reflexive, it is called an **equivalence**. $m \subseteq \mathcal{X}$ is an R-**clique** iff for all $x, y \in m$ such that $x \neq y$, $R(x,y)$ holds. A maximal R-clique is a **ken** of R. We denote the class of R-cliques by $\mathcal{CLI}(R)$ and that of R-kens by $\mathcal{KEN}(R)$.

2.2 Acyclic Partial orders

Partial orders are among the most important cases of relational structures. A binary relation R over a non-empty set $\mathcal{X}$ is a **partial order** iff R is transitive, simple and (therefore) asymmetric. If, in addition, $\mathcal{X}^2 = R \cup R^T \cup id_{\mathcal{X}}$, R is a **total order** over $\mathcal{X}$. Let $\Gamma = (\mathcal{X}, <)$ be a partial order; we write $<(x,y)$ or $x < y$ for $(x,y) \in <$, and set $li := < \cup <^T$ and $co := \mathcal{X}^2 - (id_{\mathcal{X}} \cup li)$. A ken of $li^{(2)}$ is called a **line**, and the set of lines in Γ is denoted by $\mathcal{LINES}(\Gamma)$; the elements of $\mathcal{CUTS}(\Gamma) := \mathcal{KEN}(co_\Gamma)$ are the **cuts** of Γ.

The **intervals** between x and y are $]x,y[:= \{z : x < z < y\}$, $[x,y[:=]x,y[\cup \{x\}$, $]x,y] :=]x,y[\cup \{y\}$, and $[x,y] :=]x,y[\cup \{x,y\}$. An **edge** of Γ is a li-clique $\mathcal{E}$ such that there exist $a, b \in \mathcal{X}$ satisfying $li(a,b)$ and $\mathcal{E} \subseteq [a,b]$, and $\mathcal{E}$ is maximal relative $[a,b]$, i.e. for any $u \in [a,b]$ such that $\forall\, v \in \mathcal{E} : li(v,u)$ one has $u \in \mathcal{E}$. Set $\mathfrak{sta}(\mathcal{E}) := a$, $\mathfrak{end}(\mathcal{E}) := b$. Note that, if $\mathcal{E}$ is an edge, $\mathfrak{sta}(\mathcal{E}) \in \mathcal{E}$ and $\mathfrak{end}(\mathcal{E}) \in \mathcal{E}$ since one has $li(\mathfrak{sta}(\mathcal{E}), \mathfrak{end}(\mathcal{E}))$ and, for all $v \in \mathcal{E} - \{\mathfrak{sta}(\mathcal{E}), \mathfrak{end}(\mathcal{E})\}$, $li(\mathfrak{sta}(\mathcal{E}), v)$ and $li(\mathfrak{end}(\mathcal{E}), v)$. Moreover, every edge $\mathcal{E}$ can be represented as the intersection of an appropriate line $l_{\mathcal{E}}$ with $[\mathfrak{sta}(\mathcal{E}), \mathfrak{end}(\mathcal{E})]$. Γ is **weakly discrete** iff for all $l \in \mathcal{LINES}(\mathcal{N})$ and all x, y, $|[x,y] \cap l| \in \mathbb{N}$, i.e. iff all edges are finite.

2.3 Cyclic orders and Orientability

After the above preliminaries, we start with a definition of cyclic orders that is self-contained, i.e. does not in itself depend on partial orders or quotients. After that, it will be shown that quotients of partial orders yield special cases of cyclic orders, in fact, just the *orientable* ones; this class can be considered the most interesting one since it has a physical meaning and, as we will see, contains the relational models of live and 1-safe synchronization graphs as special cases.

Definition 2.1 (Cyclic Orders) *Let $\prec$ be a ternary relation over the set $\mathcal{X}$. $\Phi = (\mathcal{X}, \prec)$ is a* **cyclic order** (CyO) *iff it satisfies*

1. **inversion asymmetry**: $\prec (x,y,z) \Rightarrow \neg \prec (y,x,z)$,
2. **rotational symmetry**: $\prec (x,y,z) \Rightarrow \prec (z,x,y)$,
3. **ternary transitivity**: $\prec (a,b,c) \wedge \prec (a,c,d) \Rightarrow \prec (a,b,d)$.

Lemma 2.2 *A* CyO *is* **simple**: $\prec (x,y,z) \Rightarrow x \neq y,\ y \neq z$, *and* $z \neq x$.

> **Proof:** Suppose $\prec$ is not simple. Then rotational symmetry permits to assume without loss of generality that $\prec (x,x,z)$; but then, inversion asymmetry is violated. $\square$

Definition 2.3 *Let* $\Phi = (\mathcal{X}, \prec)$ *be a* CyO *. Set* $li := \{(x,y) :\ \exists\, z : \prec (x,y,z) \vee \prec (x,z,y)\}$ *and* $co := \mathcal{X}^2 - (id_{\mathcal{X}} \cup li)$, *and denote the kens of* li *as* **lines** *and those of* co *as* **cuts**.

Note that all lines have at least three elements. Because of their inversion asymmetry, CyOs actually discern the two opposite senses of rotations for cycles. As is intuitively clear, binary relations do not have that property; on the level of binary relations, *rotational* symmetry cannot be distinguished from *full* symmetry. In fact, the following shows how the passage to binary relations destroys the sense of rotation:

Lemma 2.4 *Let* $R^{(3)} \neq \emptyset$ *be a simple, inversion asymmetric and rotationally symmetric ternary relation over* $\mathcal{X}$. *Set* $R^{(2)} = \{(x,y) :\ \ \exists\, z \in \mathcal{X} : R^{(3)}(x,y,z)\}$, *and*

$$\tilde{R}^{(3)} \quad := \quad \{(a,b,c) :\ R^{(2)}(a,b) \vee R^{(2)}(a,c) \vee R^{(2)}(b,c)\}$$

Then for any $R^{(3)}(a,b,c)$, *one has* $(b,a,c), (b,c,a), (a,c,b), (c,b,a), (c,a,b) \in \tilde{R}^{(3)}$.

> **Proof:** Let $R^{(3)}(a,b,c)$; then $R^{(3)}(b,c,a)$, hence $R^{(2)}(b,a)$, and therefore $\tilde{R}^{(3)}(b,a,c)$. The rest follows from rotational symmetry. $\square$

We will see that CyOs may in fact serve to reflect cyclic concurrent behavior, i. e. that one *observes* a CyO in every life and safe SG, the model of cyclic behavior we study here. But does, on the other hand, every LOCyO also permit a physical interpretation, i.e. is there a sense of rotation corresponding to periodic behavior ? The answer is no in general; the characterization of this class of cyclic orders is given in Section 3. It is related to a completely different question: Does a counterpart to Szpilrajn's Theorem [Szp30] hold for cyclic orders, i.e. can every cyclic order be embedded into a total cyclic order ? Surprisingly, the answer is *negative* in general; additional properties are needed to ensure totalizability.

Definition 2.5 *A* CyO $\Phi_{tot} = (\mathcal{X}, \prec_{tot})$ *such that for all pairwise distinct* $a,b,c \in \mathcal{X}$, *either* $\prec (a,b,c)$ *or* $\prec (b,a,c)$, *is called* **total**.

Thus a CyO is total iff it satisfies $co = \emptyset$. As for terminology, if Φ_1 and Φ_2 are CyOs such that Φ_1 is a substructure of Φ_2, then Φ_1 is a SubCyO of Φ_2 and Φ_2 a SuperCyO of Φ_1. Let $\Phi = (\mathcal{X}, \prec)$ be a CyO. If there exists a total CyO Φ_{tot} on $\mathcal{X}$ such that Φ_{tot} **embeds** Φ, that is, $\prec \subseteq \prec_{tot}$, then Φ is called **totalizable** and Φ_{tot} a **totalization** of Φ. The existence of a totalization for Φ is equivalent to Φ having a graphical representation by *clock cycles*, i.e. as a collection of directed loops in the two-dimensional plane such that the origin is avoided and such that all loops run clockwise around the origin[1]; totalizable CyOs are therefore also called **globally oriented**.

Now, the CyO on the system of Figure 1 is already given in clock cycles, but not the structure shown in Fig. 2; in fact, it can be shown to have no such representation at all (Stehr [Ste98], Genrich [Gen71]; for a different example of a non-orientable CyO,cf. Megiddo [Meg76]). Such CyOs, called **non-orientable**, can not even have a totalizable SuperCyO since any CyO is totalizable iff all its SubCyOs are. In particular, orientability of a CyO Φ on a *net* $\mathcal{N}$ is equivalent to Φ representing the behavior of a live and safe SG. Therefore, structures like those in Figure 2 have no physical interpretation in terms of periodic processes.

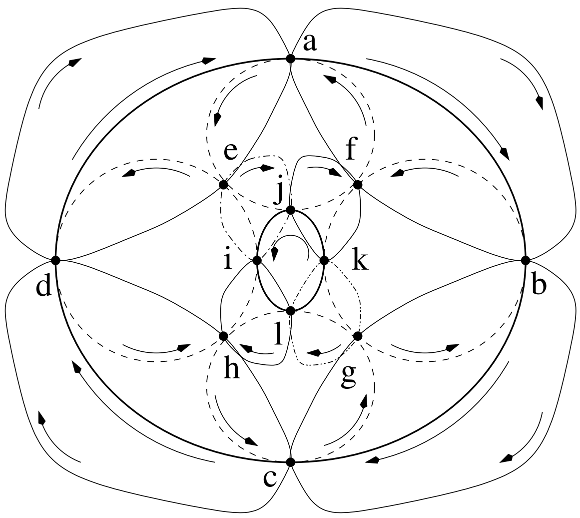

Figure 2: A non-totalizable CyO

The characterizations of orientability proved by Jakubík [Jak94], Chajda and Novák [ChN83] Quilliot [Qui89], [Qui91], differ from the one given here in two respects: no use is made of cuts, and the connection to partial orders, if any, is not in terms of the dynamics of concurrent systems. The approach of Stehr [Ste98] does not use windings, yet proves the validity of the separation criterion below in the case of finite CyOs ; our results here are more general and come with a more general framework that encompasses abstractions.

Before presenting our approach, we note first that a necessary condition for global orientability is the orientation of every li-triplet:

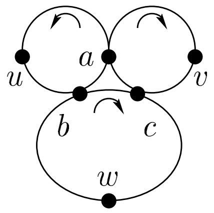

Figure 3:

Definition 2.6 *A* CyO $\Phi = (\mathcal{X}, \prec)$ *is said to be* li-**oriented** *iff, for any* $\{a, b, c\} \in \mathcal{CLI}(li)$*, either* $\prec(a, b, c)$ *or* $\prec(b, a, c)$.

This condition is stronger than the requirements in Quilliot [Qui91], [Qui89], Jakubík [Jak94] etc.; cf. Stehr [Ste98]. The CyO in Figure 3 is not li-oriented (consider $\{a, b, c\}$); by contrast, the one in Figure 2 *is* li-oriented, thus showing that li-orientation is insufficient for global orientation. We will henceforth call a li-oriented CyO a LOCyO and speak of SuperLOCyO etc. accordingly.

[1] cf. the *arc orders* in Alles, Nešetřil, Poljak [ANP91].

3 Characterization of orientable CyOs

This section is devoted to the proof of Theorem 3.14 which gives a two-fold characterization of orientable CyOs: as those LOCyOs that are representable by windings, and as those that can be saturated into a superstructure satisfying a separation property (*weak Q-density*). Subsection 3.1 introduces quotients of relational structures with some basic properties. The following subsection 3.2 defines windings of posets to cyclic orders; *co* and *li* will be seen to correspond to quotients of their acyclic counterparts. Subsection 3.3 studies separation or *density* properties of acyclic and cyclic orders; in 3.4, the characterization of orientability is proved.

3.1 Quotients

Let $\mathcal{X}$, $\overline{\mathcal{X}} \neq \emptyset$, and let $\overline{R}$ be an n-ary relation over $\overline{\mathcal{X}}$ and $\rho : \overline{\mathcal{X}} \to \mathcal{X}$. If R is a binary relation over $\mathcal{X}$, ρ is called R-$\overline{R}$-**preserving** or a (R-$\overline{R}$)-**homomorphism** iff $\overline{R}^{\rho} \subseteq R$, and R-$\overline{R}$-**respecting** iff $R^{\rho^T} \subseteq \overline{R}$. A surjective homomorphism is an **epimorphism**; a bijective homomorphism ρ such that ρ^{-1} is a homomorphism is an **isomorphism**; an isomorphism such that $\overline{\mathcal{X}} = \mathcal{X}$, ρ is an **automorphism**. Note that automorphisms are special congruences in the sense of [SS93]. Let $\overline{R}_i$, $i \in \mathcal{I}$, be n-ary relations over $\overline{\mathcal{X}}$, R an n-ary relation over $\mathcal{X}$ and $\rho : \overline{\mathcal{X}} \to \mathcal{X}$; set $\overline{R} := \bigcup_{i \in \mathcal{I}} \overline{R}_i$ and $\underline{R} := \bigcap_{i \in \mathcal{I}} \overline{R}_i$. If ρ is R-$\overline{R}_i$-respecting (R-$\overline{R}_i$-preserving) for all i, it is R-$\overline{R}$-respecting (R-$\underline{R}$-preserving). We are now ready to define:

Definition 3.1 *Let* $\rho : \overline{\mathcal{X}} \to \mathcal{X}$. *The* **weak lower quotient** $\nabla^w_\rho(\overline{R})$ *of* $\overline{R}$ *under* ρ *is* $\nabla^w_\rho(\overline{R}) := \overline{R}^{\rho}$*; the* **lower quotient** *of* $\overline{R}$ *under* ρ *is* $\nabla_\rho(\overline{R}) := simp(\nabla^w_\rho(\overline{R}))$. *The* **weak upper quotient** $\triangle^w_\rho(\overline{R})$ *of* $\overline{R}$ *under* ρ *is the union of all relations* R *over* $\mathcal{X}$ *such that* ρ *is* R-$\overline{R}$-*respecting; the* **upper quotient** *of* $\overline{R}$ *under* ρ *is* $\triangle_\rho(\overline{R}) := simp(\triangle^w_\rho(\overline{R}))$.

Equivalently, one can define

$$\begin{aligned} \nabla^w_\rho(\overline{R}) &:= \{(x_1, \dots, x_n) : \exists\, \overline{x}_i \in \rho^{-1}(\{x_i\}), i \in \{1, \dots, n\} : \overline{R}(\overline{x}_1, \dots, \overline{x}_n)\} \\ \triangle^w_\rho(\overline{R}) &:= \{(x_1, \dots, x_n) : \forall\, \overline{x}_i \in \rho^{-1}(\{x_i\}), i \in \{1, \dots, n\} : \overline{R}(\overline{x}_1, \dots, \overline{x}_n)\} \end{aligned}$$

Obviously, $\triangle_\rho(\overline{R}) \subseteq \nabla_\rho(\overline{R})$. By the above, ρ is both $\triangle_\rho(\overline{R}) - \overline{R}$- preserving and $\nabla_\rho(\overline{R}) - \overline{R}$-respecting. – To the best of our knowledge, the notions of 'upper' and 'lower' quotients were first used by Ehrenfeucht and Rozenberg [ER90] in different settings. Compare also the upper and lower inverses for *correspondences* (e.g. [Cha99]).

Example 3.1 (see Figure 4) *Take* $\overline{\mathcal{X}} = \{a, b, c, d, e, f, g\}$, $\mathcal{X} = \{u, v, w, x\}$, ρ *defined by* $\rho(c) = \rho(d) = u$, $\rho(a) = \rho(b) = w$, $\rho(e) = v$, $\rho(f) = \rho(g) = x$; *let* $R = \widetilde{R} \cup \widetilde{R}^T$, *where* $\widetilde{R} = \{(a, c), (b, c), (b, d), (b, e), (a, f), (a, g), (b, f), (b, g)\}$. *Then* $\triangle_\rho(\overline{R}) = \{(w, x), (x, w)\}$ *and* $\nabla_\rho(\overline{R}) = \{(u, w), (w, u), (v, w), (w, v), (w, x), (x, w)\}$.

Quotients are defined as respecting relations; do they respect *cliques* as well ? In general, the answer is no.

Definition 3.2 (Faithfulness) *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be sets, R_1 a binary relation over $\mathcal{X}_1$, $\rho : \mathcal{X}_1 \to \mathcal{X}_2$ surjective, $R_2 \in \{\triangle_\rho(R_1), \nabla_\rho(R_1)\}$. The sets $\mathfrak{Rc}(\rho, R_2) := \{\rho(m) : m \in \mathcal{CLI}(R_1)\}$ and $\mathfrak{Sc}(\rho, R_2) := \mathcal{CLI}(R_2) - \mathfrak{Rc}(\rho, R_2)$ contain the* **regular** *and* **strange cliques** *of R_2, respectively. The maximal elements of $\mathfrak{Sc}(\rho)$ are the* **strange kens** *of ρ. If $\mathfrak{Sc}(\rho) = \emptyset$, ρ and R_2 are called* **faithful**.

Note first that no strange clique can be a subset of a regular clique, and every strange clique has at least three elements. Since ρ maps R_1-cliques to $\nabla_\rho(R_1)$-cliques, we have:

Lemma 3.3 *Let $\mathcal{X}_1$, $\mathcal{X}_2$, ρ as in Definition 3.2; then $\mathfrak{Rc}(\rho) \subseteq \mathcal{CLI}(\nabla_\rho(R_1))$.*

The converse of Lemma 3.3 is **not** true in general, cf. Figure 5 (a):

Example 3.2 *Let $\mathcal{X}_1 = \{a, b, c, d\}$, $\mathcal{X}_2 = \{u, v, w\}$, $R = \{(a, b), (b, a), (b, c), (c, b), (c, d), (d, c)\}$, $\rho(a) = \rho(d) = u$, $\rho(b) = v$, $\rho(c) = w$; cf. Figure 5 (a). Then $\kappa := \{u, v, w\}$ is a strange clique of $\nabla^\rho(R)$. The example also shows that '$\mathcal{CLI}$' in Lemma 3.3 can not be replaced by '$\mathcal{KEN}$': $k' := \{a, b\}$ is a ken of R but $\rho(k') = \{u, v\}$ is not a ken of $\nabla^\rho(R)$.*

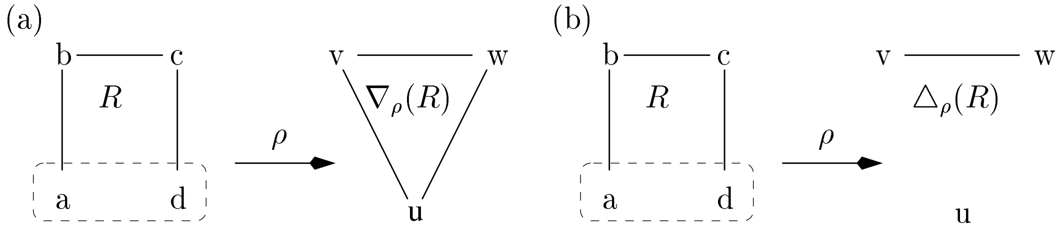

Figure 5: Examples 3.2 (a) and 3.3 (b)

On the other hand, $\mathcal{CLI}(\triangle_\rho(R_1)) \subseteq \mathcal{CLI}^\rho(R_1)$, so upper quotients are faithful; but the inclusion is proper in general, as the following shows:

Example 3.3 *Let $\mathcal{X}_1$, $\mathcal{X}_2$, ρ and R be as in Example 3.2. Then $k := \{a, b\}$ is a clique (even a ken) of R, but $\rho(k) = \{u, v\}$ is not a clique of $\triangle^\rho(R)$ (cf. Figure 5 (b)).*

So, non-faithfulness is possible only for lower quotients. Moreover, a strange clique can exist if there is a '*triple jump*', i.e. a path of three R-edges, leading from one ρ-equivalence class back into it; this is, of course, not sufficient for non-faithfulness. For every $n \geqslant 2$, there are quotient structures with a minimal strange clique of $(n+1)$ elements. However, if additional structure can be exploited, *effective* criteria for faithfulness can be obtained, viz. the *triple jump criterion* of Theorem 4.17.

3.2 Windings and unwindings of posets

We return to acyclic partial orders. The causal net $\overline{\mathcal{N}}$ in Figure 1. and its associated poset are *periodic*, i.e. display translational symmetries; so we define:

Definition 3.4 *Let $\Gamma = (\mathcal{X}, <)$ be a poset. An automorphism $\tau : \mathcal{X} \to \mathcal{X}$ of Γ is an* **order translation** *of Γ if either $<(x, \tau(x))$ for all $x \in \mathcal{X}$* (**forward**) *or $<(\tau(x), x)$ for all $x \in \mathcal{X}$* (**backward order translation**).

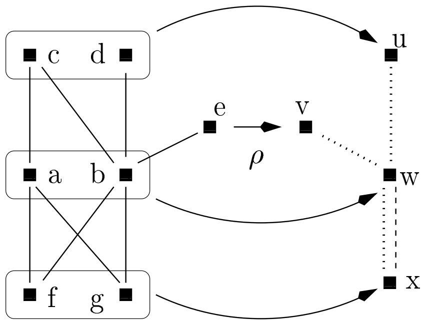

Figure 4: Lower quotient (dotted lines) and upper quotient (dashed lines)

The group of order translations thus consists of symmetries that *shift* the entire poset *along* the directions of its order relation. Obviously, f is a forward order translation iff its inverse is a backward order translation and vice versa. Let $\overline{\Gamma} = (\overline{\mathcal{X}}, <)$ be a poset with a forward order translation τ, and $\mathcal{G}$ the group generated by τ acting on $\overline{\Gamma}$; $\mathcal{G}$ is isomorphic to $(\mathbb{Z}, +)$. Write $\overline{x} \equiv_\tau \overline{y}$ iff there exists $k \in \mathbb{Z}$ such that $\tau^k(\overline{x}) = \overline{y}$; then $\equiv_\tau$ is an equivalence relation on $\overline{\mathcal{X}}$. The equivalence class $[x]_{\equiv_\tau}$ of x is the $\mathcal{G}$-orbit of x. Let $\mathcal{X} := \overline{\mathcal{X}}_{/\equiv_\tau}$ and $\phi_\tau : \overline{\mathcal{X}} \to \mathcal{X}$, $\overline{x} \mapsto [x]_{\equiv_\tau}$ the associated quotient map. We will see that ϕ_τ induces both upper and lower quotients; the most important quotient relation is the following, which belongs to neither class and even entails a change of arity:

Definition 3.5 *Let $\overline{\Gamma} = (\overline{\mathcal{X}}, <)$ a poset with forward order translation τ, and let $\mathcal{X} := \overline{\mathcal{X}}_{\equiv_\tau}$. The* **cyclic order quotient (COQ)** $\prec = \prec_<$ *of $<$ on $\mathcal{X}$ is given by:*

$$\begin{aligned} \prec(x, y, z) \quad :\Leftrightarrow \quad & \exists\, \overline{x} \in \phi_\tau^{-1}(x),\ \overline{y} \in \phi_\tau^{-1}(y),\ \overline{z} \in \phi_\tau^{-1}(z) : \\ & (\overline{x} < \overline{y} < \overline{z} < \tau(\overline{x}_1)). \end{aligned}$$

For an example cf. Figure 1. There, $\prec(\alpha, c, \gamma)$ holds since $\alpha_0 < c_0 < \gamma_0 < \alpha_1$. Note that the method in Alles, Nešetřil, Poljak [ANP91] of generating a CyO from a poset by simply taking the rotational closure is not at all equivalent to windings. Obviously, no abstraction takes place in the rotational closure process, so the cyclic order has as many elements as its generating poset. But even the restriction to one section of the wound poset does not yield an isomorphic cyclic orders: in Figure 1, consider only the elements with index 0. Then the cyclic order generated by rotational closure contains the triplet $(\alpha_0, \gamma_0, i_0)$, but (α, γ, i) does not belong to the cyclic order quotient since $co(i_0, \alpha_1)$.

If Φ is generated according to Definition 3.5, we say that $\overline{\Gamma}$ is **wound** to Φ and call ϕ_τ the associated **winding**. We note:

Lemma 3.6 *In the situation of Definition 3.5, $co = \nabla_\phi(co_{\overline{\Gamma}})$ and $li = \triangle_\phi(li_{\overline{\Gamma}})$.*

One finds that all COQs are LOCyOs. So the question arises which property characterizes those LOCyOs that have a representation as a COQ. Before preparing Theorem 3.14 which will give the answer, we note the following inheritance:

Lemma 3.7 *Let Φ_1 be a* LOCyO *and Φ_2 a* SubCyO *of Φ_1. If Φ_1 is a* COQ *then so is Φ_2.*

Proof: An unwinding for Φ_2 is obtained by restricting an unwinding for Φ_1. □

3.3 Density Properties

We now study the third – and last – aspect of LOCyOs dealt with in Theorem 3.14 below, that of density or separation properties.

3.3.1 K-Density Of Partial Orders

A cut c can be viewed as a *global* state of the set of local processes that are represented by lines. The intersection of c with a line l then yields the local state that l is in when the 'snapshot' c is taken. From the semantic point of view, one is therefore interested to know whether l and c *do* intersect in the first place; more generally, density properties provide deeper insight into the order structure.

Definition 3.8 *Let $\Gamma = (\mathcal{X}, <)$ be a partial order. Then we say that $c \in \mathcal{CUTS}(\Gamma)$ is* **K-separating**[2] *iff for all $l \in \mathcal{LINES}(\Gamma)$, $c \cap l \neq \emptyset$. Γ is* **weakly K-dense** *iff it contains a K-separating cut, and* **(strongly) K-dense** *iff every cut of Γ is K-separating.*

Density properties of causal nets have been extensively studied; the most comprehensive account is given in [BF88]. We are interested in the following property:

Theorem 3.9 *Let $\mathcal{X}$ be a non-empty set and $\Gamma = (\mathcal{X}, <)$ a partial order. Then there exist a* SuperPO *$\overline{\Gamma}$ of Γ such that $\overline{\Gamma}$ is weakly K-dense.*

Proof: Consider a totalization $\Gamma_{tot} = (\mathcal{X}, <_{tot})$ of Γ and fix $x \in \mathcal{X}$. For every line $l \in \mathcal{LINES}(\Gamma)$ such that $x \notin l$, add a new element x_l. Then, for all $l \in \mathcal{LINES}(\Gamma)$ and $u \in \mathcal{X}$, add the relational atoms $<(u, x_l)$ iff $<_{tot} (u, x_l)$ and $<(x_l, u)$ iff $<_{tot} (x_l, u)$. One verifies that $\overline{\Gamma}$ thus obtained on $\overline{\mathcal{X}} := \mathcal{X} \cup \{x_l : l \in \mathcal{LINES}(\Gamma)\}$ is a SuperPO of Γ. By construction, $c_x := \{x_l : l \in \mathcal{LINES}(\Gamma)\}$ is a K-separating cut of $\overline{\Gamma}$. □

3.3.2 Density for LOCyOs

In the context of cyclic orders, the notions of *intervals* and *edges* carry over from posets:

Definition 3.10 *Let $\Phi = (\mathcal{X}, \prec)$ be a* LOCyO*. For $li(a, b)$, define* **intervals** *as follows:* $]a, b[:= \{x \in \mathcal{X} : \prec (a, x, b)\}$ *and* $[a, b[:=]a, b[\cup \{a\}$, $]a, b] :=]a, b[\cup \{b\}$, $[a, b] :=]a, b[\cup \{a, b\}$. *An* **edge** *of Φ is a lii2-clique $\mathcal{E}$ such that there exist $a, b \in \mathcal{X}$ satisfying $li(a, b)$ and $\mathcal{E} \subseteq [a, b]$, and $\mathcal{E}$ is maximal relative $[a, b]$, i.e. for any $u \in [a, b]$ such that $\forall\, v \in \mathcal{E} : li(v, u)$ one has $u \in \mathcal{E}$. Set $\mathfrak{sta}(\mathcal{E}) := a$, $\mathfrak{end}(\mathcal{E}) := b$.*

[2] called a **true cut** in[Ste98]

Note that, as in the acyclic case, if $\mathcal{E}$ is an edge, $\mathfrak{sta}(\mathcal{E}) \in \mathcal{E}$ and $\mathfrak{end}(\mathcal{E}) \in \mathcal{E}$, and every edge $\mathcal{E}$ can be represented as the intersection of an appropriate line $l_{\mathcal{E}}$ with $[\mathfrak{sta}(\mathcal{E}), \mathfrak{end}(\mathcal{E})]$. The following definition generalizes the graph theoretic notion of a cycle or circuit:

Definition 3.11 (Cycles of a LOCyO) *Let $\Phi = (\mathcal{X}, \prec)$ be a* LOCyO*. $\mathcal{C} \subseteq \mathcal{X}$ is a* **cycle** *of Φ iff there exists $n \geqslant 2$ and edges $\mathcal{E}_1, \ldots, \mathcal{E}_n$ such that $start(\mathcal{E}_1) = \mathfrak{end}(\mathcal{E}_n)$, $start(\mathcal{E}_{i+1}) = \mathfrak{end}(\mathcal{E}_i)$ for $1 \leqslant i \leqslant n-1$, and $\mathcal{C} = \bigcup_{i=1}^{n-1} \mathcal{E}_i$. $\mathcal{CYCLE}(\Phi)$ denotes the set of cycles of Φ.*

Note that every line in Φ is a cycle (no line contains fewer than three elements). In general, however, not every cycle is a line of Φ: in Figure 1, the cycle whose places are a, d, f, h, c, and i, is not a line (since $co(a, h)$ etc.). Lines are those cycles that complete exactly one "round"; in general, a cycle may wind several times around the LOCyO.

Definition 3.12 *Let Φ be a* LOCyO*. A cut c is called* **Q-separating** *iff, for all $q \in \mathcal{CYCLE}(\Phi)$, $c \cap q \neq \emptyset$; Φ is called* **weakly Q-dense**[3] *iff there exists a Q-separating $c \in \mathcal{CUTS}$, and* **strongly Q-dense** *iff all $c \in \mathcal{CUTS}(\Phi)$ are Q-separating. If Φ has a weakly (strongly) Q-dense* SuperCyO*, it is called* **(strongly) saturable.**

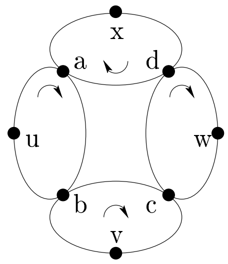

Figure 6: On K- and Q-separation

Thus Q-separation implies K-separation etc., but not vice versa. Figure 6 shows a cyclic order with a cut $\{u, v, w, x\}$ that is obviously K-separating but fails to intersect the cycle $\{a, b, c, d\}$; note that this structure is nonetheless weakly Q-dense since $\{a, c\}$ is a Q-separating cut.

Since a total CyO contains only one cycle, every singleton is a Q-separating cut; hence:

Lemma 3.13 *Every total* CyO *is strongly Q-dense.*

3.4 Characterization of Orientable LOCyOs

The following is the main theorem of this section. It characterizes the connection between Q-density, COQ representability, and totalizability. The result holds for arbitrarily large structures and thus characterizes *all* totalizable LOCyOs irrespective of their cardinality.

Theorem 3.14 *Let $\mathcal{X} \neq \emptyset$, and $\Phi = (\mathcal{X}, \prec)$ a* LOCyO*. Then the following are equivalent:*

1. *Φ is saturable;*
2. *Φ has a representation as a* COQ*;*
3. *Φ has a totalization.*

[3] In the special case of nets, Q-density has been introduced as 'F-density' in [KS97], [Ste98]

Proof of (1) $\Rightarrow$ (2): By Lemma 3.7, we only have to consider the case where Φ is itself weakly Q-dense. So it suffices to construct an unwinding for Φ given a Q-separating $c \in \mathcal{CUTS}(\Phi)$. Set $\overline{\mathcal{X}} := \mathcal{X} \times \mathbb{Z}$; we write x_k for (x, k). Let $\tau : \overline{\mathcal{X}} \to \overline{\mathcal{X}}$, $x_k \mapsto x_{k+1}$. Define a relation $\tilde{\mathfrak{u}}$ on $\overline{\mathcal{X}}$ as follows:

1. If either ($a \in c \ \wedge \ li(a, b)$ or $\exists\, x \in c : \prec (a, b, x)$, set $\tilde{\mathfrak{u}}(a_0, b_0)$ and $\tilde{\mathfrak{u}}(b_0, a_1)$;
2. if $\tilde{\mathfrak{u}}(a_l, b_m)$, set $\tilde{\mathfrak{u}}(a_{k+l}, b_{k+m})$ for all $k \in \mathbb{Z}$;
3. set $\tilde{\mathfrak{u}}(a_k, a_{k+l})$ for all $a \in \mathcal{X}$ and $k \in \mathbb{Z}$.

Now let $<$ be the transitive closure of $\tilde{\mathfrak{u}}$; then $<$ is a partial order. In fact, assume $u_k < u_k$ for some $u \in \mathcal{X}$, $k \in \mathbb{Z}$. Then there exist $n \in \mathbb{N}$, $y^1, \ldots, y^n \in \mathcal{X}$, and $k_1, \ldots, k_n \in \mathbb{N}$ such that $\tilde{\mathfrak{u}}(y^i_{k_i}, y^{i+1}_{k_{i+1}})$ for $i \in \{1, \ldots, n-1\}$ and $\tilde{\mathfrak{u}}(y^n_{k_n}, u_k)$. But that implies $k_i = k$ for all $1 \leqslant i \leqslant n$ and thus, since $y_i \notin c$ for all i, the existence of $x^i \in c$, $1 \leqslant i \leqslant n+1$, satisfying $\prec (u, y^1, x^1)$ and $\prec (y^i, y^{i+1}, x^{i+1})$ for $1 \leqslant i \leqslant n-1$, and such that $\prec (y^n, u, x^{n+1})$. So for any choice of edges $\mathcal{E}_j$, $j = 1, \ldots, n$, for which $\mathfrak{sta}(\mathcal{E}_1) = u$, $\mathfrak{end}(\mathcal{E}_n) = u$, $\mathfrak{end}(\mathcal{E}_j) = y^j$, and $\mathfrak{sta}(\mathcal{E}_{j+1}) = y^j$ for $1 \leqslant j \leqslant n-1$, the cycle $\mathcal{C}_u := \bigcup_{j=1}^n \mathcal{E}_j$ does not intersect c, contradicting the assumption. Hence $\overline{\Gamma} = (\overline{\mathcal{X}}, <)$ is a poset; moreover, τ is a forward order translation for $\overline{\Gamma}$. Finally, $\overline{\Gamma}$ is an unwinding of Φ, and $\phi_\tau : (\mathcal{X} \times \mathbb{Z}) \to \mathcal{X}$, $(x, z) \mapsto x$ winds $\overline{\Gamma}$ back to Φ. □

Proof of (2) $\Rightarrow$ (3): Let $\Phi = (\mathcal{X}, \prec)$ have an unwinding $\overline{\Gamma} = (\overline{\mathcal{X}}, <)$ with respect to a forward order translation τ. Let $\overline{\Gamma}_*$ be a weakly K-dense SuperPO of $\overline{\Gamma}$, and c_0 a K-separating cut of $\overline{\Gamma}_*$. Let $c_k := \tau^k(c_0)$, and set

$$\mathcal{U}^*_k \quad := \quad c_k \cap \{\overline{y} \in \overline{\mathcal{X}}_* : \ \exists\, \overline{y}_k \in c_k, \overline{y}_{k+1} \in c_{k+1} : \prec_* (\overline{y}_k, \overline{y}, \overline{y}_{k+1})\}.$$

Then the **unit sections** $\mathcal{U}_k := \mathcal{U}^*_l \cap \overline{\mathcal{X}}$ are pairwise disjoint and convex; moreover, they cover $\overline{\mathcal{X}}$ and induce SubPOs Φ_z of $\overline{\Gamma}$ such that there exists, for every $n, m \in \mathbb{Z}$, an order isomorphism $\theta_{n,m} : \mathcal{U}_n \to \mathcal{U}_m$ from $\overline{\Gamma}_n$ to $\overline{\Gamma}_m$. By Szpilrajn's Theorem, there exists, for every z, a total ordering $\overline{\Gamma}^{tot}_z$ on $\mathcal{U}_z$, and the identity mappings η_z on $\mathcal{U}_z$ are order homomorphisms from $\overline{\Gamma}|_{\mathcal{U}_z}$ to $\overline{\Gamma}|_{\mathcal{U}^{tot}_z}$. Now, the mapping $\sigma : \overline{\mathcal{X}} \to \overline{\mathcal{X}}$ given for $z \in \mathbb{Z}$ by $\sigma_{|\mathcal{U}_z} := \theta_z^{-1} \circ \eta_0 \circ \theta_z$ is a well-defined bijective order homomorphism which one can easily show to embed $\overline{\Gamma}$ into a total order $\overline{\Gamma}^{tot}$ (whose restriction to $\mathcal{U}_z$ is $\overline{\Gamma}$) on $\overline{\mathcal{X}}$. Then, by construction, $\tau \circ \sigma = \sigma \circ \tau$, and the quotient of $\overline{\Gamma}^{tot}$ under $\rho_{\tau\circ\sigma}$ is a totalization of Φ. □

Proof of (3) $\Rightarrow$ (1): If Φ is total, we are done by Lemma 3.13. Otherwise, choose a totalization $\Phi_{tot} = (\mathcal{X}, \prec_{tot})$ of Φ. Let $x \in \mathcal{X}$. Add pairwise distinct new elements x_l for $l \in \mathcal{LINES}(\Phi)$; on $\mathcal{X}_x := \mathcal{X} \cup \{x_l : l \in \mathcal{LINES}(\Phi),\ x \notin l\}$, define a new LOCyO Φ_x as follows: let $\iota : \mathcal{X} \to \mathcal{X}_x$ be the insertion of $\mathcal{X}$ into $\mathcal{X}_x$. Now there exists, on $\mathcal{X}_x$, a smallest SuperLOCyO $\prec_x$ of $(\prec)^\iota$ such that for all $l \in \mathcal{LINES}(\Phi)$, $\prec_x (x, x_l, y)$ whenever $li(x, y)$ and $\prec^{(3)}_{tot} (x, x_l, y)$. One obtains a cut c_x of $\mathcal{X}_x$ by setting $c_x := \{x_l : l \in \mathcal{LINES}(\Phi)\}$;

we claim that c_x is Q-separating for $\mathcal{X}_x$. Let $\mathcal{C} = \bigcup_{i=1}^{k} \mathcal{E}_i$ be a cycle. If there is an index $1 \leqslant j \leqslant k$ such that one of the boundary points of $\mathcal{E}_j$ lies in c_x, we are already done. Otherwise, there exists an index $1 \leqslant j' \leqslant k$ such that $\prec_{tot}$ $(\mathfrak{sta}(\mathcal{E}'_j), x, \mathfrak{end}(\mathcal{E}_l))$. Let $l \in \mathcal{LINES}(\Phi)$ such that $\mathcal{E}_{j'}$; then $\prec_x$ $(\mathfrak{sta}(\mathcal{E}'_j), x_l, \mathfrak{end}(\mathcal{E}_l))$, so $x_l \in \mathcal{E}_{j'}$, contradicting the assumption that $c_x \cap \mathcal{C} = \emptyset$. □

By Theorem 3.14, the non-totalizable LOCyO of Figure 2 can not be enriched to a weakly Q-dense SuperLOCyO and, a fortiori, is *not* weakly Q-dense itself; this can be verified by explicitly inspecting all cycles and cuts. So Theorem 3.9 has no counterpart w.r.t. Q-density in LOCyOs.

4 Petri Nets

We will show here how the definitions and results given thus far apply to the context of Petri Nets. Subsection 4.1 introduces nets, processes, and net quotients; Subsection 4.2 shows the connection between Theorem 3.14 and Synchronization Graphs; Subsections 4.3 and 4.3.2 characterize action refinements in causal nets and synchronization graphs. This study will motivate an investigation of *faithful* windings that follows in Subsection 4.4.

4.1 Definitions

Here, a triple $\mathcal{N} = (S,T,F)$ is a **(Petri) net** if S (**places**) and T (**transitions**) are disjoint sets, and $F \subset [(S \times T) \cup (T \times S)]$. The **inverse net** of $\mathcal{N}$ is $\mathcal{N}^{-1} = (S,T,F^T)$. Set $\mathcal{X} := S \cup T$; for any $x \in \mathcal{X}$, set $x^\bullet := F[x]$, ${}^\bullet x := F^{-1}[x]$, and, by extension, for $M \subseteq (S \cup T)$, ${}^\bullet M := \bigcup_{x \in M} {}^\bullet x$ and $M^\bullet := \bigcup_{x \in M} x^\bullet$. A **T-net** is a net such that, for all places $s \in S$, $|{}^\bullet s| \leqslant 1$ and $|s^\bullet| \leqslant 1$. An acyclic T-net is an **causal net**[4] (CN), a disjoint union of strongly connected T-nets is a **synchronization graph** (SG). If $\mathcal{N} = (S,T,F)$ is an CN, $< := F^+$ defines a binary partial order on $\mathcal{X}$. Denote as $\mathcal{CASE}(\mathcal{N}) := \mathcal{CUTS}(\mathcal{N}) \cap \mathcal{P}(S)$ the set of $\mathcal{N}$'s **S-cuts** or **cases**.

For any CN $\mathcal{N} = (S,T,F)$, S and T, with the respective restriction of the order relations, each form a partial order substructure or SubPO. Conversely, every weakly discrete (*event ordering*) $(T,<)$ has natural SuperPOs on some CNs $\overline{\mathcal{N}}_T = (S,T,F)$; among those, there is a unique (up to isomorphism) $\mathcal{N}_T$ with S of minimal cardinality, the **net version** of $(T,<)$.

If $\mathcal{N} = (S,T,F)$ is a net, any $c \subseteq S$ is a **marking** of $\mathcal{N}$. If c is a marking of $\mathcal{N}$, $\Sigma = (\mathcal{N}, c)$ is an **elementary net system** (ENS) with **initial marking** c. A transition $t \in T$ is **enabled** in a marking c, written $c \rhd t$, iff $({}^\bullet t \subseteq c) \wedge (t^\bullet \cap c = \emptyset)$. If t is enabled in c and $c' = (c - {}^\bullet t) \cup t^\bullet$, we say that c is transformed to c' by the **firing** of t, for short: $r(c_1, c_2)$. $\looparrowright := r^+$ is the **(forward) reachability relation** on $\mathcal{P}(S)$; set $\mathfrak{r} := symm(\looparrowright)$. For $c \subseteq S$ and $t \in T$, there is a **contact** at t in c iff ${}^\bullet t \in c$ and $t^\bullet \cap c \neq \emptyset$. An ENS $(\mathcal{N}, c)$ is **safe** iff no markings with contacts are reachable from c and **live** iff for all $t \in T$ and

[4] Many texts, e.g. [BF88], use '*occurrence net*'; we do not follow that terminology here, but rather reserve that term for the net model of *branching processes* ([Eng91], [Haa98a]).

$c' \in \looparrowright [c]$, there exists $c'' \in \looparrowright [c']$ such that $c'' \rhd t$. $(\mathcal{N}, c)$ is **weakly live** iff this condition holds with $\looparrowright$ replaced by $\leftrightarrows := symm(\looparrowright)$. We will abbreviate "live and 1-safe" as **l.s.** For safe systems, *processes* according to (cf. Reisig [Rei85]) are defined as follows:

Definition 4.1 *Let $\overline{\mathcal{N}} = (\overline{S}, \overline{T}, \overline{F})$ be a weakly discrete causal net such that the set $min(\overline{\mathcal{N}})$ of the minimal elements of $\overline{\mathcal{N}}$ is a cut of $\overline{\mathcal{N}}$. If $\Sigma = (\mathcal{N}, m)$ with $\mathcal{N} = (S, T, F)$ is a safe* ENS *and if, with $\mathcal{X} := S \cup T$ and $\overline{\mathcal{X}} := \overline{S} \cup \overline{T}$, $p : \overline{\mathcal{X}} \to \mathcal{X}$ satisfies*

1. $p(min(\overline{\mathcal{N}})) = m$,
2. $p(\overline{S}) \subseteq S$ *and* $p(\overline{T}) \subseteq T$, *and*
3. *for all $\overline{T} \in \overline{T}$, p induces a net isomorphism from $\overline{\mathcal{N}}[{}^\bullet\overline{t} \cup \overline{t}^\bullet \cup \{\overline{t}\}]$ to $\mathcal{N}[{}^\bullet t \cup t^\bullet \cup \{t\}]$,*

then $(\overline{\mathcal{N}}, p)$ is called a (one-sided) **process** *of Σ.*

Reachable markings of an ENSare represented by S-Cuts of its process CN; moreover, in typical cases, e.g. if all S-cuts are finite, $(\overline{\mathcal{N}}, c)$ is itself a safe and weakly live ENS for all $c \in \mathcal{CAAE}(\overline{\mathcal{N}})$, and c and all $c' \in \mathcal{CAAE}(\overline{\mathcal{N}})$ are in $\mathfrak{r}$ ([BF88], [FT85]). CNs resulting from unwindings as in Figure 1 can be thought of as the *two-sided* process of a suitable $\mathcal{N} = (S, T, F, m_0)$, i.e. combined from the processes of $\mathcal{N}$ and its inverse net $\mathcal{N}^{-1}$.

For the formalization of **net quotients**, we make use of the relation A as used in Petri [Pet96]:

Definition 4.2 *For a net $\mathcal{N} = (S, T, F)$, $A = A(\mathcal{N}) := (F \cup F^T) \cap (S \times T)$.*

Obviously, $domain(A) \subseteq S$ and $range(A) \subseteq T$ with equality iff no place (or transition, respectively) is isolated in $\mathcal{N}$. So A allows to encode the *topological* distinction between places and transitions in a *relational* structure.

Definition 4.3 *Let $\mathcal{N}_1 = (S_1, T_1, F_1)$ be a net, $A_1 = A(\mathcal{N})$ and $\equiv$ an equivalence on $\mathcal{X}_1$. With $\mathcal{X}_2 := \mathcal{X}_1/_\equiv$, let $\rho : \mathcal{X}_1 \to \mathcal{X}_2$ be the surjective mapping $x \mapsto [x]_\equiv$. Let A_2 and F_2 be the lower quotients under ρ of A_1 and F_1, respectively, and set $S_2 := domain(A_2)$, $T_2 := range(A_2)$. $\mathcal{N}_2 := (S_2, T_2, F_2)$ is called a* **quotient net** *of $\mathcal{N}_1$ and ρ the corresponding* **quotient map** *iff*

1. $(F_2 \cup F_2^T) \subseteq (A_2 \cup A_2^T)$ *and*
2. $S_2 \cup T_2 = \mathcal{X}_2$.

Remark 4.1 *The crucial point in the net quotient construction is to actually obtain a* **net** *from $\equiv$; this is ensured by conditions 1 and 2, which prevent F-arcs between transitions or between places. Definition 4.3 also rules out, mappings that interchange the roles of state and transition elements. We have, for simplicity, also ensured that $\mathcal{N}_2$ contains no isolated elements.*

Note that quotient maps are special cases of *morphisms* in the sense of Petri [Pet96].

4.2 Weakly discrete LOCyOs and SGs

We take another look at orientable LOCyOs to see how they are related to physical concurrent systems modeled as synchronization graphs (cf. [Ste98] for a different approach). A LOCyO is called **weakly discrete** iff all its lines are finite. Consider a weakly discrete LOCyO Φ and its successor relation $\lessdot := \{x, y : \ li(x,y) \wedge]x,y[= \emptyset\}$. Obviously, $li \subseteq \lessdot^+$ holds in every weakly discrete LOCyO. Similar to the acyclic case, every weakly discrete LOCyO $\Phi = (\mathcal{X}, <)$ has a **net version** $\mathcal{N}_\Phi = (T_\Phi, S_\Phi, F_\Phi)$, where $T_\Phi = \mathcal{X}$, $S_\Phi = \lessdot$, and

$$F_\Phi \quad := \quad \{(x, (x,y)) : \ x \lessdot y\} \cup \{((x,y), y) : \ x \lessdot y\}.$$

In other words, net versions of LOCyO are obtained by "filling open gaps" between adjacent (transition) elements by places. $\mathcal{N}_\Phi$ is the smallest "net-shaped" SuperLOCyO of Φ.

Return now to the characterization of orientable LOCyOs given by Theorem 3.14. For $\mathcal{X}$ finite, we retrieve Genrich and Lautenbach's Theorem [GL73]: An SG is l.s. iff

1. all cycles contain at least one token, and
2. the net is covered by cycles containing exactly one token each.

In fact, consider the non-sequential process nets (in the sense of Definition 4.1) $\overline{\mathcal{N}}$ and $\overline{\mathcal{N}}_\ominus$ for a live and safe SG $\mathcal{N}$ and for $\mathcal{N}^{-1}$, respectively, with the same initial marking; then, glue together $\overline{\mathcal{N}}$ and $\overline{\mathcal{N}}_\ominus^{-1}$ at the minimal/maximal cut. One obtains a two-way infinite causal net, which gives the unfolding of a LOCyO that has all reachable markings as S-cuts. For the dynamics, the LOCyO structure ensures that every place is contained in at least one line l, and since l can intersect any cut in at most one element, there is never more than one token on l; weak Q-density, on the other hand, corresponds to the absence of unmarked cycles. So we are led to ask whether, for weakly discrete LOCyOs on **nets**, the existence of a weakly Q-dense SuperLOCyO implies weak Q-density of the original LOCyO. The answer, given by Theorem 4.4 below, is affirmative; so, Genrich and Lautenbach's result is in fact the dynamic interpretation of the finite case of Theorem 3.14.

Theorem 4.4 *Let $\Phi = (\mathcal{X}, \prec)$ be the net version of a weakly discrete* LOCyO*. If Φ is saturable, then Φ is itself weakly Q-dense.*

Proof: Let $\Phi_* = (\mathcal{X}_*, \prec_*)$ be a weakly Q-dense SuperLOCyO of Φ and $c_* \in \mathcal{CUTS}(\Phi_*)$ a Q-separating cut for Φ_*; further, let γ be any cycle of Φ with edges $\mathcal{E}_1, \ldots, \mathcal{E}_n$. For $i \in \{1, \ldots, n\}$, let $\mathcal{E}_i^* := \mathcal{E}_i \cup \{v \in \mathcal{X}_* : \prec (\mathfrak{sta}(\mathcal{E}_i), \mathfrak{end}(\mathcal{E}_i)) \wedge li(u,v) \ \forall u \in \mathcal{E}_i\}$. Then there exists a line $l_*(i)$ of Φ_* such that $\mathcal{E}_i^* = l_*(i) \cap [\mathfrak{sta}(\mathcal{E}_i), \mathfrak{end}(\mathcal{E}_i)]$; hence $\gamma_* := \bigcup_{i=1}^n \mathcal{E}_i^*$ is a cycle of Φ_* containing γ, and, by assumption, there exists $x_{\gamma_*} \in c_* \cap \gamma_*$. If $x_{\gamma_*} \in \gamma$, we are done; otherwise there exist $a, b \in \gamma$ such that $a \lessdot b$ and $\prec_* (a, x_{\gamma_*}, b)$. Suppose a is a place (the case when b is a place is analogous). Then $co_*(a,x)$ for all $x \in c_* - \{x_{\gamma_*}\}$, since $li_*(a,x)$ implies $\prec_*(a, x_{\gamma_*}, b)$, and so, by ternary transitivity, $\prec_*(a, x_{\gamma_*}, x)$, a contradiction. Hence $c := c_* \cap \mathcal{X}$ is Q-separating for Φ; it is a clique of co by construction, and it is maximal since it intersects every line, so we are done. $\square$

Theorem 3.14 generalizes a result by Stehr for **finite** LOCyOs; it shows that the connection between Q-density and orientation is intrinsical and independent of the interpretation as Petri nets; moreover, it includes infinite nets like Petri's [Pet96] **R-orthoids**.

4.3 Coarsenings in CNs and SGs

With relational quotients, it is also possible to characterize abstractions that preserve CN semantics, i.e. such that: (i) the quotient net is again a CN, (ii) the relational structure is reflected, and (iii) local contractions do not change relations *outside* the contracted set ('black box'). We will give a characterization of admissible action refinements (more precisely, of the dual operation which we call *coarsening*) CNs in terms of quotients.

4.3.1 Coarsenings in Causal Nets

There exist several approaches to Petri net abstractions and refinements in the literature; cf the morphism approaches in Merceron [Mer94], Desel and Merceron [DM96], Valk [Ge98], or in the calculus of *Petri Boxes* ([BDE93]). Best and Randell [BR81] present a notion of *atomic occurrence*. Since their underlying process model is (essentially) CNs, their atomic subnets correspond to the *oblong sets* we will introduce below; we will show that and how, in the common relational framework, the results carry over to l.s. synchronization graphs and their cyclic orders. Figure 7 (a) shows an example of an intuitively admissible coarsening. We define:

Definition 4.5 *Let $\overline{\mathcal{N}}_i = (S_i, T_i, F_i)$, $i \in \{1,2\}$, be* CN*s, and $\xi : \mathcal{X}_1 \to \mathcal{X}_2$ a surjection. $\mathcal{N}_2$ is a* **coarsened version** *of $\mathcal{N}_1$ with* **coarsening** *ξ iff ξ is a quotient map and $<_2 = \nabla_\xi(<_1)$.*

Obviously, the coarsened version of a weakly discrete net is itself weakly discrete. Moreover:

Lemma 4.6 *If $\mathcal{N}_2$ is a coarsened version of $\mathcal{N}_1$ under ξ, then $li_2 = \nabla^\xi(li_1)$ and $co_2 = \nabla^\xi(co_1)$.*

So, concurrency relations in coarsened versions hold iff they hold among *any* set of representatives in the original CN. Consider again Figure 7 (a): In the net $\mathcal{N}_1$ on the left, $co_1(a,b)$, whereas $<_2(\xi(a),\xi(b))$ in $\mathcal{N}_2$. Now, b is a detail of D not *perceptible* (by causal interrelation) from a; however, there is *some* interrelation between D and a, so D as a whole *is* perceptible for a. This justifies $li_2(\xi(a),\xi(x))$ for all $x \in D$. Conversely, if *no* detail of M is perceptible for a, then D as a whole is not; so, in fact, choosing the *upper* quotient for *co*-relations agrees with the intuitive notion of coarsening above. To formalize matters, define, for a net $\mathcal{N} = (S,T,F)$ and $D \subseteq \mathcal{X} = S \cup T$, the **boundaries** of D as

$$\partial^- D := \{x \in D : \neg({}^\bullet x \ \subseteq D)\};\ \partial^+ D := \{x \in D : \neg(x^\bullet \ \subseteq D)\};\ \partial D := \partial^+ D \cup \partial^- D.$$

The most elementary requirement for coarsening is that $|\xi^{-1}(x)| > 1$ implies that $x \in T_2$ and $\partial\xi^{-1}(x) \subseteq T_1$; this will be tacitly required. As Figure 7 shows, more is needed:

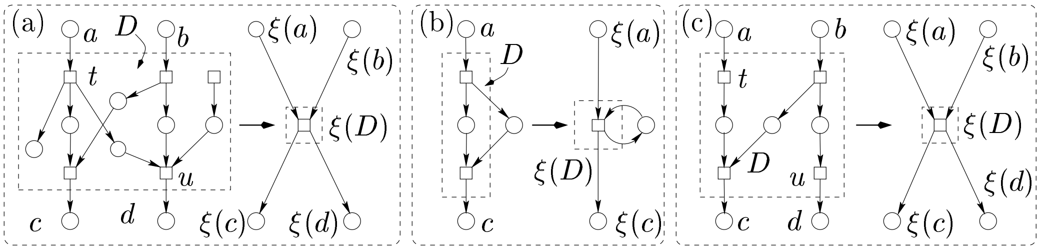

Figure 7: (a): a coarsening; (b), (c): non-coarsening quotients of CNs

Definition 4.7 *Let $\mathcal{N} = (S, T, F)$ be a weakly discrete* CN *and $D \subseteq \mathcal{X}$. D is called* **oblong** *iff $(\partial^- M \times \partial^+ M) \subseteq (<^{(2)} \cup id_{\mathcal{X}})$.*

All oblong sets are convex, yet Figure 7 (c) shows that the converse is not true: M is not oblong since $co_1(u,v)$. The quotient shown in Figure 7 (c) is not a coarsening since $co_1(a,b)$ and $li_2(\xi(a), \xi(b))$; so convexity is not sufficient for atomicity, but oblongness is:

Theorem 4.8 *Let $\overline{\mathcal{N}}_1$ be a weakly discrete* CN*, $\mathcal{X}_2$ a set, $\xi : \mathcal{X}_1 \to \mathcal{X}_2$ surjective, and $\overline{\mathcal{N}}_2$ the quotient net of $\overline{\mathcal{N}}_1$ under ξ. Then $\xi : \mathcal{X}_1 \to \mathcal{X}_2$ is a coarsening from $\mathcal{N}_1$ to $\mathcal{N}_2$, iff for all $x_2 \in \overline{T}_2$, $M(x_2) := \xi^{-1}(x_2) \subset \mathcal{X}_1$ is oblong (in $\overline{\mathcal{N}}_1$).*

Proof: For the "if" part, let $a \in \partial^- M$, $b \in \partial^+ M$, and $u \in {}^\bullet a - M$ and $v \in b^\bullet - M$; then $u, v \in S_1$. Assume $\neg < (a,b)$. Then $co_1(u,v)$, for $<_1 (u,v)$ implies $<_2 (a,b)$ contradicting the assumption, and $<_1 (v,u)$ or $u = v$ both imply $<_2 (x_2, x_2)$ which contradicts the simplicity of $<_2$. Now, $<_2 (\xi(u), x_2$ and $<_2 (\xi(v))$ and thus $<_2 (\xi(u), \xi(v))$; but then ξ cannot be a coarsening. For "only if", it suffices to prove that $\mathcal{N}_2$ is an CN and that $<_2 = \nabla_\xi(<_1)$ and $co_2 = \triangle_\xi(<_1)$. First, only acyclicity has to be shown to prove that $\mathcal{N}_2$ is a CN. Suppose there exist $a \in \mathcal{X}_2$ and an F_2-chain $u_1, \ldots, u_n$, $n \geqslant 2$, such that $a = u_1 = u_n$. Let, for $1 \leqslant i \leqslant n$, $M_i := \xi^{-1}(\{u_i\})$. Then there exist for each $1 \leqslant i \leqslant n-1$ an $x_i \in \partial^+ M_i$ and a $y_{i+1} \in \partial^- M_{i+1}$ such that $<_1 (x_i, y_{i+1})$. Since the M_i's are oblong, $<_1 (x_1, x_1)$ follows, contradicting the simplicity of $<_1$; hence $\mathcal{N}_2$ is a CN. It remains to show that $<_2 = \nabla_\xi(<_1)$. Let $a, b \in \mathcal{X}_1$ and $<_2 (\xi(a), \xi(b))$. Then there is an F_2-chain $u_1, \ldots, u_n$ such that $u_1 = \xi(a)$ and $u_n = \xi(b)$. For $1 \leqslant i \leqslant n$, set again $M_i := \xi^{-1}(\{u_i\})$. By assumption, there exists, for every $1 \leqslant i \leqslant n-1$, an $x_i \in \partial^+ M_i$ and a $y_{i+1} \in \partial^- M_{i+1}$ such that $<_1 (x_i, x_{i+1})$. Since every M_i is oblong, $<_2 (\xi(a), \xi(b))$. Therefore, $u, v \in \mathcal{X}_2$, $<_2 (u,v)$ implies the existence of $a \in \xi^{-1}(\{u\})$ and $b \in \xi^{-1}(\{v\})$ such that $<_1 (a,b)$. $\square$

Before passing to the cyclic case, we note:

Lemma 4.9 *Any* CN *coarsening ξ from $\mathcal{N}_1$ to $\mathcal{N}_2$ is faithful for li and co.*

Proof: Assume $\mathcal{N}_2$ is the coarsened version of $\mathcal{N}_1$ as above. For co, the result follows since upper quotients are faithful by Lemma 4.6. For li, suppose $m = \{x_1, \dots, x_n\}$ is a clique of li, $n \geqslant 3$; without loss of generality, $x_1 <_2 \dots <_2 x_n$. Let $M_i := \xi^{-1}(\{x_i\})$. Then there exist $\overline{x}_i^+ \in \partial^+ M_i$ and $x_-^i \in \partial^+ M_{i+1}$ such that $x_i^+, <_1, x_i^-$ for $1 \leqslant i \leqslant n-1$, and since the M_i are oblong, we have $x_1^+ <_1 x_1^- <_1 x_2^- <_1 \dots <_1 x_{n-1}^-$, so $m \in \mathfrak{Rc}(\xi, li)$. $\square$

4.3.2 Coarsening in l.s. synchronization graphs

Essentially, coarsening in cyclic orders follows the same lines as in posets; yet once again, the presence of cycles makes things more involved. Let us start with two examples of intuitively admissible coarsenings. Figure 8 shows a cyclic coarsening reflected by an acyclic one in the unwinding; the example in Figure 9 shows that it is also possible to contract a cyclic set, but there is, in general, no corresponding coarsening in the unwinding. So consider an SG $\mathcal{N}_1 = (S_1, T_1, F_1)$, and let $\mathcal{N}_2 = (S_2, T_2, F_2)$ be the quotient net of $\mathcal{N}_1$ under the mapping $\xi : S_1 \cup T_1 \to S_2 \cup T_2$. Then ξ is called a **coarsening** iff Definition 4.5 is satisfied with $<$ replaced by $\prec$. Once again, the atomic sets will be the oblong ones, with the definition adapted to cyclic orders as follows:

Definition 4.10 *Let $\Phi = (\mathcal{X}, \prec)$ be a weakly discrete* LOCyO. *$M \subset \mathcal{X}$ is called* **convex** *iff for all $x \in \partial^- D$ and $y \in \partial^+ D$, $li(x,y)$ implies $[x,y] \subset D$; if, in addition, all $x \in \partial^- D$ and $y \in \partial^+ D$ satisfy $li(x,y)$, D is called* **oblong**.

Note that here, oblong sets may contain cycles, cf. Figure 9. We will treat that general case below and start with the elementary case of a single *acyclic* contraction set as in Figure 8. Its image under the unwinding decomposes into oblong components without being oblong itself.

Theorem 4.11 *If there exists $t \in T_2$ such that $\forall\, x \in \mathcal{X}_2 - \{t\} : |\xi^{-1}(\{x\})| = 1$, and the set $M_t := \xi^{-1}(\{t\})$ contains no F-cycles, then ξ is a coarsening iff M_t is oblong.*

Proof: The connected components $\overline{M}_t^i$, $i \in \mathbb{Z}$, of $\overline{M}_t := \phi_1^{-1}(M_t)$ are oblong in $\overline{\mathcal{N}}_1$. Thus, one obtains an CN $\overline{\mathcal{N}}_2$ and a quotient map $\overline{\xi} : \overline{\mathcal{X}}_1 \to \overline{\mathcal{X}}_2$ such that: $\overline{\xi}|_{(\overline{\mathcal{X}}_1 - \overline{M}_t)}$ is injective, and for all $i \in \mathbb{Z}$, $\overline{\xi}(\overline{\mathcal{X}}_1 - \overline{M}_t) \cap \overline{\xi}(\overline{M}_t) = \emptyset$, and also $\overline{\xi}(\overline{M}_t^i) = \{t_i\}$, where the t_i, $i \in \mathbb{Z}$, are mutually distinct and in T_2. Thus $\overline{\xi}$ is a coarsening from $\overline{\mathcal{N}}_1$ to $\overline{\mathcal{N}}_2$. Moreover, one obtains a winding ϕ_2 from $\overline{\mathcal{N}}_2$ to $\mathcal{N}_2$ such that $\xi \circ \phi_1 = \phi_2 \circ \overline{\xi}$, and the theorem follows. $\square$

For more general situations, Figure 9 warns us that coarsening and winding need not commute; the pre-image under the winding of the contracted set does not split into convex connected components, thus permitting no corresponding coarsening on the causal net. To see how coarsenings of cyclic sets are possible, we distinguish, in a LOCyO $(\mathcal{X}, \prec)$, *visible* and *invisible* elements w.r.t. causality: Call $\mathcal{VI}(D) := \{x \in D :\ \exists\, y \in (\mathcal{X} - D) : li(x,y)\}$ the **visible** and $\mathcal{IV}(D) := D - \mathcal{VI}(D)$ the **invisible part** of D. Obviously, $\partial D \subseteq \mathcal{VI}(D)$. One then has:

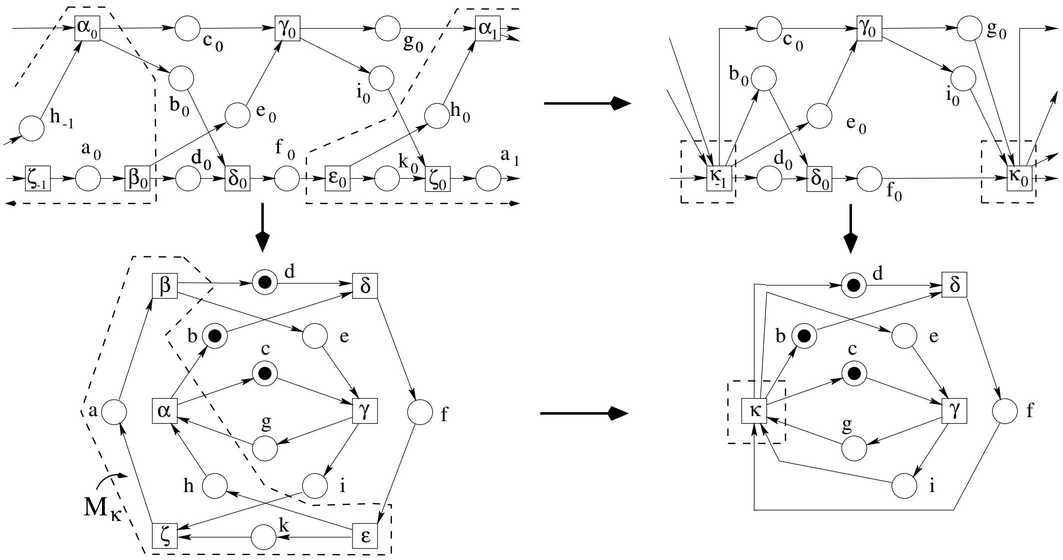

Figure 8: A cyclic coarsening. Contracted sets are indicated by dotted lines.

Theorem 4.12 *Suppose there exists $t \in T_2$ such that, with $M_y := \xi^{-1}(\{y\})$, $|M_x| = 1$ for all $x \in \mathcal{X}_2 - \{t\}$, and there exists $c_t \in \mathcal{CUTS}(\mathcal{N})$ Q-separating such that $c_t \cap \mathcal{VI}(M_t) = \emptyset$. Then ξ is a coarsening iff M_t is oblong.*

Proof: c_t defines a unique (up to translation) unwinding into a $\mathsf{CN}\,\overline{\mathcal{N}}_1$; let τ_1 and ϕ_1 be a forward order translation and corresponding winding, respectively. From $\overline{\mathcal{N}}_1$, remove the set $\mathcal{IV}_1 := \mathcal{IV}(\phi_1^{-1}(\mathcal{IV}(M)))$, i.e. let $\overline{\mathcal{N}}_1^* = (\overline{S}_1', \overline{T}_1', F_1')$ be given by $\overline{S}_1^* := \overline{S}_1 - \mathcal{IV}_1$, $\overline{T}_1^* := \overline{T}_1 - \mathcal{IV}_1$, $\overline{\mathcal{X}}_1^* := \overline{S}_1^* \cup \overline{T}_1^*$ and $\overline{F}_1^* := \overline{F}_1 \cap \left(\overline{\mathcal{X}}_1^* \times \overline{\mathcal{X}}_1^*\right)$. Then $\overline{\mathcal{N}}_1^*$ is an CN; we have to show that, for $x, y \in (\overline{S}_1^* \cup \overline{T}_1^*)$ such that $<_1 (x,y)$, also $<_1^* (x,y)$. Suppose this is not the case; then for all lines l containing x and y, there is $z_l \in \mathcal{IV}(M_t)$ such that $x <_1 z_l <_1 y$. Without loss of generality, $\neg <_1 (\tau_1(x), y)$. If $<_1 (y, \tau_1(x))$, we have $li_1(x,z)$ contradicting $z \in \mathcal{IV}(M_t)$; so $co_1(y, \tau_1(x))$ and $co_1(z, \tau_1(x))$. On the other hand, we may assume without loss of generality that $\phi_1(x) \in^\bullet M_t$ and $\phi_2(y) \in M_t^\bullet$; let $u \in \partial^-(M_t) \cap x^\bullet$ and $v \in \partial^-(M_t) \cap y^\bullet$. Then $u <_1 z <_1 v$ and $co_1(v, \tau_1(u))$; but this implies $co_1(\phi_1(u), \phi_2(v))$, which contradicts M_t being oblong. Hence $(\overline{\mathcal{N}}_1^*, <_1^*)$ is a SubPO of $(\overline{\mathcal{N}}_1, <_1)$. Now, every $c_1 \in \mathcal{CUTS}(\overline{\mathcal{N}}_1)$ such that $\phi_1(c) = c_1$ satisfies $c_1^* := c_1 \cap \overline{\mathcal{X}}_1^* \in \mathcal{CUTS}(\overline{\mathcal{N}}_1^*)$. Set $\phi_1^* := \phi_{1_{|\overline{\mathcal{X}}_1^*}}$, $S_2^* := \phi(S_1)$ and $T_2^* := \phi(T_1)$; we obtain a subnet $\mathcal{N}_2^* := \mathcal{N}_{2_{|\mathcal{X}_2^*}}$ of $\mathcal{N}_2$ satisfying $\phi^*(\mathcal{X}_1^*) = \mathcal{X}_2^*$. $\mathcal{N}_2^*$ is an SG whose LOCyO is a SubCyO of that on $\mathcal{N}_2$. By Theorem 4.8, $\xi_{|\mathcal{X}_1^*}$ is a coarsening from $\mathcal{N}_1$ to $\mathcal{N}_2$; by construction of $\mathcal{N}_1$, ξ is thus a coarsening from $\mathcal{N}_1$ to $\mathcal{N}_2$. $\square$

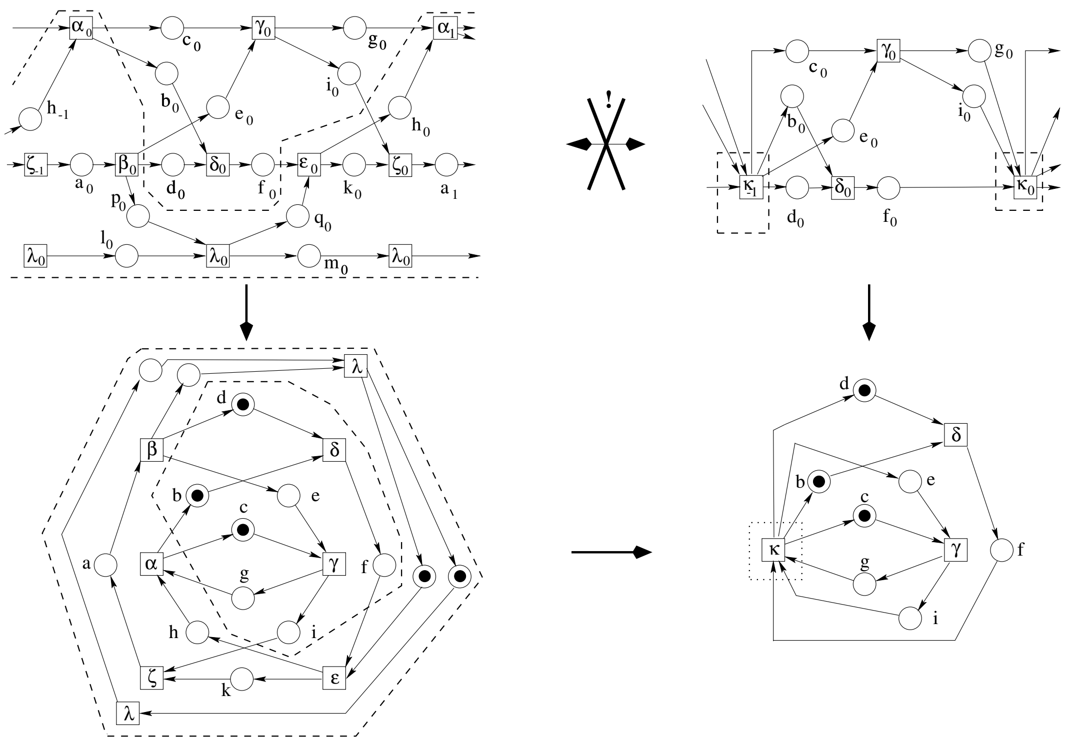

Figure 9: Coarsening of a set with cycles

Obviously, we wish to extend Theorem 4.12 to the simultaneous contraction of several disjoint M_t^k (in CNs, this was immediate). This will be done in Theorem 4.14 below; some preparations first. Theorem 4.12 can immediately be extended in two situations: (i) there exists $c \in \mathcal{CUTS}(\mathcal{N}_1)$ playing the role of c_{t^k} in Theorem 4.12 for all k simultaneously, or (ii) one can choose all the c_{t^k} from the same reachability class, so that the unwinding obtained is unique up to translation. We start with an auxiliary result:

Lemma 4.13 *If $\mathcal{N}$ is a weakly discrete* SG *then, for every convex T-bordered set $D \subset \mathcal{X}$, ${}^\bullet D := \bigcup_{x \in D} {}^\bullet x$ and $D^\bullet := \bigcup_{x \in D} x^\bullet$ are cliques of $co_{\mathcal{N}}$.*

Proof: Let $x \neq y$, $x, y \in {}^\bullet M$; the case $x, y \in M^\bullet$ is analogous. By assumption, $x, y \in S$ and thus $|x^\bullet| = |y^\bullet| = 1$. So assume $li(x, y)$, and let $x^\bullet = \{t_x\}$ and $y^\bullet = \{t_y\}$ be such that $t_x, t_y \in T$ and $t_x \neq t_y$ (note that, if $x^\bullet = y^\bullet$, we are done); then also $li(x, t_y)$, $li(t_x, y)$ and $li(t_x, t_y)$, and thus $li^{(4)}(x, y, t_x, t_y)$. Because of $F(x, t_x)$ and $F(y, t_y)$, this implies that $\prec (x, t_x, y) \wedge \prec (x, y, t_y)$ or $\prec (y, t_y, x) \wedge \prec (y, x, t_x)$, contradicting, since $x, y \notin M$, the convexity of M. $\square$

Using this result, we obtain:

Theorem 4.14 *Let $\mathcal{N}_1$, $\mathcal{N}_2$ be* SG*s and $\xi : S_1 \cup T_1 \to S_2 \cup T_2$ a net quotient map.*

1. *If $\mathcal{N}_1$ is strongly Q-dense and $|M_x| = 1$ for all $x \in \mathcal{X}_2 - \{t\}$, then ξ is a coarsening iff M_t is oblong in $\mathcal{N}_1$.*

2. *If $\mathfrak{r} = \mathcal{CAAE}(\mathcal{N}) \times \mathcal{CAAE}(\mathcal{N})$, then ξ is a coarsening iff, for all $x \in \mathcal{X}_2 \cap T_1$, M_x is oblong.*

Proof: For Part 1, it suffices, by Lemma 4.13, to show that, for any cuts $c_+ \supseteq^\bullet M_t$ and $c_- \supseteq M_t^\bullet$, one has $c_+ \cap (\mathcal{VI}(M_t)) = \emptyset$ and $c_- \cap (\mathcal{VI}(M_t)) = \emptyset$. Suppose $x \in (c_+ \cup c_-) \cap (\mathcal{VI}(M_t))$, and let $u \in \mathcal{X} - M_t$ be such that $li(u, x)$; but any line containing u and x intersects both ${}^\bullet M_t$ and $M_t^\bullet$, so there exist $a_+ \in c_+$ and $a_- \in c_-$ satisfying $\prec (a_-, x, a_+)$, contradicting the assumption. For Part 2, note first that Q-density follows from the assumption; the construction described in the proof of Theorem 4.12 can be repeated for all contraction sets using a single unwinding, and the claim follows. $\square$

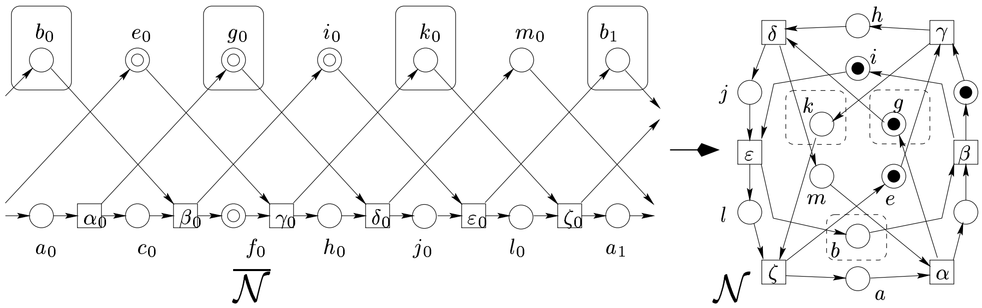

Figure 10: A winding with a strange cut

4.4 Faithful Windings

One is thus led to ask for sufficient conditions under which Theorem 4.14 can be applied. We prove here that this is case when $\mathcal{CUTS}(\overline{\mathcal{N}})$ is connected w.r.t. reachability and ϕ_1 is *faithful* for *co*. Thus, we will first have a closer look at causal nets again and then turn to faithfulness. Let $\overline{\mathcal{N}}$ be a CN. The following results are well-known ([BF88], [FT85]):

Lemma 4.15 *If $\overline{\mathcal{N}}$ is weakly discrete and all cuts are finite, then $\overline{\mathcal{N}}$ is K-dense, and $\mathfrak{r} = (\mathcal{CAAE}(\overline{\mathcal{N}})) \times (\mathcal{CAAE}(\overline{\mathcal{N}}))$.*

Now, under windings, separation carries over:

Lemma 4.16 *If* Γ *is wound to* Φ *by* ϕ *and* $\overline{c} \in \mathcal{CUTS}(\Gamma)$ *is K-separating, then* $c := \phi(\overline{c})$ *is Q-separating in* Φ.

Proof: First, c is a *co*-clique by construction. Let $\mathcal{C}$ be a cycle such that $\mathcal{C} \cap c = 0$, with Φ-edges $\mathcal{E}_1, \ldots, \mathcal{E}_n$. Then there exist Γ-edges $\overline{\mathcal{E}}_1, \ldots, \overline{\mathcal{E}}_n$ such that $\mathfrak{sta}(\overline{\mathcal{E}}_1) = x$, $\mathfrak{end}(\overline{\mathcal{E}}_i) = \mathfrak{sta}(\overline{\mathcal{E}}_{i+1})$ for $i \in \{1, \ldots, n-1\}$, and $\rho(\overline{\mathcal{E}}_i) = \mathcal{E}_i$ for $i \in \{1, \ldots, n\}$. One verifies that $\mathfrak{end}(\overline{\mathcal{E}}_n) = \tau^k(x)$ for some $k \geq 1$. Set $l := \bigcup_{j \in \mathbb{Z}} \bigcup_{i=1}^{n} (\tau^k)^j(\overline{\mathcal{E}}_i)$; then l is a line and, since $\overline{c} \cap l \neq \emptyset$ by assumption, c intersects $\mathcal{C} = \rho(l)$, so we are done. □

It is easy to see that reachability is lifted by windings: if $\looparrowright (\overline{c}_1, \overline{c}_2)$ in $\overline{\mathcal{N}}$, then $\looparrowright (\phi(\overline{c}_1), \phi(\overline{c}_2))$ in $\mathcal{N}$. Since the $\mathcal{CAAE}$-classes are inherited under faithful windings, so is, e.g., K-density; this property, as Figure 10 shows, may otherwise be lost. So, it remains to ask under what conditions a winding is faithful. Consider the winding of Figure 10. It is *too tight* to be faithful; a strange cut $c = \{b, g, k\}$ exists in $\mathcal{N}$. Here, accidentally, c is *not* K-separating; it also corresponds to a dead marking. In general, not all strange S-cuts in windings have that property: Figure 12 shows an SG obtained by winding from the CN in Figure 12 (see [Pet96], [KS97] or [Haa98b] for more details on this net). On the left, the SG bears a *canonical marking*, i.e. all tokens lie on an S-cut that is the image of a cut from the net of Figure 12; on the right, the tokens mark a *strange* S-cut. Both markings are live and have strongly connected reachability classes but are not reachable from one another. To complete the already complicated picture, isomorphic finite graphs may carry non-isomorphic cyclic orders.

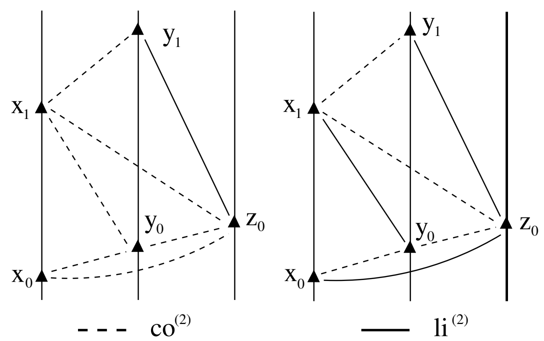

Figure 11: Improper (left) and proper triple jump

Now, back to the technical question: how tight can a faithful winding be ? As indicated in Figure 10, the net sections between subsequent representatives b_0 and b_1 of b are so 'short' that it only takes three *co*-"steps" from b_0 to b_1. Let Φ be the COQ on $\mathcal{X}$ of a partial order $\overline{\Gamma}$ on $\overline{\mathcal{X}}$ w.r.t. τ. A triplet $(x, y, z) \in \mathcal{X}^3$, $y \neq z$, for which there exist $\overline{x} \in \phi_\tau^{-1}(\{x\})$, $\overline{y} \in \phi_\tau^{-1}(\{y\})$, $\overline{z} \in \phi_\tau^{-1}(\{z\})$ such that $co_{\overline{\Gamma}}(\overline{x}, \overline{y})$, $co_{\overline{\Gamma}}(\overline{y}, \overline{z})$ and $co_{\overline{\Gamma}}(\overline{z}, \tau(\overline{x}))$, is called a **triple jump**, and **proper** iff $li_{\overline{\Gamma}}(x_b, z_b)$; cf. Figure 11. A **double jump** is a pair $(x, y) \in \mathcal{X}^2$ such that there are $\overline{x} \in \phi_\tau^{-1}(\{x\})$ and $\overline{y} \in \phi_\tau^{-1}(\{y\})$ satisfying $co(\overline{x}, \overline{y})$ and $co(\tau(\overline{x}), \overline{y})$.

Theorem 4.17 (Triple jump criterion) *With the above notation, let* $\overline{\Gamma}$ *be a partial order wound to* Φ *w.r.t.* τ. *If there exists a strange clique* $\kappa \in \mathcal{CLI}(co_\Phi)$, $\phi := \phi_\tau$ *has at least one double or triple jump.*

Proof: Choose κ minimal, let $x \in \kappa$ and $x_0 \in \phi^{-1}(\{x\})$; set $x_k := \tau^k(x_0)$ and $\kappa_k := co[x_k] \cap \kappa$ for $k \in \mathbb{Z}$. If $\kappa_0 \cap \kappa_1 \neq \emptyset$, we have a double jump and are done. So assume $\kappa_0 \cap \kappa_1 = \emptyset$; then

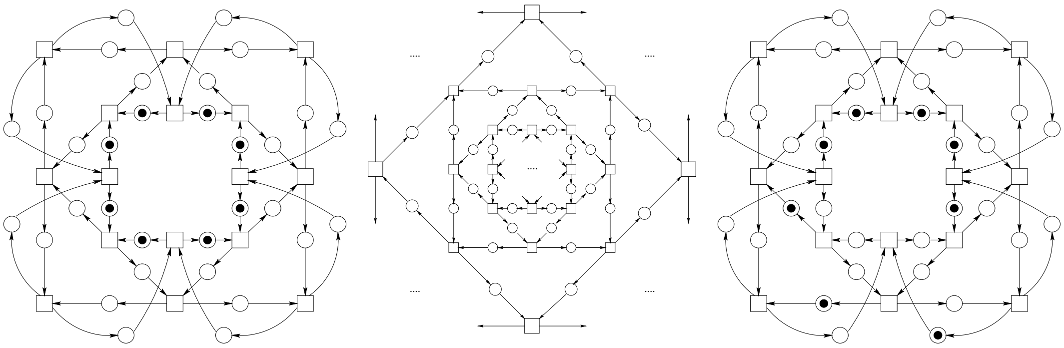

Figure 12: A net marked in a regular (left; center: An unwinding) and in a strange cut (right)

for all $y, z \in \kappa$, κ_0 contains exactly one element from $\phi^{-1}(y)$ and $\phi^{-1}(z)$, which we denote y_0 and z_0, respectively. As a consequence, $< (y_{-j}, x_0)$ and $< (x_0, y_j)$ for all $0 \neq j \in \mathbb{N}$. Since κ is strange, there exist $y, z \in \kappa$ such that $< (y_0, z_0)$. But then, with $y_k := \tau^k(x_0)$, it must hold that $co(z_0, y_1)$; for $< (z_0, y_1)$ implies $li(z, y)$, and $< (y_1, z_0)$ entails $< (x_0, z_0)$, both contradicting the assumptions. Therefore, we have a triple jump. □

Note that the triple jump criterion states a sufficient but not necessary condition for the existence of strange cuts. In particular, no strange *co*-cliques are possible, even with double or triple jumps, if all cuts have only two elements. It should be noted that faithfulness can actually be achieved by "repairing"the winding according to the following construction: For every weakly discrete SG $\mathcal{N}$ with an unwinding $\overline{\mathcal{N}}$, there is a number[5] $\kappa = \kappa_{\mathcal{N}}$ such that for all $x \in \mathcal{X}$ and k positive,, neither (x_k, x_0) nor $(x{-}k, x_0)$ are connected to x_0 by fewer than four *co*-steps. Now, ϕ_{4n} is faithful.

Returning to the question of cyclic coarsenings again, one finds (with the above notation):

Lemma 4.18 *If ϕ is faithful and ξ is a coarsening, then ξ is also faithful for co.*

Proof: Under the assumption, the conditions of Part 2 of Theorem 4.14 are satisfied. The rest follows as in Lemma 4.9. □

Acknowledgments: I would like to thank three unknown referees for their comments, which greatly helped to improve on an earlier version of this article. Moreover, I wish to thank the Net Theory group at Hamburg University for discussions and help with an earlier version; special thanks to O. Kummer nad M.-O.Stehr.

[5]This number, which is bounded above by the size of $\mathcal{N}$, depends on the structure of $\mathcal{N}$; we know of no good *general* bounds on it.

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ISSN 0249-6399