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Maximum Level and Hitting Probabilities in Stochastic Fluid Flows Using Matrix Differential Riccati Equations

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Abstract In this work, we expose a clear methodology to analyze maximum level and hitting probabilities in a Markov driven fluid queue for various initial condition scenarios and in both cases of infinite and finite buffers. Step by step we build up our argument that finally leads to matrix differential Riccati equations for which there exists a unique solution. The power of the methodology resides in the simple probabilistic argument used that permits to obtain analytic solutions of these differential equations. We illustrate our results by a comprehensive fluid model that we exactly solve.

Keywords Fluid queues · Matrix differential Riccati equations · Markov chains

AMS 2000 Subject Classifications 60K25 · 60J27

1 Introduction

In the last decade, stochastic fluid models and in particular Markov driven fluid process, have received a lot of attention in various context of system modeling, e.g. manufacturing systems (Aggarwal et al. 2005), communication systems (as TCP (vanForeest et al. 2003) or more recently peer to peer file sharing process (Kumar et al. 2007)) and economic systems (risk analysis; Badescu et al. 2005). Many techniques exist to analyze such systems. To cite but a few, we refer to van Dorn and Scheinhardt (1997) for the analysis of a fluid queue fed by a general birth and

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death process, using spectral theory; or (Guillemin and Sericola 2007) that considers a more general case of infinite state space Markov process that drives the fluid queue.

Using the Wiener–Hopf factorization of finite Markov chains, it has been shown in Rogers (1994) that the distribution of the buffer level has a matrix exponential form. Ramaswami (1999), da Silva Soares and Latouche (2002), Ahn and Ramaswami (2003, 2004) and da Silva Soares and Latouche (2006) respectively exhibit and exploit the similarity between stationary fluid queues in a finite Markovian environment and quasi-birth-and-death processes.

Various papers, see for instance Badescu et al. (2005), Bean et al. (2005b), Ramaswami (2006) and Remiche (2005) and references therein, have analyzed first passage time probabilities for Markov driven fluid processes. One key parameter in their analysis was the matrix Ψ characterizing the distribution of the phase visited at the end of a busy period in an infinite buffer or finite buffer case Markov fluid queue. In a recent paper, Ramaswami (2006) derive formulae for a variety of passage time distributions, with application to insurance risk modeling. The passage time distributions are obtained explicitly in terms of Laplace–Stieltjes transforms. His analysis makes extensively use of the so-called $K(s)$ matrix as defined in Ahn and Ramaswami (2004) that permits to determine the expected upcrossings of various levels in a busy period. We propose in this paper to identify and to analyze several other but related performance measures. Our objective is to characterize the maximum peak and minimum trough observed within a busy period that has to be managed by the queue. Such performance measures are of interest when optimizing policies in flow control systems or when realizing perturbation analysis. Maximum and minimum backlogs indeed permit to estimate worst case performance bounds (like safe bounds for encountered delays for example). Another example in the telecommunication domain, is found in the dimensioning of a token bucket: The knowledge of the minimum trough ever reached (before loosing a packet in the system) may help in renegotiating bucket filling rate with the provider. In the same manner, maximum profit or minimum revenues (avoiding bankrupt) need to be estimated when managing risk in stock companies.

The methodology we use in this work, makes no use of the so-called Laplace–Stieltjes transform, but rather exploits another probabilistic argument to directly derive measures of interest. Quantities of interest are obtained through an appropriate conditioning on the number of peaks observed during a busy period. Moreover, obtained equations can be re-interpreted as matrix differential equations that exhibit the so-called Riccati format and for which there exists a unique and explicit solution. We consider both finite and infinite buffer cases for various initial scenarios. The paper is organized as follows.

Section 2 clearly identifies the model we work with and some of the key performance measures of interest in the sequel. We derive in Section 3 an expression for the distribution of the minimum trough visited in a busy period for an infinite buffer size fluid queue. This approach also allows us to give another probabilistic interpretation for the matrix Ψ as defined by Ramaswami (1999), that is, as previously mentioned, the probability of the visited phase at first return to initial level. Next we extend our analysis to first the joint distribution of both the minimum and maximum level visited in such a busy period and secondly, to the distribution of the maximum level visited. Our approach naturally leads to a Riccati differential system of equations for which there exists an explicit solution. In Section 5, the initial fluid level is positive and decreasing. We pursue our analysis by considering the distribution of the maximum peak reached before the fluid level comes back to its initial level. A level-reversed argument leads to the analysis of the finite buffer case for the distribution of minimum level ever visited before going back to the initial level. Throughout the whole paper, we clearly identify the relations that link the different performance measures. Finally, we illustrate the efficiency of the method by considering a simple and rather formal example but that clearly shows the power of the method defined in this work.

2 The Model

We consider a classical fluid queue with an infinite buffer and in which the input and service rates are controlled by a finite homogeneous Markov process $\varphi = \{\varphi_t, t \geq 0\}$ on the finite state space S with infinitesimal generator T. The process φ is also called the phase process and we denote by X_t the amount of fluid in the buffer at time t. It is well-known that the pair (φ_t, X_t) forms a Markov process having a pair of discrete and continuous states. Let ρ_i be the input rate and η_i be the service rate when the Markov process φ is in state i. We denote by r_i the effective input rate of state i, that is $r_i = \rho_i - \eta_i$ and we define the diagonal matrix of effective input rates $R = \mathrm{diag}(r_i, i \in S)$. We denote by S^0, S^- and S^+ the subsets of states $i \in S$ such that $r_i = 0$, $r_i < 0$ and $r_i > 0$, respectively. In the same way, we denote by R_0, R_- and R_+ the diagonal matrices $R_0 = \mathrm{diag}(r_i, i \in S^0)$, $R_- = \mathrm{diag}(-r_i, i \in S^-)$ and $R_+ = \mathrm{diag}(r_i, i \in S^+)$. We clearly have $R_0 = 0$, the null matrix. The number of states in S^0, S^+ and S^- are denoted by n_0, n_- and n_+, respectively. We partition matrix T in a manner conformant to that decomposition of S, by writing

$$T = \begin{pmatrix} T_{00} & T_{0-} & T_{0+} \\ T_{-0} & T_{--} & T_{-+} \\ T_{+0} & T_{+-} & T_{++} \end{pmatrix}.$$

The Markov process φ is supposed to be irreducible and we denoted by π its stationary distribution. We thus have $\pi T = 0$ and $\pi \mathbb{1} = 1$, where $\mathbb{1}$ is the column vector with all its entries equal to 1, its dimension being specified by the context of its use.

We suppose that the stability condition for the fluid queue is satisfied, which means that

$$\sum_{i \in S} r_i \pi_i < 0.$$

This condition ensures that all the busy periods are of finite length a.s. and that the maximum level of the queue during every busy period is finite a.s.

We consider a busy period of that fluid queue and we are interested in several distributions of the fluid queue during a busy period. We denote by M the maximum level in the queue during a busy period. We denote by H the fluid level at the minimum trough and by L the integer-valued random variable representing the index of the smallest trough when they are numbered in the order of their occurrences, if there is indeed a trough. For any $x \geq 0$, let $\theta(x)$ denote the first time greater than x at which the fluid level is equal to x, that is, $\theta(x) = \inf\{t > x | X_t = x\}$. When $X_0 = 0$ and

$\varphi(0) \in S^0 \cup S^-$ then, by definition, we have $\theta(0) = 0$. When $X_0 = 0$ and $\varphi(0) \in S^+$, the instants 0 and $\theta(0)$ are respectively the initial and the final instants of the busy period. These variables are shown in Fig. 1 in the case where the effective input rates are either equal to -1 or equal to 1. We have numbered the successive peaks and troughs arising during a busy period.

For every $n \geq 2$, $\ell = 1, \ldots, n-1$ and $x \geq 0$, we consider the $n_+ \times n_-$ matrix $F_{i,j}(n, \ell, x)$ whose entries are defined, for $i \in S^+$ and $j \in S^-$, by

$$F_{i,j}(n, \ell, x) = \Pr\left\{\varphi(\theta(0)) = j, N = n, L = \ell, H \leq x \middle| \varphi(0) = i, X_0 = 0\right\},$$

where N is the number of peaks arising during a busy period. In the case $N = 1$, there is only one peak during the busy period and so, since there is no trough, we do not define the matrix for $n = 1$. In the path described in Fig. 1, we have $N = 7$ and $L = 3$. $F_{i,j}(\ell, n, x)$ is the probability that a busy period, starting in phase i, contains n peaks, ℓ troughs with its minimum less than or equal to x and ends in phase j.

We also consider the $n_+ \times n_-$ matrix $\Psi_{i,j}(x, y)$ whose entries are defined, for $i \in S^+$, $j \in S^-$, $x \geq 0$ and $0 \leq y \leq x$, by

$$\Psi_{i,j}(x, y) = \Pr\left\{\varphi(\theta(0)) = j, M \leq x, H \leq y \middle| \varphi(0) = i, X_0 = 0\right\},$$

and the $n_- \times n_+$ matrix $\Theta_{i,j}(x)$ whose entries are defined, for $i \in S^-$, $j \in S^+$ and $x > 0$, by $\Theta_{i,j}(x) = \Pr\{\varphi(\gamma(x)) = j | \varphi(0) = i, X_0 = x\}$, where, for any $x \geq 0$, $\gamma(x)$ denotes the first positive instant at which the fluid level is equal x, i.e. $\gamma(x) = \inf\{t > 0 | X_t = x\}$. $\Psi_{i,j}(x, y)$ is the probability that a busy period, starting in phase i, has its maximum peak less than or equal to x, its minimum trough less than or equal to y and ends in phase j. $\Theta_{i,j}(x)$ is the probability that a busy period, starting in phase i, ends in phase j.

Note that $\gamma(x)$ is the first time at which the fluid level is equal to x. The function $\theta(x)$ is used to detect the time at which the minimum trough H when $X_0 = 0$ (see Fig. 1) or the maximum peak M when $X_0 > 0$ (see Fig. 2a) occur. By definition of H and M, when $X_0 = 0$, $\theta(H)$ is the second time at which the fluid level is equal to H and, when $X_0 > 0$, $\theta(M)$ is the second time at which the fluid level is equal to M. We also consider the finite buffer case.

Let us introduce the $(n_- + n_+) \times (n_- + n_+)$ matrix

$$Q = \begin{pmatrix} Q_{--} & Q_{-+} \\ Q_{+-} & Q_{++} \end{pmatrix},$$

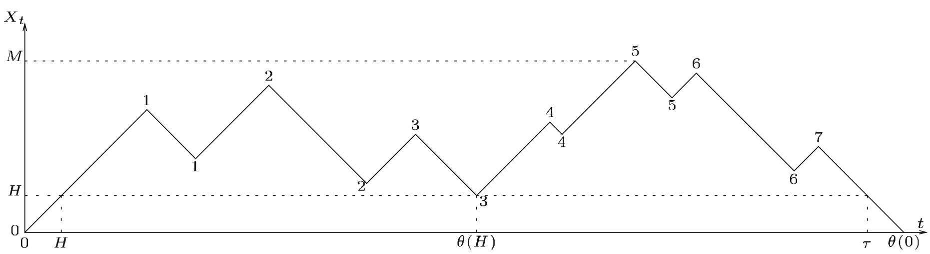

Fig. 1 A busy period

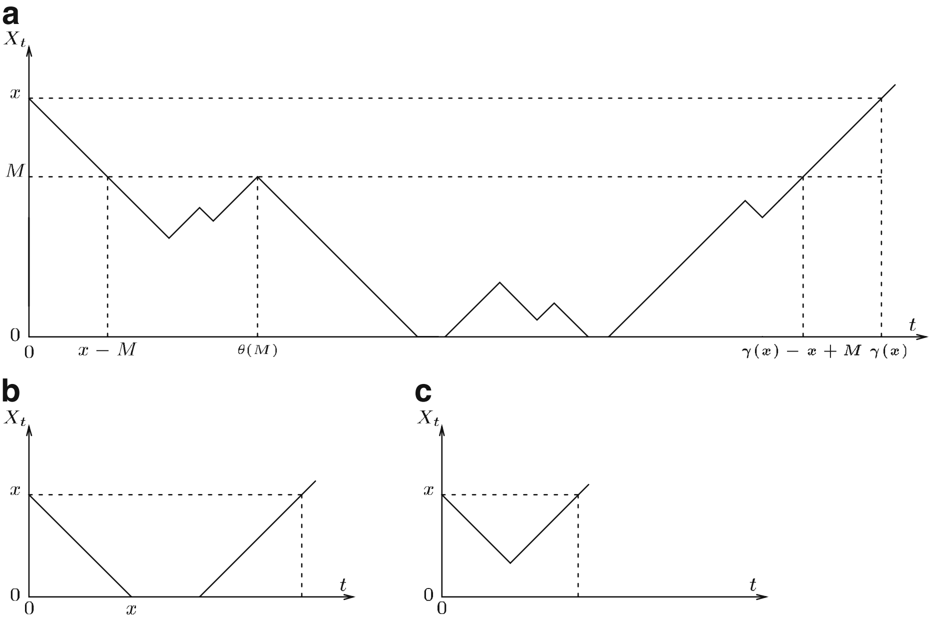

Fig. 2 A period of level $< x$ (**a**–**c**)

where $Q_{--} = R_-^{-1}\left(T_{--} - T_{-0}T_{00}^{-1}T_{0-}\right)$, $Q_{-+} = R_-^{-1}\left(T_{-+} - T_{-0}T_{00}^{-1}T_{0+}\right)$, $Q_{+-} = R_+^{-1}\left(T_{+-} - T_{+0}T_{00}^{-1}T_{0-}\right)$ and $Q_{++} = R_+^{-1}\left(T_{++} - T_{+0}T_{00}^{-1}T_{0+}\right)$.

These matrices allow us to restrict our problem to a Markov chain with state space $S^- \cup S^+$ with infinitesimal generator Q and effective input rates equal to -1 or $+1$. For instance the matrix Q_{--} governs the transitions from $i \in S^-$ to $j \in S^-$ without any visit to S^+. More formally, as shown in Rubino and Sericola (1989) and Bean et al. (2005b), $e^{Q_{--}x}(i, j)$ is the probability, starting from state i, to reach state j with an accumulated reward equal to x and without leaving the set $S^0 \cup S^-$. Here the accumulated reward corresponds to the amount of fluid generated from the effective input rates of the matrix R_-. A symmetric interpretation holds for matrix Q_{++}. Concerning the matrix Q_{-+}, the entry $Q_{-+}(i, j)$ is the rate, rescaled according to matrix R_-, at which state j is reached from state i either directly from state i or after some time spent in subset S^0. A symmetric interpretation holds for matrix Q_{+-}. Such a transformation consists in considering that the time spent in the zero rate states is immaterial.

Since we are concerned by quantities such as the minimum trough, the maximum level and hitting probabilities, the fluid queue with parameters (T, R) is equivalent to the fluid queue with parameters (Q, C) where C is the $(n_- + n_+) \times (n_- + n_+)$ matrix

$$C = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix},$$

where I is the identity matrix whose dimension is specified by the context of its use. This transformation has been also suggested in Asmussen (1995) and Rogers (1994).

Thus, in the following, we will consider a fluid queue driven by a Markov chain $\varphi = \{\varphi_t, t \geq 0\}$ with state space $S = S^- \cup S^+$, infinitesimal generator Q and effective input rates given by matrix C, i.e. equal to -1 or equal 1.

3 Minimum Trough

For $i \in S^+$, $j \in S^-$, $n \geq 2$, $\ell = 1, \ldots, n-1$ and $y \geq 0$, we denote by $f_{i,j}(n, \ell, y)$ the joint density associated with the distribution

$$F_{i,j}(n, \ell, y) = \Pr\left\{\varphi(\theta(0)) = j, N = n, L = \ell, H \leq y \middle| \varphi(0) = i, X_0 = 0\right\}$$

and defined by $f_{i,j}(n, \ell, y) = dF_{i,j}(n, \ell, y)/dy$. We also introduce, for $n \geq 1$, the $n_+ \times n_-$ matrix $W(n)$ whose entries are defined, for $i \in S^+$ and $j \in S^-$, by

$$W_{i,j}(n) = \Pr\left\{\varphi(\theta(0)) = j, N = n \middle| \varphi(0) = i, X_0 = 0\right\}.$$

$W_{i,j}(n)$ is the probability that a busy period, starting in phase i, contains n peaks and ends in phase j. Summing and integrating the density $f_{i,j}(\ell, n, y)$ over the location and the value of the minimum trough, we obtain, for $n \geq 2$,

$$W_{i,j}(n) = \int_0^\infty \sum_{\ell=1}^{n-1} f_{i,j}(n, \ell, y) dy.$$

The $n_+ \times n_-$ matrix containing the terms $f_{i,j}(n, \ell, y)$ is denoted by $f(n, \ell, y)$ and is given together with matrix $W(1)$ by the following theorem.

Theorem 1 *For $n \geq 2$, $\ell = 1, \ldots, n-1$ and $y \geq 0$, we have*

$$W(1) = \int_0^\infty e^{Q_{++}y} Q_{+-} e^{Q_{--}y} dy \quad \text{and} \quad f(n, \ell, y) = e^{Q_{++}y} W(\ell) Q_{-+} W(n-\ell) e^{Q_{--}y}.$$

Proof The proof follows the same arguments developed in Bean et al. (2005b). To obtain the expression for $W(1)$ we consider a sample path starting from fluid level 0 in phase $i \in S^+$ and returning to level 0 in phase $j \in S^-$ with only one peak of height y in between. Such a sample path can be broken up into three stages which are:

1. The phase process φ starts in phase $i \in S^+$ and reaches some state $k \in S^+$ at time y without leaving subset S^+. This means that the fluid level increases from level 0 to level y. As seen in the previous section, the corresponding probability is equal to $e^{Q_{++}y}(i, k)$.
2. Since a peak (of height equal to y) occurs, this means that a transition of the phase process φ occurs from state k to some state $h \in S^-$. The corresponding transition rate is equal to $Q_{+-}(k, h)$.
3. Starting from state $h \in S^-$, the process φ reaches state $j \in S^-$ at time y without leaving subset S^-. This means that the fluid level decreases from level y to level 0. As seen in the previous section, the corresponding probability is equal to $e^{Q_{--}y}(h, j)$.

We thus obtain

$$W_{i,j}(1) = \int_0^\infty \sum_{k\in S^+} \sum_{h\in S^-} e^{Q_{++}y}(i,k)\, Q_{+-}(k,h) e^{Q_{--}y}(h,j)dy,$$

that is

$$W(1) = \int_0^\infty e^{Q_{++}y} Q_{+-} e^{Q_{--}y} dy.$$

Let now $n \geq 2$ and $1 \leq \ell \leq n-1$. A typical example of such a sample path is shown in Fig. 1. It is a path starting from fluid level 0 in phase $i \in S^+$ and returning to level 0 in phase $j \in S^-$ with $N = n$ peaks, with the minimum trough of height $H = y$ at position $L = \ell$. Such a sample path can be broken up into five stages which are:

1. As we observe, H denotes the minimum level as well as the amount of time needed to reach it. The phase process φ starts in phase $i \in S^+$ and reaches some state $k \in S^+$ at time $H = y$ without leaving subset S^+. This means that the fluid level increases from level 0 to level y. As seen in the previous section, the corresponding probability is equal to $e^{Q_{++}y}(i,k)$.
2. Starting from phase $k \in S^+$ and level y, the fluid process returns for the first time to level y in some phase $h \in S^-$. By definition of H and L, the number of peaks between instants H and $\theta(H)$ is equal to ℓ. By the spatial homogeneity of the process, the corresponding probability is equal to $W_{k,h}(\ell)$.
3. Since a trough (of height equal to y at time $\theta(H)$) occurs, this means that a transition of the phase process φ occurs from state $h \in S^-$ to some state $m \in S^+$. The corresponding transition rate is equal to $Q_{-+}(h,m)$.
4. Once again, starting from phase $m \in S^+$ and level y, the fluid process returns for the first time to level y in some phase $v \in S^-$. By definition of H and L, the number of peaks between instants $\theta(H)$ and $\tau = \theta(0) - H$ is equal to $n - \ell$. By the spatial homogeneity of the process, the corresponding probability is equal to $W_{m,v}(n-\ell)$.
5. The phase process φ starts in phase $v \in S^-$ and level y. It reaches level 0 in state j without leaving subset S^-, at time $\varphi(\theta(0))$. This means that the fluid level decreases from level y to level 0. As seen in the previous section, the corresponding probability is equal to $e^{Q_{--}y}(v,j)$.

We thus obtain

$$f_{i,j}(n,\ell,y) = \sum_{k\in S^+} \sum_{h\in S^-} \sum_{m\in S^+} \sum_{v\in S^-} e^{Q_{++}y}(i,k) W_{k,h}(\ell) Q_{-+}(h,m) W_{m,v}(n-\ell) e^{Q_{--}y}(v,j),$$

that is $f(n,\ell,y) = e^{Q_{++}y} W(\ell) Q_{-+} W(n-\ell) e^{Q_{--}y}$. □

The distribution of the number of peaks is given in the following corollary.

Corollary 1 *For every $n \geq 2$, we have*

$$W(n) = \int_0^\infty e^{Q_{++}y} \sum_{\ell=1}^{n-1} W(\ell) Q_{-+} W(n-\ell) e^{Q_{--}y} dy.$$

Proof It suffices to write

$$W(n)=\int_0^{\infty}\sum_{\ell=1}^{n-1} f_{i,j}(n,\ell,y)dy$$

to get the result. □

Let Ψ denote the $n_+\times n_-$ stochastic matrix whose entries are defined, for $i\in S^+$ and $j\in S^-$, by $\Psi_{i,j}=\Pr\{\varphi(\theta(0))=j|\varphi(0)=i,\,X_0=0\}$. $\Psi_{i,j}$ is the probability that the phase at the instant of the first return to the initial level is equal to j. By spatial homogeneity, this probability is independent of the value of the initial level. We have

$$\Psi=\sum_{n=1}^{\infty}W(n).$$

In the next theorem, we give a relation satisfied by matrix Ψ.

Theorem 2

$$\Psi=\int_0^{\infty}e^{Q_{++}y}Q_{+-}e^{Q_{--}y}dy+\int_0^{\infty}e^{Q_{++}y}\Psi Q_{-+}\Psi e^{Q_{--}y}dy.$$

Proof From Corollary 1, we have, for $n\geq 2$,

$$W(n)=\int_0^{\infty}e^{Q_{++}y}\sum_{\ell=1}^{n-1}W(\ell)Q_{-+}W(n-\ell)e^{Q_{--}y}dy.$$

By summation over n, we get, using the monotone convergence theorem,

$$\begin{aligned}
\Psi &= W(1)+\sum_{n=2}^{\infty}\int_0^{\infty}e^{Q_{++}y}\sum_{\ell=1}^{n-1}W(\ell)Q_{-+}W(n-\ell)e^{Q_{--}y}dy\\
&= W(1)+\int_0^{\infty}e^{Q_{++}y}\sum_{n=2}^{\infty}\sum_{\ell=1}^{n-1}W(\ell)Q_{-+}W(n-\ell)e^{Q_{--}y}dy\\
&= W(1)+\int_0^{\infty}e^{Q_{++}y}\sum_{\ell=1}^{\infty}\sum_{n=\ell+1}^{\infty}W(\ell)Q_{-+}W(n-\ell)e^{Q_{--}y}dy\\
&= W(1)+\int_0^{\infty}e^{Q_{++}y}\sum_{\ell=1}^{\infty}W(\ell)Q_{-+}\sum_{n=\ell+1}^{\infty}W(n-\ell)e^{Q_{--}y}dy\\
&= W(1)+\int_0^{\infty}e^{Q_{++}y}\Psi Q_{-+}\Psi e^{Q_{--}y}dy,
\end{aligned}$$

which is the desired result. □

It is easily checked that matrix Ψ is solution to the following matrix algebraic Riccati equation $Q_{++}\Psi+\Psi Q_{--}+\Psi Q_{-+}\Psi+Q_{+-}=0$. This equation has been considered in several papers and several algorithms have been developed to compute

Ψ which is the minimal solution of this equation. See for instance Bean et al. (2005a, b) and the references therein.

Let $F(x)$ denote the $n_+ \times n_-$ the matrix whose entries are defined, for $i \in S^+$ and $j \in S^-$, by $F_{i,j}(x) = \Pr\{\varphi(\theta(0)) = j, H \leq x|\varphi(0) = i\}$. $F_{i,j}(x)$ is the probability that the phase at the instant of the first return to the initial level is equal to j with a minimum trough of height less than or equal to x. This probability is defined only when the minimum trough exists. The minimum trough exists if and only if the number of peaks N is greater than or equal to 2. Moreover, when it exists, we have $H > 0$. Thus we define the event $H = 0$ to represent the non-existence of the minimum trough, i.e. the case where the number of peaks is equal to 1. We then have the following result.

Corollary 2 *For every $x \geq 0$,*

$$F(x) = \int_0^\infty e^{Q_{++}y} Q_{+-} e^{Q_{--}y} dy + \int_0^x e^{Q_{++}y} \Psi Q_{-+} \Psi e^{Q_{--}y} dy.$$

Proof By definition, we have

$$\begin{aligned} F_{i,j}(x) &= \Pr\left\{\varphi(\theta(0)) = j, H = 0\middle|\varphi(0) = i\right\} + \Pr\left\{\varphi(\theta(0)) = j, 0 < H \leq x\middle|\varphi(0) = i\right\} \\ &= \Pr\left\{\varphi(\theta(0)) = j, N = 1\middle|\varphi(0) = i\right\} + \Pr\left\{\varphi(\theta(0)) = j, N \geq 2, H \leq x\middle|\varphi(0) = i\right\} \\ &= W_{i,j}(1) + \int_0^x \sum_{n=2}^\infty \sum_{\ell=1}^{n-1} f_{i,j}(n, \ell, y) dy. \end{aligned}$$

Following the same lines used in the proof of Theorem 2, we get the result. □

4 Maximum Level

We consider in this section the maximum fluid level M reached during a busy period. A typical path is shown in Fig. 1. For $i \in S^+$, $j \in S^-$, $x \geq 0$ and $0 \leq y \leq x$, we denote by $\psi_{i,j}(x, y)$ the marginal density associated with the distribution

$$\Psi_{i,j}(x, y) = \Pr\left\{\varphi(\theta(0)) = j, M \leq x, H \leq y\middle|\varphi(0) = i, X_0 = 0\right\},$$

and defined by $\psi_{i,j}(x, y) = \partial\Psi_{i,j}(x, y)/\partial y$. For $x \geq 0$, we denote by $\Psi(x)$ the $n_+ \times n_-$ matrix containing the $\Psi_{i,j}(x)$ defined, for $i \in S^+$ and $j \in S^-$, by

$$\Psi_{i,j}(x) = \Pr\left\{\varphi(\theta(0)) = j, M \leq x\middle|\varphi(0) = i, X_0 = 0\right\}. \tag{1}$$

$\Psi_{i,j}(x)$ is the probability that, starting from phase $i \in S^+$ and any level $u \geq 0$, the phase at the instant of the first return to the initial level u is equal to j and the maximum level is less than or equal to $x + u$. By the spatial homogeneity, this probability is independent of u, that is why we define $\Psi(x)$ only for $u = 0$. The $n_+ \times n_-$ matrix containing the terms $\psi_{i,j}(x, y)$ is denoted by $\psi(x, y)$ and is given by the following theorem.

Theorem 3 *For $x \geq 0$ and $0 \leq y \leq x$, we have*

$$\Psi(x, 0) = \int_0^x e^{Q_{++}y} Q_{+-} e^{Q_{--}y} dy \quad \text{and} \quad \psi(x, y) = e^{Q_{++}y} \Psi(x - y) Q_{-+} \Psi(x - y) e^{Q_{--}y}.$$

Proof We proceed as for the proof of Theorem 1. The term $\Psi(x, 0)$ corresponds to the case where there is only one peak, and thus no trough, which means, as defined in the previous section, that $H = 0$. To obtain the expression for $\Psi(x, 0)$ we consider a sample path starting from fluid level 0 in phase $i \in S^+$ and returning to level 0 in phase $j \in S^-$ with only one peak of height y (with $y \le x$) in between. Such a sample path can be broken up into three stages which are:

1. The phase process φ starts in phase $i \in S^+$ and reaches some state $k \in S^+$ at time y without leaving subset S^+. This means that the fluid level increases from level 0 to level y. As seen in the previous section, the corresponding probability is equal to $e^{Q_{++}y}(i, k)$.
2. Since a peak (of height equal to y) occurs, this means that a transition of the phase process φ occurs from state k to some state $h \in S^-$. The corresponding transition rate is equal to $Q_{+-}(k, h)$.
3. Starting from state $h \in S^-$, the process φ reaches state $j \in S^-$ at time y without leaving subset S^-. This means that the fluid level decreases from level y to level 0. As seen in the previous section, the corresponding probability is equal to $e^{Q_{--}y}(h, j)$.

We thus obtain

$$\Psi_{i,j}(x, 0) = \int_0^x \sum_{k \in S^+} \sum_{h \in S^-} e^{Q_{++}y}(i, k) Q_{+-}(k, h) e^{Q_{--}y}(h, j) dy,$$

that is

$$\Psi(x, 0) = \int_0^x e^{Q_{++}y} Q_{+-} e^{Q_{--}y} dy.$$

We consider now the case where a minimum trough exists, i.e. $H > 0$. A typical example of such a sample path is shown in Fig. 1. It is a path starting from fluid level 0 in phase $i \in S^+$ and returning to level 0 in phase $j \in S^-$ with at least two peaks (i.e. $H > 0$), with the minimum trough and the maximum level less than or equal to x. Such a sample path can be broken up into five stages which are:

1. The phase process φ starts in phase $i \in S^+$ and reaches some state $k \in S^+$ at time $H = y$ (with $y \le x$) without leaving subset S^+. This means that the fluid level increases from level 0 to level y. As seen in the previous section, the corresponding probability is equal to $e^{Q_{++}y}(i, k)$.
2. Starting from phase $k \in S^+$ and level y, the fluid process returns for the first time to level y in some phase $h \in S^-$, without exceeding level x. By the spatial homogeneity of the process, the corresponding probability is equal to $\Psi(x - y)$.
3. Since a trough (of height equal to y at time $\theta(H)$) occurs, this means that a transition of the phase process φ occurs from state $h \in S^-$ to some state $m \in S^+$. The corresponding transition rate is equal to $Q_{-+}(h, m)$.
4. Once again, starting from phase $m \in S^+$ and level y, the fluid process returns for the first time to level y in some phase $v \in S^-$ without exceeding level x. By the spatial homogeneity of the process, the corresponding probability is equal to $\Psi(x - y)$.

5. The phase process φ starts in phase $v \in S^-$ and level y. It reaches level 0 in state j without leaving subset S^-, at time $\varphi(\theta(0))$. This means that the fluid level decreases from level y to level 0. As seen in the previous section, the corresponding probability is equal to $e^{Q_{--}y}(v, j)$.

We thus obtain

$$\psi_{i,j}(x, y) = \sum_{k\in S^+} \sum_{h\in S^-} \sum_{m\in S^+} \sum_{v\in S^-} e^{Q_{++}y}(i, k)\Psi_{k,h}(x-y)Q_{-+}(h, m)\Psi_{m,v}(x-y)e^{Q_{--}y}(v, j),$$

that is $\psi(x, y) = e^{Q_{++}y}\Psi(x-y)Q_{-+}\Psi(x-y)e^{Q_{--}y}$. □

The matrices $\Psi(x, z)$ and $\Psi(x)$ are given by the following corollary.

Corollary 3 *For $x \geq 0$ and $0 \leq z \leq x$, we have*

$$\Psi(x, z) = \int_0^x e^{Q_{++}y}Q_{+-}e^{Q_{--}y}dy + \int_0^z e^{Q_{++}y}\Psi(x-y)Q_{-+}\Psi(x-y)e^{Q_{--}y}dy. \quad (2)$$

$$\Psi(x) = \int_0^x e^{Q_{++}y}Q_{+-}e^{Q_{--}y}dy + \int_0^x e^{Q_{++}y}\Psi(x-y)Q_{-+}\Psi(x-y)e^{Q_{--}y}dy. \quad (3)$$

Proof It suffices to write

$$\Psi(x, z) = \Psi(x, 0) + \int_0^z \psi(x, y)dy,$$

and $\Psi(x) = \Psi(x, x)$. □

Note that we have $\lim_{x\longrightarrow\infty} \Psi(x, z) = F(z)$ and $\lim_{x\longrightarrow\infty} \Psi(x) = \Psi$. We denote by $\Psi'(x)$ the derivative of $\Psi(x)$ with respect to x. We then have the following result.

Theorem 4 *The function $\Psi(x)$ satisfies the following matrix differential Riccati equation*

$$\Psi'(x) = Q_{++}\Psi(x) + \Psi(x)Q_{--} + \Psi(x)Q_{-+}\Psi(x) + Q_{+-}, \quad (4)$$

with $\Psi(0) = 0$ as initial condition.

Proof By definition of function $\Psi(x)$ in relation (1), the initial condition is trivially given by $\Psi(0) = 0$, since the maximum level M during a busy period is positive. Using a variable change, relation (3) can be written as

$$\Psi(x) = \int_0^x e^{Q_{++}y}Q_{+-}e^{Q_{--}y}dy + e^{Q_{++}x}\int_0^x e^{-Q_{++}y}\Psi(y)Q_{-+}\Psi(y)e^{-Q_{--}y}dye^{Q_{--}x}.$$

In order to avoid too long expressions in the derivation of $\Psi'(x)$, we introduce the following notation

$$\alpha(x) = \int_0^x e^{Q_{++}y}Q_{+-}e^{Q_{--}y}dy \text{ and } \beta(x) = \int_0^x e^{-Q_{++}y}\Psi(y)Q_{-+}\Psi(y)e^{-Q_{--}y}dy.$$

It is easy to check that $e^{Q_{++}x}Q_{+-}e^{Q_{--}x} - Q_{++}\alpha(x) - \alpha(x)Q_{--} = Q_{+-}$. Differentiating Eq. 3, we get

$$\begin{aligned}\Psi'(x) &= e^{Q_{++}x}Q_{+-}e^{Q_{--}x} + Q_{++}e^{Q_{++}x}\beta(x)e^{Q_{--}x} + e^{Q_{++}x}\left[\beta'(x)e^{Q_{--}x} + \beta(x)e^{Q_{--}x}Q_{--}\right]\\ &= e^{Q_{++}x}Q_{+-}e^{Q_{--}x} + Q_{++}\left[\Psi(x) - \alpha(x)\right] + e^{Q_{++}x}\beta'(x)e^{Q_{--}x} + \left[\Psi(x) - \alpha(x)\right]Q_{--}\\ &= Q_{++}\Psi(x) + \Psi(x)Q_{--} + e^{Q_{++}x}\beta'(x)e^{Q_{--}x} + Q_{+-}\\ &= Q_{++}\Psi(x) + \Psi(x)Q_{--} + \Psi(x)Q_{-+}\Psi(x) + Q_{+-},\end{aligned}$$

which is the desired result. □

With the initial condition $\Psi(0) = 0$, the Cauchy–Lipschitz Theorem ensures that the matrix differential Riccati equation (4) has a unique solution. This solution is thus given, for $x \geq 0$, by Eq. 1. By definition of diagonal matrix C, we have

$$CQ = \begin{pmatrix} -Q_{--} & -Q_{-+} \\ Q_{+-} & Q_{++} \end{pmatrix}.$$

According to the decomposition $S = S^- \cup S^+$, we define the four matrices $A(x)$, $B(x)$, $C(x)$ and $D(x)$ occurring in the matrix e^{CQx} by writing

$$e^{CQx} = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}. \tag{5}$$

The following theorem gives an expression of the solution $\Psi(x)$ to the matrix differential Riccati equation (4).

Theorem 5 *For every $x \geq 0$, we have $\Psi(x) = C(x)A(x)^{-1}$.*

Proof Let us consider the following linear differential equation

$$Y'(x) = \big(-Q_{--} - Q_{-+}\Psi(x)\big)Y(x) \quad \text{and} \quad Y(0) = I. \tag{6}$$

The function Ψ being continuous, this linear system has a unique solution which is invertible. We now define the $n_+ \times n_-$ matrix $Z(x)$ by $Z(x) = \Psi(x)Y(x)$. Using this definition, Eq. 6 becomes

$$Y'(x) = -Q_{--}Y(x) - Q_{-+}Z(x). \tag{7}$$

Differentiating $Z(x)$ with respect to x, we obtain from Eqs. 4 and 7

$$\begin{aligned}Z'(x) &= \Psi'(x)Y(x) + \Psi(x)Y'(x)\\ &= \big(Q_{++}\Psi(x) + \Psi(x)Q_{--} + \Psi(x)Q_{-+}\Psi(x) + Q_{+-}\big)Y(x)\\ &\quad + \Psi(x)\big(-Q_{--}Y(x) - Q_{-+}Z(x)\big)\\ &= Q_{++}Z(x) + \Psi(x)Q_{-+}\Psi(x)Y(x) + Q_{+-}Y(x) - \Psi(x)Q_{-+}Z(x)\\ &= Q_{++}Z(x) + Q_{+-}Y(x).\end{aligned} \tag{8}$$

Putting together Eqs. 7 and 8 we obtain

$$\begin{pmatrix} Y'(x) \\ Z'(x) \end{pmatrix} = \begin{pmatrix} -Q_{--} & -Q_{-+} \\ Q_{+-} & Q_{++} \end{pmatrix}\begin{pmatrix} Y(x) \\ Z(x) \end{pmatrix} = CQ\begin{pmatrix} Y(x) \\ Z(x) \end{pmatrix},$$

with $Y(0) = I$ and $Z(0) = 0$. The solution to that equation is given by

$$\begin{pmatrix} Y(x) \\ Z(x) \end{pmatrix} = e^{CQx} \begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} A(x) \\ C(x) \end{pmatrix},$$

which means that $Y(x) = A(x)$, $Z(x) = C(x)$ and thus, since $A(x)$ is invertible, we have $\Psi(x) = C(x)A(x)^{-1}$. □

5 Maximum Peak

We consider the first return to the initial level x when $x > 0$ and the initial phase is in S^-. More formally, we introduce the $n_- \times n_+$ matrix $G(x, y)$ whose entries are defined, for $i \in S^-$, $j \in S^+$, $x > 0$ and $0 \leq y \leq x$, by

$$G_{i,j}(x, y) = \Pr\left\{\varphi(\gamma(x)) = j, M \leq y \middle| \varphi(0) = i, X_0 = x\right\},$$

where M denotes the maximum peak between instants 0 and $\gamma(x)$. When there are no peaks, we take as convention $M = 0$. The $n_- \times n_+$ matrix $\Theta(x)$ whose entries are defined, for $i \in S^-$, $j \in S^+$ and $x > 0$, by

$$\Theta_{i,j}(x) = \Pr\left\{\varphi(\gamma(x)) = j \middle| \varphi(0) = i, X_0 = x\right\},$$

satisfies $\Theta(x) = G(x, x)$. By spatial homogeneity, we have, for $i \in S^+$ and $j \in S^-$, $\Pr\{\varphi(\gamma(x)) = j | \varphi(0) = i, X_0 = x\} = \Psi_{i,j}$. For $i \in S^-$ and $j \in S^+$, the problem is more complicated because of the influence of the boundary level zero which makes the hitting probability $\Theta_{i,j}(x)$ dependent of x. A typical example of such a sample path is shown in Fig. 2.

We denote by $g_{i,j}(x, y)$ the density associated with the distribution $G_{i,j}(x, y)$, i.e. $g_{i,j}(x, y) = \partial G_{i,j}(x, y)/\partial y$.

Theorem 6 *For $x > 0$, we have*

$$G(x, 0) = \int_0^x e^{Q_{--}y} Q_{-+} e^{Q_{++}y} dy + \left(-Q_{--}\right)^{-1} e^{Q_{--}x} Q_{-+} e^{Q_{++}x} \tag{9}$$

and, for $0 \leq y \leq x$,

$$g(x, y) = e^{Q_{--}y} \Theta(x - y) Q_{+-} \Theta(x - y) e^{Q_{++}y} \tag{10}$$

Proof We proceed as for the proof of Theorems 1 and 3. The term $G(x, 0)$ corresponds to the case where there are no peaks in the interval $[0, \gamma(x)]$, i.e. where $M = 0$. This situation corresponds to the paths shown in Fig. 2b and c. Figure 2b corresponds to the case where the height of the unique trough, denoted by H, is zero and Fig. 2c corresponds to the case where H is positive. We thus have

$$\begin{aligned} G(x, 0) = &\Pr\left\{\varphi(\gamma(x)) = j, M = 0, H > 0 \middle| \varphi(0) = i, X_0 = x\right\} \\ &+ \Pr\left\{\varphi(\gamma(x)) = j, M = 0, H = 0 \middle| \varphi(0) = i, X_0 = x\right\}. \end{aligned}$$

The first term is the symmetric term of $\Psi(x, 0)$, so we easily get

$$\Pr\left\{\varphi(\gamma(x)) = j, M = 0, H > 0 \middle| \varphi(0) = i, X_0 = x\right\} = \int_0^x e^{Q_{--}y} Q_{-+} e^{Q_{++}y} dy.$$

For the second term, which corresponds to the sample path of Fig. 2b, the phase process φ starts in phase $i \in S^-$ with a level x. It stays in subset S^- for a duration $y \geq x$, reaching some state $k \in S^-$ and thus with a level 0. Next a transition occurs from state $k \in S^-$ to state $h \in S^+$, with rate $Q_{+-}(k,h)$, and the process φ reaches state $j \in S^+$ at time x without leaving subset S^+. We thus have

$$\begin{aligned}\Pr\left\{\varphi(\gamma(x)) = j, M = 0, H = 0 \middle| \varphi(0) = i, X_0 = x\right\} &= \int_x^\infty e^{Q_{--}y} dy\ Q_{-+} e^{Q_{++}x} \\ &= \left(-Q_{--}\right)^{-1} e^{Q_{--}x} Q_{-+} e^{Q_{++}x}\end{aligned}$$

We consider now the case where there is at least one peak, i.e. $M > 0$. A typical example of such a sample path is shown in Fig. 2a. Such a sample path can be broken up into five stages which are:

1. The phase process φ starts in phase $i \in S^-$ and reaches some state $k \in S^-$ at time $x - M = y$ without leaving subset S^-. This means that the fluid level decreases from level x to level $x - y$. As seen in the previous section, the corresponding probability is equal to $e^{Q_{--}y}(i,k)$.
2. Starting from phase $k \in S^-$ and level $x - y$, the fluid process returns for the first time to level $x - y$ in some phase $h \in S^+$, without exceeding level $x - y$. The corresponding probability is equal to $\Theta(x - y)$.
3. Since a peak (of height equal to $x - y$ at time $\theta(M)$) occurs, this means that a transition of the phase process φ occurs from state $h \in S^+$ to some state $m \in S^-$. The corresponding transition rate is equal to $Q_{-+}(h,m)$.
4. Once again, starting from phase $m \in S^-$ and level $x - y$, the fluid process returns for the first time to level $x - y$ in some phase $v \in S^+$ without exceeding level $x - y$. By the spatial homogeneity of the process, the corresponding probability is equal to $\Theta(x - y)$.
5. The phase process φ starts in phase $v \in S^+$ and level $x - y$. It reaches level x in state j without leaving subset S^+, at time $\varphi(\gamma(x))$. This means that the fluid level increases from level $x - y$ to level x. As seen in the previous section, the corresponding probability is equal to $e^{Q_{++}y}(v, j)$.

We thus obtain

$$g_{i,j}(x,y) = \sum_{k\in S^-}\sum_{h\in S^+}\sum_{m\in S^-}\sum_{v\in S^+} e^{Q_{++}y}(i,k)\Theta_{k,h}(x-y)Q_{+-}(h,m)\Theta_{m,v}(x-y)e^{Q_{++}y}(v, j),$$

that is $g(x,y) = e^{Q_{--}y}\Theta(x-y)Q_{+-}\Theta(x-y)e^{Q_{++}y}$. □

The matrices $\Theta(x)$ and $G(x,z)$ have been defined only for $x > 0$. Clearly, if we set $x = 0$, and thus $z = 0$, in these definitions, we obtain the zero matrix. In fact, we write $\Theta(0)$ for $\Theta(0^+)$ and $G(0,0)$ for $G(0^+,0)$ which means that $\Theta(0) = \lim_{x\longrightarrow 0}\Theta(x) = \lim_{x\longrightarrow 0} G(x,0) = G(0,0)$. This simply means that $x = 0$ is not a continuity point of these two functions.

The matrices $G(x,z)$ and $\Theta(x)$ are given by the following corollary.

Corollary 4 *For $x \geq 0$ and $0 \leq z \leq x$, we have*

$$G(x,z) = \int_0^x e^{Q_{--}y} Q_{-+} e^{Q_{++}y} dy + \big(-Q_{--}\big)^{-1} e^{Q_{--}x} Q_{-+} e^{Q_{++}x} + \int_0^z e^{Q_{--}y} \Theta(x-y) Q_{+-} \Theta(x-y) e^{Q_{++}y} dy. \tag{11}$$

$$\Theta(x) = \int_0^x e^{Q_{--}y} Q_{-+} e^{Q_{++}y} dy + \big(-Q_{--}\big)^{-1} e^{Q_{--}x} Q_{-+} e^{Q_{++}x} + \int_0^x e^{Q_{--}y} \Theta(x-y) Q_{+-} \Theta(x-y) e^{Q_{++}y} dy. \tag{12}$$

Proof It suffices to write

$$G(x,z) = G(x,0) + \int_0^z g(x,y) dy,$$

and $\Theta(x) = G(x,x)$. □

The following result shows that $\Theta(x)$ satisfies a matrix differential Riccati equation.

Theorem 7 $\Theta(0) = (-Q_{--})^{-1} Q_{-+}$ *and, for $x > 0$, we have*

$$\Theta'(x) = Q_{--}\Theta(x) + \Theta(x) Q_{++} + \Theta(x) Q_{+-} \Theta(x) + Q_{-+} \tag{13}$$

Proof The proof is based on Eq. 12 and is thus quite similar to the proof of Theorem 4. □

Again, the Cauchy–Lipschitz Theorem ensures that the matrix differential Riccati equation (13) with the initial condition $\Theta(0) = (-Q_{--})^{-1} Q_{-+}$ has a unique solution. The following theorem gives an expression of the solution $\Theta(x)$ to that equation.

Theorem 8 *For every $x \geq 0$, we have $\Theta(x) = Z(x) Y(x)^{-1}$, where the matrices $Y(x)$ and $Z(x)$ are given by*

$$\begin{pmatrix} Z(x) \\ Y(x) \end{pmatrix} = e^{-CQx} \begin{pmatrix} (-Q_{--})^{-1} Q_{-+} \\ I \end{pmatrix} = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}^{-1} \begin{pmatrix} (-Q_{--})^{-1} Q_{-+} \\ I \end{pmatrix}. \tag{14}$$

Proof Let us consider the following linear differential equation

$$Y'(x) = \big(-Q_{++} - Q_{+-}\Theta(x)\big) Y(x) \quad \text{and} \quad Y(0) = I. \tag{15}$$

The function $\Theta(x)$ being continuous, this linear system has a unique solution which is invertible. We now define the $n_- \times n_+$ matrix $Z(x)$ by $Z(x) = \Theta(x) Y(x)$. Using this definition, Eq. 15 becomes

$$Y'(x) = -Q_{++} Y(x) - Q_{+-} Z(x). \tag{16}$$

Differentiating $Z(x)$ with respect to x, we obtain from (13) and (16)

$$\begin{aligned}
Z'(x) &= \Theta'(x)Y(x) + \Theta(x)Y'(x) \\
&= \big(Q_{--}\Theta(x) + \Theta(x)Q_{++} + \Theta(x)Q_{+-}\Theta(x) + Q_{-+}\big)Y(x) \\
&\quad +\Theta(x)\big(-Q_{++}Y(x) - Q_{+-}Z(x)\big) \\
&= Q_{--}Z(x) + \Theta(x)Q_{+-}\Theta(x)Y(x) + Q_{-+}Y(x) - \Theta(x)Q_{+-}Z(x) \\
&= Q_{--}Z(x) + Q_{-+}Y(x). \qquad (17)
\end{aligned}$$

Putting together Eqs. 16 and 17 we obtain

$$\begin{pmatrix} Z'(x) \\ Y'(x) \end{pmatrix} = \begin{pmatrix} Q_{--} & Q_{-+} \\ -Q_{+-} & -Q_{++} \end{pmatrix} \begin{pmatrix} Z(x) \\ Y(x) \end{pmatrix} = -CQ \begin{pmatrix} Z(x) \\ Y(x) \end{pmatrix},$$

with $Y(0) = I$ and $Z(0) = (-Q_{--})^{-1}Q_{-+}$. The solution to that equation is given by

$$\begin{pmatrix} Z(x) \\ Y(x) \end{pmatrix} = e^{-CQx} \begin{pmatrix} (-Q_{--})^{-1}Q_{-+} \\ I \end{pmatrix} = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}^{-1} \begin{pmatrix} (-Q_{--})^{-1}Q_{-+} \\ I \end{pmatrix}.$$

□

6 Finite Buffer Case

We suppose now that the fluid queue is of finite capacity and we denote that capacity by x. We are then interested to the evaluation of the probability, starting from a phase $i \in S^+$ and a level z, to reach level z in phase j while staying above level z. This probability is denoted by $\Gamma_{i,j}(x, z)$ which is defined below.

We first introduce the $n_+ \times n_-$ matrix $K(x, y, z)$ whose entries are defined, for $i \in S^+$, $j \in S^-$, $x > 0$ and $0 \le z \le y \le x$, by

$$K_{i,j}(x, y, z) = \Pr\left\{\varphi(\gamma(z)) = j, H \le y \middle| \varphi(0) = i, X_0 = z\right\},$$

where H denotes the minimum trough between instants 0 and $\gamma(z)$. When there are no troughs, we take as convention $H = 0$. This corresponds either to the situation of Fig. 3b, i.e. no peak, or to the situation of Fig. 3c, i.e. only one peak.

By the spatial homogeneity, we have $K_{i,j}(x, y, z) = K_{i,j}(x - z, y - z, 0)$. The $n_+ \times n_-$ matrix $\Gamma(x, z)$ whose entries are defined, for $i \in S^+$, $j \in S^-$, $x > 0$ and $0 \le z \le x$, by

$$\Gamma_{i,j}(x, z) = \Pr\left\{\varphi(\gamma(z)) = j \middle| \varphi(0) = i, X_0 = z\right\},$$

satisfies $\Gamma(x, z) = K(x, x, z) = K(x - z, x - z, 0) = \Gamma(x - z, 0)$. The $n_+ \times n_-$ matrix $\Gamma(x)$ whose entries are defined, for $i \in S^+$, $j \in S^-$ and $x > 0$, by

$$\Gamma_{i,j}(x) = \Pr\left\{\varphi(\gamma(0)) = j | \varphi(0) = i, X_0 = 0\right\},$$

satisfies $\Gamma(x) = \Gamma(x, 0)$. A typical example of such paths is shown in Fig. 3.

We denote by $k_{i,j}(x, y, z)$ the joint density associated with the joint distribution $K_{i,j}(x, y, z)$, i.e. $k_{i,j}(x, y, z) = \partial K_{i,j}(x, y, z)/\partial y$.

This situation is symmetric to the one studied in Section 5. In Section 5, we have considered the maximum peak between two successive visits to level x, starting with

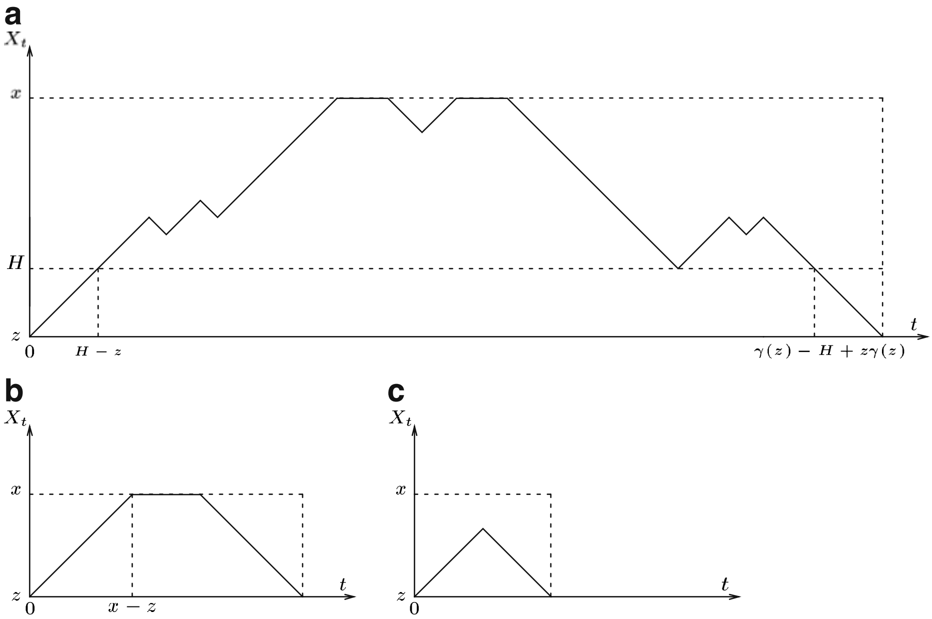

Fig. 3 A period of level $> z$ in the finite buffer case (**a**–**c**)

a negative flow and with a barrier at level 0. This barrier is natural but it is important to mention it for the comparison with the finite buffer case. Here, we consider the minimum trough between two successive visits to level z, starting with a positive flow and with a barrier at level x. When $z = 0$, if we take the symmetry of Fig. 3 with respect to the line $X_t = x/2$ we obtain a figure similar to Fig. 2. Actually, if we exchange the indices $+$ and $-$ of the submatrices of Q, in the expression of $g(x, y)$ and $\Theta(x)$ we obtain respectively, as we shall see, $k(x, y, 0)$ and $\Gamma(x)$.

Theorem 9 *For $x > 0$ and $0 \le z \le y \le x$, we have*

$$K(x, z, z) = \int_z^x e^{Q_{++}(y-z)} Q_{+-} e^{Q_{--}(y-z)} dy + \left(-Q_{++}\right)^{-1} e^{Q_{++}(x-z)} Q_{+-} e^{Q_{--}(x-z)} \tag{18}$$

and

$$k(x, y, z) = e^{Q_{++}(y-z)} \Gamma(x-y) Q_{-+} \Gamma(x-y) e^{Q_{--}(y-z)}. \tag{19}$$

Proof The proof is quasi-identical to the proof of Theorem 6. □

Using the spatial homogeneity, the matrices $K(x, y, z)$ and $\Gamma(x)$ are given by the following equations. For $x \ge 0$ and $0 \le z \le y \le x$, we have

$$K(x, y, z) = K(x, z, z) + \int_z^y k(x, u, z) du. \tag{20}$$

For $x \geq 0$, we have $\Gamma(x) = \Gamma(x, 0) = K(x, x, 0)$, that is

$$\Gamma(x) = \int_0^x e^{Q_{++}y} Q_{+-} e^{Q_{--}y} dy + \big(- Q_{++}\big)^{-1} e^{Q_{++}x} Q_{+-} e^{Q_{--}x} + \int_0^x e^{Q_{++}y} \Gamma(x-y) Q_{-+} \Gamma(x-y) e^{Q_{--}y} dy. \tag{21}$$

This equation is the symmetric version of Eq. 12. We thus obtain, in the same way we got for Eq. 12, the matrix differential Riccati equation

$$\Gamma'(x) = Q_{++}\Gamma(x) + \Gamma(x) Q_{--} + \Gamma(x) Q_{-+} \Gamma(x) + Q_{+-}. \tag{22}$$

This equation is identical to Eq. 4. The only difference concerns the initial condition. Here we have, from Eq. 21, $\Gamma(0) = (-Q_{++})^{-1} Q_{+-}$. Following the same lines used in the proof of Theorem 5, we obtain $\Gamma(x) = Z(x) Y^{-1}(x)$, where

$$\begin{pmatrix} Y(x) \\ Z(x) \end{pmatrix} = e^{CQx} \begin{pmatrix} I \\ P_{+-} \end{pmatrix}$$

and $P_{+-} = (-Q_{++})^{-1} Q_{+-}$. Note that P_{+-} is a stochastic matrix. This can be written as

$$\begin{cases} Y(x) = A(x) + B(x) P_{+-} \\ Z(x) = C(x) + D(x) P_{+-} \end{cases}$$

which gives

$$\Gamma(x) = \big(C(x) + D(x) P_{+-}\big)\big(A(x) + B(x) P_{+-}\big)^{-1}. \tag{23}$$

7 Numerical Application

In order to give a numerical example, we consider a fluid queue fed by m statistically independent and identical on-off sources. This example was proposed in Sericola (1998) to evaluate the transient distribution of the buffer content. Here we consider the distribution of the maximum fluid level during a busy period when the buffer is infinite and the distribution of the phase process at the end of a busy period.

For each source, we assume that the on periods and the off periods form an alternating renewal process and their durations are exponentially distributed with mean λ^{-1} and μ^{-1} respectively. The state space of the phase process φ is thus $S = \{0, 1, \ldots, m\}$, where φ_t represents the number of on sources at time t. When a source is in the state on, it generates a fluid flow at rate ν and we denote by η the service rate of the queue. The infinitesimal generator T is then a tridiagonal matrix whose entries are $T(i, i-1) = i\lambda$ for $i = 1, \ldots, m$, $T(i, i+1) = (m-i)\mu$ for $i = 0, \ldots, m-1$ and so $T(i, i) = -i\lambda - (m-i)\mu$ for $i = 0, \ldots, m$. For each $i \in S$, we have $\rho_i = i\nu$ and $\eta_i = \eta$. The traffic intensity ρ is then given by

$$\rho = \frac{m\nu\mu}{\eta(\lambda + \mu)}.$$

All the computations have been done using MAPLE.

7.1 The Two Sources Case

In this subsection, we fix $m = 2$, $\nu = 1$, $\lambda = 1$, $\mu = 1/2$ and $\eta = 6/5$. The traffic intensity is thus $\rho = 5/9$ and we have $r_0 = -6/5$, $r_1 = -1/5$ and $r_2 = 4/5$. We then have $n_- = 2$ and $n_+ = 1$ and the generator Q and the matrix C are given by

$$Q = \begin{pmatrix} -5/6 & 5/6 & 0 \\ 5 & -15/2 & 5/2 \\ 0 & 5/2 & -5/2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix $\Psi(x)$ is in this case the 1×2 matrix $\Psi(x) = (\Psi_{2,0}(x), \Psi_{2,1}(x))$. To compute it, we use Theorem 5, i.e. we need to solve the linear system $\Psi(x)A(x) = C(x)$. Matrices $A(x)$ and $C(x)$ are defined by relation (5), as constituting blocks of the matrix e^{CQx}, that thus need to be computed.

In the same way, if the buffer has a finite capacity equal to x, the matrix $\Gamma(x)$ is in this case the 1×2 matrix $\Gamma(x) = (\Gamma_{2,0}(x), \Gamma_{2,1}(x))$. To compute it, we use relation (23). As for $\Psi(x)$, we have first to compute the matrix e^{CQx}, to extract from it the matrices $A(x)$, $B(x)$, $C(x)$ and $D(x)$ and then to solve the linear system $\Gamma(x)(A(x) + B(x)P_{+-}) = C(x) + D(x)P_{+-}$, where the matrix P_{+-} is given in this case by $P_{+-} = (0, 1)$.

Calculating the matrix e^{CQx} leads to

$$A(x) = \begin{pmatrix} 1 + \dfrac{1}{11}e^{15x/2} - \dfrac{1}{11}e^{-5x/3} & \dfrac{1}{6} - \dfrac{4}{33}e^{15x/2} - \dfrac{1}{22}e^{-5x/3} \\[2ex] 1 - \dfrac{8}{11}e^{15x/2} - \dfrac{3}{11}e^{-5x/3} & \dfrac{1}{6} + \dfrac{32}{33}e^{15x/2} - \dfrac{3}{22}e^{-5x/3} \end{pmatrix},$$

$$B(x) = \begin{pmatrix} -\dfrac{1}{6} + \dfrac{1}{33}e^{15x/2} + \dfrac{3}{22}e^{-5x/3} \\[2ex] -\dfrac{1}{6} - \dfrac{8}{33}e^{15x/2} + \dfrac{9}{22}e^{-5x/3} \end{pmatrix},$$

$$C(x) = \begin{pmatrix} 1 - \dfrac{2}{11}e^{15x/2} - \dfrac{9}{11}e^{-5x/3} & \dfrac{1}{6} + \dfrac{8}{33}e^{15x/2} - \dfrac{9}{22}e^{-5x/3} \end{pmatrix},$$

$$D(x) = -\frac{1}{6} - \frac{2}{33}e^{15x/2} + \frac{27}{22}e^{-5x/3}.$$

Solving the linear system $\Psi(x)A(x) = C(x)$, we get

$$\Psi_{2,0}(x) = \frac{6\left(9e^{15x/2} - 11e^{35x/6} + 2e^{-5x/3}\right)}{81e^{15x/2} - 11e^{35x/6} - 4e^{-5x/3}},$$

$$\Psi_{2,1}(x) = \frac{27e^{15x/2} - 11e^{35x/6} - 16e^{-5x/3}}{81e^{15x/2} - 11e^{35x/6} - 4e^{-5x/3}}.$$

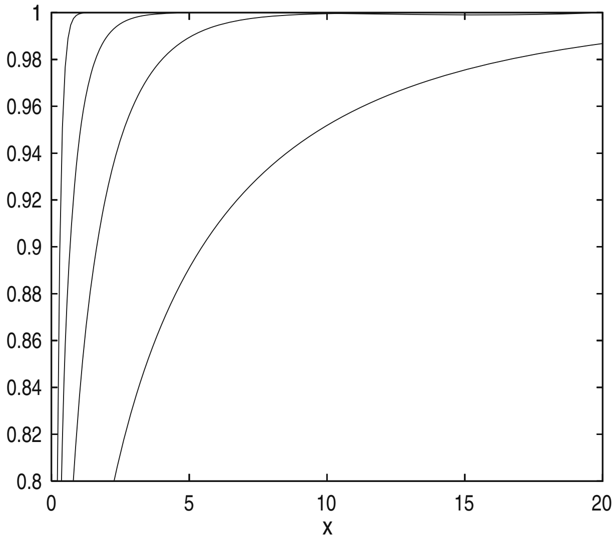

Fig. 4 From *top* to the *bottom*: $\Pr\{M \leq x\}$ for $m = 4, 6, 8, 10$

Taking the limit when x tends to infinity, we get, as expected $\Psi = (2/3, 1/3)$. Adding $\Psi_{2,0}(x)$ and $\Psi_{2,1}(x)$, we obtain the distribution of the maximum level M during a busy period, i.e.

$$\Pr\{M \leq x\} = \frac{81e^{15x/2} - 77e^{35x/6} - 4e^{-5x/3}}{81e^{15x/2} - 11e^{35x/6} - 4e^{-5x/3}}$$
$$= 1 - \frac{66}{81e^{5x/3} - 11 - 4e^{-15x/2}}$$

Suppose now that the buffer is finite with a capacity equal x. Solving the linear system $\Gamma(x)(A(x) + B(x)P_{+-}) = C(x) + D(x)P_{+-}$, we get

$$\Gamma_{2,0}(x) = \frac{6\left(e^{15x/2} - e^{-5x/3}\right)}{9e^{15x/2} + 2e^{-5x/3}} = \frac{6 - 6e^{-55x/6}}{9 + 2e^{-55x/6}},$$

$$\Gamma_{2,1}(x) = \frac{3e^{15x/2} + 8e^{-5x/3}}{9e^{15x/2} + 2e^{-5x/3}} = \frac{3 + 8e^{-55x/6}}{9 + 2e^{-55x/6}}.$$

Again, as expected, we have $\lim_{x\longrightarrow\infty} \Gamma(x) = \Psi$.

7.2 More than 2 Sources

We focus in this subsection on the distribution of the maximum fluid level M during a busy period when the buffer is infinite. We suppose that $\varphi_0 = n_-$, i.e. the phase process starts in the state with the smallest positive effective input rate. The distribution of φ_0 restricted to the states of S_+, which we denote by α_+, is thus given by $\alpha_+ = (1, 0, \ldots, 0)$, its dimension being equal to n_+. The distribution of M is then

$$\Pr\{M \leq x\} = \alpha_+ \Psi(x) \mathbb{1}.$$

Table 1 Buffer capacity as a function of ε and m

m	4	5	6	7	8	9	10
$\varepsilon = 10^{-2}$	0.62	1.26	2.04	3.19	5.09	8.91	22.37
$\varepsilon = 10^{-3}$	0.93	2.14	3.51	5.48	8.71	15.43	42.80
$\varepsilon = 10^{-4}$	1.24	3.02	5.00	7.78	12.33	22.00	63.70
$\varepsilon = 10^{-5}$	1.55	3.90	6.50	10.08	16.00	28.56	84.65

We denote by $V(x)$ the row vector $V(x) = \alpha_+\Psi(x)$. So, as mentioned above, to compute $V(x)$, we use Theorem 5. We thus have first to compute the matrix e^{CQx}, to extract from it the matrices $A(x)$ and $C(x)$ and then to solve the linear system $V(x)A(x) = \alpha_+C(x)$ and to add up the entries of vector $V(x)$.

We fix $\nu = 1$, $\lambda = 1$, $\mu = 1/2$ and $\eta = 7/2$. The stability condition $\rho < 1$ is thus equivalent to $m \leq 10$. Figure 4 shows the distribution of M for several values of the number of sources m.

This distribution is very useful to size the capacity of the buffer, i.e. to determine the smallest value of the fluid level B such that $\Pr\{M \leq B\} > 1 - \varepsilon$. We show in Table 1, the value of B for different values of m and ε.

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