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Abstract: This paper investigates structural properties of occurrence (Petri) nets and their interpretation as unfolding semantics of Petri net systems. Occurrence nets (ONs) exhibit three kinds of node relations associated with *causal ordering*, *concurrency*, and *conflict*. We show that ONs can be decomposed in a natural way into substructures in each of which one or two of these relations are empty, namely: *branches*, *trails*, *choices*, *lines*, *cuts* and *alternatives*. All finite systems will be shown to satisfy certain *density* properties, i.e. non-empty intersections of substructures as above. On the semantic level, two established, yet non-equivalent, definitions of unfolding semantics are studied: *branching processes* (introduced by Engelfriet, Winskel et.al) and *branching executions* (Vogler). To both, the structural results apply, and both support appropriate partial order logics of high expressive power. We present two such logics that can be interpreted over occurrence net semantics of either kind: the branching time logic BLC and a non-branching logic LLC whose frame is composed of *choices*, i.e. objects representing horizons of mutually exclusive *cuts* (generalized global states) compatible with the corresponding stage.

Key-words: Concurrency, Petri nets, Semantics, branching processes

(Résumé : tsvp)

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Sur les Sémantiques de Réseau d' actions des Réseaux de Petri

Résumé : Cet article étudie des proprietés structurelles de *réseaux d'actions* (*occurrence nets*) aussi que leur interprétation en tant que sémantique de dépliage de systèmes de réseaux de Petri. Les réseaux d'actions présentent trois sorte differentes de relations de sommets, associées à l'*ordre causal*, à la *concurrence*, et au *conflit*. Nous démontrons qu'ils se décomposent en des sous-structures dans chacune desquelles une ou deux de ces rélations sont vides, à voir, *branches*, *trails*, *choices*, *lines*, *cuts* et *alternatives*. Il sera démontré que tout système fini vérifie certaines proprietés de densité (c.-à-d d'intersections non-vides desdites sous-structures). Au niveau sémantique, deux définitions bien établies, bien que non équivalentes, de sémantique de dépliage sont étudiées, à voir les *processus de branchement* (introduits par Engelfriet et Winskel et al.) et les *exécutions de branchement* (Vogler). Les résultats structurels obtenus avant s'appliquent á toutes les deux, et les deux sémantiques soutiennent des logiques d'ordre partiel á force d'expression très élévée. Nous présentons deux logiques de ce type-là qui permettent d'être interprétées dans les deux types de sémantique de réseaux d'action: la logique BLC à temps branché et une logique LLC sans branchement dont le cadre consiste en des *choices*, c.-à-d objets qui représentent des horizons de *cuts* (états globaux généralisés) s'excluant l'un l'autre et compatibles avec la phase correspondante.

Mots-clé : Concurrence, Réseaux de Petri, Sémantique, Processus de Branchement

1 Introduction

This paper investigates *occurrence nets* (ONs for short) with respect both to their structural properties and their interpretation as branching processes semantics for Petri nets.

Branching Processes (BPs for short) give a partial order representation of system behaviour by means of an unfolding into an occurrence net. A first branching process semantics – which we will present here as EWMMS processes – has been introduced by Nielsen, Plotkin, and Winskel [18] and Engelfriet [4]. McMillan [15], Esparza [5] and Esparza/Römer/Vogler [12], [11] have used BPs to decide temporal logic properties efficiently (avoiding state explosion).

The second semantics, **branching executions**, was introduced in [12][1] The differences of both semantics notwithstanding, the results we derive here will be relevant for both.

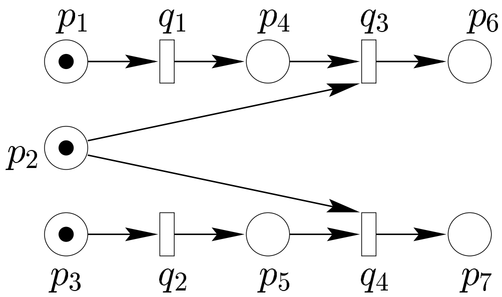

Figure 1: Interplay of conflicts and concurrency

We will introduce structural notions for ONs and propose a suitable CTL^* type temporal logic with concurrency to be interpreted over (the *cuts* of) ONs.

As an introduction, consider the system in Figure 1 (which happens to be an isomorphic copy of its branching process net under both semantics below). Several observations can be made:

1. If a state s was reached where p_6 is marked, any state in which p_7 is marked lies on a path incompatible with s.

2. It is not sure whether the state with p_4 and p_5 marked will ever be reached; e.g., q_3 may fire before q_2.

3. Under progress – which will be assumed throughout –, the system will eventually reach a state – and remain there – in which either p_6 or p_7 is marked (but not both), and in which either p_4 or p_5 is marked (but not both).

[1] where they were called ; in [8], the author introduced that semantics independently using a different terminology; here, we follow [12] because the construction is essentially a branching versions of executions in the sense of [22]

Virbitskaite [21] and others have introduced variants of CTL^* that allow to express alternatives (as in Number 1) and distinguish them from nondeterminism brought about by concurrency (Number 2). But even then, a *trans-branch*-property such as Number 3 is not expressible: within the path, trace, or process semantics, it does not correspond to a **state** of the system. At most one place out of $\{p_6, p_7\}$) can in any actual state be marked; to consider both, a state notion allowing for counter-factual properties (i.e. concerning alternatives *not* chosen) is needed. We will call objects such as $\{p_6, p_7\}$ the *choices* of the process net; a formal definition will be given below.

After studying structural properties of ONs, we will present two different partial order logics interpreted over some ON semantics of a net here (in Section 4). The first, BLC, is interpreted over the lower semi-lattice frame of *cuts* as possible worlds; by contrast, LLC is interpreted over the frame of *choices*, each of which comprises several different global states that could have been reached at this stage.Thus the frame objects, containing in general incompatible elements, are *not* possible worlds as in the case of BLC; in fact, this makes LLC a non-standard temporal logic. Another difference is that the atoms of LLC will not be derived from net nodes but *alternatives* (to be explained below).

The paper is organised as follows: in this section we give the notation and definitions used; in Section 2, two non-equivalent occurrence net semantics – EWMMS *processes* and *branching executions* – are studied, and structural properties of occurrence nets are investigated. The differences between both semantics are restricted to the handling of simultaneously enabled transitions: EWMMS processes treat such transitions as concurrent, while branching processes interleave the transitions in question unless they are structurally independent.

The results on occurrence nets thus apply to both semantics (and might do so for still others as well) since both reflect causality, although the resulting process nets are in general not isomorphic.

Section 4 and 5 then introduce the two logics mentioned above, which will be interpretable over *both* occurrence net semantics.

1.1 Notation

1.1.1 Sets and Multisets

For the following, let M be a set. A **multi-set** over M is a mapping $\chi : M \to \mathbb{N}_0$. For $x \in M$, $\chi(x)$ is the **multiplicity** of x in M. If $\chi(x) > 0$, we write $x \in \chi$. We further denote as $supp(\chi) := \{x \in M : x \in \chi\}$ the set given by χ. The set of multi-sets over a set M is denoted by $\mathcal{MULT}(M)$, the power set (set of subsets) of M by $\wp(M)$. Addition and subtraction of multi-sets are defined as

$$\begin{aligned} (\chi_1 + \chi_2)(x) &:= \chi_1(x) + \chi_2(x) \\ (\chi_1 - \chi_2)(x) &:= \chi_1(x) - \chi_2(x) \ \textit{if}\ \chi_1 \geq \chi_2. \end{aligned}$$

1.1.2 Relations

A binary relation over $\mathcal{X}$ is a set $\Re \subseteq \mathcal{X} \times \mathcal{X}$; the identity relation of $\mathcal{X}$ is

$$id_{\mathcal{X}} \quad := \quad \{(x,x) : \; x \in \mathcal{X}\}.$$

For $x \in \mathcal{X}$, we write $\Re x := \{y \in \mathcal{X} : \; (y,x) \in \Re\}$ and $x\Re := \{y \in \mathcal{X} : \; (x,y) \in \Re\}$. The concatenation $\Re_1 \circ \Re_2$ is the relation

$$\Re_1 \circ \Re_2 \quad := \quad \{(x,y) : \; \exists z \in \mathcal{X} : \; \Re_1(x,z) \wedge \Re_2(z,y)\}.$$

Further, $\Re^+ := \bigcup_{n=1}^{\infty} \Re^n$ and $\Re^* := \Re^+ \cup id_{\mathcal{X}}$. By extension, we also set, for $\theta \in \mathcal{MULT}(\mathcal{X})$:

$$\Re\theta := \bigcup_{x \in \theta} \Re x \quad \mathit{and} \quad \theta\Re := \bigcup_{x \in \theta} x\Re.$$

The **lifting** $[\Re]$ of $\Re$ to $\wp(\mathcal{X})$ is given by

$$[\Re] \quad := \quad \{(m_1, m_2) \in \wp(\mathcal{X}) : \; (m_1 \times m_2) \cap \Re \neq \emptyset\}.$$

1.1.3 Petri Nets

A **Petri net** is a tuple $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$, where

1. $\mathcal{P}$ is a countable set of places,
2. $\mathcal{Q}$ a countable set of transitions with $\mathcal{P} \cap \mathcal{Q} = \emptyset$, and
3. $pre : \mathcal{Q} \to \mathcal{MULT}(\mathcal{P})$ and $post : \mathcal{Q} \to \mathcal{MULT}(\mathcal{P})$ mappings that assign to each transition q its multi-sets $pre^q(post^q)$ of input (output) places, respectively.
4. $\mathcal{M}_0 : \mathcal{P} \to \mathbb{N}_0$ is the initial marking.

We write $\mathcal{P}(\mathcal{N})$ and $\mathcal{Q}(\mathcal{N})$ if necessary to avoid confusion, and set $\mathcal{X}(\mathcal{N}) := \mathcal{P}(\mathcal{N}) \cup \mathcal{Q}(\mathcal{N})$. The subnet of $\mathcal{N}$ spanned by $\mathcal{U} \subseteq \mathcal{X}(\mathcal{N})$ is denoted $\mathcal{N}[\mathcal{U}]$. The numbers $pre^q(p)$ and $post^q$ are the *arc weights* from p to q and from q to p, respectively. The set of **arcs** of $\mathcal{N}$ is given by *pre* and *post* according to

$$\begin{aligned} F \quad := \quad & \{(p,q) \in \mathcal{P} \times \mathcal{Q} : \; p \in pre^q\} \\ & \cup \{(q,p) \in \mathcal{Q} \times \mathcal{P} : \; p \in post^q\}. \end{aligned}$$

Set $\mathcal{Y}(\mathcal{N}) := \mathcal{X}(\mathcal{N}) \cup F$ and $G := [F \cup F^{-1}]$. For a node $x \in \mathcal{P} \cup \mathcal{Q}$, we write

$$Fx := \{y \in \mathcal{P} \cup \mathcal{Q} : \; (y,x) \in F\} \quad \text{and} \quad xF := \{y \in \mathcal{P} \cup \mathcal{Q} : \; (x,y) \in F\};$$

obviously, $Gx = Fx \cup xF$. Further, set $\bigcirc x := Gx \cup \{x\}$. By extension, we also set – for θ a multi-set of nodes –

$$F\theta := \bigcup_{x \in \theta} Fx, \quad \theta F := \bigcup_{x \in \theta} xF, \quad \bigcirc\theta := \bigcup_{x \in \theta} \bigcirc x.$$

Definition 1 *Let $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$ be a net. $\mathcal{N}$ satisfies condition (A) iff*

(A) $F(q) \neq \emptyset \neq (Fq)$ for all $q \in \mathcal{Q}$, and $\mathcal{X}(\mathcal{N})$ is finite.

Remark 1 *To avoid technical case distinctions in defining the process nets, we will henceforth assume that all nets considered satisfy (A).*

A net $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$ such that, for all $q \in \mathcal{Q}$ and $p \in Gq$, $|pre^q(p) \in \{0,1\}|$ and $|post^q(p) \in \{0,1\}|$, is called **ordinary**.

If $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post)$ is an ordinary *unmarked net*, the **dual** net $\mathcal{N}^d$ of $\mathcal{N}$ is obtained by regarding places as transitions and vice versa, i.e. $\mathcal{P}^d = \mathcal{Q}$, $\mathcal{Q}^d = \mathcal{P}$, and for some bijection $d_1 : \mathcal{P} \to \mathcal{Q}^d$,

$$\begin{aligned} pre_d^{d_1(p)} &:= 1_{Fp} \\ post_d^{d_1(p)} &:= 1_{pF}. \end{aligned}$$

A net is a topological space with the topology given by the **boundary** operator $\partial(\bullet)$: For $E \subseteq \mathcal{X}(\mathcal{N})$, set

$$\begin{aligned} \partial^{in}(E) &:= \{x \in E : \exists y \in \mathcal{X}(\mathcal{N}) - E : F(y,x)\} \\ \partial^{out}(E) &:= \{x \in E : \exists y \in \mathcal{X}(\mathcal{N}) - E : F(x,y)\} \\ \partial(E) &:= \{x \in E : \exists y \in \mathcal{X}(\mathcal{N}) - E : G(x,y)\} \\ &= \partial^{in}(E) \cup \partial^{out}(E). \end{aligned}$$

In the net topology on $\mathcal{N}$, E is **open** iff $\partial(E) \subseteq \mathcal{P}(\mathcal{N})$ and **closed** iff $\partial(E) \subseteq \mathcal{Q}(\mathcal{N})$.

Definition 2 *Let $\mathcal{N}_1 = (\mathcal{P}_1, \mathcal{Q}_1, pre_1, post_1, \mathcal{M}_0^1)$, $\mathcal{N}_2 = (\mathcal{P}_2, \mathcal{Q}_2, pre_2, post_2, \mathcal{M}_0^2)$ be nets. A mapping $\phi : \mathcal{X}(\mathcal{N}_1) \to \mathcal{X}(\mathcal{N}_2)$ is a* **morphism** *iff it is continuous in the net topology. A surjective morphism is an* **epimorphism***; an injective epimorphism is an* **isomorphism**.

We now turn towards the dynamics of the net. Any multiset θ over $\mathcal{Q}$ is called a **step**.

Definition 3 *Let $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$ be a net. A step $\theta \in \mathcal{MULT}(\mathcal{Q})$ is* **enabled** *in $\mathcal{M} \in \mathcal{MULT}(\mathcal{P})$, denoted $\mathcal{M} \xrightarrow{\theta}$. iff*

$$\mathcal{M} \geq \sum_{q \in \theta} \theta(q) pre^q.$$

θ **transforms** *$\mathcal{M}$ into $\mathcal{M}'$, denoted $\mathcal{M} \xrightarrow{\theta} \mathcal{M}'$, iff $\mathcal{M} \xrightarrow{\theta}$ and*

$$\mathcal{M}' = \left(\mathcal{M} - \sum_{q \in \theta} \theta(q) pre^q \right) + \sum_{q \in \theta} \theta(q) post^q.$$

A marking $\mathcal{M}$ is **reachable** *from $\mathcal{M}_0$ iff there exists a* **firing sequence**

$$\mathcal{M}_0 \xrightarrow{\theta_1} \mathcal{M}_1 \xrightarrow{\theta_2} \ldots \xrightarrow{\theta_{n-1}} \mathcal{M}_n = \mathcal{M} \quad .$$

1.2 Occurrence nets

For a net $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$, set $<:= F^+$ and $\leq := F^*$. For $x \in S \cup T$, set

$$^\bullet x := \{y : \ (y,x) \in F\}, \ x^\bullet := \{y : (x,y) \in F\}, \ FxF := {}^\bullet x \cup x^\bullet .$$

Definition 4 *Let $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$. The* **conflict** *relation $\#$ on $\mathcal{X}(\mathcal{N})$ is given by: $x\#y$ iff there exist $s \in S$ and $t_1, t_2 \in s^\bullet$ such that $t_1 \neq t_2$, $t_1 \leq x$, and $t_2 \leq y$. We write $\bowtie := (\mathcal{X} \times \mathcal{X}) - \#$ and, for $\mathcal{U} \subseteq \mathcal{X}$, $\max_<$ ($\max_<$) for the set of maximal (minimal) elements of $\mathcal{U}$ w.r.t. $<$.*

Definition 5 *An ordinary net $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$ is called an* **occurrence net** *(or* ON *for short) iff:*

1. **no backward branching**: $|{}^\bullet p| \leq 1$ *for all $p \in \mathcal{P}$;*
2. **Acyclicity***: $\neg(x < x)$ for all $x \in \mathcal{X}(\mathcal{N})$; and*
3. **absence of auto-conflict***: $\neg(x\#x)$ for all $x \in \mathcal{X}(\mathcal{N})$.*
4. **prefix - finite***: $<$ is well-founded, i.e. contains no infinite decreasing sequence, and*
5. **initial cut***: $c_0 := min \subseteq \mathcal{P}$*

If also $|bF| \leq 1$ for all $p \in \mathcal{P}$, $\bar{\mathcal{N}}$ is called a **causal net** *or* CN*. If $|c_0| = 1$, $|Fq| \leq 1$ and $|qF| \leq 1$ for all $q \in \mathcal{Q}$, $\bar{\mathcal{N}}$ is an* **acyclic forward state machine** *or* **AFSM**.

In ONs, more can be derived from the node relations $<$ and $\#$.

Definition 6 *For an* ON *$\bar{\mathcal{N}}$, set*

1. *x li y iff $x < y$ or $y < x$, and*
2. *x co y iff $x \neq y$ and neither x li y nor $x\#y$.*

Denote

1. *the set of maximal co $-$cliques (***cuts***) by $\mathcal{CUTS}(\bar{\mathcal{N}})$,*
2. *the elements of $\mathcal{CUTS}^{\mathcal{P}}(\bar{\mathcal{N}}) := \mathcal{CUTS}(\bar{\mathcal{N}}) \cap \mathcal{P}(S)$ as* **P-cuts***,*
3. *the set of maximal li$-$cliques,* **lines***, by $\mathcal{LINES}(\bar{\mathcal{N}})$, and*
4. *the set of maximal $\#-$cliques – called* **Alternatives** *– by $\nabla(net)$.*

In particular, c_0 is a P-cut, so our terminology above is justified.

Definition 7 *Let $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$ be an* ON*. A set $u \subseteq \mathcal{X}(\bar{\mathcal{N}})$ is called* **line-separating** *iff, for all $l \in \mathcal{LINES}(\bar{\mathcal{N}})$, $u \cap l \neq \emptyset$. $\bar{\mathcal{N}}$ is*

1. **K-dense** *iff all $c \in \mathcal{CUTS}(\bar{\mathcal{N}})$ are line-separating, and*

2. **L-dense** *iff for all $\delta \in \nabla(\bar{\mathcal{N}})$ are line-separating.*

Note that the dual of a CN is an AFSM and vice versa.

A P-cut intersecting every line can be seen as a snapshot of concurrent state(s) of all local processes. If a cut contains non-place elements, this interpretation can of course not be upheld; however, in the context of semantics for CTL^*-type logics, it remains important to locate every non-concurrent state or event either in the future or in the past of the cut considered. This comparability requirement is ensured by K-density. – Below, an analogous density requirement – K^*-density – will lead us to accept $arcs$ as possible elements of cuts.

Obviously, a K-dense ON must be a CN; an L-dense ON is an AFSM whose dual is K-dense..

Remark 2 *Under (A), all branches are K-dense ([1]).*

The following properties are easily verified:

Lemma 1 *If $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$ is an* ON*, the following holds:*

1. *$<$ is a partial order (i.e. irreflexive and transitive);*

2. *li, co , and $\#$ are symmetric and irreflexive;*

3. *$\bowtie$ is symmetric.*

4. *$(\mathcal{P} \cup \mathcal{Q}) \times (\mathcal{Q} \cup \mathcal{P})$ is the disjoint union of id, li, co , and $\#$.*

Moreover, $\bar{\mathcal{N}}$ is a CN *iff $\#$ is empty .*

1.3 CN Projections

We will often regard arcs of an ON as objects of their own, having structural relations with nodes and among each other like the nodes of the net. For this, it is convenient to project an ON $\bar{\mathcal{N}}_O = (\mathcal{P}_O, \mathcal{Q}_O, pre_O, post_O, c_0^O)$ to a CN $\bar{\mathcal{N}}_C = (\mathcal{P}_C, \mathcal{Q}_C, pre_C, post_C, c_0^C)$ in such a way that the places and transitions alike of $\bar{\mathcal{N}}_O = (\mathcal{P}_O, \mathcal{Q}_O, pre_O, post_O, c_0^O)$ become the transitions of $\bar{\mathcal{N}}_2 = (\mathcal{P}_2, \mathcal{Q}_2, pre_2, post_2, c_0^2)$, and the arcs are replaced by places. Formally:

Definition 8 *Let $\bar{\mathcal{N}}_O = (\mathcal{P}_O, \mathcal{Q}_O, pre_O, post_O, c_0^O)$ be an* ON*, $F_O \subseteq (\mathcal{P}_O \times \mathcal{Q}_O) \cup (\mathcal{P}_O \times \mathcal{Q}_O)$ its arc set, and $\bar{\mathcal{N}}_C = (\mathcal{P}_C, \mathcal{Q}_C, pre_C, post_C, c_0^C)$ a net such that there exist bijections $\iota_\mathcal{Q} : \mathcal{X}(\bar{\mathcal{N}}_O) \to \mathcal{Q}_C$ and $\iota_\mathcal{P} : F(\bar{\mathcal{N}}_O) \to \mathcal{P}_C$ such that for all $x_O \in \mathcal{X}_O$ and $p_C \in \mathcal{P}_C$*

$$pre^{\iota_O(x)} \quad = \quad 1_{x \in pr_1(\iota^{-1}(p_C))}.$$

Then $\bar{\mathcal{N}}_C$ is called the **projection net** *of $\bar{\mathcal{N}}_O$.*

One has:

Lemma 2 *If $\bar{\mathcal{N}}_C$ is the projection net of $\bar{\mathcal{N}}_O$, $\bar{\mathcal{N}}_C$ is a* CN*, and $<_O = <_C \circ \iota_Q^{-1}$. Moreover, $\bar{\mathcal{N}}_C$ is the unique* ON *– up to isomorphism – whose projection net is $\bar{\mathcal{N}}_C$.*

2 Branching Processes

Informally speaking, Branching Processes (BPs) are unfoldings of Petri net systems into occurrence nets obtained from the firing rule. This can be done in several ways, depending on the semantic interpretation of concurrent enabling.

2.1 EWMMS Processes

Engelfriet's [4] definition of branching processes requires that $\mathcal{N}$ is ordinary – the definition below does not require this– and the initial marking contains at most one token per place (see also [16]). The following

Definition 9 *Let $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$ be a net, and $\mathcal{M}_0(p) \in \{0,1\}$ for all $p \in \mathcal{P}$. An* EWMMS[2] BP *of $\mathcal{N}$ is a pair $(\mathcal{N}_1, \pi)$, where*

1. *$\mathcal{N}_1 = (\mathcal{P}_1, \mathcal{Q}_1, pre_1, post_1, \mathcal{M}_0^1)$ is an* ON *such that $min(\bar{\mathcal{N}_1}) \in \mathcal{CUTS}^{\mathcal{P}}(\bar{\mathcal{N}})$,*
2. *$\pi\mathcal{X}(\bar{\mathcal{N}_1}) \to \mathcal{X}(\mathcal{N})$ a labelling function such that*
   (a) $\pi(\mathcal{P}_1) \subset \mathcal{P}$ and $\pi(\mathcal{Q}_1) \subset \mathcal{Q}$,
   (b) for all $q_1 \in \mathcal{Q}_1$, let $\pi(q_1) = q$; then
   $$\begin{array}{rcl} \forall p \in Fq: \ pre^q(p) & = & |F_1 q_1 \cap \pi^{-1}(p)| \\ \forall p \in qF: \ post^q(p) & = & |q_1 F_1 \cap \pi^{-1}(p)|. \end{array}$$
   (c) $\forall\ q_1, \bar{q}_1 \in \mathcal{Q}_1 : (F_1 q_1 = F_1 \bar{q}_1 \wedge \pi(q_1) = \pi(\bar{q}_1)) \Rightarrow q_1 = \bar{q}_1$, and
   (d) $\forall\ p \in \mathcal{P}: \ \mathcal{M}_0(p) = |\pi^{-1}(p) \cap c_0|$.

See Figure 2 for an illustration. To extend Definition 9 to more general initial markings, the following auxiliary construction can be used:

Definition 10 *Let $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$ be an arbitrary finite net.*

1. *Add a place p_{start} and a transition q_{start} to $\mathcal{N}$ such that*
   (a) $pre^{q_{start}} = 1_{\{p_{start}\}}$ and $p_{start}F = \{q_{start}\}$, and
   (b) $\forall p \in \mathcal{P} : post^{q_{start}}(p) = \mathcal{M}_0(p)$

   and let $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)a$ be the new net with $\mathcal{M}_0^a = 1_{\{p_{start}\}}$.
2. *Compute a branching process $(\bar{\mathcal{N}}_{(1,a)}, \pi_a)$ for $\mathcal{N}_a$.*
3. *Remove the minimal place and the minimal transition from $\bar{\mathcal{N}}_{(1,a)}$ and let π be the restriction of π_a to the smaller net.*

We will call $(\bar{\mathcal{N}_1}, \pi)$ the EWMMS *(branching) process* of $\mathcal{N}$.

[2] Engelfriet/Winskel/Meseguer/Montanari/Sassone

2.2 Branching executions

The following alternative definition of branching processes, introduced in [8] and [12] , also yields an occurrence net. For 1-safe systems, the two definitions agree (in the sense that the process nets are isomorphic), in general, however, the concepts are quite different.

Definition 11 *Let $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$ be a net. A* **branching execution**[3] *of Σ is a triple $\Pi = (\bar{\mathcal{N}_1}, \pi, l)$, where $\bar{\mathcal{N}_1} = (\mathcal{P}_1, \mathcal{Q}_1, pre_1, post_1, c_0^1)$ is an* ON*, $c_0^1) \in \mathcal{CUTS}^{\mathcal{P}}(\bar{\mathcal{N}_1})$, and*

$$\pi : \mathcal{X}(\bar{\mathcal{N}_1}) \to \mathcal{X}(\mathcal{N}) \quad , \quad l : \mathcal{P}_1 \to \mathbb{N}$$

are mappings such that

1. $\pi(\mathcal{P}_1) \subset \mathcal{P}$ *and* $\pi(\mathcal{Q}_1) \subset \mathcal{Q}$,
2. $\forall\ q_1, \bar{q}_1 \in \mathcal{Q}_1 : (Fq_1 = F\bar{q}_1 \wedge \pi(q_1) = \pi(\bar{q}_1)) \Rightarrow q_1 = \bar{q}_1$,
3. π_{c_0} *is a bijection between* c_0 *and* $\mathcal{P}$,
4. $\forall\ p \in c_0$: $l(p) = \mathcal{M}_0(\pi(p))$,
5. $\forall\ q_1 \in \mathcal{Q}_1$: *with* $\pi(q_1) = q$,
   (a) *for any* $p \in Gq$, *either*
      i. $\exists!\ (p_1, \bar{p}_1) \in [F_1 q_1 \times q_1 F_1]$ *such that* $\pi(p_1) = \pi(\bar{p}_1) = p$ *and* $l(p_1) > pre^q(p)$, *or*
      ii. $\pi^{-1}(p) \cap q_1 F_1 = \emptyset$, *and* $\exists!\ p_1 \in F_1 q_1$ *such that* $\pi(p_1) = p$, *and* $l(p_1) = pre^q(p)$ *and* $post^q(p) = 0$, *or*
      iii. $\pi^{-1}(p) \cap F_1 q_1 = \emptyset$, *and* $\exists!\ \bar{p}_1 \in q_1 F_1$ *such that* $\pi(\bar{p}_1) = p$, *and* $l(\bar{p}_1) = post^q(p)$, *and* $pre^q(p) = 0$.
6. $\forall\ p_1, \bar{p}_1 \in \mathcal{P}_1 : p_1\ co\ \bar{p}_1 \Rightarrow \pi(p_1) \neq \pi(\bar{p}_1)$.

Thus, a firing of transition q is reflected in such a way that q's pre- and post-domain are jointly represented (by corresponding conditions) both in the pre- and the post-domain of the corresponding event in the branching execution, where clean places are suppressed. Thus in the 1-safe case, this and the EWMMS definition are equivalent.

The *structural* mapping π encodes the two-fold effect of any transition, forward and backward, whereas l encodes the *marking behaviour*. In particular, the BP semantics is generated by the *quantitative* change of markings in $\mathcal{N}$, where concurrent writing on the variable containing the marking of a place is prohibited. The 'identity' of a given token, the history of the way past conflicts have been resolved, is ignored by π and l, as it is ignored by the firing rule of Petri nets.

Thus branching executions allow fewer concurrent firing than EWMMS processes but use fewer place elements to represent a state, and abstract more from individual token histories. As

[3] **generalised branching process in [8]**

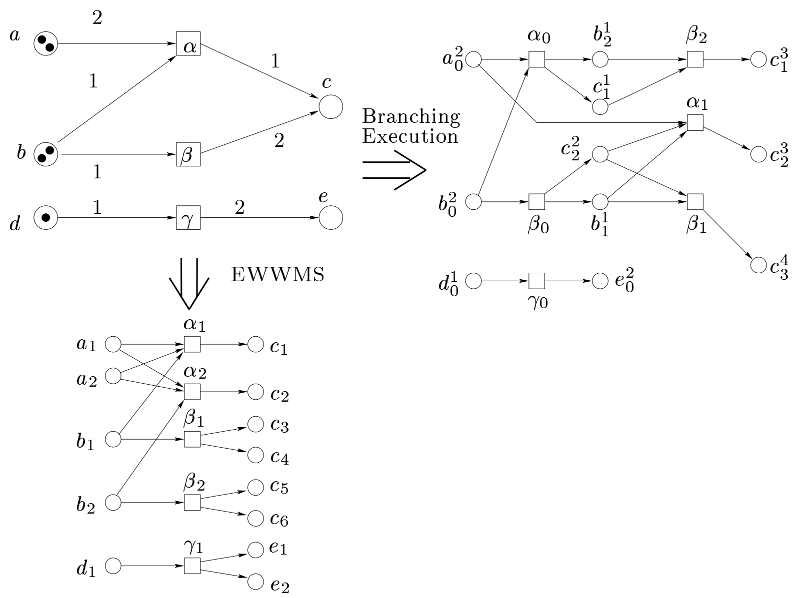

Figure 2: Maximal branching execution and EWMMS process

an example, consider figure 2 which shows the two unfoldings for the same system. The final markings reachable are $\mathcal{M}_1$ and $\mathcal{M}_2$, where

$$M_1(a) = M_1(b) = M_1(d) = 0,\ M_1(c) = 3,\ M_1(e) = 2$$
$$\textit{and} \quad M_2(a) = 2,\ M_2(b) = M_2(d) = 0, M_2(c) = 4,\ M_2(e) = 2 \quad ;$$

this corresponds to the cuts

$$C_1 = \{a_1^0, b_1^0, c_1^3, d_0^0, e_0^2\} \quad \textit{and} \quad C_2 = \{a_0^2, b_2^0, c_2^4, d_0^0, e_0^2\}$$

in the branching execution, and

$$C_1' = \{c_1, c_3, c_4, e_1, e_2\} \quad \textit{or} \quad C_2' = \{c_2, c_5, c_6, e_1, e_2\}$$

and

$$C_1' = \{c_1, c_3, c_4, e_1, e_2\} \quad \textit{or} \quad C_2' = \{c_3, c_4, c_5, c_6, e_1, e_2\},$$

respectively, in the EWMMS process.

The central difference between the two semantics presented is the following: In figure 2, transitions α and β are 'in some way' in conflict with one another, depending on the situation;

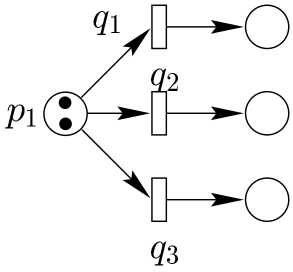

Figure 3: On Concurrency

in the initial marking depicted, they may fire in parallel, but after one of them has fired, α and β 'battle' over the remaining token on b. The *conflict* situation is, as expected, represented by conflict in the branching execution. Now, the *parallelism* is reflected by the presence of all interleavings of the corresponding events in, respectively, those *branches* of the process in which both occur.

On the other hand, true concurrency – by which we mean the *absence* of ordering between events – arises iff the extensions are disjoint: consider β and γ. All pairs of their occurrences are co - pairs in the branching execution; this is the case – and is only possible – for each pair $q, \bar{q}$ of transitions such that $Gq \cap G\bar{q} = \emptyset$.

So we have, altogether, four different ways in which two occurrences q_1 and $\bar{q}_1$ of transitions q, $\bar{q}$ can be related to one another:

1. causal ordering, reflected by li,
2. conflict (#),
3. interchangeability: $Gq \cap G\bar{q} \neq \emptyset$, but the common upstream places contain enough tokens to allow both q and $\bar{q}$ to fire; in this case, the event representing firing of q will occur before that for $\bar{q}$ in some of the branches in the branching execution, and with the order reversed in others; and, finally ,
4. concurrency: $Gq \cap G\bar{q} = \emptyset$, and hence $q_1 \;\; co \;\; \bar{q}_1$.

Note that case 3 can arise even 'between' a transition and itself (i.e. $q = \bar{q}$) whereas, obviously, case 4 cannot.

Returning to the previous discussion, we may say that a collection $q^1, ...q^n$ of two or more transitions accessing the same place (resource, variable, ...) are in proper conflict only if the resources are too limited to allow q^1 through q^n to fire. Otherwise, $q^1, ...q^n$ may fire in arbitrary order and with repetitions ("auto-concurrency"). Put less formally, conflict is interchangeability under limited resources. The semantical differences we just discussed are one justification for the differences in treating independence and parallelism; another is illustrated in figure 3.

In the system, transitions q, r, s are *pairwise* parallel, but at most two of them can actually fire. Thus parallelism lacks the relational property of *upward inheritance*, see [10], while true concurrency – independence here – is *upward hereditary*. A formal distinction between merely parallel and independent collections of transitions, is upheld in branching executions but not EWMMS processes.

To end this section, a few remarks on the relationship between EWMMS processes and branching executions are in order: both definitions coincide in the case of 1-safe net; computing a branching execution of $\mathcal{N}$ is equivalent to producing a 1-safe version $\mathcal{N}'$ of $\mathcal{N}$ by a "folklore" construction probably first described in [20] and then computing a corresponding EWMMS process for $\mathcal{N}'$; and, most importantly for our purposes, both represent reachable markings as P-cuts in a non-ambiguous way.

Figure 4 shows both kinds of processes for the same net (which is a slight variation of that in figure1). The inscriptions (n) and (n, m) at a place p indicate that $\pi(p) = p_n$ and $l(p) = m$ for the two types of processes, respectively.

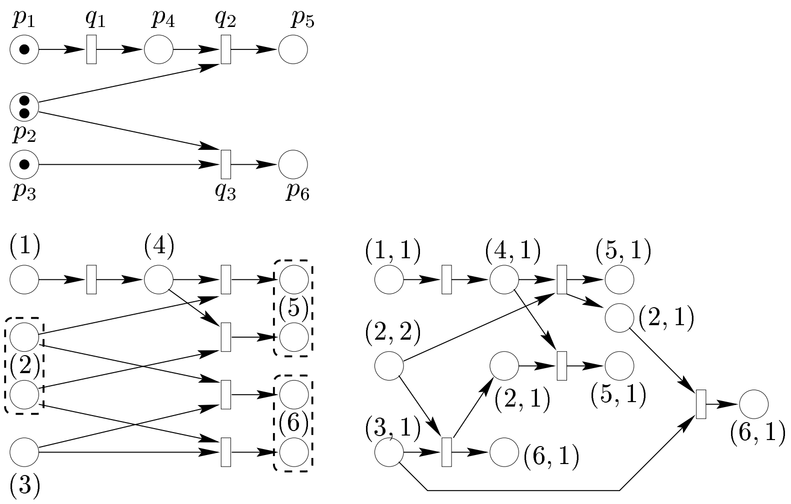

Figure 4: A net with both EWMMS process (left) and branching execution

3 Properties of Branching Processes

3.1 Node Relations

We first consider structural properties of ONs. For a partial order $(\mathcal{X}, <)$, a set $d \subseteq \mathcal{X}$ is **upward closed** iff for all $x \in d$ and $y \in \mathcal{X}$, $x < y$ implies $y \in d$, and **downward closed** iff for all $x \in d$

and $y \in \mathcal{X}$, $y < x$ implies $y \in d$ The upward closure of d is denoted as $d\uparrow$, the downward closure as $d\downarrow$.

Lemma 3 *Let $\mathcal{N}_1 = (\mathcal{P}_1, \mathcal{Q}_1, pre_1, post_1, \mathcal{M}_0^1)$ be an* ON *and $x \in \mathcal{X}_1$.*

1. *x co is open.*
2. *For all $q_1 \in \mathcal{Q}_1$ and $x \in \mathcal{X}(\bar{\mathcal{N}}_1) - \{q_1\}$ such that x co p for any $p \in F_1q_1$, x co q_1.*
3. *For all $q_1 \in \mathcal{Q}_1$ and $x \in \mathcal{X}(\bar{\mathcal{N}}_1) - \{q_1\}$ such that x co p for any $p \in q_1F_1$, x co q_1.*
4. *For all $c_1 \in \mathcal{CUTS}^{\mathcal{P}}(\bar{\mathcal{N}}_1)$ and $q_1 \in \mathcal{Q}_1$ such that $F_1q_1 \subset c$, set $c' := [c - F_1q_1] \cup q_1F_1$. Then $c' \in \mathcal{CUTS}^{\mathcal{P}}(\bar{\mathcal{N}}_1)$.*
5. *$(x\#)$ is upward closed.*
6. *$(x\#)$is (topologically) closed.*
7. *$(x \bowtie)$ is downward closed.*
8. *$(x \bowtie)$ is open.*

Proof: Part 1: We have to show that for all $q_1 \in \mathcal{Q}_1$ and $x \in \mathcal{X}(\bar{\mathcal{N}}_1) - \{q_1\}$ such that x co q_1, x co p_1 for any $p_1 \in G_1q_1$. First, let $p_1 \in F_1q_1$ such that $\neg(x$ co $p_1)$. Then we have to consider the following cases:

1. $x < p_1$: Then also $x \leq q_1$ contradicting the assumption.
2. $p_1 < x$: Then either $q_1 \leq x$ or $q_1\#x$, again a contradiction.
3. $p_1\#x$: Then there exist $\bar{p}_1 \in \mathcal{P}_1$ and $\bar{q}_1, \bar{q}_1' \in q_1F_1$ such that $\bar{q}_1 \neq \bar{q}_1'$, $\bar{q}_1 < p_1$, and $\bar{q}_1' \leq x$. But then $\bar{q}_1 < \bar{q}_1'$ and hence $\bar{q}_1\#x$.

Thus we have contradictions in all cases. For $p_1 \in q_1F_1$, the proof is analogous.

Part 2: Assume $\neg(x$ co $q_1)$.

1. $x < q_1$ implies the existence of $p_1 \in F_1q_1$ such that $x \leq p_1$.
2. $q_1 < x$ implies $p_1 < x$ for all $p_1 \in F_1q_1$.
3. $q_1\#x$ implies the existence of $p_1 \in \mathcal{P}_1$ and $\bar{q}_1, \bar{q}_1' \in p_1F_1$ such that $\bar{q}_1 \neq \bar{q}_1'$ and $\bar{q}_1 \leq q_1$, $\bar{q}_1 \leq x$. But then there exists $p_1 \in F_1q_1$ such that $x\#p_1$.

Part 3: Assume $\neg(x$ co $q_1)$. Again, we have three cases:

1. $x < q_1$ implies $x < p_1$ for all $p_1 \in q_1F_1$.
2. $q_1 < x$ implies the existence of $p_1 \in q_1F_1$ such that $x \leq p_1$.
3. $q_1\#x$ implies the existence of $p_1 \in \mathcal{P}_1$ and $\bar{q}_1, \bar{q}_1' \in b\bar{F}$ such that $\bar{q}_1 \neq \bar{q}_1'$ and $\bar{p}_1 \leq q_1$, $\bar{q}_1 \leq x$. But then $x\#p_1$ for all $p_1 \in q_1F_1$.

Part 4: a consequence of parts 1 – 3 .

Part 5 is immediate from the definitions;

Part 6 is equivalent to the claim that $m_x := (xli) \cup x\ co\ = \mathcal{X} - x\#$ is open. Suppose $q \in m_x$ and $p \in Gq \cap (x\#)$. Then there exists $\tilde{p} \in \mathcal{P}$ and $q_x, q_p \in \tilde{p}F$ such that $q_x \neq_p$ and $q_x \leq x$, $q_p \leq p$. If pFq, this implies $q_p < p < q$ and thus $q\#x$ contradicting the assumption; if qFp, one has $q_p < q < p$ since $Fp = \{q\}$, so again $q\#x$ contradicting the assumption.

Part 7: an immediate consequence of Part 5. and Part 8 is just the dual of Part 6. □

If c' arises from c as described in 4, we write $c[[e[\rangle c'$. We order branching processes according to how far the corresponding net unfolding goes:

Definition 12 *Let* $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$.

1. *If* $\Pi_1 = (\bar{\mathcal{N}_1}, \pi_1, l_1)$ *and* $\Pi_2 = (\bar{\mathcal{N}_2}, \pi_2, l_2)$ *are two branching executions of* $\mathcal{N}$*, then* $\Pi_1 \prec \Pi_2$ *iff* $\bar{\mathcal{N}_1}$ *is isomorphic to an initial segment of* $\bar{\mathcal{N}_2}$.
2. *If* $\Pi_1 = (\bar{\mathcal{N}_1}, \pi_1)$ *and* $\Pi_2 = (\bar{\mathcal{N}_2}, \pi_2,)$ *are two* EWMMS *processes of* $\mathcal{N}$*, then* $\Pi_1 \prec \Pi_2$ *iff* $\bar{\mathcal{N}_1}$ *is isomorphic to an initial segment of* $\bar{\mathcal{N}_2}$.

It can be shown that $\prec$ is a partial order; the most important result concerning $\prec$ is

Theorem 1 *For every system* Σ*, there is – up to isomorphism – a unique* $\prec$*-maximal branching execution* $\Pi_1 = (\bar{\mathcal{N}_1}, \pi_1, l_1)$ *and a unique* $\prec$*-maximal* EWMMS *process* $\Pi_2 = (\bar{\mathcal{N}_2}, \pi_2$ *of* Σ .

Proof: As in Engelfriet [4] or Meseguer, Montanari and Sassone [16]. □

By induction, one finds that both EWMMS processes and branching executions do in fact reflect the behavior of Σ:

Theorem 2 *Let* $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$, $\bar{\mathcal{N}_1} = (\mathcal{P}_1, \mathcal{Q}_1, pre_1, post_1, c_0^1)$ *and* $\bar{\mathcal{N}_2} = (\mathcal{P}_2, \mathcal{Q}_2, pre_2, post_2, c_0^2)$, *and let* $\Pi_1 = (\bar{\mathcal{N}_1}, \pi_1, l_1)$ *be the maximal branching execution and* $\Pi_2 = (\bar{\mathcal{N}_2}, \pi_2)$ *the maximal* EWMMS *process of* $\mathcal{N}$*. Then for every* $n \geq 1$ *and every firing sequence* $M_0[q^0\rangle M_1[...q^n\rangle_n$*, there exist* $c_j^0, ..., c_j^n \in \mathcal{CUTS}^{\mathcal{P}}(\bar{\mathcal{N}_j})$ *and* $q_j^i \in \pi_j^{-1}(q^i)$, $j \in \{1, 2\}$ *and* $\{0, \ldots, n\}$*, such that*

1. $c_j^0[[q_j^0[\rangle c_j^1[[..[\rangle c_j^n$,
2. *for all* $i = 0, ..., n$ *and all* $p \in c_i$, $M_i(\pi_1(p)) = l(p)$, *and*
3. *for all* $i = 0, ..., n$ *and all* $p \in c_i$, $M_i(\pi_2(p)) = |c_i \cap \pi_2^{-1}(\{p\})|$.

One therefore has a correspondence from markings in the system to S-cuts of the branching execution net, in the sense that markings reachable in Σ can be 'reached' in $\bar{\mathcal{N}}$; in fact, $[[\rangle\rangle$ defines a 'pseudo-firing' in the unfolded net that simulates the firings in Σ. If $\mathcal{N}$ is finite[4], the

[4] in fact, the claim made subsequently is valid under more general assumptions; but finiteness is already general enough to cover all practical cases

correspondence is actually two-way, i.e. for every $c \in \mathcal{CUTS}^{\mathcal{P}}(\bar{\mathcal{N}})$ there is a *reachable* marking $\mathcal{M}_c$ represented by c.

For EWMMS processes, one has a 1-1-correspondence of the P-cuts in the maximal process net to those in the maximal branching execution, translating l-values into numbers of representing places.

Note that the theory of alternatives is dual to that of cuts (replacing co by $\#$); hence we have the following property corresponding to Lemma 4:

Lemma 4 *Let $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$. For all $\delta \in \nabla(\bar{\mathcal{N}})$ and $p \in \mathcal{P} \cap \delta$, set*

$$\delta_1 := (\delta - \{p\}) \cup (\{p\} \times pF_{\bar{\mathcal{N}}}) \quad \textit{and} \quad \delta_2 := (\delta - \{p\}) \cup pF_{\bar{\mathcal{N}}}.$$

Then $\delta_1, \delta_2 \in \nabla(\bar{\mathcal{N}})$.

Proof: An application of Lemma 3. □

3.2 Branches

3.2.1 Definition and Properties

Definition 13 *Let $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$ be an* ON*, and*

$$\begin{aligned} \mathcal{V} \quad := \quad & \{\mathcal{U} \subseteq \mathcal{X}(\bar{\mathcal{N}}) : \\ & \mathcal{U} \textit{ is downward closed, and } (\mathcal{U} \times \mathcal{U}) \subseteq \bowtie\}. \end{aligned}$$

If $\mathcal{Z}$ is a maximal element of $\mathcal{V}$, $\bar{\mathcal{N}}[\mathcal{Z}]$ is a **branch** *of $\bar{\mathcal{N}}$. The set of all branches of $\bar{\mathcal{N}}$ is denoted by $\sqsupset(\bar{\mathcal{N}})$.*

As we will see, each branch represents a possible behaviour of the system, with all conflicts resolved. – For the next result, we need an additional assumption on the structure of $\sqsupset(\bar{\mathcal{N}})$:

Definition 14 *An* ON *$\bar{\mathcal{N}}$ satisfies condition (B) iff:*
(B) For all $\beta \in \sqsupset(\bar{\mathcal{N}})$ and $x \in \mathcal{X}(\beta)$, x is $<_\beta$-maximal iff it is $<$-maximal.

Note that the net in Figure 1 violates condition (B): the branch containing p_7 has p_5 as a (relative) maximal element which is not maximal for $\bar{\mathcal{N}}$. Any such *private maximal* element, i.e. maximal w.r.t. one branch but not for $\bar{\mathcal{N}}$, must be a place since branches are open. So an ON violating Condition (B) can be repaired by extending every β in question by a new transition q_β, whose input places are the private maximal places of β, and a new post-place p_β for q_β.

Now we can prove some properties of branches:

Lemma 5 *Let $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$ be an* ON.

1. *$\mathcal{X}(\bar{\mathcal{N}}) = \bigcup_{\beta \in \sqsupset(\bar{\mathcal{N}})} \mathcal{X}(\beta)$.*
2. *Every branch is a* CN.

3. *For all* $\beta \in \beth(\bar{\mathcal{N}})$, $co_\beta = (\ co_{\bar{\mathcal{N}}})_\beta$ *and* $li_\beta = (li_{\bar{\mathcal{N}}})_\beta$.
4. $c_0 \subseteq \mathcal{X}(\beta)$ *for all* $\beta \in \beth(\bar{\mathcal{N}})$.
5. *Under* (A), $\mathcal{CUTS}(\bar{\mathcal{N}}) = \bigcup_{\beta \in \beth(\bar{\mathcal{N}})} \mathcal{CUTS}(\beta)$
6. *Under* (A), $\mathcal{LINES}(net) = \bigcup_{\beta \in \beth(\bar{\mathcal{N}})} \mathcal{LINES}(\beta)$

Proof: Part 1 is immediate; Part 2 follows from the definition of $\#$. Part 3: Let $x, y \in \mathcal{X}(\beta)$. If $x\ co_\beta y$, neither $x <_{\bar{\mathcal{N}}} y$ nor $y <_{\bar{\mathcal{N}}} x$ since $\mathcal{X}(\beta)$ is downward closed. Since $x \bowtie y$, this implies $x\ co_{\bar{\mathcal{N}}} y$. Conversely, $x\ co_{\bar{\mathcal{N}}} y$ excludes $x <_{\bar{\mathcal{N}}} y$ nor $y <_{\bar{\mathcal{N}}} x$, entailing $x\ co_\beta y$. – The result for li is an immediate consequence.

For 5, we first show $\supseteq$. Let $c \in \mathcal{CUTS}(\bar{\mathcal{N}})$; then $c\downarrow$ is downward closed and a clique of $\bowtie$ (otherwise there would be $x, y \in c$ such that $x\#y$ contradicting the definitions), hence $c\downarrow$ can be extended to a branch β_c. By Part 3c is a co -clique of β_c; we have to show maximality. Let $x \in \mathcal{X}(\beta)$ such that $x\ co_{\beta_c} y$ for all $y \in c$; this implies $x\ co_{\bar{\mathcal{N}}} y$ for all $y \in c$, so $c \notin \mathcal{CUTS}(\bar{\mathcal{N}})$, which is a contradiction.

Conversely, let $c \in \mathcal{CUTS}(\beta)$ for some $\beta \in \beth(\bar{\mathcal{N}})$. By Part 3, we only have to show maximality: Let $x \in \mathcal{X}(\bar{\mathcal{N}}) - c$ such that $x\ co_{\bar{\mathcal{N}}} y$ for all $y \in c$. Now $x \in \mathcal{X}(\beta)$ is impossible by assumption, so there exists $\bar{x} \in \mathcal{X}(\beta)$ such that $x\#\bar{x}$. However, since $\bowtie$ is downward closed, $x \bowtie y$ for all $y \in c\downarrow$, hence $\bar{x} \in c\uparrow$. Let $p \in \mathcal{P}$ be such that there exist $q_x, q_{\bar{x}} \in pF$, $q_x \neq q_{\bar{x}}$, and $q_x \leq x$, $q_{\bar{x}} \leq \bar{x}$. If $p \in c\uparrow$, $x \in c\uparrow$; thus $p \in c\downarrow$. Because (A) implies of K -density of β (by Remark 2), there exists $\bar{y} \in c$ such that $q_{\bar{x}} < \bar{y} < \bar{x}$; but then $x\#\bar{y}$, contradicting the assumption.

Part 6: First, let $l \in \mathcal{LINES}(\bar{\mathcal{N}})$; thus we can extend l to a branch β_l, and by Part 3, it follows that $l \in \mathcal{LINES}(\beta)$. Now, let $\beta \in \beth(\bar{\mathcal{N}})$ and $l \in \mathcal{LINES}(\beta)$. If $l \notin \mathcal{LINES}(\beta)$, there exists $x \in (\mathcal{Y}(\bar{\mathcal{N}}) - \mathcal{Y}(\beta))$ such that $\{x\} \times l \subseteq li_\beta$. If there exists $y \in l$ such that $x < y$, $x \in \mathcal{Y}(\beta)$, since $\mathcal{Y}(\beta)$ is downward closed; since this contradicts the assumption, it must hold that $y < x$ for all $y \in l$. By assumption (A), there then exists a maximal element $\bar{y}$ of l. Now, $\bar{y}$ must be $<_\beta$-maximal, since otherwise l would not be a line of β. But by Condition (B), $\bar{y}$ is then maximal in $\bar{\mathcal{N}}$, contradicting $\bar{y} < x$; hence $l \in \mathcal{LINES}(\bar{\mathcal{N}})$. □

3.2.2 Relations between Cuts

Within branches, it is known ([6]) that the restriction to $\mathcal{CUTS}(\beta)$ of the binary relation $\sqsubset \subseteq \mathcal{X}(\beta) \times \mathcal{X}(\beta)$ given by

$$\begin{aligned} c_1 \sqsubset c_2 \quad :\Longleftrightarrow \quad & (\forall y_1 \in c_1, y_2 \in c_2 : \neg (y_2 < y_1) \\ & \wedge \exists\ x_1 \in c_1,\ x_2 \in c_2 :\ x_1 < x_2) \end{aligned}$$

is a partial order and that $(\mathcal{CUTS}(\beta), \sqsubset)$ is a conditionally complete sub-lattice with

$$\begin{aligned} \forall \mathcal{S} \subseteq \mathcal{CUTS}(\beta) : \bigvee \mathcal{S} &= \max_{<} \bigcup \mathcal{S} \\ \bigwedge \mathcal{S} &= \min_{<} \bigcup \mathcal{S}, \end{aligned}$$

and its restriction to $\mathcal{CUTS}^{\mathcal{P}}(\beta)$ is a conditionally complete sub-lattice.

Remark 3 *Note that $\sqsubset$ does in general not coincide with $[<]$.*

For any partial order $<$, the covering relation $<^\circ$ for $<$ is given by

$$x <^\circ y \quad :\iff \quad x < y \ \wedge \ \forall z : \ x \ \leq \ z \ \leq \ y \Rightarrow z \in \{x, y\}.$$

The net topology lifts to $\mathcal{CUTS}$ as follows:

Definition 15 *A set $\mathcal{O} \subseteq \mathcal{CUTS}(\bar{\mathcal{N}})$ is* **open** *iff $\tilde{\mathcal{O}} := \bigcup_{c \in \mathcal{O}} c$ is open in $\bar{\mathcal{N}}$.*

Lemma 6 *$\mathcal{O} \subseteq \mathcal{CUTS}(\beta)$ is open in $\mathcal{CUTS}(\beta)$ iff it is open in $\mathcal{CUTS}(\bar{\mathcal{N}})$.*

Proof: Follows from Part 1 of Lemma 3 and Part 5 of Lemma 5 □

Lemma 7 *Let $\bar{\mathcal{N}}$ be an* ON *and $\mathcal{U} \subseteq \beth(\bar{\mathcal{N}})$. Then $\bar{\mathcal{U}}$ is open, where*

$$\bar{\mathcal{U}} \quad := \quad \bigcap_{\beta \in \mathcal{U}} \mathcal{X}(\beta)$$

Proof: Follows from Part 8 of Lemma 3. □

We consider now the lifting $[\#]$ of $\#$ to $\mathcal{CUTS}(\bar{\mathcal{N}})$ and the relation $\hat{co}$ on $\mathcal{CUTS}(\bar{\mathcal{N}})$ defined as follows:

Definition 16 *Let $a, b \in \mathcal{CUTS}(\bar{\mathcal{N}})$. Then we write*

$$a \ \hat{co} \ b \quad \textit{iff} \quad (a \vee b \ \textit{exists} \ \wedge \ (a \vee b) \notin \{a, b\}).$$

One observes:

Lemma 8 *For $a, b \in \mathcal{CUTS}(\bar{\mathcal{N}})$, $a \ \hat{co} \ b$ iff*

1. *$(a \times b) \subseteq \bowtie$, and*
2. *$a \not\sqsubseteq b$ and $b \not\sqsubseteq a$.*

Moreover, the following properties result from the absence of auto-conflict in ONs and the order minimality of c_0:

Lemma 9 *Let $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$.*

1. *$a, b \in \mathcal{CUTS}(\bar{\mathcal{N}})$, $a[\#]\ b$ iff a and b do not have a common upper bound; in particular, $a \vee b$ does not exist.*
2. *$c_0[\#] = \emptyset$.*

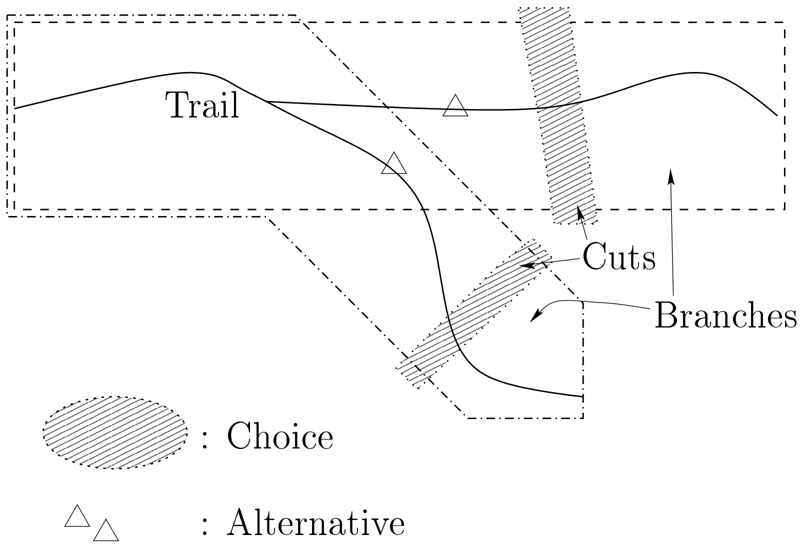

Figure 5: A schematic view of some substructures of ONs

3.3 Trails

Definition 17 *Let $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$ be an* ON. *A maximal clique of $(li \ \cup \ \#)$ is called a* **trail** *of $\bar{\mathcal{N}}$. The set of trails of $\bar{\mathcal{N}}$ is denoted $\digamma(\bar{\mathcal{N}})$.*

Theorem 3 *For all $\tau \in \digamma(\bar{\mathcal{N}})$ and all $\beta \in \beth(\bar{\mathcal{N}})$, $\tau \cap \mathcal{Y}(\beta) \in \mathcal{LINES}(\bar{\mathcal{N}})$*

Proof: Set $l := \tau \cap \mathcal{Y}(\beta)$; we will show $l \in \mathcal{LINES}(\beta)$, which suffices because of Lemma 5. By construction, $l \times l \subseteq li_\beta$, so it remains to show maximality of l. Let $x \in \mathcal{X}(\beta) - l$ such that $l \times \{x\} \subseteq li$, and $y \in \tau - l$: $(x \; co \; y)$. Assume $y \in \mathcal{P}(\bar{\mathcal{N}})$. .If $y \in c_0$, $y \in \mathcal{Y}(\beta)$ and therefore $y \in l$, a contradiction. Hence y cannot be minimal; thus there is a unique $q \in \mathcal{Q}$ such that qFy. $x\#q$ implies $x\#y$ and is therefore impossible; $x < q$ implies $x < y$ contradicting the assumption; $q < x$ implies $q \in \mathcal{X}(\beta)$ and thus $y \in \mathcal{X}(\beta)$, which also contradicts the assumption. Thus $q \; co \; x$. Let $\{p\} = \tau \cap pre^q$ (note that, as above, q cannot be minimal); then $x \; co \; p$. If $p \notin l$, repeat the above argument; we can hence assume $p \in l$. But by construction, $p \notin x$; hence l is maximal. □

Thus trails are composed of lines within different branches, rooting in a unique element of c_0.

3.4 Choices

We now turn towards *choices.*

As hinted at above, we will now consider *arcs* as possible elements of cuts (and, firstly, of choices, see below), along with nodes as before. Obviously, arcs have no "state" interpretation in themselves but mark possible intersections of the state snapshots we will consider with the lines representing local processes. As above, the comparability requirement that any element should be concurrent, in the future or in the past of the snapshot, leads to requiring line separation; this, as we will see in an example, can only be guaranteed by including arcs.

For the formal preliminaries, let $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$, and extend the relations $\#$ and co to $\mathcal{X}(\bar{\mathcal{N}}) \cup F_{\bar{\mathcal{N}}}$ as follows:

1. $(x, y,) < z$ iff $y \leq z$, $x < (y, z)$ iff $x \leq y$, $(x, y) < (z_1, z_2)$ iff $y \leq z_1$;
2. $(p, q)\#x$ (or $(q, p)\#x$) iff $q\#x$;
3. $(x_1, y_1)\#(x_2, y_2)$ iff $y_1\#y_2$;
4. $u \ co \ v$ iff neither $u \leq v$ nor $v < u$ nor $u\#v$.

The inheritance properties from Lemma 3 carry over up to the obvious modifications. Maximal cliques of co on $\mathcal{X}(\bar{\mathcal{N}}) \cup F_{\bar{\mathcal{N}}}$ are called *weak cuts*, and their set is denoted by $\mathcal{CUTS}^{\mathcal{W}}(\bar{\mathcal{N}})$. For notational convenience, we set $\mathcal{Y}(\bar{\mathcal{N}}) := \mathcal{X}(\bar{\mathcal{N}}) \cup F_{\bar{\mathcal{N}}}$.

Definition 18 (compare [9]) *Let $\bar{\mathcal{N}}_O = (\mathcal{P}_O, \mathcal{Q}_O, pre_O, post_O, c_0^O)$ be an* ON *with arc set F_O and projection net $\bar{\mathcal{N}}_C$ of $\bar{\mathcal{N}}_O$ according to Definition 8. A* **choice** *of $\bar{\mathcal{N}}$ is a set $\gamma \subseteq \mathcal{X}(\bar{\mathcal{N}}_O) \cup F_O$ such that $\iota(\gamma) \in \mathcal{CUTS}(\bar{\mathcal{N}}_C)$.*

Thus choices represent possible *state horizons* of the systems, i.e. complete collections of states the system could have reached at similar stages of its evolution, had some decisions been taken this instead of that way. In general, choices may contain arcs.

Definition 19 *Let $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$ be an* ON *and $\gamma \in \daleth(\bar{\mathcal{N}})$. γ is called* **line-separating** *iff $\forall l \in \mathcal{LINES}(\bar{\mathcal{N}}) : \ l \cap \gamma \neq \emptyset$. $\bar{\mathcal{N}}$ is called* **weakly K-dense** *iff all $\gamma \in \daleth(\bar{\mathcal{N}})$ are line-separating.*

Lemma 10 *Under (A),* EWMMS *process and branching executions nets are weakly K-dense.*

Proof: Under (A), the projection nets are of **finite width**, i.e. all cuts are finite; such CNs are known to be K-dense ([19], [1]), from which the claim follows by reversing the projection. □

Consider choices restricted to nodes; following Rozenberg and Engelfriet [2], we denote such sets as *slices*. However, in general, slices are not line-separating: consider Figure 6. While the slice $\{p_5, p_6\}$ is a choice (and line-separating), this does not hold for the slice $u := \{p_1, q_3\}$ sketched by dashed lines: u misses the line connecting p_2 and p_5.

Now, $(\daleth(\bar{\mathcal{N}}), \sqsubset)$ is obviously a partial order; one even has:

Lemma 11 *Let $\bar{\mathcal{N}}_O = (\mathcal{P}_O, \mathcal{Q}_O, pre_O, post_O, c_0^O)$ be an* ON *such that $pre^q \neq \emptyset$ for all $q \in \mathcal{Q}$, and $\bar{\mathcal{N}}_C = (\mathcal{P}_C, \mathcal{Q}_C, pre_C, post_C, c_0^C)$ the projection net of $\bar{\mathcal{N}}$. $(\daleth(\bar{\mathcal{N}}), \sqsubset)$ is a conditionally complete lattice with, for $\mathcal{C} \subseteq \daleth(\bar{\mathcal{N}})$,*

$$\bigwedge \mathcal{C} = \iota_{\mathcal{Q}}^{-1}\left(\bigwedge_{\gamma\in\mathcal{C}} \iota_{\mathcal{Q}}(\gamma)\right)$$

$$\bigvee \mathcal{C} = \iota_{\mathcal{Q}}^{-1}\left(\bigvee_{\gamma\in\mathcal{C}} \iota_{\mathcal{Q}}(\gamma)\right)$$

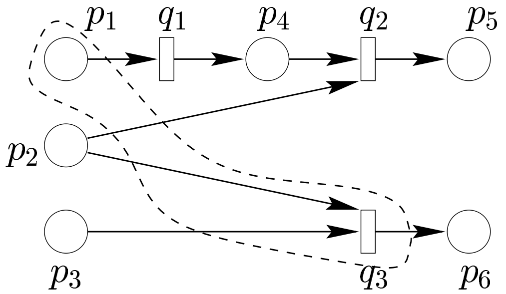

Figure 6: A non-separating slice

The interpretation of choices as *horizons of states* is supported by the following fact:

Theorem 4 *Let $\mathcal{N}$ satisfy Conditions (A) and (B). Let $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$ be a process* ON *for $\mathcal{N}$, and $\gamma \in \daleth(\bar{\mathcal{N}})$. Then for each $\beta \in \beth(\bar{\mathcal{N}})$, $(\gamma \cap \beta) \in \mathcal{CUTS}^{\mathcal{W}}(\bar{\mathcal{N}})$.*

Proof: Obviously, $c_\beta := \gamma \cap \beta$ is a clique of co_β; it remains to show that c_β is maximal. Suppose there exists $x \in \mathcal{Y}(\beta) - \gamma$ and such that $\{x\} \times c_\beta \subseteq co_\beta$. Let $l_x \in \mathcal{LINES}(\beta)$ be any line containing x. Then, by Lemmas 6 and, 10, there exists $y \in l_x \cap \gamma$; by construction, $y \in c_\beta$. But then $x\ li\ y$, which contradicts the assumption. □

Dually, choices intersect with trails in alternatives:

Theorem 5 *Let $\mathcal{N}$ satisfy Conditions (A) and (B). Let $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$ be a process* ON *for $\mathcal{N}$, and $\gamma \in \daleth(\bar{\mathcal{N}})$. Then for each trail $\tau \in \digamma(\bar{\mathcal{N}})$, $(\gamma \cap \tau) \in \nabla(\bar{\mathcal{N}})$.*

Proof: Let $\delta := (\beta \cap \tau)$; δ is a clique of $\#$ by construction; it remains to show maximality. Let $x \in \mathcal{X} - \delta$ be such that $(\{x\} \times \delta) \subseteq \#$, and let β_x be a branch containing x. By Theorem 4, there exists a cut $c_\delta := (\gamma \cup \beta_x) \in \mathcal{CUTS}^{\mathcal{W}}(\beta_x)$, and by Theorem 3, a line

$l_\delta := (\beta_x \cap \tau) \in \mathcal{LINES}(\beta)$; by K-density of β (Remark 2), $l\delta \cap c_\delta = \{u\}$ for some $u \in \beta_x$. By construction, $u \in \delta$; however $u \bowtie x$ contradicting the assumption.

□

We may thus regard choices as basic temporal entities, composed from (it exhaustive) alternatives. Alternatives will yield the atomic propositions in the "choice logic" LLC which we will introduce in Section 5.

4 BLC: A branching Partial Order Temporal Logic for ONs processes

The logics proposed here are both interpreted in an occurrence net semantics; that is, let $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$ be a finite net satisfying (A) and $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$ an occurrence net satisfying (B), which can be given by the maximal branching execution or the maximal EWMMS process of $\mathcal{N}$. We will give the definition of the state based version for both logics; by including $\mathcal{Q}(\bar{\mathcal{N}})$ into the set of elementary formulae, **events** can be considered as well. A P-cut $c_\mathcal{P} \in \mathcal{CUTS}^\mathcal{P}(\bar{\mathcal{N}})$ can be seen as a possible global state of the system, while a non-P-cut $d \in \mathcal{CUTS}(\bar{\mathcal{N}}) - \mathcal{CUTS}^\mathcal{P}(\bar{\mathcal{N}})$ is a "time slice " (compare, e.g., the logic ESTL, Kindler et al. [14]). So for BLC, Ξ^c denotes either $\mathcal{CUTS}^\mathcal{P}(\bar{\mathcal{N}})$ or $\mathcal{CUTS}(\bar{\mathcal{N}})$. However, we will not make the corresponding distinction in the context of LLC where we will often have arcs as elements of choices.

Note that, in order to ensure condition (B), it may be necessary to add new elements as above; we denote the set of places (transitions) added in this way as $\mathcal{P}_B$ $(\mathcal{Q}_B)$. The interpretation functions have to be adjusted for that.

We further assume – as is common practice in the context of CTL^* type logics ([3], [17]) – that there exist no maximal cuts; that is, if necessary, to each such a cut c we have to add a loop $c \to c$ to $\sqsubset$ in order to ensure that in c, $EG\phi$ holds for every ϕ that holds in c (accordingly for the choices). Note that we do not need to add new *elements*.

So BLC (and LLC below) will in fact be interpreted over a quasi-order $\check{\sqsubset}$ derived from $\sqsubset$; $\check{\sqsubset}$ will be a partial order iff it coincides with $\sqsubset$, i.e. iff $\bar{\mathcal{N}}$ has no maximal cuts.

4.1 BLC: A branching partial order logic on the frame of cuts

To translate elements of an ON $\bar{\mathcal{N}}$ into predicates, let $\mathcal{A}$ be a set of atomic propositions and $\lambda : (\mathcal{Y}\bar{\mathcal{N}} \cup \mathcal{P}_B \cup \mathcal{Q}_B) \to \mathcal{A}$. In general, $\lambda_{|\mathcal{Y}\bar{\mathcal{N}}} \equiv \pi$; in special cases, however, more can be coded into $\mathcal{A}$ and λ. If, for example, $\mathcal{N}$ is k-bounded, and we are given a maximal branching execution $\Pi = (\bar{\mathcal{N}}, \pi, l)$, set $\mathcal{A} := \mathcal{Q} \ \cup \ (\mathcal{P} \times \{1, \ldots, k\})$, and

$$\lambda(x) \quad := \quad \begin{cases} \pi(x) & : \quad x \in \mathcal{Q}(\bar{\mathcal{N}}) \\ (\pi\,(x)\,, l\,(x)) & : \quad x \in \mathcal{Q}(\bar{\mathcal{N}}). \end{cases}$$

In a similar way, one can, e.g., introduce counters for the number of occurrences of a given transition in some branch, provided the length of branches is bounded.

4.1.1 Syntax of BLC

Let $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$ satisfy (A) and (B), and $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$ a maximal process of $\mathcal{N}$. Further, let x range over $\mathcal{X}(\bar{\mathcal{N}})$ or $\mathcal{P}(\bar{\mathcal{N}})$, depending on Ξ^γ.

1. $\lambda(x)$ is a BLC formula.
2. If ϕ and ψ are BLC formulae, then so are
   (a) $\neg(\phi)$ and $(\phi \vee \psi)$,
   (b) $P\phi$, $H\phi$ and $\mathcal{C}\phi$,
   (c) $EF\phi$,$EX\phi$, $EG\phi$, $EG\phi$, and $E(\phi U \psi)$,
   (d) $AF\phi$, $AG\phi$,$AP\phi$, $AG\phi$, $AH\phi$ and $E(\phi U \psi)$,
   (e) $O^a\phi$ and $O^e\phi$.

As usual, $(p \wedge q)$ and $(p \to q)$ are used as abbreviations for $\neg(\phi \vee \neg\psi)$ and $(q \vee \neg q)$.

Before the formal definitions, a few informal comments on the intuitive meaning of the operators in BLC.

P can be read as "*at some stage in the past*", H as "*always in the past*". C refers to a cut in $\hat{co}$, i.e. which is not exclusive, but non-comparable with the current one. – The operators in 2c and 2d are almost as in CTL^* ([17]), the difference being that the interpretation over *paths* is replaced by that over *branches*, which correspond – in general – to several different paths in the associated transition system (reachability graph). Finally, O^a and O^e are operators connecting to alternative, i.e. inaccessible, branches (compare [21]), with O^e containing an *existential* and O^a an *all* quantification.

4.1.2 Semantics of BLC

Let c range over $\Xi^c(\mathcal{N})$ and x over $\mathcal{X}(\bar{\mathcal{N}})$ or $\mathcal{P}(\bar{\mathcal{N}})$, depending on Ξ^c .

1. $\mathcal{N}, c \models \lambda(x)$ iff $x \in c$.
2. $\mathcal{N}, c \models \phi \vee \psi$ iff $\mathcal{N}, c \models \phi$ and $\mathcal{N}, c \models \psi$.
3. $\mathcal{N}, c \models \neg\phi$ iff $\mathcal{N}, c \not\models \phi$.
4. $\mathcal{N}, c \models P\phi$ iff $\exists c' \in \Xi^c(\bar{\mathcal{N}})$: $c' \sqsubset c$ and $\mathcal{N}, c' \models \phi$.
5. $\mathcal{N}, c \models H\phi$ iff $\forall c' \in \Xi^c(\bar{\mathcal{N}})$: $c' \sqsubset c \Rightarrow \mathcal{N}, c' \models \phi$.
6. $\mathcal{N}, c \models \mathcal{C}\phi$ iff $\exists c' \in \Xi^c(\bar{\mathcal{N}})$: $c' \hat{co} c$ and $\mathcal{N}, c' \models \phi$.
7. $\mathcal{N}, c \models EF\phi$ iff $\exists c' \in \Xi^c(\bar{\mathcal{N}})$: $c \sqsubset c'$ and $\mathcal{N}, c' \models \phi$.
8. $\mathcal{N}, c \models AF\phi$ iff $\forall \beta \in \beth(\bar{\mathcal{N}})$: $c \in \Xi^c(\bar{\mathcal{N}}) \Rightarrow \exists c' \in \Xi^c(\beta)$: $\mathcal{N}, c' \models \phi$ and $c \sqsubset c'$.
9. $\mathcal{N}, c \models EX\phi$ iff $\exists c' \in \Xi^c(\bar{\mathcal{N}})$: $c \sqsubset^\circ c'$ and $\mathcal{N}, c' \models \phi$.

10. $\mathcal{N}, c \models AX\phi$ iff: $\forall c' \in \Xi^c(\bar{\mathcal{N}})$: $c \sqsubset^\circ c' \Rightarrow \mathcal{N}, c' \models \phi$
11. $\mathcal{N}, c \models EG\phi$ iff $\exists \beta \in \beth(\bar{\mathcal{N}})$: $c \in \Xi^c(\bar{\mathcal{N}})$ and $\forall c' \in \Xi^c(\beta)$: $c \sqsubset c' \Rightarrow \mathcal{N}, c' \models \phi$.
12. $\mathcal{N}, c \models AG\phi$ iff: $\forall c' \in \Xi^c(\bar{\mathcal{N}})$: $c \sqsubset c' \Rightarrow \mathcal{N}, c' \models \phi$
13. $\mathcal{N}, c \models E(\phi U \psi)$ iff there exists $c' \in \Xi^c(\bar{\mathcal{N}})$:
    $c \sqsubset c'$ and $c' \models \psi$ and $\forall c'' \in \Xi^c(\bar{\mathcal{N}})$: $c \sqsubseteq c'' \sqsubset c' \Rightarrow c'' \models \phi$.
14. $\mathcal{N}, c \models A(\phi U \psi)$ iff for all $\beta \in \beth(\bar{\mathcal{N}})$ such that $c \subseteq \mathcal{X}(\beta)$, there exists $c' \in \Xi^c(\bar{\mathcal{N}})$: $c \sqsubset c'$ and $c' \models \psi$ and $\forall c'' \in \Xi^c(\bar{\mathcal{N}})$: $c \sqsubseteq c'' \sqsubset c' \Rightarrow c'' \models \phi$.
15. $\mathcal{N}, c \models O^a\phi$ iff $\forall c' \in \Xi^c(\bar{\mathcal{N}})$:
    $c'[\#]c \Rightarrow \mathcal{N}, c' \models \phi$
16. $\mathcal{N}, c \models O^e\phi$ iff $\exists c' \in \Xi^c(\bar{\mathcal{N}})$:
    $c'[\#]c$ and $\mathcal{N}, c' \models \phi$.

4.1.3 Rules

Let ϕ and ψ be BLC formulae.

1. $\mathcal{N}, c \models \phi$ implies
   (a) $\mathcal{N}, c \models EG\phi$ iff $\mathcal{N}, c \models EGEG\phi$
   (b) $\mathcal{N}, c \models AG\phi$ iff $\mathcal{N}, c \models AGAG\phi$
   (c) $\mathcal{N}, c \models EF\phi$ iff $\mathcal{N}, c \models EFEF\phi$
   (d) $\mathcal{N}, c \models AF\phi$ iff $\mathcal{N}, c \models AFAF\phi$
   (e) $\mathcal{N}, c \models H\phi$ iff $\mathcal{N}, c \models HH\phi$
   (f) $\mathcal{N}, c \models P\phi$ iff $\mathcal{N}, c \models PP\phi$
   (g) $\mathcal{N}, c \models \phi$ implies $\mathcal{N}, c \models HEF\phi$
   (h) $\mathcal{N}, c \models \phi$ implies $\mathcal{N}, c \models AGP\phi$
   (i) $\mathcal{N}, c \models H\phi$ implies $\mathcal{N}, c \models P\phi$
   (j) $\mathcal{N}, c \models EG\phi$ implies $\mathcal{N}, c \models EF\phi$
   (k) $\mathcal{N}, c \models AF\phi$ implies $\mathcal{N}, c \models EF\phi$
   (l) $\mathcal{N}, c \models AG\phi$ implies $\mathcal{N}, c \models EG\phi$ and $\mathcal{N}, c \models AF\phi$
   (m) $\mathcal{N}, c \models C\phi$ implies $\mathcal{N}, c \models PEF\phi$ and $\mathcal{N}, c \models AGP\phi$
   (n) $\mathcal{N}, c \models A(\phi U \psi)$ implies $\mathcal{N}, c \models E(\phi U \psi)$
   (o) $\mathcal{N}, c \models O^a\phi$ implies $\mathcal{N}, c \models O^e\phi$

Further, let $p, p_1, p_2 \in \mathcal{P}(\bar{\mathcal{N}})$, $q, q_1, q_2 \in \mathcal{Q}(\bar{\mathcal{N}})$, $x, x_1, x_2 \in \mathcal{X}(\bar{\mathcal{N}})$, $m \subseteq \mathcal{X}(\bar{\mathcal{N}})$ finite, and $c \in \Xi^c(\bar{\mathcal{N}})$.

Theorem 6 *Let ϕ, ψ be* BLC *formulae.*

1. *If $x_1 \# x_2$ or $x_1 < x_2$ then $\mathcal{N}, c \models H\neg(x_1 \wedge x_2) \wedge \neg(x_1 \wedge x_2) \wedge AG\neg(x_1 \wedge x_2)$.*
2. *$\mathcal{N}, c_0 \models (x \vee EFx)$*
3. *If $x_1 \# x_2$ and $\mathcal{N}, c \models x_1$ then $\mathcal{N}, c \models ((\neg x_2 \wedge AG\neg x_2) \wedge H\neg x_2)$*

Remark 4 *An asymmetry between past and future results from the forward-branching-only of occurrence nets. It is reflected by the fact that P and H are not E/A - quantified over: from any cut c, there is only one non-branching (if partial order) past leading up to c.*

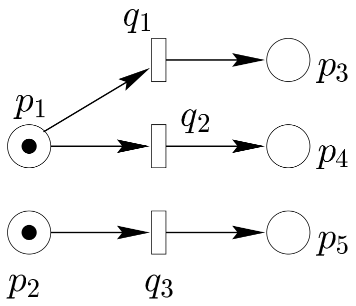

Figure 7: On the C-Operator of BLC

Note that statements containing C formulae are of a particular kind, belonging neither to the past nor to an alternative branch. It is possible, if it is desired to express such properties, to allow E/A quantification over C, since there are nontrivial situations as in figure 7: on has

$$\begin{aligned} \mathcal{N}, \{p_1, p_5\} &\models C\{p_2, p_3\} \\ \mathcal{N}, \{p_1, p_5\} &\models C\{p_2, p_4\}, \end{aligned}$$

and the cuts $\{p_2, p_3\}$ and $\{p_2, p_4\}$ belong to alternative branches. This leads to variants of BLC that are of independent interest.

4.2 Limits of Expressiveness: trans-branch requirements

Consider again figure 1. Informally, it holds that

$$\textit{it will always be the case that eventually either } p_6 \textit{ or } p_7 \textit{ is marked.} \tag{1}$$

In BLC, we have (suppressing λ)

$$\mathcal{N}, c_0 \models AG(EFp_6 \vee EFp_7)$$

which is weaker than (1) since it allows runs that never actually reach a state with p_6 or with p_7. The statement

$$\mathcal{N}, c_0 \models AGAF(p_6 \vee p_7) \quad , \tag{2}$$

has lesser power of distinction than (1): it holds, e.g., also if p_6 inevitably holds eventually.

While E/A operators cannot express any *trans-branch properties*, we have , in BLC, the operators O^e and O^a which permit that: in the present example, we have

$$\mathcal{N}, c_0 \models EF(p_6 O^a AF p_7). \tag{3}$$

Statement 3 connects p_6 and p_7 in a way that shows the quasi-simultaneity of the two nodes across the boundaries of alternative branches, yet not the symmetry of the situation. The fact of the matter is that BLC does not cover counter-factual properties, i.e. invariant properties across branches.

However, such *trans-branch* requirements may arise as intended "no-matter-what" properties (which need not be safety ones); imagine for instance an arbiter modelled as in Figure 1.

In order to account for such properties, we propose to change frames: the logic LLC defined in the next section is interpreted over *choices* rather than cuts, i.e. instead of *possible worlds* we look at "*horizons*" of alternative worlds of the system. The idea is to identify *stages* of the system behaviour in which, returning to the example, *exactly one* out of $\{p_6, p_7\}$ hold, i. e. their simultaneous alternative.

5 LLC: A Lattice Logic on the Frame of Choices

When passing from the frame $(\Xi^c(\bar{\mathcal{N}}), \sqsubset)$ to $(\Xi^\gamma(\bar{\mathcal{N}})))$, all branchings are eliminated since the choices of $\bar{\mathcal{N}}$ form a single "flow" or "stream" of time. The logics LLC to be proposed below will therefore not be a branching time one and "more linear" than BLC because of the absence of E,A and O quantification. However, the order relation $\sqsubset$ is not entirely linear but a lattice; in fact, lattices are the smallest possible generalisation of linear orders allowing for proper independence of local subprocesses.

An important difference resulting from the change of frames is the need to work with alternatives rather than nodes within one frame object; this reflects the fact that these objects are no longer *possible worlds* since, in general, they contain mutually exclusive (i.e. $\#$) elements. Instead, we use as atoms the **alternatives**: by asserting an alternative δ in γ, one states that in any realization, whenever a cut contained in γ is reached, exactly one element of δ is valid in the sense of BLC validity for that cut. So LLC fits into the context of non-sequential algorithms in which several choices (which may, e.g., be representations of correct versus failure behaviour) are subsequently made, and one wishes to ensure progress to certain stages of the program irrespective of the way these choices are made.

The interpretation function λ, see above, is extended to arcs in the obvious way:

$$\lambda((x,y)) \quad := \quad (\lambda(x), \lambda(y)),$$

and lifts to an interpretation function λ on alternatives.

5.0.1 Syntax of LLC

Let $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, pre, post, \mathcal{M}_0)$ satisfy (A) and (B), and $\bar{\mathcal{N}} = (\bar{\mathcal{P}}, \bar{\mathcal{Q}}, pre, post, \bar{c}_0)$ a maximal process of $\mathcal{N}$. Further, let τ range over $F(\mathcal{N})$.

1. $\lambda(\tau)$ is an LLC formula.
2. If ϕ and ψ are LLC formulae, then so are
   (a) $\neg(\phi)$ and $(\phi \vee \psi)$,
   (b) $\triangle\ \phi$, $\nabla\phi$, $\blacktriangle\phi$ and $\blacktriangledown\phi$,
   (c) $\triangle^\circ\ \phi$, $\nabla^\circ\phi$, $\blacktriangle^\circ\phi$ and $\blacktriangledown^\circ\phi$,
   (d) $(\phi\mathcal{U}\psi)$ and $(\phi\mathcal{S}\psi)$.

The operators here are quasi-modal ones: $\triangle$ and ∇ stand for "at some stage in the past" or "at some stage in the future", respectively, while $\blacktriangle$ and $\blacktriangledown$ denote "*always in the past*" or "*always in the future*", respectively. Again, the interpretation is not taken over possible worlds, which is why LLC is not a modal logic in the conventional sense.

5.0.2 Semantics of LLC

Here, let γ range over $\Xi^\gamma(\bar{\mathcal{N}})$, x over $\mathcal{Y}(\bar{\mathcal{N}})$, and m over the finite subsets of $\mathcal{Y}(\bar{\mathcal{N}})$.

1. $\mathcal{N}, \gamma \models \|\ m$ iff $m \in \gamma$ and $m \times m \subseteq\ co$.
2. $\mathcal{N}, \gamma \models \oplus m$ iff $m \in \gamma$ and $m \times m \subseteq\ \#$.
3. $\mathcal{N}, \gamma \models \phi \vee \psi\ \mathcal{N}, \gamma \models \phi$ or $\mathcal{N}, \gamma \models \psi$.
4. $\mathcal{N}, \gamma \models \neg\phi$ iff $\mathcal{N}, \gamma \not\models \phi$.
5. $\mathcal{N}, \gamma \models\triangle\ \phi$ iff $\exists\gamma' :\in \Xi^\gamma(\bar{\mathcal{N}})\ \gamma \sqsubset \gamma'$ and $\mathcal{N}, \gamma' \models \phi$
6. $\mathcal{N}, \gamma \models \nabla\phi$ iff $\exists\gamma' :\in \Xi^\gamma(\bar{\mathcal{N}})\ \gamma' \sqsubset \gamma$ and $\mathcal{N}, \gamma' \models \phi$.
7. $\mathcal{N}, \gamma \models \blacktriangle\phi$ iff $\forall\gamma \in \Xi^\gamma(\bar{\mathcal{N}})$: $\gamma' \sqsubset \gamma \Rightarrow \mathcal{N}, \gamma' \models \phi$
8. $\mathcal{N}, \gamma \models \blacktriangledown\phi$ iff $\forall\gamma' \in \Xi^\gamma(\bar{\mathcal{N}})$: $\gamma \sqsubset \gamma' \Rightarrow \mathcal{N}, \gamma' \models \phi$
9. $\mathcal{N}, \gamma \models (\phi\mathcal{U}\psi)$ iff $\exists\gamma' \in \Xi^\gamma(\bar{\mathcal{N}})$: $\gamma \sqsubset \gamma'$ and $\mathcal{N}, \gamma' \models \psi$ and $\forall\gamma'' \in \Xi^\gamma(\bar{\mathcal{N}})$: $\gamma \sqsubseteq \gamma'' \sqsubset \gamma' \Rightarrow \mathcal{N}, \gamma' \models \phi$
10. $\mathcal{N}, \gamma \models (\phi\mathcal{S}\psi)$ iff $\exists\gamma' \in \Xi^\gamma(\bar{\mathcal{N}})$: $\gamma' \sqsubset \gamma$ and $\gamma' \models \psi$ and $\forall\gamma'' \in \Xi^\gamma(\bar{\mathcal{N}})$: $\gamma' \sqsubset \gamma'' \sqsubseteq \gamma \Rightarrow \mathcal{N}, \gamma' \models \phi$.

Here, we have greater past-future-symmetry than in BLC; there is, however, still an underlying asymmetry in the *unfolding* which

5.0.3 Rules

Theorem 7 *Let ϕ be an* LLC *formula.*

1. $\mathcal{N}, \gamma' \models \blacktriangle\phi$ *implies* $\mathcal{N}, \gamma' \models\vartriangle \phi$.
2. $\mathcal{N}, \gamma' \models \blacktriangledown\phi$ *implies* $\mathcal{N}, \gamma' \models \triangledown\phi$.
3. $\mathcal{N}, \gamma' \models \phi$ *implies* $\mathcal{N}, \gamma' \models \blacktriangle\triangledown\phi$ *and* $\mathcal{N}, \gamma' \models \blacktriangledown \vartriangle \phi$

5.1 Examples

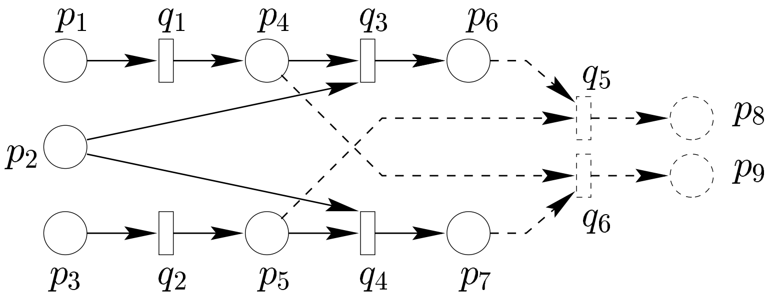

Figure 8: The modified net from Figure 1

Returning to our introductory example, we first notice we have to extend the net to satisfy Condition (B); Figure 8 shows the resulting ON, which we refer to as $\bar{\mathcal{N}}$. Let $\mathcal{N}$ be a net whose maximal process (EWMMS or branching execution) net is $\bar{\mathcal{N}}$; the straightforward solution is to take $\bar{\mathcal{N}}$ itself with $\mathcal{M}_0 = 1c_o$. Then we can state, with $\gamma_0 = \{p_1, p_2, p_3\}$, $\gamma_1 = \{p_2, p_4, p_5\}$, and $\gamma_2 = \{p_8, p_9\}$

$$\begin{array}{lcl} \mathcal{N}, \gamma_0 & \models & \{p_1\} \wedge \{p_2\} \\ \mathcal{N}, \gamma_1 & \models & \vartriangle \{p_1\} \wedge \{p_4\} \wedge \triangledown\{p_6, p_7\} \\ \mathcal{N}, \gamma_2 & \models & \blacktriangledown \vartriangle \{p_6, p_7\} \end{array}$$

Notice that $\{p_1, p_2\}$ is not an alternative, hence

$$\mathcal{N}, \gamma_0 \quad \not\models \quad \{p_1, p_2\}.$$

The fact 1 – adapted to the extended structure – now reads

$$\mathcal{N}, \gamma_0 \quad \models \quad \blacktriangledown\triangledown\{p_8, p_9\}$$

and is valid and derivable.

5.2 BLC Theorems

The BLC theory of $\mathcal{N}$ contains – with $c_0 = \{p_1, p_2, p_3\}$ and $c_1 = \{p_4, p_2, p_5\}$

$$\begin{aligned}\mathcal{N}, c_0 &\models EF((p_3 \wedge p_4) \wedge C(p_1, p_5)\\ \mathcal{N}, c_1 &\models Hp_2\end{aligned}$$

etcetera.

Theorem 8 *For $q \in \mathcal{P}(\bar{\mathcal{N}})$ and $q \in c \in \mathcal{CUTS}(\bar{\mathcal{N}})$,*

$$\begin{aligned}\mathcal{N}, c &\models EX \bigwedge_{p \in post^q}\\ \mathcal{N}, c &\models AF \bigwedge_{p \in post^q}.\end{aligned}$$

Proof: Both statements follow from the fact that $post^q \neq \emptyset$ under (A) and the openness of branches. $\square$

Theorem 9 *For $q \in \mathcal{P}(\bar{\mathcal{N}})$ and $pre^q \subseteq c \in \mathcal{CUTS}(\bar{\mathcal{N}})$,*

$$\mathcal{N}, c \models EXq$$

Proof: Follows from the maximality of the unfolding since in any marking $\mathcal{M}_c$ corresponding to q, $\mathcal{M}_q \xrightarrow{\pi(q)}$. $\square$

5.2.1 LLC Theorems

In LLC, we are also able to express firing operations.

Theorem 10 *Let $p \in \mathcal{P}(\bar{\mathcal{N}})$ such that $pF_{\bar{\mathcal{N}}} \neq \emptyset$, and $p \in \delta \in \nabla(\bar{\mathcal{N}})$ and $\delta \subseteq \gamma \in \daleth(\bar{\mathcal{N}})$. With*

$$\begin{aligned}\delta_1 &:= (\delta - \{p\}) \cup (\{p\} \times pF_{\bar{\mathcal{N}}})\\ \gamma_1 &:= (\gamma - \{p\}) \cup (\{p\} \times pF_{\bar{\mathcal{N}}})\\ \delta_2 &:= (\delta - \{p\}) \cup pF_{\bar{\mathcal{N}}}.\end{aligned}$$

Then $\mathcal{N}, \gamma \models \nabla^\circ \delta_1$, $\mathcal{N}, \gamma \models \nabla \delta_2$ and $\mathcal{N}, \gamma_1 \models \nabla^\circ \delta_2$.

Proof: Follows, under (A) and (B), from Lemma 4. $\square$

Of course, a dual result can be obtained with $\triangle$ instead of ∇.

6 Closing Remarks and Perspectives

We have given two partial order logics on different frames yet on the same underlying structure, that of (cuts of) occurrence nets. The semantic equivalences induced by the two process concepts are strictly incomparable; the equivalence(s) given by BLC and LLC correspond, respectively, to relational isomorphism for $(\mathcal{CUTS}, \sqsubset)$ or $(\daleth, \sqsubset)$.

This frame being a conditionally complete lattice, it can be viewed as a *near-linear time* semantics, in which the execution history contains all alternatives, chosen or not, that have occurred along the way (however, the model is more general than that of bipolar synchronisation schemes in [7].)

Occurrence nets semantics have been used for model-checking of S_4 formulas; this requires, of course, that the system can be checked using a finite model. This is the case iff the net $\mathcal{N}$ is finite and bounded. For the 1-safe case, an efficient reduction to a *finite prefix* has been developed and exploited; McMillan's approach [15] has been improved upon and used by Esparza et al. [12] also extend their approach to the general bounded case using branching executions. Unfolding techniques are particularly efficient in systems whose degree of concurrency is high compared to the degree of branching.

For BLC, essentially the same holds; model checking a CTL^* type logic on a partial order structure will, however, be a considerable gain in expressive power. Care must be taken not to loose efficiency in BLC by the overhead incurred by computing and inserting $\mathcal{P}_B$ and $\mathcal{P}_B$.

For LLC, no investigation has thus far been made into dedicated model checking algorithms and their efficiency. It would also be of interest to study the equivalences obtained from possible fragments or extensions of LLC, and to further explore the change in expressiveness brought about by working with alternatives and choices rather than nodes and possible worlds.

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