Chapter 1
Sojourn Times in Dependability Modeling

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Abstract We consider Markovian models of computing or communication systems, subject to failures and, possibly, repairs. The dependability properties of such systems lead to metrics that can all be described in terms of the time the Markov chain spends in subsets of its state space. Some examples of such metrics are MTTF (Mean Time To Failure) and MTTR (Mean Time To Repair), reliability or availability at a point in time, the mean or the distribution of the interval availability in a fixed time interval, and, more generally, different performability versions of these measures. This paper reviews this point of view and its consequences, and discusses some new related results.

1.1 Introduction

In this chapter, we are interested in Markovian models of systems subject to failures and, possibly, repairs. A typical framework is that of a system made of several independently operating components or units, each of them belonging to a given class. When the system starts operating, each component has a life-time distributed according to the Exponential distribution whose parameter (the component failure rate) is the same for all components in the same class. At the beginning, the system starts with a given number of operational components in each of the classes. Let K be the number of classes, indexed from 1 to K, and N_k the number of class k units in the system. If n_k is the number of class k units that are operational at any point in time, $0 \le n_k \le N_k$, a basic case is when the vector $(n_1, \ldots, n_K)$ suffices

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to obtain a Markovian evolution for the model. For instance, assume that the system is non repairable, and that the only transitions are the individual components' failures. Assume, for instance, that the system works when, for each class k, at least m_k units are operational (we say that, from the dependability viewpoint, it works as a series of m_k-out-of-N_k modules). The initial state for the Markovian model $X = \{X_t,\ t \geq 0\}$ is $X_0 = (N_1, \ldots, N_K)$. For this simple structure, we need to evaluate the system's reliability at time t, $R(t)$, which is here the probability that the system is alive at time t, or the mean system's life-time, typically called Mean Time To Failure, MTTF. For this purpose, we consider that the state space of X is $S = \{m_1, m_1+1, \ldots, N_1\} \times \cdots \times \{m_K, m_K+1, \ldots, N_K\} \cup \{a\}$, where state a is absorbing and the remaining $(N_1 - m_1 + 1) \cdots (N_K - m_K + 1)$ states are transient. If we denote $B = S \setminus \{a\}$, and if T is the absorption time of the chain, we have that T is the sojourn time of X in B, there is only one such sojourn and $R(t) = \mathbb{P}\{T > t\}$ and $\mathrm{MTTF} = \mathbb{E}\{T\}$.

With the same state representation as before, and the same system structure (the series of m_k-out-of-N_k modules), we can handle the case where any failed unit is immediately repaired with some rate μ_k when it belongs to class k, the repair times being also independent of anything else in the model. In that case, process X live in $S = \{0, \ldots, N_1\} \times \cdots \times \{0, \ldots, N_K\}$. The Markov chain X has a single recurrent class, and the same B defined above, $B = \{(n_1, \ldots, n_K) \in S \mid n_1 \geq m_1, \ldots, n_K \geq m_K\}$, together with its complement B^c define a partition of S. The model will spend some random time working, during what we will call its first *operational period*. This is referred to as the first sojourn time of X in B, whose duration will be denoted by $S_{B,1}$. Then, some failure will put it in B^c, that is, in a failed system situation. At that point in time, a sojourn in B^c starts, that from the system point of view, can be seen as a system repair. This length in time will be denoted by $S_{B^c,1}$. At some later point in time, the system will come back to subset B and a new operational period will start. This alternate sequence of operational periods and repair or unoperational ones will thus continue forever. In this setting, it can be of interest to evaluate not only the previously defined metrics (the reliability at t defined by $R(t) = \mathbb{P}\{S_{B,1} > t\}$ and the MTTF = $\mathbb{E}\{S_{B,1}\}$) but also other ones, such as the mean availability on the interval $[0,t]$, defined as the mean fraction of that interval spent by X in B. Asymptotically, we can consider the widely used asymptotic availability, which can be seen as the limit of the previous fraction when $t \to \infty$.

As noted in the abstract, most dependability metrics can be associated with the time the system's model spends in subsets of its state space. The examples discussed before illustrate the point, but they can get much more complex. For instance, assume that in the repairable case described above, there is a single repair facility that can handle one component at a time, connected to a buffering area where failed units wait (say, in FIFO order) until the repair server is available. Then, we need to put more information in the states' definition to be able to build a Markovian evolution on this state space, we need to know how many components of each class are working, but also some information about what happens at the repair facility. In all cases, the same connection with sojourn times will hold.

Generally speaking, we often have a stochastic process representing the system, living in some state space S, and associated with each state x we have a function Φ to indicate if the system is working or not when its state is x. The typical convention is to use $\Phi(x) = 1$ if the system works when its state is x, and 0 if it is failed. This function Φ is called the system's *structure function*, which defines the set of operational states $B = \Phi^{-1}(1)$ and its complement, where the system doesn't work, $B^c = \Phi^{-1}(0)$. The main dependability metrics are defined as in the initial examples: MTTF = $\mathbb{E}\{S_{B,1}\}$, $R(t) = \mathbb{P}\{S_{B,1} > t\}$, Mean Time To Repair, MTTR = $\mathbb{E}\{S_{B^c,1}\}$, etc., and they correspond to different aspects of the sojourns of X in B and/or in B^c.

For a global background on these topics, an appropriate reference is the blue (and yellow) book [20]. All the basic mathematical objects used here are introduced and carefully explained there. Another directly relevant reference is the Introduction of the authors' recent book [17], in particular Section 1.3. The reader can also find in Kishor Trivedi's Web pages many general presentations of the topics discussed in this chapter. As an example, see [9] for an introduction to Markov models in dependability and extensions to performability, or [8] for generalities about dependability analysis.

The analysis of sojourn times in Markov chains, both in discrete and continuous time, was started in dependability in [10]. In that paper, the distribution of the nth sojourn time of the chain in a subset of its state space, in the irreducible and finite case, is derived, and its asymptotic behavior is analyzed. The connections with state lumping and the so-called "pseudo-aggregation" in [12], [16], are also discussed. In [11], the corresponding absorbing case is analyzed. At the end of the chapter, supplementary bibliographical notes are provided.

It appears that some of the results published in this area, for instance the basic distributions, were known in a specific biological field called "ion channel analysis", as reported in [6]. As stated in [17], the analysis of sojourn times is also relevant in queueing theory (think of the busy period of a single queue, for instance).

This chapter reviews part of the basic material concerning the times spent by a Markovian process in proper subsets of its state space, and adds some new elements and guidelines for obtaining more results. Since we focus here on dependability applications, all our developments are in continuous time, but they have counterparts in discrete time, not discussed here (see the references). We had to make some choices because of the amount of material available. For this reason, until Section 1.5, we limit the discussion to irreducible models. Section 1.2 reviews the basic facts when studying these objects, namely, the distribution of the sojourn time of a Markov process in a subset of its state space. The asymptotic behavior of this distribution is also discussed. Section 1.3 presents the joint distribution of sojourn times. In Section 1.4, we focus on a specific function of them, the sum of the first n sojourn times of the chain in a given subset of states, its distribution, the computation of its moments, etc. Section 1.5 illustrates how to deal with absorbing models, and at the same time, with Markov models whose states are weighted by rewards (Markov reward models). In Section 1.6, bibliographical notes are provided, together with some supplementary comments and discussions.

1.2 Successive Sojourn Times Distribution

We consider a homogeneous irreducible continuous-time Markov chain $X = \{X_t,\, t \geq 0\}$ with finite state space denoted by S. Its initial probability distribution is given by the row vector α and its transition rate matrix by A. The stationary distribution of X, denoted by π, is the row vector satisfying $\pi A = 0$ and $\pi \mathbb{1} = 1$ where $\mathbb{1}$ is the column vector with all its entries equal to 1, its dimension being given by the context of its use. For every $i \in S$, we denote by λ_i the output rate of state i, that is $\lambda_i = -A_{i,i}$. Let λ be a positive real number such that $\lambda \geq \max\{\lambda_i,\, i \in S\}$ and let $\{N_t,\, t \geq 0\}$ be a Poisson process with rate λ. We then define matrix P by $P = I + A/\lambda$, where I is the identity matrix with dimension also determined by the context of its use. We introduce the discrete-time Markov chain $Z = \{Z_n,\, n \geq 0\}$ on the state space S, with transition probability matrix P and with initial probability distribution α. Assuming that the processes $\{N_t\}$ and Z are independent, the stochastic processes X and $\{Z_{N_t},\, t \geq 0\}$ are equivalent, i.e. they have the same finite-dimensional distributions. This well-known construction is called the uniformization technique. The Markov chain Z is called the discrete-time Markov chain associated with the uniformized Markov chain of X with respect to the uniformization rate λ.

Let B be a proper subset of S, i.e. $B \neq \emptyset$ and $B \neq S$. We denote by B^c the subset $S \setminus B$. Subset B contains the operational states and subset B^c contains the non operational ones. The subsets B, B^c form a partition of the state space S. Ordering the states such that those in B appear first, then those in B^c, the partition induces the following decomposition of matrices A and P and vectors α and π:

$$A = \begin{pmatrix} A_B & A_{BB^c} \\ A_{B^cB} & A_{B^c} \end{pmatrix},\, P = \begin{pmatrix} P_B & P_{BB^c} \\ P_{B^cB} & P_{B^c} \end{pmatrix},\, \alpha = \begin{pmatrix} \alpha_B & \alpha_{B^c} \end{pmatrix} \text{ and } \pi = \begin{pmatrix} \pi_B & \pi_{B^c} \end{pmatrix}.$$

Lemma 1. *The matrices $I - P_B$ and $I - P_{B^c}$ are invertible.*

Proof. Consider the auxiliary discrete-time Markov chain Z' on the state space $B \cup a$, where a is an absorbing state, with transition probability matrix P' given by

$$P' = \begin{pmatrix} P_B & u \\ 0 & 1 \end{pmatrix},$$

where u is the column vector defined by $u = \mathbb{1} - P_B\mathbb{1}$. The Markov chain X being irreducible, the states of B are transient for Markov chain Z'. A well known result about Markov chains says that if state j is transient, then for all state i, we have $(P^k)_{i,j} \longrightarrow 0$ as $k \longrightarrow \infty$, if P is the transition probability matrix of the chain, see for instance [18]. Here, this translates into the fact that

$$\lim_{k \longrightarrow \infty} (P_B)^k = 0.$$

From the immediate identity

$$(I-P_B)\sum_{k=0}^{K}(P_B)^k = \left(\sum_{k=0}^{K}(P_B)^k\right)(I-P_B) = I-(P_B)^{K+1},$$

we obtain, taking the limit as $K \longrightarrow \infty$, that the series $\sum_{k\geq 0}(P_B)^k$ converges, and calling M its limit, that $(I-P_B)M = M(I-P_B) = I$. This means that $I-P_B$ is invertible and that $(I-P_B)^{-1} = \sum_{k\geq 0}(P_B)^k$. Replacing B by B^c and using again the same argument, we obtain that $I-P_{B^c}$ is invertible as well. □

Consider the successive instants at which the Markov chain X enters subsets B and B^c. We define $T_{B,1} = \inf\{t \geq 0 \mid X_t \in B\}$ and $T_{B^c,1} = \inf\{t \geq 0 \mid X_t \in B^c\}$ and we define, for every $n \geq 2$,

$$T_{B,n} = \inf\{t > T_{B,n-1} \mid X_{t^-} \in B^c, X_t \in B\},$$
$$T_{B^c,n} = \inf\{t > T_{B^c,n-1} \mid X_{t^-} \in B, X_t \in B^c\}.$$

Note that if $X_0 \in B$ (resp. $X_0 \in B^c$) then we have $T_{B,1} = 0$ (resp. $T_{B^c,1} = 0$). The nth sojourn time of X in B is denoted by $S_{B,n}$ and we have, for every $n \geq 1$,

$$S_{B,n} = \begin{cases} T_{B^c,n} - T_{B,n} & \text{if } X_0 \in B \\ \\ T_{B^c,n+1} - T_{B,n} & \text{if } X_0 \in B^c. \end{cases}$$

Let $V_{B,n}$ (resp. $V_{B^c,n}$), for $n \geq 1$, be the random variable representing the state of B (resp. B^c) in which the nth sojourn of X in B (resp. B^C) starts. With the usual convention saying that the paths of X are right-continuous we have, for $n \geq 1$, $V_{B,n} = X_{T_{B,n}}$ and $V_{B^c,n} = X_{T_{B^c,n}}$. We introduce the matrices R and H defined by

$$R = (I-P_B)^{-1}P_{BB^c} \text{ and } H = (I-P_{B^c})^{-1}P_{B^cB}.$$

Note that both matrices R and H are stochastic matrices.

Theorem 1. *The process $V_B = \{V_{B,n},\, n \geq 1\}$ is a homogeneous discrete-time Markov chain with state space B. Its initial probability distribution, denoted by $v^{(1)}$, and its transition probability matrix, denoted by G, are given by*

$$v^{(1)} = \alpha_B + \alpha_{B^c}H \textit{ and } G = RH.$$

Proof. The sequence of instants $T_{B,n}$ being an increasing sequence of stopping times, the process $V_B = \{V_{B,n},\, n \geq 1\}$ is a homogeneous discrete-time Markov chain with state space B. For every $i, j \in B$, we have

$$\mathbb{P}\{V_{B,1} = j \mid X_0 = i\} = \mathbb{P}\{X_0 = j \mid X_0 = i\} = 1_{\{i=j\}}.$$

For every $i \in B^c$ and $j \in B$, we have, using the Markov property,

$$\begin{aligned}
\mathbb{P}\{V_{B,1}=j \mid X_0=i\} &= \mathbb{P}\{X_{T_{B,1}}=j \mid X_0=i\} \\
&= \sum_{k\in S} P_{i,k}\mathbb{P}\{X_{T_{B,1}}=j \mid Z_1=k, X_0=i\} \\
&= \sum_{k\in B} P_{i,k}1_{\{k=j\}} + \sum_{k\in B^c} P_{i,k}\mathbb{P}\{X_{T_{B,1}}=j \mid X_0=k\} \\
&= P_{i,j} + \sum_{k\in B^c} P_{i,k}\mathbb{P}\{V_{B,1}=j \mid X_0=k\}.
\end{aligned}$$

Denoting by M the matrix defined, for $i\in B^c$ and $j\in B$ by $M_{i,j}=\mathbb{P}\{V_{B,1}=j \mid X_0=i\}$, this last equality leads to $M=P_{B^cB}+P_{B^c}M$, that is, using Lemma 1, $M=(I-P_{B^c})^{-1}P_{B^cB}$, i.e. $M=H$. This leads, for every $j\in B$, to

$$v_j^{(1)}=\mathbb{P}\{V_{B,1}=j\}=\sum_{i\in B}\alpha_i 1_{\{i=j\}}+\sum_{i\in B^c}\alpha_i H_{i,j}=\alpha_j+(\alpha_{B^c}H)_j,$$

that is $v^{(1)}=\alpha_B+\alpha_{B^c}H$.

Using the same arguments and denoting by K the matrix defined, for $i\in B$ and $j\in B^c$, by $K_{i,j}=\mathbb{P}\{V_{B^c,1}=j \mid X_0=i\}$, we obtain symmetrically $K=R$. For every $i,j\in B$, we get, using the Markov property,

$$\begin{aligned}
G_{i,j} &= \mathbb{P}\{V_{B,2}=j \mid V_{B,1}=i)=\mathbb{P}\{V_{B,2}=j \mid X_0=i\} \\
&= \sum_{k\in B^c}\mathbb{P}\{V_{B^c,1}=k \mid X_0=i\}\mathbb{P}\{V_{B,2}=j \mid V_{B^c,1}=k, X_0=i\} \\
&= \sum_{k\in B^c}\mathbb{P}\{V_{B^c,1}=k \mid X_0=i\}\mathbb{P}\{V_{B,1}=j \mid X_0=k\} \\
&= \sum_{k\in B^c} R_{i,k}H_{k,j},
\end{aligned}$$

that is $G=RH$. □

The Markov chain V_B contains only one recurrent class, which we denote by B', containing the states of B that are directly accessible from B^c. More precisely,

$$B'=\{j\in B \mid \exists i\in B^c,\ P_{i,j}>0\}.$$

If $B'\neq B$, we denote by B'' the set $B\setminus B'$. The subsets B', B'' form a partition of B which induces the following decomposition of matrices G and H,

$$G=\begin{pmatrix} G' & 0 \\ G'' & 0 \end{pmatrix} \text{ and } H=\begin{pmatrix} H' & 0 \end{pmatrix}. \tag{1.1}$$

In the same way, the partition B', B'', B^c of S leads to the following decomposition of P,

$$P=\begin{pmatrix} P_{B'} & P_{B'B''} & B_{B'B^c} \\ P_{B''B'} & P_{B''} & B_{B''B^c} \\ P_{B^cB'} & 0 & P_{B^c} \end{pmatrix}.$$

The matrices H', G' and G'' are given by the following theorem.

Theorem 2. *We have*

$$\begin{aligned} H' &= (I - P_{B^c})^{-1} P_{B^c B'}, \\ G' &= (I - Q_{B'B''} P_{B''B'})^{-1} \left(Q_{B'B''} P_{B''B^c} + (I - P_{B'})^{-1} P_{B'B^c} \right) H', \\ G'' &= (I - Q_{B''B'} P_{B'B''})^{-1} \left(Q_{B''B'} P_{B'B^c} + (I - P_{B''})^{-1} P_{B''B^c} \right) H', \end{aligned}$$

where

$$Q_{B'B''} = (I - P_{B'})^{-1} P_{B'B''} (I - P_{B''})^{-1} \text{ and } Q_{B''B'} = (I - P_{B''})^{-1} P_{B''B'} (I - P_{B'})^{-1}.$$

Proof. Since $P_{B^c B''} = 0$, we get

$$H = (I - P_{B^c})^{-1} P_{B^c B} = \left(H' \ 0 \right),$$

with $H' = (I - P_{B^c})^{-1} P_{B^c B'}$. Next, from $G = RH$ or equivalently

$$(I - P_B) G = G - P_B G = P_{BB^c} H,$$

we have $G = P_B G + P_{BB^c} H$. Using now the decomposition of matrices G, P_B, P_{BB^c} and H with respect to the partition $\{B', B''\}$ of B, we obtain

$$\begin{cases} G' = P_{B'} G' + P_{B'B''} G'' + P_{B'B^c} H', \\ \\ G'' = P_{B''B'} G' + P_{B''} G'' + P_{B''B^c} H'. \end{cases}$$

This gives

$$\begin{cases} G' = (I - P_{B'})^{-1} P_{B'B''} G'' + (I - P_{B'})^{-1} P_{B'B^c} H', \\ \\ G'' = (I - P_{B''})^{-1} P_{B''B'} G' + (I - P_{B''})^{-1} P_{B''B^c} H'. \end{cases}$$

Putting the second relation in the first one leads to the expression of G' and putting the first relation in the second one leads to the expression of G''. $\square$

We denote by $v^{(n)}$ the distribution of the state from which the nth sojourn of X in B starts, which means that $v^{(n)}$ is the distribution of $V_{B,n}$. Since V_B is a Markov chain, we have, for every $n \geq 1$,

$$v^{(n)} = v^{(1)} G^{n-1}.$$

Using these results, we derive the distribution of $S_{B,n}$, the nth sojourn time of X in subset B.

Theorem 3. *For every $n \geq 1$ and for all $t \geq 0$, we have*

$$\mathbb{P}\{S_{B,n} > t\} = v^{(n)} e^{A_B t} \mathbb{1} = v^{(1)} G^{n-1} e^{A_B t} \mathbb{1}.$$

Proof. Using a classical backward renewal argument, as for instance in [18], we have, for every $i \in B$,

$$\mathbb{P}\{S_{B,1} > t \mid X_0 = i\} = \sum_{j \in S} P_{i,j} \int_0^\infty \mathbb{P}\{S_{B,1} > t \mid Z_1 = j, T_1 = x, X_0 = i\} \lambda e^{-\lambda x} dx,$$

where T_1 is the first occurrence instant of the Poisson process $\{N_t, t \geq 0\}$. We then have

$$\mathbb{P}\{S_{B,1} > t \mid X_0 = i\} = \int_t^\infty \lambda e^{-\lambda x} dx + \sum_{j \in B} P_{i,j} \int_0^t \mathbb{P}\{S_{B,1} > t - x \mid X_0 = j\} \lambda e^{-\lambda x} dx.$$

The change of variable $x := t - x$ in the second integral leads to

$$\mathbb{P}\{S_{B,1} > t \mid X_0 = i\} = e^{-\lambda t} \left(1 + \sum_{j \in B} P_{i,j} \int_0^t \mathbb{P}\{S_{B,1} > x \mid X_0 = j\} \lambda e^{\lambda x} dx \right).$$

Introducing the column vector $w(t)$ defined by $w_i(t) = \mathbb{P}\{S_{B,1} > t \mid X_0 = i\}$, for every $i \in B$, we get $w(0) = \mathbb{1}$ and

$$w(t) = e^{-\lambda t} \left(\mathbb{1} + \lambda P_B \int_0^t w(x) e^{\lambda x} dx \right).$$

Differentiating with respect to t leads to

$$w'(t) = -\lambda w(t) + \lambda P_B w(t) = A_B w(t),$$

that is

$$w(t) = e^{A_B t} w(0) = e^{A_B t} \mathbb{1}. \tag{1.2}$$

For every $n \geq 1$ and for all $t \geq 0$, we have, using now the Markov property, the homogeneity of X and (1.2),

$$\begin{aligned} \mathbb{P}\{S_{B,n} > t\} &= \sum_{i \in B} v_i^{(n)} \mathbb{P}\{S_{B,n} > t \mid V_{B,n} = i\} \\ &= \sum_{i \in B} v_i^{(n)} \mathbb{P}\{S_{B,1} > t \mid X_0 = i\} \\ &= v^{(n)} e^{A_B t} \mathbb{1}, \end{aligned}$$

which completes the proof. $\square$

The next theorem gives the limiting distribution of $S_{B,n}$ when n tends to infinity. To determine the limiting distribution, we need the following lemma.

Lemma 2. *The row vector π_B satisfies*

$$\pi_B = \pi_B P_{BB^c} \left(I - P_{B^c}\right)^{-1} P_{B^c B} \left(I - P_B\right)^{-1}.$$

Proof. Since $\pi A = 0$, we also have $\pi P = \pi$ that is

$$\begin{cases} \pi_B = \pi_B P_B + \pi_{B^c} P_{B^c B} \\ \\ \pi_{B^c} = \pi_B P_{BB^c} + \pi_{B^c} P_{B^c}. \end{cases}$$

The second equation gives

$$\pi_{B^c} = \pi_B P_{BB^c} \left(I - P_{B^c}\right)^{-1}.$$

The first equation gives

$$\pi_B = \pi_{B^c} P_{B^c B} \left(I - P_B\right)^{-1}.$$

Replacing the value of π_{B^c} in this last relation, we get

$$\pi_B = \pi_B P_{BB^c} \left(I - P_{B^c}\right)^{-1} P_{B^c B} \left(I - P_B\right)^{-1},$$

which completes the proof. □

Theorem 4. *For every $t \geq 0$, the sequence $\mathbb{P}\{S_{B,n} > t\}$ converges in the Cesàro sense when n tends to infinity to $ve^{A_B t}\mathbb{1}$, where*

$$v = \frac{\pi_{B^c} P_{B^c B}}{\pi_{B^c} P_{B^c B} \mathbb{1}}$$

is the stationary distribution of the Markov chain V_B. The convergence is simple if and only if matrix G' is aperiodic.

Proof. It suffices to prove that the sequence $v^{(n)}$ converges in the Cesàro sense. The Markov chain V_B has a finite state space B and a single recurrent class B', so it has a unique invariant distribution which we denote by v. We thus have $vG = v$, with $v\mathbb{1} = 1$. According to the partition B', B'' of B, we write $v = (v' \;\; 0)$ and the decomposition of matrix G described in (1.1) leads, for every $n \geq 1$, to

$$G^n = \begin{pmatrix} G'^n & 0 \\ G'' G'^{n-1} & 0 \end{pmatrix}.$$

The general properties of Markov chains tell us that G'^n converges in the Cesàro sense to $\mathbb{1}v'$ and that the convergence is simple if and only if G' is aperiodic. In the same way, $G''G'^{n-1}$ also converges to $\mathbb{1}v'$, since $G''\mathbb{1} = \mathbb{1}$. Using Lemma 2, we obtain

$$\pi_B(I - P_B) = \pi_B P_{BB^c} \left(I - P_{B^c}\right)^{-1} P_{B^c B},$$

which implies that $\pi_B(I - P_B) = \pi_B(I - P_B)G$. Normalizing this vector, we have

$$v = \frac{\pi_B(I - P_B)}{\pi_B(I - P_B)\mathbb{1}} = \frac{\pi_{B^c} P_{B^c B}}{\pi_{B^c} P_{B^c B} \mathbb{1}},$$

which completes the proof. □

Note that there is no relation between the periodicity of matrix P and the periodicity of matrix G'. The four situations in which each matrix is either periodic or aperiodic is possible as shown in Figures 1.1 and 1.2. In all cases, the state space is $S = \{1,2,3,4\}$, and we have $B = \{1,2\}$ and $B^c = \{3,4\}$. An arrow between two states means that the corresponding transition probability is positive.

Fig. 1.1 On the left graph P and G' are both aperiodic. On the right graph both P and G' are periodic.

Fig. 1.2 On the left graph P is periodic and G' is aperiodic. On the right graph P aperiodic and G' is periodic.

1.3 Joint Distributions of Sojourn Times

In many situations, we are interested in functions of several sojourn times of X in a subset of states B, which means that we need the joint distribution of these random times. For instance, when B is composed of the operational states only, we would like to evaluate the random variable $\min_{n \le N} S_{B,n}$, or we may want to move control variables in the model in order to obtain that for some $N \in \mathbb{N}$, $t > 0$ and $\varepsilon > 0$, we have $\mathbb{P}(S_{B,1} > t, S_{B,2} > t, \ldots, S_{B,N} > t) > 1 - \varepsilon$. The point is that the random variables $S_{B,1}, S_{B,2}, \ldots$ are in general dependent. To get a feeling of this, just consider the example depicted in Figure 1.3, where $B = \{1,2\}$. If ε is small, the sojourn times of X in state 1 are "very short" and in state 2 are "very long". The topology of the chain says that knowing that previous sojourn was a short one, it is highly probable that the next one will be short as well.

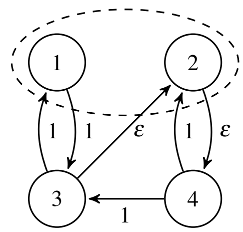

Fig. 1.3 Illustrating the dependence in the sequence of sojourn times of X in a subset B of states; here, $B = \{1,2\}$. If $S_{B,n-1}$ was small, it is highly probable that the $(n-1)$th sojourn in B was a holding time in state 1. We then expect that the next one, $S_{B,n}$, will be small as well.

If we are interested in studying the correlations between successive operational times, we need to evaluate second order moments, and, again, we need the joint distribution of sojourn times. This is the topic of this subsection.

Recall that the matrices R and H are defined by $R = (I - P_B)^{-1} P_{BB^c}$ and $H = (I - P_{B^c})^{-1} P_{B^cB}$ and that we have $G = RH$. We also recall that $V_{B^c,1} = X_{T_{B^c,1}}$ and that for every $i, \ell \in B$ and $j, k \in B^c$,

$$\mathbb{P}\{V_{B^c,1} = j \mid X_0 = i\} = R_{i,j} \text{ and } \mathbb{P}\{V_{B,1} = \ell \mid X_0 = k\} = H_{k,\ell}.$$

We first give a lemma that will be used in the next theorem.

Lemma 3. *For all $s, t \geq 0$, we have*

$$\mathbb{P}\{S_{B,1} > t, V_{B^c,1} = j \mid X_0 = i\} = \left(e^{A_B t} R\right)_{i,j}, \textit{ for } i \in B, j \in B^c,$$

$$\mathbb{P}\{S_{B^c,1} > s, V_{B,1} = j \mid X_0 = i\} = \left(e^{A_{B^c} s} H\right)_{i,j}, \textit{ for } i \in B^c, j \in B,$$

$$\mathbb{P}\{S_{B,1} > t, S_{B^c,1} > s, V_{B,2} = j \mid X_0 = i\} = \left(e^{A_B t} R e^{A_{B^c} s} H\right)_{i,j}, \textit{ for } i \in B, j \in B.$$

Proof. We introduce the matrix $L(t)$ defined, for every $i \in B$ and $j \in B^c$, by

$$L_{i,j}(t) = \mathbb{P}\{S_{B,1} > t, V_{B^c,1} = j \mid X_0 = i\}.$$

Using again classical backward renewal arguments, as for instance in [18], we have

$$\begin{aligned}
L_{i,j}(t) &= \mathbb{P}\{S_{B,1} > t, Z_1 = j \mid X_0 = i\} + \sum_{k\in B} \mathbb{P}\{S_{B,1} > t, V_{B^c,1} = j, Z_1 = k \mid X_0 = i\} \\
&= P_{i,j} e^{-\lambda t} + \sum_{k\in B} P_{i,k} e^{-\lambda t} \mathbb{P}\{V_{B^c,1} = j \mid X_0 = k\} + \sum_{k\in B} P_{i,k} \int_0^t L_{k,j}(t-x) \lambda e^{-\lambda x} dx \\
&= e^{-\lambda t} \left(P_{i,j} + \sum_{k\in B} P_{i,k} R_{k,j} + \sum_{k\in B} P_{i,k} \int_0^t L_{k,j}(x) \lambda e^{\lambda x} dx \right).
\end{aligned}$$

This gives in matrix notation

$$L(t) = e^{-\lambda t} \left(P_{BB^c} + P_B R + \lambda P_B \int_0^t L(x) e^{\lambda x} dx \right).$$

Differentiating with respect to t leads to

$$L'(t) = -\lambda L(t) + \lambda P_B L(t) = A_B L(t),$$

which gives $L(t) = e^{A_B t} L(0)$ and since $L(0) = R$, we get

$$L(t) = e^{A_B t} R,$$

which completes the proof of the first relation. The second relation follows immediately from the first one by interchanging the role played by subsets B and B^c. The third relation is easily obtained from the first two. Indeed, for every $i, j \in B$, we have, using the Markov property and the homogeneity of X,

$$\begin{aligned}
&\mathbb{P}\{S_{B,1} > t, S_{B^c,1} > s, V_{B,2} = j \mid X_0 = i\} \\
&\qquad = \sum_{k\in B^c} \mathbb{P}\{S_{B,1} > t, V_{B^c,1} = k, S_{B^c,1} > s, V_{B,2} = j \mid X_0 = i\} \\
&\qquad = \sum_{k\in B^c} \mathbb{P}\{S_{B^c,1} > s, V_{B,2} = j \mid S_{B,1} > t, V_{B^c,1} = k, X_0 = i\} \left(e^{A_B t} R\right)_{i,k} \\
&\qquad = \sum_{k\in B^c} \mathbb{P}\{S_{B^c,1} > s, V_{B,1} = j \mid X_0 = k\} \left(e^{A_B t} R\right)_{i,k} \\
&\qquad = \sum_{k\in B^c} \left(e^{A_B t} R\right)_{i,k} \left(e^{A_{B^c} s} H\right)_{k,j} \\
&\qquad = \left(e^{A_B t} R e^{A_{B^c} s} H\right)_{i,j},
\end{aligned}$$

which completes the proof. $\square$

Theorem 5. *For every $n \geq 1$, for all $t_1, \ldots, t_n \geq 0$ and $s_1, \ldots, s_n \geq 0$, we have*

$$\begin{aligned}
&\mathbb{P}\{S_{B,1} > t_1, S_{B^c,1} > s_1, \ldots, S_{B,n} > t_n, S_{B^c,n} > s_n\} \\
&= \alpha_B \left[\prod_{k=1}^{n-1} e^{A_B t_k} R e^{A_{B^c} s_k} H \right] e^{A_B t_n} R e^{A_{B^c} s_n} \mathbb{1} + \alpha_{B^c} \left[\prod_{k=1}^{n-1} e^{A_{B^c} s_k} H e^{A_B t_k} R \right] e^{A_{B^c} s_n} H e^{A_B t_n} \mathbb{1}.
\end{aligned} \tag{1.3}$$

Proof. The proof is made by recurrence over integer n. Consider first the case $i \in B$. For $n = 1$, we have, using the third relation of Lemma 3 and the fact that H is a stochastic matrix,

$$\begin{aligned}\mathbb{P}\{S_{B,1} > t_1, S_{B^c,1} > s_1 \mid X_0 = i\} &= \sum_{j\in B} \mathbb{P}\{S_{B,1} > t_1, S_{B^c,1} > s_1, V_{B,2} = j \mid X_0 = i\}\\ &= \sum_{j\in B} \left(e^{A_B t_1} R e^{A_{B^c} s_1} H\right)_{i,j}\\ &= \left(e^{A_B t_1} R e^{A_{B^c} s_1} \mathbb{1}\right)_i .\end{aligned}$$

which is Relation (1.3) since the product is equal to the identity matrix when $n = 1$. Suppose that Relation (1.3) is true for integer $n-1$. We have, using the Markov property, the homogeneity of X, the third relation of Lemma 3 and the recurrence hypothesis,

$$\begin{aligned}&P\{S_{B,1} > t_1, S_{B^c,1} > s_1, \ldots, S_{B,n} > t_n, S_{B^c,n} > s_n \mid X_0 = i\}\\ &\quad= \sum_{j\in B} \mathbb{P}\{V_{B,2} = j, S_{B,1} > t_1, S_{B^c,1} > s_1, \ldots, S_{B,n} > t_n, S_{B^c,n} > s_n \mid X_0 = i\}\\ &\quad= \sum_{j\in B} \mathbb{P}\{S_{B,1} > t_1, S_{B^c,1} > s_1, V_{B,2} = j \mid X_0 = i\}\\ &\qquad\quad \times \mathbb{P}\{S_{B,2} > t_2, S_{B^c,2} > s_2, \ldots, S_{B,n} > t_n, S_{B^c,n} > s_n \mid V_{B,2} = j\}\\ &\quad= \sum_{j\in B} \left(e^{A_B t_1} R e^{A_{B^c} s_1} H\right)_{i,j}\\ &\qquad\quad \times \mathbb{P}\{S_{B,1} > t_2, S_{B^c,1} > s_2, \ldots, S_{B,n-1} > t_n, S_{B^c,n-1} > s_n \mid X_0 = j\}\\ &\quad= \sum_{j\in B} \left(e^{A_B t_1} R e^{A_{B^c} s_1} H\right)_{i,j} \left(\left[\prod_{k=2}^{n-1} e^{A_B t_k} R e^{A_{B^c} s_k} H\right] e^{A_B t_n} R e^{A_{B^c} s_n} \mathbb{1}\right)_j\\ &\quad= \left(\left[\prod_{k=1}^{n-1} e^{A_B t_k} R e^{A_{B^c} s_k} H\right] e^{A_B t_n} R e^{A_{B^c} s_n} \mathbb{1}\right)_i .\end{aligned}$$

In the same way, interchanging the role played by subsets B and B^c, we obtain, for every $i \in B^c$,

$$\begin{aligned}&\mathbb{P}\{S_{B,1} > t_1, S_{B^c,1} > s_1, \ldots, S_{B,n} > t_n, S_{B^c,n} > s_n \mid X_0 = i\}\\ &\qquad= \left(\left[\prod_{k=1}^{n-1} e^{A_{B^c} s_k} H e^{A_B t_k} R\right] e^{A_{B^c} s_n} H e^{A_B t_n} \mathbb{1}\right)_i .\end{aligned}$$

Unconditioning with respect to the initial state gives the result. □

Corollary 1. *For every $n \ge 1$ and for all $t_1, \ldots, t_n \ge 0$, we have*

$$\mathbb{P}\{S_{B,1} > t_1, \ldots, S_{B,n} > t_n\} = v^{(1)} \left[\prod_{k=1}^{n-1} e^{A_B t_k} G\right] e^{A_B t_n} \mathbb{1}, \tag{1.4}$$

Proof. Putting $s_1 = \cdots = s_n = 0$ in Theorem 1.3 gives the result. □

We obtain, in the same way, the joint distribution of the first n sojourn times in subset B^c by interchanging the role played by subsets B and B^c.

For instance, as illustrated at the beginning of the subsection, if we want to make sure that the first N operational periods are long enough, that is, formally, if we want that the probability that each of these periods lasts at least t units of time is, at least, $1-\varepsilon$, we must check that

$$\mathbb{P}\{S_{B,1} > t, \ldots, S_{B,N} > t\} = v^{(1)} \left(e^{A_B t} G\right)^{N-1} e^{A_B t} \mathbb{1} \geq 1 - \varepsilon.$$

The independence of the sequences $(S_{B,n})$ and $(S_{B^c,n})$ is discussed in [13].

1.4 Sum of the First n Sojourn Times

In this section, we focus on the distribution of the sum of the first n sojourn times. We denote this random variable by $TS_{B,n}$. We then have

$$TS_{B,n} = \sum_{\ell=1}^{n} S_{B,\ell}.$$

The distribution of $TS_{B,n}$ is given by the following theorem which uses the next lemma. We first introduce the column vectors $w_B(n,t)$ and $w_{B^c}(n,t)$ defined by

$$w_B(n,t) = (\mathbb{P}\{TS_{B,n} > t \mid X_0 = i\}, i \in B)$$

and

$$w_{B^c}(n,t) = (\mathbb{P}\{TS_{B,n} > t \mid X_0 = i\}, i \in B^c).$$

Lemma 4. *For every $n \geq 1$ and for all $t \geq 0$, we have*

$$w_{B^c}(n,t) = H w_B(n,t).$$

Proof. For $i \in B^c$, we have, using the Markov property and since $X_0 = Z_0$,

$$\begin{aligned}
\mathbb{P}\{TS_{B,n} > t \mid X_0 = i\} &= \sum_{j \in S} \mathbb{P}\{TS_{B,n} > t, Z_1 = j \mid Z_0 = i\} \\
&= \sum_{j \in B} P_{i,j} \mathbb{P}\{TS_{B,n} > t \mid X_0 = j\} \\
&\quad + \sum_{j \in B^c} P_{i,j} \mathbb{P}\{TS_{B,n} > t \mid X_0 = j\}.
\end{aligned}$$

This gives, in matrix notation,

$$w_{B^c}(n,t) = P_{B^c B} w_B(n,t) + P_{B^c} w_{B^c}(n,t),$$

that is

$$w_{B^c}(n,t) = (I - P_{B^c})^{-1} P_{B^cB} w_B(n,t) = H w_B(n,t),$$

which completes the proof. □

Theorem 6. *For every $n \geq 1$ and for all $t \geq 0$, we have*

$$\mathbb{P}\{TS_{B,n} > t\} = w_n e^{M_n t} \mathbb{1},$$

where $w_n = \begin{pmatrix} v^{(1)} & 0 & \cdots & 0 \end{pmatrix}$ is the row vector with length $n|B|$ (each 0 represents here the null vector with length $|B|$) and M_n is the $(n|B|, n|B|)$ matrix given by

$$M_n = \begin{pmatrix} Q_1 & Q_2 & 0 & 0 & & \cdots & & & 0 & 0 \\ 0 & Q_1 & Q_2 & 0 & 0 & & \cdots & & 0 & 0 \\ 0 & 0 & Q_1 & Q_2 & 0 & 0 & & \cdots & 0 & 0 \\ & & & & & & & & & \\ \vdots & & & \ddots & \ddots & \ddots & & & \vdots & \vdots \\ \vdots & & & & & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & & & & & & & \\ 0 & 0 & & & \cdots & & & 0 & Q_1 & Q_2 \\ 0 & 0 & & & \cdots & & & & 0 & Q_1 \end{pmatrix}$$

with $Q_1 = A_B$ and $Q_2 = -A_{BB^c}(A_{B^c})^{-1} A_{B^cB}$ (each 0 represents here the $(n|B|, n|B|)$ null matrix).

Proof. For $n = 1$, the result is immediate since $w_1 = v^{(1)}$ and $M_1 = Q_1 = A_B$. Let $n \geq 2$. We use classical backward renewal arguments, see for instance [18]. For every $i \in B$, we have

$$\begin{aligned} \mathbb{P}\{TS_{B,n} > t \mid X_0 = i\} &= \sum_{j \in S} P_{i,j} \int_0^\infty \mathbb{P}\{TS_{B,n} > t \mid Z_1 = j, T_1 = x, X_0 = i\} \lambda e^{-\lambda x} dx \\ &= \sum_{j \in S} P_{i,j} \int_0^t \mathbb{P}\{TS_{B,n} > t \mid Z_1 = j, T_1 = x, X_0 = i\} \lambda e^{-\lambda x} dx \\ &\quad + \int_t^\infty \lambda e^{-\lambda x} dx \\ &= \sum_{j \in B} P_{i,j} \int_0^t \mathbb{P}\{TS_{B,n} > t - x \mid X_0 = j\} \lambda e^{-\lambda x} dx \\ &\quad + \sum_{j \in B^c} P_{i,j} \int_0^t \mathbb{P}\{TS_{B,n-1} > t - x \mid X_0 = j\} \lambda e^{-\lambda x} dx + e^{-\lambda t}. \end{aligned}$$

where T_1 is the first occurrence instant of the Poisson process $\{N_t,\ t \geq 0\}$. The change of variable $x := t - x$ leads to

$$\mathbb{P}\{TS_{B,n} > t \mid X_0 = i\} = e^{-\lambda t}\left(1 + \sum_{j\in B} P_{i,j}\int_0^t \mathbb{P}\{TS_{B,n} > x \mid X_0 = j\}\lambda e^{\lambda x}dx \right.$$
$$\left. + \sum_{j\in B^c} P_{i,j}\int_0^t \mathbb{P}\{TS_{B,n-1} > x \mid X_0 = j\}\lambda e^{\lambda x}dx\right).$$

Using the column vectors $w_B(n,t)$ and $w_{B^c}(n,t)$ defined above, we obtain

$$w_B(n,t) = e^{-\lambda t}\left(\mathbb{1} + \lambda P_B\int_0^t w_B(n,x)e^{\lambda x}dx + \lambda P_{BB^c}\int_0^t w_{B^c}(n-1,x)e^{\lambda x}dx\right).$$

Differentiating with respect to t we get

$$w_B'(n,t) = -\lambda w_B(n,t) + \lambda\left(P_B w_B(n,t) + P_{BB^c}w_{B^c}(n-1,t)\right).$$

Using Lemma 4, we have

$$\begin{aligned} w_B'(n,t) &= -\lambda w_B(n,t) + \lambda\left(P_B w_B(n,t) + P_{BB^c}(I-P_{B^c})^{-1}P_{B^cB}w_B(n-1,t)\right) \\ &= -\lambda\,(I-P_B)\,w_B(n,t) + \lambda P_{BB^c}(I-P_{B^c})^{-1}P_{B^cB}w_B(n-1,t). \end{aligned}$$

Note that since $-\lambda\,(I-P) = A$, we have $-\lambda\,(I-P_B) = A_B$, $\lambda P_{BB^c}(I-P_{B^c})^{-1}P_{B^cB} = -A_{BB^c}\,(A_{B^c})^{-1}A_{B^cB}$ and thus

$$\begin{aligned} w_B'(n,t) &= A_B w_B(n,t) - A_{BB^c}\,(A_{B^c})^{-1}A_{B^cB}w_B(n-1,t) \\ &= Q_1 w_B(n,t) + Q_2 w_B(n-1,t). \end{aligned}$$

Introducing the column vector $u_B(n,t)$ defined by

$$u_B(n,t) = \left(w_B(n,t), w_B(n-1,t), \ldots, w_B(1,t)\right),$$

this gives

$$u_B'(n,t) = M_n u_B(n,t)$$

and thus, since $u_B(n,0) = \mathbb{1}$,

$$u_B(n,t) = e^{M_n t}u_B(n,0) = e^{M_n t}\mathbb{1}.$$

Finally,

$$\mathbb{P}\{TS_{B,n} > t\} = \alpha_B w_B(n,t) + \alpha_{B^c}w_{B^c}(n,t).$$

Using Lemma 4, we get

$$\mathbb{P}\{TS_{B,n} > t\} = (\alpha_B + \alpha_{B^c}H)\,w_B(n,t) = v^{(1)}w_B(n,t) = w_n u_B(n,t) = w_n e^{M_n t}\mathbb{1},$$

which completes the proof. □

To compute the distribution of $TS_{B,n}$ for a fixed n we proceed as follows. Let β be a positive real number such that $\beta \geq \max\{\lambda_i,\ i \in B\}$. The matrix T_n defined by

$T_n = I + M_n/\beta$ is substochastic and given by

$$T_n = \begin{pmatrix} P_1 & P_2 & 0 & 0 & \cdots & & & 0 & 0 \\ 0 & P_1 & P_2 & 0 & 0 & & \cdots & 0 & 0 \\ 0 & 0 & P_1 & P_2 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & & \cdots & & & 0 & P_1 & P_2 \\ 0 & 0 & & \cdots & & & & 0 & P_1 \end{pmatrix},$$

where $P_1 = I + Q_1/\beta$ and $P_2 = Q_2/\beta$. We then have

$$\mathbb{P}\{TS_{B,n} > t\} = w_n e^{M_n t} \mathbb{1} = \sum_{k=0}^{\infty} e^{-\beta t} \frac{(\beta t)^k}{k!} w_n (T_n)^k \mathbb{1}.$$

The special form of matrix T_n leads to the following form of its kth power, i.e.

$$(T_n)^k = \begin{pmatrix} (P_1)^k & W_{n-1,k} \\ 0 & (T_{n-1})^k \end{pmatrix},$$

where, by writing $(T_n)^k = T_n (T_n)^{k-1}$, the matrix $W_{n-1,k}$, which is a $(|B|, (n-1)|B|)$ matrix, is given for $k, n \geq 2$ by

$$W_{n-1,k} = P_1 W_{n-1,k-1} + W_{n-1,1} (T_{n-1})^{k-1}$$

with $W_{n-1,1} = (P_2\, 0 \cdots 0)$. If $x_B(n,k)$ denotes the column vector composed of the first $|B|$ entries of vector $(T_n)^k \mathbb{1}$, we have

$$x_B(n,0) = \mathbb{1} \text{ for } n \geq 1,\ x_B(1,k) = (P_1)^k \mathbb{1} \text{ for } k \geq 0$$

and, for $n \geq 2$ and $k \geq 1$,

$$\begin{aligned} x_B(n,k) &= (P_1)^k \mathbb{1} + W_{n-1,k} \mathbb{1} \\ &= (P_1)^k \mathbb{1} + P_1 W_{n-1,k-1} \mathbb{1} + W_{n-1,1} (T_{n-1})^{k-1} \mathbb{1} \\ &= (P_1)^k \mathbb{1} + P_1 W_{n-1,k-1} \mathbb{1} + P_2 x_B(n-1,k-1) \\ &= P_1 \left((P_1)^{k-1} \mathbb{1} + W_{n-1,k-1} \mathbb{1} \right) + P_2 x_B(n-1,k-1) \\ &= P_1 x_B(n,k-1) + P_2 x_B(n-1,k-1). \end{aligned} \tag{1.5}$$

We then have

$$\mathbb{P}\{TS_{B,n} > t\} = \sum_{k=0}^{\infty} e^{-\beta t} \frac{(\beta t)^k}{k!} v^{(1)} x_B(n,k).$$

Let ε be a given specified error tolerance associated with the computation of the distribution of $TS_{B,n}$ and let K be the integer defined by

$$K = \min\left\{ k \in \mathbb{N} \,\middle|\, \sum_{j=0}^{k} e^{-\beta t} \frac{(\beta t)^j}{j!} \geq 1 - \varepsilon \right\}.$$

This gives

$$\mathbb{P}\{TS_{B,n} > t\} = \sum_{k=0}^{K} e^{-\beta t} \frac{(\beta t)^k}{k!} v^{(1)} x_B(n,k) + e(K),$$

where

$$e(K) = \sum_{k=K+1}^{\infty} e^{-\beta t} \frac{(\beta t)^k}{k!} v^{(1)} x_B(n,k) \leq \sum_{k=0}^{K} e^{-\beta t} \frac{(\beta t)^k}{k!} = 1 - \sum_{k=0}^{K} e^{-\beta t} \frac{(\beta t)^k}{k!} \leq \varepsilon.$$

We consider now the moments of the sum of the first n sojourn times of X in subset B. From Theorem 6, we have

$$\mathbb{E}\left\{(TS_{B,n})^k\right\} = (-1)^k k! w_n (M_n)^{-k} \mathbb{1}.$$

The expected value of $TS_{B,n}$ is given by

$$\mathbb{E}\{TS_{B,n}\} = \sum_{\ell=1}^{n} \mathbb{E}\{S_{B,\ell}\} = -v^{(1)} \left(\sum_{\ell=0}^{n-1} G^{\ell}\right) (A_B)^{-1} \mathbb{1}.$$

For the higher moments of $TS_{B,n}$ we need the following lemma. We denote by $M_n[i,j]$ the submatrix of matrix M_n, of dimension $(|B|,|B|)$ defined for $i,j = 1,\ldots,n$ by

$$M_n[i,j] = \begin{cases} Q_1 & \text{if } i = j, \\ Q_2 & \text{if } i = j-1, \\ 0 & \text{otherwise.} \end{cases}$$

We define in the same way the blocks $(M_n)^{-k}[i,j]$ of matrix $(M_n)^{-k}$. For $k = 1$, these blocks are given by the following result.

Lemma 5. *For every $i,j = 1,\ldots,n$, we have*

$$(M_n)^{-1}[i,j] = \begin{cases} (-1)^{j-i} (Q_1)^{-1} \left(Q_2 (Q_1)^{-1}\right)^{j-i} = G^{j-i} (Q_1)^{-1} & \textit{if } i \leq j, \\ 0 & \textit{if } i > j. \end{cases}$$

Proof. The matrix M_n being upper triangular, the matrix $(M_n)^{-1}$ is also upper triangular. For $i = j$, we clearly have $(M_n)^{-1}[i,i] = (Q_1)^{-1}$. For $i < j$, by writing

$I = M_n (M_n)^{-1}$, we have $0 = Q_1 (M_n)^{-1} [i, j] + Q_2 (M_n)^{-1} [i+1, j]$, that is

$$(M_n)^{-1} [i, j] = -(Q_1)^{-1} Q_2 (M_n)^{-1} [i+1, j].$$

Since $(M_n)^{-1} [i,i] = (Q_1)^{-1}$ we easily get the first equality recursively. The second one follows immediately from the first one. $\square$

For every $k \geq 2$, it is easily checked that the matrix $(M_n)^{-k}$ has the same structure as matrices M_n and $(M_n)^{-1}$, that is, that the blocks $(M_n)^{-k} [i, j]$ only depend on the difference $j - i$. We thus get

$$\begin{aligned}
\mathbb{E}\left\{(TS_{B,n})^2\right\} &= 2w_n (M_n)^{-2} \mathbb{1} \\
&= 2v^{(1)} \sum_{j=1}^{n} (M_n)^{-2} [1, j] \mathbb{1} \\
&= 2v^{(1)} \sum_{j=1}^{n} \sum_{h=1}^{j} (M_n)^{-1} [1, h] (M_n)^{-1} [h, j] \mathbb{1} \\
&= 2v^{(1)} \sum_{j=1}^{n} \sum_{h=1}^{j} G^{h-1} (Q_1)^{-1} G^{j-h} (Q_1)^{-1} \mathbb{1}.
\end{aligned}$$

For $k \geq 2$, we obtain

$$\begin{aligned}
(M_n)^{-k} [1, j] &= \sum_{h=1}^{j} (M_n)^{-1} [1, h] (M_n)^{-k+1} [h, j] \\
&= \sum_{h=1}^{j} G^{h-1} (Q_1)^{-1} (M_n)^{-k+1} [1, j-h+1].
\end{aligned}$$

Let us introduce the column vectors $\theta_n(k, j)$ of dimension $|B|$, defined by $\theta_n(k, j) = (M_n)^{-k} [1, j] \mathbb{1}$. To compute these vectors, we have the following recurrence relation.

$$\begin{aligned}
\theta_n(k, j) &= \sum_{h=1}^{j} G^{h-1} (Q_1)^{-1} (M_n)^{-k+1} [1, j-h+1] \mathbb{1} \\
&= \sum_{h=1}^{j} G^{h-1} (Q_1)^{-1} \theta_n(k-1, j-h+1) \\
&= \sum_{h=1}^{j} G^{j-h} (Q_1)^{-1} \theta_n(k-1, h),
\end{aligned}$$

with $\theta_n(1, j) = G^{j-1} (Q_1)^{-1} \mathbb{1}$, for $j = 1, \ldots, n$. We then have

$$\begin{aligned}\mathbb{E}\left\{(TS_{B,n})^k\right\} &= (-1)^k k! w_n (M_n)^{-k} \mathbb{1}\\ &= (-1)^k k! v^{(1)} \sum_{j=1}^{n} (M_n)^{-k}[1,j]\mathbb{1}\\ &= (-1)^k k! v^{(1)} \sum_{j=1}^{n} \theta_n(k,j).\end{aligned}$$

1.5 Extension to absorbing chains with rewards

We consider now the case where the state space S of X is composed of transient states and an absorbing state denoted by a. The subset of operational states is denoted by B and we denote by B' the set of the other transient states. We then have the partition $S = B \cup B' \cup \{a\}$. The set of non operational states is thus $B^c = B' \cup \{a\}$. A reward rate or performance level r_i is associated with each state $i \in S$. We suppose that we have $r_i > 0$ for every $i \in B$ and we denote by R_B the $(|B|,|B|)$ diagonal matrix with entries r_i, for $i \in B$. As we will see, the value of the rewards associated with the other states has no influence on the sojourn times considered here. The partition B, B', $\{a\}$ of the state space S induces the following decomposition of matrices A and $P = I + A/\lambda$ and vector α.

$$A = \begin{pmatrix} A_B & A_{BB'} & A_{Ba} \\ A_{B'B} & A_{B'} & A_{B'a} \\ 0 & 0 & 0 \end{pmatrix}, P = \begin{pmatrix} P_B & P_{BB'} & P_{Ba} \\ P_{B'B} & P_{B'} & P_{B'a} \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \alpha = \begin{pmatrix} \alpha_B & \alpha_{B'} & \alpha_a \end{pmatrix}.$$

For $n \geq 1$, we denote by $S_{i,B,n}$ the total time spent by X in state $i \in B$ during the nth sojourn of X in B, if it exists. If the process gets absorbed before the nth sojourn of X in B, we set $S_{i,B,n} = 0$. For $n \geq 1$, the random variable $S_{B,n}$ representing the accumulated reward during the nth sojourn of X in B is defined by

$$S_{B,n} = \sum_{i \in B} r_i S_{i,B,n}.$$

Following the same steps used for the irreducible case, it is easily checked that the distribution $v^{(n)}$ of the random variable $V_{B,n}$ representing the state of B in which the nth sojourn of X in B starts is given, for every $n \geq 1$, by $v^{(n)} = v^{(1)} G^{n-1}$, where

$$v^{(1)} = \alpha_B + \alpha_{B'}(I - P_{B'})^{-1} P_{B'B} = \alpha_B - \alpha_{B'}(A_{B'})^{-1} A_{B'B},$$

$$G = (I - P_B)^{-1} P_{BB'} (I - P_{B'})^{-1} P_{B'B} = (A_B)^{-1} A_{BB'} (A_{B'})^{-1} A_{B'B}.$$

The distribution of $S_{B,n}$ is given by the following theorem.

Theorem 7. *For every $n \geq 1$ and for all $t \geq 0$, the distribution of the accumulated reward in B during the nth sojourn of X in B, is given by*

$$\mathbb{P}\{S_{B,n} > t\} = v^{(1)} G^{n-1} e^{(R_B)^{-1} A_B t} \mathbb{1}.$$

Proof. The proof is quite similar to the proof of Theorem 3. Using a classical backward renewal argument, as for instance in [18], we have, for every $i \in B$,

$$\mathbb{P}\{S_{B,1} > t \mid X_0 = i\} = \sum_{j \in S} P_{i,j} \int_0^\infty \mathbb{P}\{S_{B,1} > t \mid Z_1 = j, T_1 = x, X_0 = i\} \lambda e^{-\lambda x} dx,$$

where T_1 is the first occurrence instant of the Poisson process $\{N_t, t \geq 0\}$. We then have

$$\mathbb{P}\{S_{B,1} > t \mid X_0 = i\} = \int_{t/r_i}^\infty \lambda e^{-\lambda x} dx + \sum_{j \in B} P_{i,j} \int_0^{t/r_i} \mathbb{P}\{S_{B,1} > t - r_i x \mid X_0 = j\} \lambda e^{-\lambda x} dx.$$

The change of variable $x := t - r_i x$ in the second integral leads to

$$\mathbb{P}\{S_{B,1} > t \mid X_0 = i\} = e^{-\lambda t / r_i} \left(1 + \sum_{j \in B} P_{i,j} \int_0^t \mathbb{P}\{S_{B,1} > x \mid X_0 = j\} \frac{\lambda}{r_i} e^{\lambda x / r_i} dx \right).$$

Introducing the column vector $w(t)$, defined by $w_i(t) = \mathbb{P}\{S_{B,1} > t \mid X_0 = i\}$, for every $i \in B$, we get $w(0) = \mathbb{1}$ and

$$w(t) = e^{-\lambda (R_B)^{-1} t} \left(\mathbb{1} + \int_0^t e^{\lambda (R_B)^{-1} x} \lambda (R_B)^{-1} P_B w(x) dx \right).$$

Differentiating with respect to t leads to

$$w'(t) = -\lambda (R_B)^{-1} w(t) + \lambda (R_B)^{-1} P_B w(t) = (R_B)^{-1} A_B w(t),$$

that is,

$$w(t) = e^{(R_B)^{-1} A_B t} w(0) = e^{(R_B)^{-1} A_B t} \mathbb{1}. \tag{1.6}$$

For every $n \geq 1$ and for all $t \geq 0$, we have, using now the Markov property, the homogeneity of X and (1.6),

$$\begin{aligned} \mathbb{P}\{S_{B,n} > t\} &= \sum_{i \in B} v_i^{(n)} \mathbb{P}\{S_{B,n} > t \mid V_{B,n} = i\} \\ &= \sum_{i \in B} v_i^{(n)} \mathbb{P}\{S_{B,1} > t \mid X_0 = i\} \\ &= v^{(n)} e^{(R_B)^{-1} A_B t} \mathbb{1}, \end{aligned}$$

which completes the proof. □

In the same way, the distribution of the accumulated reward over the first n sojourn times of X in B is given by the following result.

Theorem 8. *For every $n \geq 1$ and for all $t \geq 0$, we have*

$$\mathbb{P}\{TS_{B,n} > t\} = w_n e^{M_n t} \mathbb{1},$$

where $w_n = \begin{pmatrix} v^{(1)} \ 0 \ \cdots \ 0 \end{pmatrix}$ *is the row vector with length* $n|B|$ *(each* 0 *represents here the null vector with length* $|B|$*) and* M_n *is the* $(n|B|, n|B|)$ *matrix given by*

$$M_n = \begin{pmatrix} Q_1 & Q_2 & 0 & 0 & & \cdots & & & 0 & 0 \\ 0 & Q_1 & Q_2 & 0 & 0 & & \cdots & & 0 & 0 \\ 0 & 0 & Q_1 & Q_2 & 0 & 0 & & \cdots & 0 & 0 \\ & & & & & & & & & \\ \vdots & & & & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & & & & & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & & & & & & & \\ 0 & 0 & & & \cdots & & & 0 & Q_1 & Q_2 \\ 0 & 0 & & & \cdots & & & & 0 & Q_1 \end{pmatrix}$$

with $Q_1 = (R_B)^{-1} A_B$ *and* $Q_2 = -(R_B)^{-1} A_{BB'} (A_{B'})^{-1} A_{B'B}$ *(each* 0 *represents here the* $(n|B|, n|B|)$ *null matrix).*

Proof. For $n = 1$, the result is immediate since $w_1 = v^{(1)}$ and $M_1 = Q_1 = (R_B)^{-1} A_B$. Let $n \geq 2$. We use the same classical backward renewal arguments already used in the proof of Theorem 6. For every $i \in B$, we have

$$\begin{aligned} \mathbb{P}\{TS_{B,n} > t \mid X_0 = i\} = & \sum_{j \in B} P_{i,j} \int_0^{t/r_i} \mathbb{P}\{TS_{B,n} > t - r_i x \mid X_0 = j\} \lambda e^{-\lambda x} dx \\ & + \sum_{j \in B'} P_{i,j} \int_0^{t/r_i} \mathbb{P}\{TS_{B,n-1} > t - r_i x \mid X_0 = j\} \lambda e^{-\lambda x} dx \\ & + P_{i,a} e^{-\lambda t / r_i}. \end{aligned}$$

The rest of the proof is identical to the proofs of Theorems 7 and 6. □

Note that unlike the irreducible case, the matrix G is substochastic and if we denote by N_B the total number of visits to the subset B until absorption, the events $\{N_B > k\}$ and $\{S_{B,k+1} > 0\}$ are equal for every $k \geq 0$. It follows that

$$\mathbb{P}\{N_B > k\} = \mathbb{P}\{S_{B,k+1} > 0\} = v^{(1)} G^k \mathbb{1}.$$

In particular, we have

$$\mathbb{E}\{N_B\} = \sum_{k=0}^{\infty} v^{(1)} G^k \mathbb{1} = v^{(1)} (I - G)^{-1} \mathbb{1}.$$

Observe that if we know that the process has visited the set B at least n times, for $n \geq 1$, that is, given that $S_{B,n} > 0$, the evaluation of the accumulated reward during the nth sojourn in B changes. The conditional distribution of $S_{B,n}$ given that $S_{B,n} > 0$

can be written, for $n \geq 1$ and $t \geq 0$, as

$$\mathbb{P}\{S_{B,n} > t \mid S_{B,n} > 0\} = \frac{\mathbb{P}\{S_{B,n} > t\}}{\mathbb{P}\{S_{B,n} > 0\}} = \frac{v^{(1)} G^{n-1} e^{(R_B)^{-1} A_B t} \mathbb{1}}{v^{(1)} G^{n-1} \mathbb{1}}.$$

The total accumulated reward in subset B until absorption is defined by

$$TS_{B,\infty} = \sum_{n=1}^{\infty} S_{B,n}.$$

Its distribution is given in [3] for semi-Markov reward processes. In the case of Markov reward processes, it becomes

$$\mathbb{P}\{TS_{B,\infty} > t\} = v^{(1)} e^{(Q_1+Q_2)t} \mathbb{1}.$$

Finally the distribution of the sojourn times $S_{B,n}$ for semi-Markov reward processes has been obtained in [15].

1.6 Notes

As stated in the introduction, the analysis of sojourn times of Markov models in subsets of their state spaces, in the dependability area, apparently started in [10], in the irreducible case. In biology, close related work, with differences however, was known before (see [6]). There are other papers related to the analysis of these objects. For instance, [1] and [4] provide bounds on reliability metrics by exploiting the fact that in many dependability models, when the system is highly reliable, there is a huge difference in the holding times the chain spends in its states.

Perhaps even closer to this chapter, we can mention [2], where the authors define a "conditional MTTF" in the case of a system subject to very different failures. If the chain has, say, two absorbing states a and a', representing two different situations where the system has failed, depending on the causes of such a failure, it makes sense to analyze the mean time the system operates given that it will be absorbed in state a, that is, the metric $\mathbb{E}\{T \mid X_\infty = a\}$, where T is the system's life-time. If, more generally, we are interested in some subset B of transient states, we can look at the nth sojourn time of the process X in B, if it exists.

Using the notation in this chapter, let us briefly outline how to derive the distribution of the nth sojourn time of X in B, given that $X_\infty = a$. As in Section 1.5, we simplify the presentation avoiding the fact that we must define $S_{B,n}$ and $V_{B,n}$ for all n, in particular when the nth sojourn in B does not exist (in that case, $S_{B,n} = 0$ and $V_{B,n}$ is assigned a specific extra state). After doing that and simplifying, we can write

$$\begin{aligned}
\mathbb{P}\{S_{B,n} > t \mid X_\infty = a\} &= \sum_{i \in B} \mathbb{P}\{S_{B,n} > t, V_{B,n} = i \mid X_\infty = a\} \\
&= \sum_{i \in B} \mathbb{P}\{S_{B,n} > t \mid V_{B,n} = i, X_\infty = a\} \mathbb{P}\{V_{B,n} = i \mid X_\infty = a\} \\
&= \sum_{i \in B} \mathbb{P}\{S_{B,1} > t \mid X_0 = i\} \mathbb{P}\{V_{B,n} = i \mid X_\infty = a\}.
\end{aligned}$$

Then, for every $i, j \in B$,

$$\begin{aligned}
\mathbb{P}\{V_{B,n+1} = j \mid V_{B,n} = i, X_\infty = a\} &= \frac{\mathbb{P}\{X_\infty = a \mid V_{B,n+1} = j, V_{B,n} = i\} G_{i,j}}{\mathbb{P}\{X_\infty = a \mid V_{B,n} = i\}} \\
&= \frac{\mathbb{P}\{X_\infty = a \mid V_{B,n+1} = j\} G_{i,j}}{\mathbb{P}\{X_\infty = a \mid V_{B,n} = i\}} \\
&= \frac{\mathbb{P}\{X_\infty = a \mid X_0 = j\} G_{i,j}}{\mathbb{P}\{X_\infty = a \mid X_0 = i\}}.
\end{aligned}$$

The quantities $\mathbb{P}\{X_\infty = a \mid X_0 = k\}$ are well-known in Markov chain theory, and the distributions $\mathbb{P}\{S_{B,1} > t \mid X_0 = k\}$ were given in Section 1.2. This type of computation also appears in [7], where bounds on the asymptotic availability, and more generally of the asymptotic reward, are derived. The Markov model considered is irreducible on a state space partitioned into three classes, say B, C, C'. The object of interest is the sojourn time of the process in B, given that when leaving B the process will visit C next. Concerning Section 1.2, see that we can get more information about sojourn times following the same lines as described there. For instance, assume that we are interested in the last state of B visited by X during its nth visit to that subset. Call it $W_{B,n}$, and call $w^{(n)}$ its distribution: $\mathbb{P}\{W_{B,n} = i\} = w_i^{(n)}$. Following the same path as in Theorem 1, we have for $n \geq 1$,

$$w^{(n)} = w^{(1)} M^{n-1} = v^{(n)} (I - P_B)^{-1},$$

where $M = P_{BB^c} (I - P_{B^c})^{-1} P_{B^cB} (I - P_B)^{-1}$. Note that this is the matrix appearing in Lemma 2, where it is stated that $\pi_B = \pi_B M$.

In [13], we analyze conditions under which the sequence of sojourn times of a Markov chain X in a subset B of states is i.i.d., with applications always in the analysis of dependability properties. A particularly important metric appearing in dependability, more complex to analyze than previously considered ones, is the interval availability over an interval $[0,t]$, which is the random variable

$$IA_t = \frac{1}{t} \int_0^t 1_{\{X_s \in B\}} ds,$$

where B is the set of operational states. In words, IA_t is the fraction of $[0,t]$ during which the system is operational. The first paper where the distribution of this variable is proposed using the uniformization techniques (as we do in this chapter) is [19]. We proposed some improvements to the algorithms (including the possibility of dealing with infinite state spaces) in [14]. Many of the results described here can be extended to semi-Markov processes, and also to the case where the states in the model are weighted by rewards, or costs. Some of these extensions have been presented in Section 1.5. See also [11] or [15]. The monograph [5] also discusses many of these results together with several other related issues.

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