Behavioural Approximations for Restricted Linear Differential Hybrid Automata

Manindra Agrawal*[1], Frank Stephan†[2], P. S. Thiagarajan[3], Shaofa Yang‡[4]

[1]Department of Computer Science and Engineering,
Indian Institute of Technology, Kanpur, India
Email: manindra@cse.iitk.ac.in

[2]School of Computing and Department of Mathematics,
National University of Singapore (NUS), Singapore
Email: fstephan@comp.nus.edu.sg

[3]School of Computing,
National University of Singapore, Singapore
Email: thiagu@comp.nus.edu.sg

[4]IRISA/INRIA, Rennes, France
Email: shaofa.yang@irisa.fr

Abstract: We show the regularity of the discrete time behaviour of hybrid automata in which the rates of continuous variables are governed by linear differential operators in a diagonal form and in which the values of the continuous variables can be observed only with finite precision. We do not demand resetting of the values of the continuous variables during mode changes. We can cope with polynomial guards and we can tolerate bounded delays both in sampling the values of the continuous variables and in effecting changes in their rates required by mode switchings. We also show that if the rates are governed by diagonalizable linear differential operators with rational eigenvalues *and* there is no delay in effecting rate changes, the discrete time behaviour of the hybrid automaton is recursive. However, the control state reachability problem in this setting is undecidable.

1 Introduction

We study the behaviour of hybrid automata in which the rate functions associated with the modes are restricted linear differential equations. We show that if the values of the continuous variables can be observed only with finite precision, then the discrete time behaviour of a large class of hybrid automata is regular. Further, these behaviours can be effectively computed. The key feature of our setting is that we do not demand that the value of a continuous variable be reset during a mode switch. Our results suggest that focusing on discrete time semantics and the realistic assumption of finite precision can lead to effective analysis methods for hybrid automata whose continuous dynamics is governed by (linear) differential equations.

In the related literature, one often assumes that the rates are piecewise constant. This is so, at least in settings where one obtains positive verification results [6, 9, 12]. Even here, since

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‡This work was done when the author was at School of Computing, NUS.

the mode changes can take place over continuous time (a transition may be taken any time its guard is satisfied), basic verification problems often become undecidable [3, 8]. In contrast, it was shown in [7] that one can go much further in the positive direction for piecewise constant rate automata, if one defines their behaviour using a discrete time semantics. As argued in [7], if the hybrid automaton models the closed loop system consisting of a digital controller interacting with a continuous plant, then the discrete time semantics is the natural one; the controller will observe via sensors, the states of the plant and effect, via actuators, changes in the plant dynamics at discrete time points determined by its internal clock. In [1] it was shown that, in this setting, one can in fact tolerate bounded delays both in the observation of the plant states and in effecting changes in the plant dynamics.

Both in [7] and [1], the transition guards were required to be rectangular; conjunctions of simple linear inequalities involving just one variable. We showed in [2] that one can cope with much more expressive guards—essentially all effectively computable guards—if one assumes that the values of the continuous variables can be observed only with finite precision. In many settings including the one where the hybrid automaton models a digital controller interacting with a continuous plant, this is a natural assumption.

Here our goal is to show that the combination of discrete time semantics and finite precision can not only allow more expressive guards but can also take us beyond piecewise constant rates. One of our main results is that under finite precision, the discrete time behaviour of a hybrid automaton is regular and effectively computable even when the rate of a continuous variable in the control state q is governed by an equation of the form $dx/dt = c_q \cdot x(t)$. This holds even though we do not demand resetting of the values of the continuous variables during mode changes. Further, we can cope with arbitrary computable guards. We can also tolerate bounded delays in sampling the values of the continuous variables and in effecting changes in their rates required by mode switchings.

We also show that the discrete time behaviours of hybrid automata in a much richer setting are *recursive*. Specifically, the rates of continuous variables at the control state q are governed by a linear differential operator represented by a diagonalizable ([10]) matrix A_q with rational eigenvalues. Further, we allow polynomial guards but do not permit delays in effecting rates changes. A consequence of this positive result is that one can effectively solve a variety of bounded model checking problems [5] in this rich setting. However, we show that the control state reachability problem is undecidable for this class of automata; this is so, even if the guards are restricted to be rectangular.

The proofs of the above two results seem to suggest that one can hope to go much further if update delays are allowed. This will prevent the hybrid automaton from retaining an unbounded amount of information as its dynamics evolves. The key obstacle is that we do not know at present how to take advantage of this observation since we lack suitable techniques for tracking rational approximations of exponential terms with *real* exponents. In this connection, the fundamental theory presented in [4] may turn to be important. We also feel that the techniques presented in [13, 14] will turn out to be useful even though they are developed under a regime where continuous variables are reset during mode changes.

In the next two sections, we define our hybrid automata and develop their discrete time semantics. In Section 4, we present our main result concerning hybrid automata whose discrete time behaviours are regular. In Section 5, we study a subclass of hybrid automata whose discrete time behaviours are recursive but whose control state reachability problem is undecidable.

2 Hybrid Automata Preliminaries

Through the rest of the paper, we fix a positive integer n and one function symbol x_i for each i in $\{1, 2, \ldots, n\}$. We will often refer to the x_i's as "continuous" variables and will view each x_i as a function (of time) $x_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ with $\mathbb{R}$ being the set of reals and $\mathbb{R}_{\geq 0}$, the set of non-negative reals. We let $\mathbb{Q}$ denote the set of rationals.

The transitions of the hybrid automaton will have associated guards that need to be satisfied by the values of the continuous variables for the transitions to be enabled. A *polynomial constraint* is an inequality of the form $p(x_1, x_2, \ldots, x_n) \leq 0$ or $p(x_1, x_2, \ldots, x_n) < 0$ where $p(x_1, x_2, \ldots, x_n)$ is a polynomial over $x_1, x_2, \ldots, x_n$ with integer coefficients. A *polynomial guard* is a finite conjunction of polynomial constraints. We let *Grd* denote the set of polynomial guards. Unless otherwise stated, by a guard we will mean a polynomial guard.

A *valuation* V is just a member of $\mathbb{R}^n$. It will be viewed as prescribing the value $V(i)$ to each variable x_i. The notion of a valuation satisfying a guard is defined in the obvious way.

A *lazy finite-precision differential hybrid automaton* is a structure $\mathcal{A} = (Q, q_{in}, V_{in}, \mathit{Delay}, \epsilon, \{\rho_q\}_{q \in Q}, \{\gamma_{min}, \gamma_{max}\}, \longrightarrow)$ where:

- Q is a finite set of *control states* with q, q' ranging over Q.
- $q_{in} \in Q$ is the initial control state.
- $V_{in} \in \mathbb{Q}^n$ is the initial valuation.
- $\mathit{Delay} = \{\delta^0_{ob}, \delta^1_{ob}, \delta^0_{up}, \delta^1_{up}\} \subseteq \mathbb{Q}$ is the set of *delay parameters* such that $0 \leq \delta^0_{up} \leq \delta^1_{up} < \delta^0_{ob} \leq \delta^1_{ob} \leq 1$.
- ϵ, a positive rational, is the *precision of measurement.*
- $\{\rho_q\}_{q \in Q}$ is a family of rate functions associated with the control states. In the general case, ρ_q will be of the form $\dot{x} = A_q x + b_q$ where A_q is an $n \times n$ matrix with rational entries and $b_q \in \mathbb{Q}^n$. For each i in $\{1, 2, \ldots, n\}$ this specifies the rate function of x_i as the differential equation $dx_i/dt = \sum_{j=1}^{n} A_q(i, j) \cdot x_j(t) + b_q(i)$ where $A_q(i, j)$ is the (i, j)-th entry of A_q.
- $\gamma_{min}, \gamma_{max} \in \mathbb{Q}$ are *range parameters* such that $0 < \gamma_{min} < \gamma_{max}$.
- $\longrightarrow \subseteq Q \times \mathit{Grd} \times Q$ is a transition relation such that $q \neq q'$ for every (q, g, q') in $\longrightarrow$.

We shall study the discrete time behaviour of our automata. At each time instant T_k, the automaton receives a measurement regarding the current values of the x_i's. However, the value of x_i that is observed at time T_k is the value that held at some time $t \in [T_{k-1} + \delta^0_{ob},\ T_{k-1} + \delta^1_{ob}]$. Further, the value is observed with a precision of ϵ. More specifically, any value of x_i in the half-open interval $[(m - 1/2)\epsilon,\ (m + 1/2)\epsilon)$ is reported as $m\epsilon$ where m is an integer. For a real number v, we will denote this rounded-off value relative to ϵ as $\langle v \rangle_\epsilon$ and often just write $\langle v \rangle$. More sophisticated rounding-off functions can be considered as in [2] but for ease of presentation, we shall not do so here.

If at T_k, the automaton is in control state q and the observed n-tuple of values $(\langle v_1 \rangle, \langle v_2 \rangle, \ldots, \langle v_n \rangle)$ satisfies the guard g with (q, g, q') being a transition, then the automaton may perform this transition instantaneously and move to the control state q'. As a result, the x_i's will cease to evolve according to the rate function ρ_q and instead start evolving according to the rate function $\rho_{q'}$. However, for each x_i, this change in the rate of evolution of each x_i will not kick in at T_k but at some time $t \in [T_k + \delta^0_{up},\, T_k + \delta^1_{up}]$. In this sense, both the sensing of the x_i's and the rate changes associated with mode switching take place in a lazy fashion but with bounded delays. We expect $\delta^0_{ob}, \delta^1_{ob}$ to be close to 1 and $\delta^0_{up}, \delta^1_{up}$ to be close to 0 while both $\delta^1_{ob} - \delta^0_{ob}$ and $\delta^1_{up} - \delta^0_{up}$ to be small compared to 1.

In the idealized setting, the value observed at T_k is the value that holds at exactly T_k ($\delta^0_{ob} = 1 = \delta^1_{ob}$) and the change in rates due to mode switching would kick in immediately ($\delta^0_{up} = 0 = \delta^1_{up}$). In addition, assuming perfect precision would boil down to setting $\langle v \rangle = v$ for every real number v.

The parameters $\gamma_{min}, \gamma_{max}$ specify the relevant range of the absolute values of the continuous variables. The automaton gets stuck if $|x_i|$ gets outside the allowed range $[\gamma_{min}, \gamma_{max}]$ for any i. Loosely speaking, the γ_{max} bound is used to restrict the amount of information carried by a continuous variable evolving at a (positive or negative) constant rate ($\dot{x} = c$) and a continuous variable increasing at an exponential rate ($\dot{x} = c \cdot x(t)$, $c > 0$). On the other hand, γ_{min} is used to restrict the amount of information carried by a continuous variable decreasing at an exponential rate ($\dot{x} = c \cdot x(t)$, $c < 0$). We note that our setting is quite different from the classical continuous setting. Hence the standard control objective of driving a system variable to 0 is not relevant here and thus does not pose a serious limitation.

We will be mainly interested in the setting that each A_q is a diagonal matrix and in the more general case where each A_q is a diagonalizable matrix having n distinct rational eigenvalues. In the former setting we show that the control state sequence languages generated by our hybrid automata are regular and can be effectively computed provided every continuous variable either evolves at (possibly different) constant rates in all the control states or at (possibly different) exponential rates in all the control states. In the latter setting, with the additional restriction that there are no delays associated with rates update ($\delta^0_{up} = 0 = \delta^1_{up}$), we show that the control state sequence languages generated by our hybrid automata are recursive, but the control state reachability problem is undecidable.

3 The Transition System Semantics

Through the rest of this paper we fix a lazy finite-precision differential hybrid automaton $\mathcal{A}$ and assume its associated notations and terminology as defined in the previous section. We shall often refer to "lazy finite-precision differential hybrid automata" simply as "hybrid automata". The behaviour of $\mathcal{A}$ will be defined in terms of an associated transition system. A *configuration* is a triple (q, V, q') where q, q' are control states and V is a valuation. q is the current control state, q' is the control state that held at the previous time instant and V captures the *actual* values of the variables at the current time instant. The valuation V is said to be *feasible* if $\gamma_{min} \leq |V(i)| \leq \gamma_{max}$ for every i in $\{1, 2, \ldots, n\}$. The configuration (q, V, q') is *feasible* iff V is a feasible valuation. The initial configuration is (q_{in}, V_{in}, q_{in}) and is assumed to be feasible. We let $Conf_{\mathcal{A}}$ denote the set of configurations. We assume that the unit of time has been fixed at some suitable level of granularity and that the rate functions $\{\rho_q\}_{q \in Q}$ have been scaled accordingly.

Suppose the automaton $\mathcal{A}$ is in the configuration (q_k, V_k, q'_k) at time T_k. Then one unit of time will be allowed to pass and at time instant T_{k+1}, the automaton $\mathcal{A}$ will make an instantaneous move by executing a transition or the silent action τ and move to a configuration $(q_{k+1}, V_{k+1}, q'_{k+1})$. The silent action τ will be used to record that no mode change has taken place during this move. The action μ will be used to record that a transition has been taken and as a result, a mode change has taken place. As is common, we will collapse the unit-time-passage followed by an instantaneous transition into one "time-abstract" transition labelled by τ or μ. We wish to formalize the transition relation $\Longrightarrow \subseteq Conf_{\mathcal{A}} \times \{\tau, \mu\} \times Conf_{\mathcal{A}}$. For doing so, we note that given a matrix $A \in \mathbb{Q}^{n \times n}$, a vector $b \in \mathbb{Q}^n$, a positive real T and a valuation V, we can find a unique family of curves (see [10]) $\{x_i\}_{1 \leq i \leq n}$ with $x_i : [0,T] \to \mathbb{R}$ such that for every i we have $x_i(0) = V(i)$ and for every $t \in [0,T]$ we have $dx_i/dt = \sum_{j=1}^{n} A_q(i,j) \cdot x_j(t) + b_q(i)$. In what follows, we shall denote the valuation $(x_1(T), x_2(T), \ldots, x_n(T))$ thus obtained as $Val(A,b,T,V)$ without explicitly displaying the curves x_i's.

Let (q, V, q'), $(q1, V1, q1')$ be in $Conf_{\mathcal{A}}$. Suppose there exist reals t_i^{up}, $i = 1, 2, \ldots, n$, in $[\delta_{up}^0, \delta_{up}^1]$ such that $V1$ is related to V as follows: Let $t_{\pi_1}^{up} \leq t_{\pi_2}^{up} \leq \cdots \leq t_{\pi_n}^{up}$ with $\pi_1, \pi_2, \ldots, \pi_n$ being a permutation of the indices $1, 2, \ldots, n$. Then there exist valuations U_i, $i = 1, 2, \ldots, n$, such that $U_1 = Val(A_{q'}, b_{q'}, t_{\pi_1}^{up}, V)$; $U_{i+1} = Val(A_i, b_i, t_{\pi_{i+1}}^{up} - t_{\pi_i}^{up}, U_i)$ for $i = 1, 2, \ldots, n-1$; and $V1 = Val(A_n, b_n, 1 - t_{\pi_n}^{up}, U_n)$, where for $i = 1, 2, \ldots, n$, the matrix $A_i \in \mathbb{Q}^{n \times n}$ and the vector $b_i \in \mathbb{Q}^n$ are given by: if $j \in \{\pi_1, \pi_2, \ldots, \pi_i\}$, then the j-th row of A_i (b_i) equals the j-th row of A_q (b_q); otherwise the j-th row of A_i (b_i) equals the j-th row of $A_{q'}$ ($b_{q'}$).

The intuition is that at time T_{k+1} the continuous variables have valuation $V1$ while at time T_k, the continuous variables have valuation V and $\mathcal{A}$ resides at control state q. Further, at time T_{k-1}, the automaton was at control state q'. For each i, the real number $T_k + t_i^{up}$ is the time at which x_i ceases to evolve at the rate $dx_i/dt = \sum_{j=1}^{n} A_{q'}(i,j) \cdot x_j + b_{q'}(i)$ and starts to evolve at the rate $dx_i/dt = \sum_{j=1}^{n} A_q(i,j) \cdot x_j + b_q(i)$.

Now we state the condition that $\Longrightarrow$ must fulfil. Let (q, V, q'), $(q1, V1, q1')$ be in $Conf_{\mathcal{A}}$. Suppose there exist reals t_i^{up}, $i = 1, 2, \ldots, n$, in $[\delta_{up}^0, \delta_{up}^1]$ such that $V1$ is related to V as dictated above.

- Suppose $q1 = q1' = q$. Then $(q, V, q') \stackrel{\tau}{\Longrightarrow} (q1, V1, q1')$.
- Suppose $q1' = q$ and there exists a transition $(q, g, q1)$ in $\longrightarrow$ and reals t_i^{ob}, $i = 1, 2, \ldots, n$, in $[\delta_{ob}^0, \delta_{ob}^1]$ such that $(\langle w_1 \rangle, \langle w_2 \rangle, \ldots, \langle w_n \rangle)$ satisfies g, where w_i is the i-th component of the valuation $Val(A_n, b_n, t_i^{ob} - t_{\pi_n}^{up}, U_n)$ for $i = 1, 2, \ldots, n$. Then $(q, V, q') \stackrel{\mu}{\Longrightarrow}_{\mathcal{A}} (q1, V1, q1')$.

As might be expected, the real $T_k + t_i^{ob}$ is the time at which the value of x_i was observed for each $i = 1, 2, \ldots, n$.

Basically there are four possible transition types depending on whether $q = q'$ and whether τ or μ is the action label. For convenience, we have collapsed these four possibilities into two cases according to τ or μ being the action label, and in each case have handled the subcases $q = q'$ and $q \neq q'$ simultaneously.

Now define the transition system $TS_{\mathcal{A}} = (RC_{\mathcal{A}}, (q_{in}, V_{in}, q_{in}), \{\tau, \mu\}, \Longrightarrow_{\mathcal{A}})$ via:

- $RC_{\mathcal{A}}$, the set of *reachable configurations* of $\mathcal{A}$ is the least subset of $Conf_{\mathcal{A}}$ that contains the initial configuration (q_{in}, V_{in}, q_{in}) and satisfies: Suppose (q, V, q') is in $RC_{\mathcal{A}}$ and is a feasible configuration. Suppose further, $(q, V, q') \stackrel{\alpha}{\Longrightarrow} (q1, V1, q1')$ for some $\alpha \in \{\tau, \mu\}$. Then $(q1, V1, q1') \in RC_{\mathcal{A}}$.
- $\Longrightarrow_{\mathcal{A}}$ is $\Longrightarrow$ restricted to $RC_{\mathcal{A}} \times \{\tau, \mu\} \times RC_{\mathcal{A}}$.

We note that a reachable configuration can be the source of a transition in $TS_{\mathcal{A}}$ only if it is feasible. Thus infeasible reachable configurations will be deadlocked in $TS_{\mathcal{A}}$. A *run* of $TS_{\mathcal{A}}$

is a finite sequence of the form

$$\sigma = (q_0, V_0, q_0') \, \alpha_0 \, (q_1, V_1, q_1') \, \alpha_1 \, (q_2, V_2, q_2') \ldots (q_\ell, V_\ell, q_\ell')$$

where (q_0, V_0, q_0') is the initial configuration and $(q_k, V_k, q_k') \overset{\alpha_k}{\Longrightarrow}_{\mathcal{A}} (q_{k+1}, V_{k+1}, q_{k+1}')$ for $k = 0, 1, \ldots, \ell - 1$. The *state sequence* induced by the run σ above is the sequence $q_0 q_1 \ldots q_\ell$. We define the state sequence language of $\mathcal{A}$ denoted $\mathcal{L}(\mathcal{A})$ to be the set of state sequences induced by runs of $TS_{\mathcal{A}}$.

4 Diagonal Rate Matrices

We first study the setting where each A_q is a diagonal matrix and where every continuous variable either evolves at constant rates in all the modes or at exponential rates in all the modes. It turns out that the language of state sequences in this setting is always regular. More precisely:

Theorem 1. *Let $\mathcal{A}$ be a lazy finite-precision differential hybrid automaton such that A_q is a diagonal matrix for every control state q. Suppose there exists a* fixed *partition $\{DIF, CON\}$ of the indices $\{1, 2, \ldots, n\}$ such that for each control state q, $\dot{x}_i = A_q(i,i) \cdot x_i$ if $i \in DIF$ and $\dot{x}_i = b_q(i)$ if $i \in CON$. Then $\mathcal{L}(\mathcal{A})$ is a regular subset of $Q^\star$. Further, a finite state automaton accepting $\mathcal{L}(\mathcal{A})$ can be effectively computed from $\mathcal{A}$.*

Proof of Theorem 1: The basic strategy is to generalize the proof of the main result in [2]. As before, the proof consists of two major steps. The first one is to quotient the set of reachable configurations $RC_{\mathcal{A}}$ into a *finite number* of equivalence classes using a suitably chosen equivalence relation $\approx$. The crucial property required of $\approx$ is that it should be a congruence with respect to the transition relation of $TS_{\mathcal{A}}$. In other words, if $(q1, V1, q1') \approx (q2, V2, q2')$ and $(q1, V1, q1') \overset{\alpha}{\Longrightarrow}_{\mathcal{A}} (q3, V3, q3')$, then we require that there exists a configuration $(q4, V4, q4')$ such that $(q2, V2, q2') \overset{\alpha}{\Longrightarrow}_{\mathcal{A}} (q4, V4, q4')$ *and* $(q3, V3, q3') \approx (q4, V4, q4')$. The second step is to show that we can effectively compute these equivalence classes and a transition relation over them such that the resulting finite state automaton generates the language of state sequences.

For notational convenience, we assume $V_{in}(i) > 0$ for every $i \in DIF$. It will become clear that this involves no loss of generality. The key consequence of this assumption is that in any reachable configuration, the value of x_i for $i \in DIF$ will be positive.

We also assume without loss of generality that for each guard g in $\mathcal{A}$, the valuation V satisfies g only if V is feasible.

Let Δ be the largest positive rational number that *integrally* divides every number in the set of rational numbers $\{\delta^0_{ob}, \delta^1_{ob}, \delta^0_{up}, \delta^1_{up}, 1\}$. Define Γ to be the largest rational which *integrally* divides every number in the finite set of rational numbers $\{A_q(i,i) \cdot \Delta \mid q \in Q, i \in DIF\} \bigcup \{b_q(j) \cdot \Delta \mid q \in Q, j \in CON\} \bigcup \{\gamma_{min}, \gamma_{max}\} \bigcup \{\epsilon/2\}$.

Let $\mathbb{Z}$ denote the set of integers. Define Θ_{con} to be the *finite* set of rational numbers $\{h\Gamma \in [-\gamma_{max}, \gamma_{max}] \mid h \in \mathbb{Z}\}$. In other words, Θ_{con} contains integral multiples of Γ in the interval $[-\gamma_{max}, \gamma_{max}]$.

Let Θ_{IR} be the set of irrational numbers $\{\ln((m + 1/2)\epsilon) \mid m \in \mathbb{Z}, \langle\gamma_{min}\rangle \leq m\epsilon \leq \langle\gamma_{max}\rangle\}$ $\bigcup \{\ln \gamma_{min}, \ln \gamma_{max}\}$. Define Θ_{dif} to be the *finite* set of real numbers $\{h\Gamma \in [\ln \gamma_{min}, \ln \gamma_{max}] \mid h \in \mathbb{Z}\} \bigcup \{\ell\Gamma + \theta \in [\ln \gamma_{min}, \ln \gamma_{max}] \mid \ell \in \mathbb{Z},\, \theta \in \Theta_{IR}\}$. In other words, Θ_{dif} contains rational numbers of the form $h\Gamma$ in the interval $[\ln \gamma_{min}, \ln \gamma_{max}]$ where h is a (positive) integer, and irrational numbers of the form $\ell\Gamma + \theta$ in the interval $[\ln \gamma_{min}, \ln \gamma_{max}]$ where ℓ is an integer (that can be positive, zero or negative) and θ is a member of Θ_{IR}.

Loosely speaking, the set Θ_{con} (respectively Θ_{dif}) contains bounds relevant to the values of continuous variables x_i's for $i \in CON$ (respectively $i \in DIF$). The points in Θ_{con} (Θ_{dif}) cut the real line into a finite number of segments. We shall use this segmentation to in turn partition the set of reachable configurations into finitely many equivalence classes. The simple but key observation that enables this is, in the (natural) logarithmic scale, exponential rates get represented as *constant* rates.

In this light, let the members of Θ_{dif} be $\{\theta_1, \theta_2, \ldots, \theta_{|\Theta_{dif}|}\}$ where $\theta_1 < \theta_2 < \cdots < \theta_{|\Theta_{dif}|}$. We define the finite set of intervals

$$\mathcal{I}_{dif} = \{(-\infty, \theta_1), (\theta_1, \theta_2), \ldots, (\theta_{|\Theta_{dif}|-1}, \theta_{|\Theta_{dif}|}), (\theta_{|\Theta_{dif}|}, \infty)\} \bigcup \{[\theta_i, \theta_i] \mid i = 1, 2, \ldots, |\Theta_{dif}|\} .$$

In the same way, we define $\mathcal{I}_{con}$ from Θ_{con}.

Let $\mathbb{R}^+$ be the set of positive reals. Define the map $\|\cdot\|_{dif} : \mathbb{R}^+ \to \mathcal{I}_{dif}$ via: $\|v\| = I$ if $\ln v \in I$. Define $\|\cdot\|_{con} : \mathbb{R} \to \mathcal{I}_{con}$ via: $\|v\| = I$ if $v \in I$. Finally we define $\|\cdot\| : RC_{\mathcal{A}} \to (\mathcal{I}_{dif} \cup \mathcal{I}_{con})^n$ by: $\|V\| = (I_1, I_2, \ldots, I_n)$ where $I_i = \|V(i)\|_{dif}$ for $i \in DIF$ and $I_i = \|V(i)\|_{con}$ for $i \in CON$. We can now define the equivalence relation $\approx \subseteq RC_{\mathcal{A}} \times RC_{\mathcal{A}}$ by: $(q1, V1, q1') \approx (q2, V2, q2')$ iff $q1 = q2$, $\|V1\| = \|V2\|$ and $q1' = q2'$. The crucial property of $\approx$ is that it is a congruence relation with respect to the transition relation $\Longrightarrow_{\mathcal{A}}$.

Claim 2. *Suppose* $(q1, V1, q1') \approx (q2, V2, q2')$ *and* $(q1, V1, q1') \stackrel{\alpha}{\Longrightarrow}_{\mathcal{A}} (q3, V3, q3')$*, then there exists a reachable configuration* $(q4, V4, q4')$ *such that* $(q2, V2, q2') \stackrel{\alpha}{\Longrightarrow}_{\mathcal{A}} (q4, V4, q4')$ and $(q3, V3, q3') \approx (q4, V4, q4')$.

Proof of Claim 2: Clearly $q1 = q2$ and $q1' = q2'$. Set $q4 = q3$ and $q4' = q3'$. We show that $(q2, V2, q2')$ is a feasible configuration and there exists a valuation $V4$ such that $(q2, V2, q2') \stackrel{\alpha}{\Longrightarrow}_{\mathcal{A}} (q4, V4, q4')$ and $\|V4\| = \|V3\|$.

We first note that the configuration $(q2, V2, q2')$ is feasible. Fix an $i \in DIF$. Since the configuration $(q1, V1, q1')$ is feasible, we have $\ln \gamma_{min} \leq \ln V1(i) \leq \ln \gamma_{max}$. Since $\ln \gamma_{min}$, $\ln \gamma_{max}$ are members of Θ_{dif} and $\|V1(i)\|_{dif} = \|V2(i)\|_{dif}$, we conclude $\ln \gamma_{min} \leq \ln V2(i) \leq \ln \gamma_{max}$ and so $\gamma_{min} \leq |V2(i)| \leq \gamma_{max}$. Similarly it is easy to see that $\gamma_{min} \leq |V2(i)| \leq \gamma_{max}$ for $i \in CON$.

We show the existence of $V4$ by considering two cases according to $\alpha = \tau$ or $\alpha = \mu$.

—**Case 1:** $\alpha = \tau$**.**

It follows from the definition of $TS_{\mathcal{A}}$ that there exist reals $t_i^{up} \in [\delta_{up}^0, \delta_{up}^1]$, $i = 1, 2, \ldots, n$, such that $\ln V3(i) = \ln V1(i) + A_{q'}(i,i) \cdot t_i^{up} + A_q(i,i) \cdot (1 - t_i^{up})$ for $i \in DIF$ and $V3(i) = V1(i) + b_{q'}(i) \cdot t_i^{up} + b_q(i) \cdot (1 - t_i^{up})$ for $i \in CON$. It suffices to show that there exist reals $s_i^{up} \in [\delta_{up}^0, \delta_{up}^1]$, $i = 1, 2, \ldots, n$, such that $\|V4\| = \|V3\|$, where $\ln V4(i) = \ln V2(i) + A_{q'}(i,i) \cdot s_i^{up} + A_q(i,i) \cdot (1 - s_i^{up})$ for $i \in DIF$ and $V4(i) = V2(i) + b_{q'}(i) \cdot s_i^{up} + b_q(i) \cdot (1 - s_i^{up})$ for $i \in CON$.

In what follows, we will often need to give similar arguments for $i \in DIF$ and $i \in CON$. To avoid repetition, we will omit the latter.

Fix an $i \in DIF$. We show the existence of s_i^{up}. Assume $\|V3(i)\|_{dif} = (\theta, \theta')$ where $\theta, \theta' \in \Theta_{dif}$ and $A_{q'}(i,i) > A_q(i,i)$. It will become clear that other cases can be similarly handled. For any real u, let $\Phi^{\tau}(u)$ be the condition

$$\begin{aligned} \exists\, t^{up} \in \mathbb{R}. \quad & \delta_{up}^0 \leq t^{up} \leq \delta_{up}^1 \\ \bigwedge \quad & \theta < u + A_{q'}(i,i) \cdot t^{up} + A_q(i,i) \cdot (1 - t^{up}) < \theta' . \end{aligned}$$

It is easy to see that $\Phi^\tau(u)$ holds iff $\eta < u < \eta'$ where $\eta = \theta - A_{q'}(i,i)\cdot\delta^1_{up} - A_q(i,i)\cdot(1-\delta^1_{up})$ and $\eta' = \theta' - A_{q'}(i,i)\cdot\delta^0_{up} - A_q(i,i)\cdot(1-\delta^0_{up})$.

Since $\Phi^\tau(\ln V1(i))$ holds, we have $\eta < \ln V1(i) < \eta'$. Note that η, η' are members of Θ_{dif} (if $\eta, \eta' \in [\ln\gamma_{min}, \ln\gamma_{max}]$). Applying $\|V2(i)\|_{dif} = \|V1(i)\|_{dif}$ then yields $\eta < \ln V2(i) < \eta'$ and consequently $\Phi^\tau(\ln V2(i))$ holds. This establishes the existence of s^{up}_i for $i \in \mathit{DIF}$.

—**Case 2:** $\alpha = \mu$**.**

As in Case 1, it follows from the definition of $TS_{\mathcal{A}}$ that there exist reals t^{up}_i in $[\delta^0_{up}, \delta^1_{up}]$, $i = 1,2,\ldots,n$, such that $\ln V3(i) = \ln V1(i) + A_{q'}(i,i)\cdot t^{up}_i + A_q(i,i)\cdot(1-t^{up}_i)$ for $i \in \mathit{DIF}$ and $V3(i) = V1(i) + b_{q'}(i)\cdot t^{up}_i + b_q(i)\cdot(1-t^{up}_i)$ for $i \in \mathit{CON}$. Further there exist reals $t^{ob}_i \in [\delta^0_{ob}, \delta^1_{ob}]$, $i = 1,2,\ldots,n$, and a guard g such that the following conditions are satisfied: Firstly, $(q1, g, q3) \in \longrightarrow$. Secondly, $(\langle U(1)\rangle, \langle U(2)\rangle, \ldots, \langle U(n)\rangle)$ satisfies g, where U is the valuation with $\ln U(i) = \ln V1(i) + A_{q'}(i,i)\cdot t^{up}_i + A_q(i,i)\cdot(t^{ob}_i - t^{up}_i)$ for $i \in \mathit{DIF}$; $U(i) = V1(i) + b_{q'}(i)\cdot t^{up}_i + b_q(i)\cdot(t^{ob}_i - t^{up}_i)$ for $i \in \mathit{CON}$. We shall show the existence of reals $s^{up}_i \in [\delta^0_{up}, \delta^1_{up}]$, $s^{ob}_i \in [\delta^0_{ob}, \delta^1_{ob}]$, $i = 1,2,\ldots,n$, such that $\|V4\| = \|V3\|$ and $\|U'\| = \|U\|$ where $V4$ is the valuation given by $\ln V4(i) = \ln V2(i) + A_{q'}(i,i)\cdot s^{up}_i + A_q(i,i)\cdot(1-s^{up}_i)$ for $i \in \mathit{DIF}$ and $V4(i) = V2(i) + b_{q'}(i)\cdot s^{up}_i + b_q(i)\cdot(1-s^{up}_i)$ for $i \in \mathit{CON}$. And U' is the valuation given by $\ln U'(i) = \ln V2(i) + A_{q'}(i,i)\cdot s^{up}_i + A_q(i,i)\cdot(s^{ob}_i - s^{up}_i)$ for $i \in \mathit{DIF}$ and $U'(i) = V2(i) + b_{q'}(i)\cdot s^{up}_i + b_q(i)\cdot(s^{ob}_i - s^{up}_i)$ for $i \in \mathit{CON}$. First we argue that the existence of U' satisfying $\|U'\| = \|U\|$ will guarantee $\langle U'(i)\rangle = \langle U(i)\rangle$ for $i = 1,2,\ldots,n$. This follows from the fact $\ln((m+1/2)\epsilon) \in \Theta_{dif}$ for integers m with $\langle\gamma_{min}\rangle \le m\epsilon \le \langle\gamma_{max}\rangle$ and $(m+1/2)\epsilon \in \Theta_{con}$ for integers m with $\langle -\gamma_{max}\rangle \le m\epsilon \le \langle\gamma_{max}\rangle$. Thus U' also satisfies the guard g since U satisfies g. So the existence of s^{up}_i, s^{ob}_i, $i = 1,2,\ldots,n$, suffices to establish the claim.

Fix an $i \in \mathit{DIF}$. Assume $\|V3(i)\|_{dif} = (\theta, \theta')$, $\|U(i)\|_{dif} = (\vartheta, \vartheta')$ where $\theta, \theta', \vartheta, \vartheta' \in \Theta_{dif}$ and $A_{q'}(i,i) > A_q(i,i) > 0$. Other cases can be similarly handled. For any real u, let $\Phi^\mu(u)$ be the condition

$$\begin{array}{rl} \exists\, t^{up} \in \mathbb{R}.\, \exists\, t^{ob} \in \mathbb{R}. & \delta^0_{up} \le t^{up} \le \delta^1_{up} \\ \bigwedge & \theta < u + A_{q'}(i,i)\cdot t^{up} + A_q(i,i)\cdot(1-t^{up}) < \theta' \\ \bigwedge & \delta^0_{ob} \le t^{ob} \le \delta^1_{ob} \\ \bigwedge & \vartheta < u + A_{q'}(i,i)\cdot t^{up} + A_q(i,i)\cdot(t^{ob}-t^{up}) < \vartheta' \;. \end{array}$$

As in Case 1, it is easy to see that $\Phi^\mu(u)$ holds iff $\eta < u < \eta'$, where η is the larger of $\theta - A_{q'}(i,i)\cdot\delta^1_{up} - A_q(i,i)\cdot(1-\delta^1_{up})$ and $\vartheta - A_{q'}(i,i)\cdot\delta^1_{up} - A_q(i,i)\cdot(\delta^1_{ob} - \delta^1_{up})$. On the other hand, η' is the smaller of $\theta' - A_{q'}(i,i)\cdot\delta^0_{up} - A_q(i,i)\cdot(1-\delta^0_{up})$ and $\vartheta' - A_{q'}(i,i)\cdot\delta^0_{up} - A_q(i,i)\cdot(\delta^0_{ob} - \delta^0_{up})$. It follows that η, η' are members of Θ_{dif} (if $\eta, \eta' \in [\ln\gamma_{min}, \ln\gamma_{max}]$). Thus, as in Case 1, one concludes that $\Phi^\mu(\ln V2(i))$ holds and the existence of s^{up}_i, s^{ob}_i for $i \in \mathit{DIF}$ is established.

By filling in similar but simpler arguments for $i \in \mathit{CON}$, we can complete the proof of Claim 2. □

Having established the claim that $\approx$ is a congruence with respect to $\Longrightarrow_{\mathcal{A}}$, we now argue that one can effectively construct a finite automaton which accepts $\mathcal{L}(\mathcal{A})$. Clearly, the members of Θ_{dif} and Θ_{con} can be effectively represented. Further, the members of Θ_{dif} (Θ_{con}) can be effectively ordered and thus the finitely many equivalence classes of $\approx$ can be effectively represented. Note that, to compare two members of Θ_{dif} one just needs to determine whether $e^{m_1} < m_2$ for integers m_1, m_2. This can be done by approximating e sufficiently precisely using for instance the power series expansion of e. Now construct a finite transition system $\mathcal{B}$ whose states are the finitely many equivalence classes of $\approx$. Further, there is a transition from $C1$ to $C2$ with label α iff there exists (q, V, q') in $C1$, $(q1, V1, q1')$ in $C2$ such that $(q, V, q') \stackrel{\alpha}{\Longrightarrow}_{\mathcal{A}} (q1, V1, q1')$. From the proof of Claim 2, to determine whether there exists a transition from $C1$ to $C2$ with label α amounts to comparing members of Θ_{dif} (and Θ_{con}). Hence the transition system $\mathcal{B}$ can be effectively computed. It is now straightforward to construct from $\mathcal{B}$ a finite state automaton which accepts $\mathcal{L}(\mathcal{A})$. This completes the proof of Theorem 1. □

It is clear that the proof of Theorem 1 also holds for any effectively computable language of guards instead of just polynomial guards.

As usual, a variety of verification and controller synthesis problems become decidable for hybrid automata satisfying the conditions set out in Theorem 1 above. One basic verification problem in this context is the *control state reachability problem*; to decide, for a designated state q_f, whether there exists a state sequence whose last letter is q_f.

5 Diagonalizable Rate Matrices

The regularity result of the previous section requires the matrices A_q to be diagonal. A natural way to relax this requirement is just to demand that every A_q be *diagonalizable* [10]. We recall that the $n \times n$ matrix A is diagonalizable in case there is a basis of eigenvectors $\{f_1, f_2, \ldots, f_n\}$ so that under the associated coordinate transformation, A can be represented as the diagonal matrix $diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with the λ_i's being the eigenvalues of A. Given our concern for effective computations, it then seems reasonable to demand that, in addition to being diagonalizable, every matrix A_q should also have (n distinct) rational eigenvalues.

We further restrict ourselves to the case that there is no delay associated with the update of rates of the continuous variables ($\delta^0_{up} = 0 = \delta^1_{up}$). This is due to the fact at present we don't know how to deal with differential equations of the form $\dot{x} = Ax + b$. One will have to deal with such equations if update delays are present ($\delta^0_{up} < \delta^1_{up}$). This is due to the fact that the rate changes of the continuous variables may kick in at *different* times in the interval $[T_k + \delta^0_{up}, T_k + \delta^1_{up}]$.

Assuming there are no update delays we first show that the state sequence language of every lazy finite-precision differential hybrid automaton is recursive. This result may be intuitively obvious it still requires an argument. This is so, since the decidability of the first order theory of the reals extended with the exponential operator is still open [16] and the results developed in [13] crucially exploit the resetting property. We then show that the control state reachability problem is undecidable in this setting.

Theorem 3. *Suppose $\mathcal{A}$ is a lazy finite-precision differential hybrid automaton such that $\delta^0_{up} = 0 = \delta^1_{up}$ and for every control state q, A_q is a diagonalizable matrix having n distinct rational eigenvalues. Then $\mathcal{L}(\mathcal{A})$ is a recursive subset of $Q^\star$.*

Proof. First we note that the first order theory of the reals augmented with the *constant* e is decidable. For convenience we shall denote this augmented structure as $(\mathbb{R}, +, \cdot, <, 0, 1, e)$ but emphasize that e, the base of the natural logarithm is being used as a constant and *not* as an operator. To see that the augmented theory is decidable, we observe that one can effectively determine whether $p(e) < 0$ for any given polynomial $p(e)$ with integer coefficients.

Since $e = 1 + \sum_{h=1}^{\infty} 1/h!$, we have

$$1 + \sum_{h=1}^{k} \frac{1}{h!} < e < 1 + \sum_{h=1}^{k} \frac{1}{h!} + \sum_{h=k+1}^{\infty} \frac{1}{k^{h-k}} = 1 + \sum_{h=1}^{k} \frac{1}{h!} + \frac{1}{k-1} .$$

Note that the polynomial $p(u)$ with one variable has finitely many real roots. Hence for sufficiently large k, $p(u)$ has no root in the interval $[1+\sum_{h=1}^{k} 1/h!,\ 1+\sum_{h=1}^{k} 1/h! + 1/(k-1)]$ and so $p(e)$ has the same sign as $p(1+\sum_{h=1}^{k} 1/h!)$. Clearly such a k can be effectively found. Now, given a sentence φ in $(\mathbb{R}, +, \cdot, <, 0, 1, e)$, one can apply Tarski's quantifier elimination algorithm [15] to obtain a quantifier-free sentence φ' such that φ is true iff φ' is true, and φ' is a boolean combination of formulas of the form $p(e) < 0$.

Next we show that given control states $q, q', q1, q1'$ and $\alpha \in \{\tau, \mu\}$, one can construct in $(\mathbb{R}, +, \cdot, <, 0, 1, e)$ a formula $\Phi_{q,q',q1,q1',\alpha}(V, V1)$ with free variables $V(i)$, $V1(i)$, $i = 1, 2, \ldots, n$, that asserts $(q, V, q') \stackrel{\alpha}{\Longrightarrow}_{\mathcal{A}} (q1, V1, q1')$. In what follows, we fix $q, q', q1, q1' \in Q$ and $\alpha \in \{\tau, \mu\}$.

Clearly we can effectively compute the rational eigenvalues λ_i, $i = 1, 2, \ldots, n$, of A_q, and for each $i = 1, 2, \ldots, n$ find a rational eigenvector $f_i \in \mathbb{Q}^n$ corresponding to λ_i (i.e. $A_q \cdot f_i = \lambda_i \cdot f_i$). Let $F = (f_1\ f_2\ \ldots\ f_n)$ be the matrix in $\mathbb{Q}^{n\times n}$ whose i-th column is f_i for $i = 1, 2, \ldots, n$. From [10] it is easy to see that for a real $T \in [0, 1]$, $\mathit{Val}(A_q, b_q, T, V) = H(e^T)$ where $H : \mathbb{R} \to \mathbb{R}^n$ is given by

$$H(u) = F\ \mathit{diag}(u^{\lambda_1}, u^{\lambda_2}, \ldots, u^{\lambda_n})\ F^{-1}(V + A_q^{-1} b_q) - A_q^{-1} b_q .$$

It is easy to see that for $\alpha = \tau$, the formula $\Phi_{q,q',q1,q1',\alpha}(V, V1)$ can be constructed. The only point to note is that constants of the form e^r where $r \in \mathbb{Q}$ are definable in $(\mathbb{R}, +, \cdot, <, 0, 1, e)$. The case $\alpha = \mu$ will follow from two observations that we now outline.

Let (q, V, q'), $(q1, V1, q1')$ be reachable configurations of $\mathcal{A}$ such that $(q, V, q') \stackrel{\mu}{\Longrightarrow}_{\mathcal{A}} (q1, V1, q1')$. It follows from the definition of $TS_{\mathcal{A}}$ that $(q, V, q') \stackrel{\mu}{\Longrightarrow}_{\mathcal{A}} (q1, V1, q1')$ iff $V1 = \mathit{Val}(A_q, b_q, 1, V)$ and there exist reals t_i^{ob} in $[\delta_{ob}^0, \delta_{ob}^1]$, $i = 1, 2, \ldots, n$, and a guard g such that $(q, g, q1)$ is a transition in $\longrightarrow$. Further, $(\langle w_1\rangle, \langle w_2\rangle, \ldots, \langle w_n\rangle)$ satisfies g, where w_i is the i-th component of $\mathit{Val}(A_q, b_q, t_i^{ob}, V)$ for $i = 1, 2, \ldots, n$. Firstly we note that the function $t \in [\delta_{ob}^0, \delta_{ob}^1] \to e^t \in [e^{\delta_{ob}^0}, e^{\delta_{ob}^1}]$ is continuous, increasing and onto. Thus there exist reals t_i^{ob}, $i = 1, 2, \ldots, n$, satisfying the desired condition iff there exist reals $u_i \in [e^{\delta_{ob}^0}, e^{\delta_{ob}^1}]$, $i = 1, 2, \ldots, n$, such that w_i is the i-th component of $H(u_i)$ for each $i = 1, 2, \ldots, n$.

Secondly, we note that $-\gamma_{max} \leq w_i \leq \gamma_{max}$ for every $i = 1, 2, \ldots, n$. For a guard g in $\mathcal{A}$, let $\mathit{Valuations}(g)$ be the *finite* set of valuations given by: $(v_1, v_2, \ldots, v_n)$ is in $\mathit{Valuations}(g)$ iff for each i, $v_i = m_i\epsilon$ where m_i is an integer with $\langle -\gamma_{max}\rangle \leq m_i\epsilon \leq \langle \gamma_{max}\rangle$, and $(v_1, v_2, \ldots, v_n)$ satisfies g. It follows that $(w_1, w_2, \ldots, w_n)$ satisfies g iff $(\langle w_1\rangle, \langle w_2\rangle, \ldots, \langle w_n\rangle)$ is in $\mathit{Valuations}(g)$.

Putting together the above two observations, it is now clear how the formula $\Phi_{q,q',q1,q1',\mu}(V, V1)$ can be constructed. It is then also straightforward to see that given a state sequence $q_0 q_1 \ldots q_\ell$ one can construct a sentence $\Phi_{q_0 q_1 \ldots q_\ell}$ such that $\Phi_{q_0 q_1 \ldots q_\ell}$ is true iff $q_0 q_1 \ldots q_\ell$ is in $\mathcal{L}(\mathcal{A})$. □

Theorem 3 implies that one can in principle solve *bounded model checking* problems [5] for the class of hybrid automata satisfying the conditions set out in the statement of the theorem. The next result shows that one can not hope to do much better in this setting.

Theorem 4. *There is no effective procedure which can, given a lazy finite-precision differential hybrid automaton $\mathcal{A}$ satisfying the restrictions stated in Theorem 3 and a control state q_f of $\mathcal{A}$, determine whether q_f is reachable in $\mathcal{A}$. In other words, whether there exists a reachable configuration (q, V, q') of $\mathcal{A}$ such that $q = q_f$.*

Proof. We shall reduce the halting problem of two-counter automata ([11]) to the control state reachability problem of the class of hybrid automata stated in the theorem.

Let $\mathcal{C} = (S, s_{in}, s_{halt}, \rightsquigarrow)$ be a two-counter automaton where S is a finite set of states, $s_{in} \in S$ the initial state, $s_{halt} \in S$ the halting state and $\rightsquigarrow\ \subseteq S \times \{ZERO, POS\}^2 \times \{INC, DEC\}^2 \times S$ the instruction table. The instruction $(s, O_1, O_2, \alpha_1, \alpha_2, s')$ indicates that at state s, if the sign of the integer stored in counter i is O_i then $\mathcal{C}$ can perform action α_i (increment or decrement) on counter i and move to state s'. For example, the instruction $(s, ZERO, POS, INC, DEC, s')$ specifies that at state s, if counter 1 is zero and counter 2 is positive, then $\mathcal{C}$ can increment counter 1, decrement counter 2 and move to state s'. The semantics of $\mathcal{C}$ is defined in the obvious way.

In what follows, we construct a lazy finite-precision differential hybrid automaton $\mathcal{A} = (Q, q_{in}, V_{in}, \{\delta^0_{ob}, \delta^1_{ob}, \delta^0_{up}, \delta^1_{up}\}, \epsilon, \{\rho_q\}_{q\in Q}, \{\gamma_{min}, \gamma_{max}\}, \longrightarrow)$ over continuous variables $x_1, \ldots, x_n$ such that $\delta^0_{up} = 0 = \delta^1_{up}$ and every ρ_q is of the form $\dot{x} = A_q x + b_q$, where A_q is a diagonalizable matrix having n distinct rational eigenvalues. Further, a designated control state $q_f \in Q$ is reachable in $\mathcal{A}$ iff the halting state $s_{halt} \in S$ is reachable in $\mathcal{C}$. In fact, we will construct $\{\rho_q\}_{q\in Q}$ in such a way that every A_q is a diagonal matrix.

We set $n = 3$ and hence $\mathcal{A}$ will be over x_1, x_2, x_3. We first outline the construction of $Q, q_{in}, V_{in}, \{\rho_q\}_{q\in Q}, \longrightarrow$ and later discuss the choice of the parameters $\delta^0_{ob}, \delta^1_{ob}, \epsilon, \gamma_{min}, \gamma_{max}$.

The set of control states Q is $S \bigcup \{s^{\sharp}_{\xi}, s^{\sharp\sharp}_{\xi} \mid \xi \in \rightsquigarrow\}$ where for $\xi = (s, O_1, O_2, \alpha_1, \alpha_2, s')$ in $\rightsquigarrow$, $s^{\sharp}_{\xi} = (s, O_1, O_2, \alpha_1, \alpha_2, s', \sharp)$ and $s^{\sharp\sharp}_{\xi} = (s, O_1, O_2, \alpha_1, \alpha_2, s', \sharp\sharp)$. Intuitively, the continuous variable x_1 (x_2) will represent values of counter 1 (2). A counter having value h will be represented by the corresponding continuous variable taking the value $1 + e^{-1} + e^{-2} + \cdots + e^{-h}$. In particular, a counter with value zero will be represented by the corresponding continuous variable taking the value 1.

Suppose at time T_k, the hybrid automaton $\mathcal{A}$ is at control state s and wants to "execute" the instruction $(s, O_1, O_2, \alpha_1, \alpha_2, s')$. This is to be done by moving first to $(s, O_1, O_2, \alpha_1, \alpha_2, s', \sharp)$ at time T_{k+1}, and then to $(s, O_1, O_2, \alpha_1, \alpha_2, s', \sharp\sharp)$ at *exactly* time T_{k+2}, and finally to land at s' at *exactly* time T_{k+3}. In this process, the variable x_3 will be used to control that $\mathcal{A}$ "stays" for exactly one time unit at each of $(s, O_1, O_2, \alpha_1, \alpha_2, s', \sharp)$, $(s, O_1, O_2, \alpha_1, \alpha_2, s', \sharp\sharp)$.

The initial control state is s_{in}. The initial valuation is $(1, 1, 1)$.

The rate functions are as follows. For $s \in S$, we set ρ_s to be $\dot{x}_1 = 0 = \dot{x}_2 = \dot{x}_3$. Suppose $(s, O_1, O_2, \alpha_1, \alpha_2, s') \in \rightsquigarrow$ is an instruction of $\mathcal{C}$ and $step \in \{\sharp, \sharp\sharp\}$, then the rate function of $(s, O_1, O_2, \alpha_1, \alpha_2, s', step)$ is: $\dot{x_1} = F^{step}_{\alpha_1}(x_1)$, $\dot{x_2} = F^{step}_{\alpha_2}(x_2)$, $\dot{x_3} = H^{step}(x_3)$ where:

- $F^{\sharp}_{INC}(x_i) = -x_i$ and $F^{\sharp\sharp}_{INC}(x_i) = 1$ for $i = 1, 2$.
- $F^{\sharp}_{DEC}(x_i) = -1$ and $F^{\sharp\sharp}_{DEC}(x_i) = x_i$ for $i = 1, 2$.
- $H^{\sharp}(x_3) = 1$ and $H^{\sharp\sharp}(x_3) = -1$.

The transition relation $\longrightarrow$ of $\mathcal{A}$ is $\bigcup_{\xi\in\rightsquigarrow} TR_{\xi}$, where for each $\xi = (s, O_1, O_2, \alpha_1, \alpha_2, s')$ in $\rightsquigarrow$, the members of TR_ξ are

$$(s,\ g_\xi^s,\ (s, O_1, O_2, \alpha_1, \alpha_2, s', \sharp))\ ,$$
$$((s, O_1, O_2, \alpha_1, \alpha_2, s', \sharp),\ g_\xi^\sharp,\ (s, O_1, O_2, \alpha_1, \alpha_2, s', \sharp\sharp))\ ,$$
$$((s, O_1, O_2, \alpha_1, \alpha_2, s', \sharp\sharp),\ g_\xi^{\sharp\sharp},\ s')\ ,$$

with the guards g_ξ^s, $g_\xi^\sharp$, $g_\xi^{\sharp\sharp}$ being specified as follows. The guard g_ξ^s is $\Phi_{O_1}(x_1) \wedge \Phi_{O_2}(x_2)$ where $\Phi_{ZERO}(x_i)$ is $x_i \leq 1$ and $\Phi_{POS}(x_i)$ is $x_i > 1$ for $i = 1, 2$. The guard $g_\xi^\sharp$ is $x_3 \leq 2$ and $g_\xi^{\sharp\sharp}$ is $x_3 \geq 1$.

It remains to choose the parameters $\delta_{ob}^0, \delta_{ob}^1, \epsilon, \gamma_{min}, \gamma_{max}$ appropriately. Recall that a valuation $(v_1, v_2, \ldots, v_n)$ satisfies a polynomial constraint $p(x_1, x_2, \ldots, x_n) < 0$ iff $p(\langle v_1 \rangle_\epsilon, \langle v_2 \rangle_\epsilon, \ldots, \langle v_n \rangle_\epsilon) < 0$. Thus the main technicality is to ensure that the guards are "stable" even with finite precision measurement of values. The only restriction we need for the choice of $\delta_{ob}^0, \delta_{ob}^1, \epsilon, \gamma_{min}, \gamma_{max}$ is that ϵ integrally divides every member of $\{1, \delta_{ob}^0, \delta_{ob}^1\}$, $\langle 1 + e^{-1} \rangle_\epsilon > 1$, $\gamma_{min} \leq 1$, $\gamma_{max} \geq 2$. We emphasize that we need not demand $\delta_{ob}^0 = 1 = \delta_{ob}^1$.

It is now straightforward to establish that the halting state s_{halt} is reachable in the two-counter automaton $\mathcal{C}$ iff the control state s_{halt} is reachable in the hybrid automaton $\mathcal{A}$. □

We note that the above proof shows that the undecidability result goes through even if we restrict ourselves to just rectangular guards. This is not surprising since we have the undecidability result of [9]. From the above proof, it is also easy to construct a lazy finite-precision hybrid automaton $\mathcal{A}_1$ satisfying the conditions in Theorem 4 such that $\mathcal{L}(\mathcal{A}_1)$ is *not* regular. For example, let $\mathcal{C}_1$ be the two-counter automaton $(\{s_{DEC}, s_{INC}, s_{halt}\}, s_{DEC}, s_{halt}, \rightsquigarrow)$ where the members of $\rightsquigarrow$ are: $(s_{DEC}, ZERO, ZERO, INC, INC, s_{INC})$, $(s_{INC}, POS, POS, INC, INC, s_{INC})$, $(s_{INC}, POS, POS, DEC, DEC, s_{DEC})$, $(s_{DEC}, POS, POS, DEC, DEC, s_{DEC})$. Let $\mathcal{A}_1$ be the hybrid automaton constructed from $\mathcal{C}_1$ as in the proof of Theorem 4. It is easy to show that $\mathcal{L}(\mathcal{A}_1)$ is not regular.

6 Summary

We have shown here that the twin features of discrete time semantics and finite precision can be used to cope with hybrid automata whose dynamics are governed by restricted linear differential operators and whose transitions have polynomial guards. It is easy to show (see the appendix) that each of our results, namely Theorem 1, 3, 4, also holds if the combination of finite precision and polynomial guards is replaced by that of perfect precision and rectangular guards.

Our results seem to suggest that once observational and update delays are included to further reduce the expressive power of these automata, one may be able to handle much richer continuous dynamics. The key obstacle here is the lack of means for constructing suitable rational approximations of the continuous dynamics. Here, the mathematical foundations provided in [4] and the logical underpinnings developed in [13, 14] promise to be good starting points.

Appendix

Here we show that each of Theorem 1, 3, 4, also holds if the combination of finite precision and polynomial guards is replaced by that of perfect precision and rectangular guards.

As usual, a rectangular constraint is of the form $x_i \sim c$ where $\sim \in \{<, \leq, >, \geq\}$ and $c \in \mathbb{Q}$. A rectangular guard is a finite conjunction of rectangular constraints. A *lazy perfect-precision rectangular-guard differential hybrid automaton* $\mathcal{A} = (Q, q_{in}, V_{in}, \{\delta^0_{up}, \delta^1_{up}, \delta^0_{ob}, \delta^1_{ob}\}, \{\rho_q\}_{q \in Q}, \{\gamma_{min}, \gamma_{max}\}, \longrightarrow)$ is defined in the same way as a lazy finite-precision differential hybrid automaton except that the precision parameter ϵ is not present and we set $\langle v \rangle = v$ for every $v \in \mathbb{R}$. Further, for every transition (q, g, q') in $\longrightarrow$, g is a rectangular guard. As before, for each $q \in Q$, ρ_q is of the form $\dot{x} = A_q x + b_q$ where A_q is an $n \times n$ matrix with rational entries and $b_q \in \mathbb{Q}^n$.

In what follows, we fix a lazy perfect-precision rectangular-guard differential hybrid automaton $\mathcal{A}$ and assume its associated notations and terminology as defined above.

Theorem 5. *Let $\mathcal{A}$ be a lazy perfect-precision rectangular-guard differential hybrid automaton such that A_q is a diagonal matrix for every control state q. Suppose there exists a* fixed *partition $\{DIF, CON\}$ of the indices $\{1, 2, \ldots, n\}$ such that for each control state q, $\dot{x}_i = A_q(i,i) \cdot x_i$ if $i \in DIF$ and $\dot{x}_i = b_q(i)$ if $i \in CON$. Then $\mathcal{L}(\mathcal{A})$ is a regular subset of $Q^\star$. Further, a finite state automaton accepting $\mathcal{L}(\mathcal{A})$ can be effectively computed from $\mathcal{A}$.*

Proof Sketch: As in the proof of Theorem 1, we shall first quotient the set of reachable configurations $RC_{\mathcal{A}}$ into a *finite number* of equivalence classes using a suitably chosen equivalence relation $\approx$. The crucial property required of $\approx$ is that it should be a congruence with respect to the transition relation of $TS_{\mathcal{A}}$. In other words, if $(q1, V1, q1') \approx (q2, V2, q2')$ and $(q1, V1, q1') \overset{\alpha}{\Longrightarrow}_{\mathcal{A}} (q3, V3, q3')$, then we require that there exists a configuration $(q4, V4, q4')$ such that $(q2, V2, q2') \overset{\alpha}{\Longrightarrow}_{\mathcal{A}} (q4, V4, q4')$ *and* $(q3, V3, q3') \approx (q4, V4, q4')$. We then complete the proof by showing that one can effectively compute these equivalence classes and a transition relation over them such that the resulting finite state automaton generates the language of state sequences.

For notational convenience, we assume $V_{in}(i) > 0$ for every $i \in DIF$. It will become clear that this involves no loss of generality. The key consequence of this assumption is that in any reachable configuration, the value of x_i for $i \in DIF$ will be positive.

Assume without loss of generality that for each rectangular constraint $x_i \sim c$ in $\mathcal{A}$, we have $\gamma_{min} \leq c \leq \gamma_{max}$ if $i \in DIF$ and $-\gamma_{max} \leq c \leq \gamma_{max}$ if $i \in CON$.

Define Δ as in the proof of Theorem 1. Define Γ to be the largest rational which *integrally* divides every number in the finite set of rational numbers $\{A_q(i,i) \cdot \Delta \mid q \in Q, i \in DIF\} \bigcup \{b_q(j) \cdot \Delta \mid q \in Q, j \in CON\} \bigcup \{\gamma_{min}, \gamma_{max}\} \bigcup \{c \mid x_i \sim c$ is a rectangular constraint in $\mathcal{A}$ for some $i \in \{1, 2, \ldots, n\}$ $\}$.

Define Θ_{con} to be the finite set of rational numbers $\{h\Gamma \in [-\gamma_{max}, \gamma_{max}] \mid h \in \mathbb{Z}\}$ (where $\mathbb{Z}$ is the set of integers). In other words, Θ_{con} contains integral multiples of Γ in the interval $[-\gamma_{max}, \gamma_{max}]$.

Let Θ_{IR} be the set of irrational numbers $\{\ln \gamma_{min}, \ln \gamma_{max}\} \bigcup \{\ln c \mid x_i \sim c$is a rectangular constraint in $\mathcal{A}$ for some $i \in DIF\}$. Define Θ_{dif} to be the *finite* set of real numbers $\{h\Gamma \in [\ln \gamma_{min}, \ln \gamma_{max}] \mid h \in \mathbb{Z}\} \bigcup \{\ell\Gamma + \theta \in [\ln \gamma_{min}, \ln \gamma_{max}] \mid \ell \in \mathbb{Z}, \theta \in \Theta_{IR}\}$. In other words, Θ_{dif} contains rational numbers of the form $h\Gamma$ in the interval $[\ln \gamma_{min}, \ln \gamma_{max}]$ where h is a (positive) integer, and irrational numbers of the form $\ell\Gamma + \theta$ in the interval $[\ln \gamma_{min}, \ln \gamma_{max}]$ where ℓ is an integer (that can be positive, zero or negative) and θ is a member of Θ_{IR}.

Loosely speaking, the set Θ_{con} (respectively Θ_{dif}) contains bounds relevant to the values of continuous variables x_i's for $i \in CON$ (respectively $i \in DIF$). The points in Θ_{con} (Θ_{dif}) cut the real line into a finite number of segments. We shall use this segmentation to in turn partition the set of reachable configurations into finitely many equivalence classes.

In this light, let the members of Θ_{dif} be $\{\theta_1, \theta_2, \ldots, \theta_{|\Theta_{dif}|}\}$ where $\theta_1 < \theta_2 < \cdots < \theta_{|\Theta_{dif}|}$. We define the finite set of intervals

$$\mathcal{I}_{dif} = \{(-\infty, \theta_1), (\theta_1, \theta_2), \ldots, (\theta_{|\Theta_{dif}|-1}, \theta_{|\Theta_{dif}|}), (\theta_{|\Theta_{dif}|}, \infty)\} \bigcup \{[\theta_i, \theta_i] \mid i = 1, 2, \ldots, |\Theta_{dif}|\} .$$

In the same way, we define $\mathcal{I}_{con}$ from Θ_{con}.

Define the map $\|\cdot\|_{dif} : \mathbb{R}^+ \to \mathcal{I}_{dif}$ via: $\|v\| = I$ if $\ln v \in I$. Define $\|\cdot\|_{con} : \mathbb{R} \to \mathcal{I}_{con}$ via: $\|v\| = I$ if $v \in I$. Finally we define $\|\cdot\| : RC_{\mathcal{A}} \to (\mathcal{I}_{dif} \cup \mathcal{I}_{con})^n$ by: $\|V\| = (I_1, I_2, \ldots, I_n)$ where $I_i = \|V(i)\|_{dif}$ for $i \in DIF$ and $I_i = \|V(i)\|_{con}$ for $i \in CON$. We can now define the equivalence relation $\approx \subseteq RC_{\mathcal{A}} \times RC_{\mathcal{A}}$ by: $(q1, V1, q1') \approx (q2, V2, q2')$ iff $q1 = q2$, $\|V1\| = \|V2\|$ and $q1' = q2'$.

As anticipated at the beginning of the proof, we can show, similarly to the proof of Claim 2, that $\approx$ is a congruence relation with respect to the transition relation $\Longrightarrow_{\mathcal{A}}$.

By similar arguments to the proof of Theorem 1, it is easy to see that the equivalence classes of $\approx$ can be effectively computed. Further, one can effectively compute the transition relation $\hookrightarrow_{\mathcal{A}}$ where, for any two equivalence classes $C1, C2$ of $\approx$, $C1 \overset{\alpha}{\hookrightarrow}_{\mathcal{A}} C2$ iff there exists (q, V, q') in $C1$, $(q1, V1, q1')$ in $C2$ such that $(q, V, q') \overset{\alpha}{\Longrightarrow}_{\mathcal{A}} (q1, V1, q1')$. From this, it is straightforward to construct a finite state automaton which accepts $\mathcal{L}(\mathcal{A})$. □

Theorem 6. *Suppose $\mathcal{A}$ is a lazy perfect-precision rectangular-guard differential hybrid automaton such that $\delta^0_{up} = 0 = \delta^1_{up}$ and for every control state q, A_q is a diagonalizable matrix having n distinct rational eigenvalues. Then $\mathcal{L}(\mathcal{A})$ is a recursive subset of $Q^\star$.*

Proof Sketch: As in the proof of Theorem 3, the main technicality is to show that given control states $q, q', q1, q1'$ and $\alpha \in \{\tau, \mu\}$, one can construct a formula $\Phi_{q,q',q1,q1',\alpha}(V, V1)$ with free variables $V(i)$, $V1(i)$, $i = 1, 2, \ldots, n$, which asserts that $(q, V, q') \overset{\alpha}{\Longrightarrow}_{\mathcal{A}} (q1, V1, q1')$. And $\Phi_{q,q',q1,q1',\alpha}(V, V1)$ is a formula in $(\mathbb{R}, +, \cdot, <, 0, 1, e)$, the first order theory of reals augmented with the constant e. In what follows, we fix $q, q', q1, q1' \in Q$ and $\alpha \in \{\tau, \mu\}$.

Compute the rational eigenvalues λ_i, $i = 1, 2, \ldots, n$, of A_q, and for each $i = 1, 2, \ldots, n$ find a rational eigenvector $f_i \in \mathbb{Q}^n$ corresponding to λ_i (i.e. $A_q \cdot f_i = \lambda_i \cdot f_i$). Define the function H as in the proof of Theorem 3.

The formula $\Phi_{q,q',q1,q1',\tau}(V, V1)$ can be constructed exactly as in the proof of Theorem 3. We now outline the construction of $\Phi_{q,q',q1,q1',\mu}(V, V1)$.

Let (q, V, q'), $(q1, V1, q1')$ be reachable configurations of $\mathcal{A}$ such that $(q, V, q') \overset{\mu}{\Longrightarrow}_{\mathcal{A}} (q1, V1, q1')$. It follows from the definition of $TS_{\mathcal{A}}$ that $(q, V, q') \overset{\mu}{\Longrightarrow}_{\mathcal{A}} (q1, V1, q1')$ iff $V1 = Val(A_q, b_q, 1, V)$ and there exist reals t^{ob}_i in $[\delta^0_{ob}, \delta^1_{ob}]$, $i = 1, 2, \ldots, n$, and a rectangular guard g such that $(q, g, q1)$ is a transition in $\longrightarrow$. Further, $(w_1, w_2, \ldots, w_n)$ satisfies g, where w_i is the i-th component of $Val(A_q, b_q, t^{ob}_i, V)$ for $i = 1, 2, \ldots, n$. As in the proof of Theorem 3, there exist t^{ob}_i, $i = 1, 2, \ldots, n$, satisfying the desired condition iff there exist reals $u_i \in [e^{\delta^0_{ob}}, e^{\delta^1_{ob}}]$, $i = 1, 2, \ldots, n$, such that w_i is the i-th component of $H(u_i)$ for each $i = 1, 2, \ldots, n$. It is now clear that the formula $\Phi_{q,q',q1,q1',\mu}(V, V1)$ can be constructed.

It follows that given a state sequence $q_0 q_1 \ldots q_\ell$, one can construct a sentence $\Phi_{q_0 q_1 \ldots q_\ell}$ such that $\Phi_{q_0 q_1 \ldots q_\ell}$ is true iff $q_0 q_1 \ldots q_\ell$ is in $\mathcal{L}(\mathcal{A})$. □

We note that the above proof shows that Theorem 6 holds even if we allow polynomial guards.

Theorem 7. *There is no effective procedure which can, given a lazy perfect-precision rectangular-guard differential hybrid automaton $\mathcal{A}$ satisfying the restrictions stated in Theorem 3 and a control state q_f of $\mathcal{A}$, determine whether q_f is reachable in $\mathcal{A}$. In other words, whether there exists a reachable configuration (q, V, q') of $\mathcal{A}$ such that $q = q_f$.*

Proof Sketch: As in the proof of Theorem 4, we reduce the halting problem of two-counter automata ([11]) to the control state reachability problem of the class of hybrid automata stated in the Theorem 7.

Let $\mathcal{C} = (S, s_{in}, s_{halt}, \rightsquigarrow)$ be a two-counter automaton. We construct a lazy perfect-precision rectangular-guard differential hybrid automaton $\mathcal{A} = (Q, q_{in}, V_{in}, \{\delta^0_{ob}, \delta^1_{ob}, \delta^0_{up}, \delta^1_{up}\}, \{\rho_q\}_{q\in Q}, \{\gamma_{min}, \gamma_{max}\}, \longrightarrow)$ over continuous variables $x_1, \ldots, x_n$ such that $\delta^0_{up} = 0 = \delta^1_{up}$ and every ρ_q is of the form $\dot{x} = A_q x + b_q$, where A_q is a diagonalizable matrix having n distinct rational eigenvalues. Further, a designated control state $q_f \in Q$ is reachable in $\mathcal{A}$ iff the halting state $s_{halt} \in S$ is reachable in $\mathcal{C}$.

We set $n = 3$ and hence $\mathcal{A}$ will be over x_1, x_2, x_3. We construct $Q, q_{in}, V_{in}, \{\rho_q\}_{q\in Q}, \longrightarrow$ exactly as in the proof of Theorem 4. As for the choice of the parameters $\delta^0_{ob}, \delta^1_{ob}, \gamma_{min}, \gamma_{max}$, we need only require that $\gamma_{min} \leq 1$ and $\gamma_{max} \geq 2$. We emphasize that we need not demand $\delta^0_{ob} = 1 = \delta^1_{ob}$.

It is now straightforward to establish that the halting state s_{halt} is reachable in the two-counter automaton $\mathcal{C}$ iff the control state s_{halt} is reachable in $\mathcal{A}$. $\square$

From the proof of Theorem 7, it is also easy to construct a lazy perfect-precision rectangular-guard differential hybrid automaton $\mathcal{A}_1$ satisfying the conditions in Theorem 7 such that $\mathcal{L}(\mathcal{A}_1)$ is *not* regular.

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