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Performance Evaluation of a Distributed Synchronization Protocol

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Abstract

This paper presents a solution to the well-known problem of synchronization in a distributed asynchronous system prone to process crashes. This problem is also known as the Test&Set problem. The Test&Set is a distributed synchronization protocol that, when invoked by a set of processes, returns a unique winning process. This unique process is then allowed to use, for instance, a shared resource. Recently many advances in implementing Test&Set objects have been achieved, however all of them uniquely target the shared memory model. In this paper we propose an implementation of a Test&Set object for a message passing distributed system. This implementation can be invoked by any number $n \leq N$ of processes where N is the total number of processes in the system. We show in this paper, using a Markov model, that our implementation has an expected step complexity in $O(\log n)$ and we give an explicit formula for the distribution of the number of steps needed to solve the problem. We also analyze the expected value and the distribution of the number of operations invoked by the n processes to determine the winning process.

Keywords : Asynchronous message-passing system, Crash failures, Distributed synchronization, Markov chain, Randomized algorithm, Test&Set

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1 Introduction

The Test&Set is a classical distributed synchronization protocol that, when invoked by a set of processes, returns yes to a unique process (the winner) and returns no to all the others (the losers). It allows to solve competition problems. In this paper we propose a randomized implementation of the Test&Set operation. Indeed, according to the hierarchy of agreement problems based on consensus numbers given by Herlihy [1], Test&Set is easier to solve than consensus, but this operation does not have a deterministic implementation as soon as one crash may occur [2]. Thus in order to implement it in an asynchronous system, it is necessary to add synchrony assumptions or to use randomization. We focus on the latter option and we consider the *adaptive adversary model*, that is the model in which the adversary makes all its scheduling decisions based on the knowledge of the full history of the events [3, 4].

Specifically, our implementation of the Test&Set operation can be invoked by any number $n \leq N$ of processes, where N is the total number of processes in the system. We analyze its step complexity, *i.e.* the number of steps needed to complete the Test&Set execution, assuming that the scheduling of the worst adversary is taken from the adaptive adversary family.

1.1 The Test&Set Algorithm

The implementation we propose of the Test&Set protocol goes through a sequence of calls to a basic building block that we call in the following the *selector*. The selector is a distributed service, invoked by a set of processes, that allows to select a winning group among two competing groups. We proposed in [5] a message-passing implementation of the selector in presence of an oblivious adversary[1]. The expected complexity for each process of our implementation is constant. A variant of the *GroupElect* object proposed by Woelfel and Giakouppis [6] would provide a shared memory implementation of the *selector* object when one considers an oblivious adversary.

Suppose that n processes of a distributed system all want to access a common resource. The selector algorithm works as follows. They all toss a local coin, with probability $1/2$ and independently of one another, and invoke the selector with the result of the toss, which is 0 or 1. Let n_0 and n_1 be respectively the number 0 and 1 obtained. We have $n = n_0 + n_1$. The output of the distributed selector algorithm is as follows. If $n_0 = 0$ or $n_1 = 0$ then all the n processes restart tossing a local coin until $n_0 \neq 0$ and $n_1 \neq 0$. If $n_0 \geq 1$ and $n_1 \geq 1$ then we have two groups of processes, denoted by G_0 and G_1, containing respectively the processes who got 0 and the processes who got 1. At this point, the selector object selects randomly, with probability $1/2$, the group which will proceed the competition. All the processes of the other group are losers. Next, the procedure restarts from the beginning with the selected

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[1]An oblivious adversary makes all its scheduling decisions at the beginning of the execution independently of the random values tossed by the processes in the course of the execution. In contrast, an adaptive adversary makes its decisions based on the full history of the events. The adaptive adversary is thus stronger than the oblivious one.

group, G_0 or G_1, of processes *i.e.* with $n := n_0$ or $n := n_1$. The procedure ends when the value $n := 1$ is reached and the corresponding process is the winning process receiving a yes while all the others received a no. We refer to [5] for a detailed description of the pseudo-code of the Test&Set algorithm and references on the subject.

In the following section, we provide a Markov model of the Test&Set algorithm described above. We show, Section 3, that the expected step complexity, *i.e.* the expected number of steps needed to determinine the winning process, is in $O(\log n)$ and we give an explicit formula of the distribution of this number of steps. Note that having a step complexity that depends on n and not on the total number N of processes in the system makes our solution *adaptive*. This is quite interesting not only for the theoretical aspect but also in practice since the cost of the implementation only depends on the number of processes that concurrently invokes it. We are not aware of any adaptive implementation of the Test&Set protocol in message-passing systems. We consider in the Section 4 the total contention needed to determine the winning process. This random variable represents the total number of times the processes have to compete until the winning process is determined. We give in this section the expected value and the distribution of this total contention. Section 5 concludes the paper.

2 Modelling the Test&Set Algorithm

The loop of the protocol is executed until a unique process remains with the value yes, meaning that it is the termination of the protocol. We suppose that $n \geq 2$ processes initially execute the algorithm. For $\ell \geq 1$, we denote by X_ℓ the random variable representing the number of processes in competition at the ℓth transition, *i.e.* the number of processes that executed the ℓth step. We consider the discrete-time stochastic process $X = \{X_\ell, \ \ell \geq 0\}$ with state space $\{1, 2, \ldots, n\}$ where the initial state of X is state n, with probability 1, that is $\mathbb{P}\{X_0 = n\} = 1$.

We assume that the events *a process returns* 0 and *a process returns* 1 occur with the same probability $1/2$ and that the behaviors of the processes at each instant are independent of each other. This means that X is a homogeneous Markov chain. We denote by P its transition probability matrix.

The Test&Set protocol terminates when there is only one process executing the protocol, *i.e.* when the Markov chain X reaches state 1. Note that the probability $P_{i,j}$ to go from state i to state j in one transition is equal to 0 if $i < j$. Indeed, at each step the number of processes in competition either remains the same or decreases. If all the processes in competition return the same value (all 0 or all 1) then they all restart the competition.

It follows that, for $i = 1, \ldots, n$, $P_{i,i}$ is the probability that all the i processes in competition get the same value, that is either 0 or 1. We thus have

$$P_{i,i} = \frac{1}{2^i} + \frac{1}{2^i} = \frac{1}{2^{i-1}}.$$

For $1 \leq j < i \leq n$, $P_{i,j}$ is the probability that exactly j processes among i obtain the same
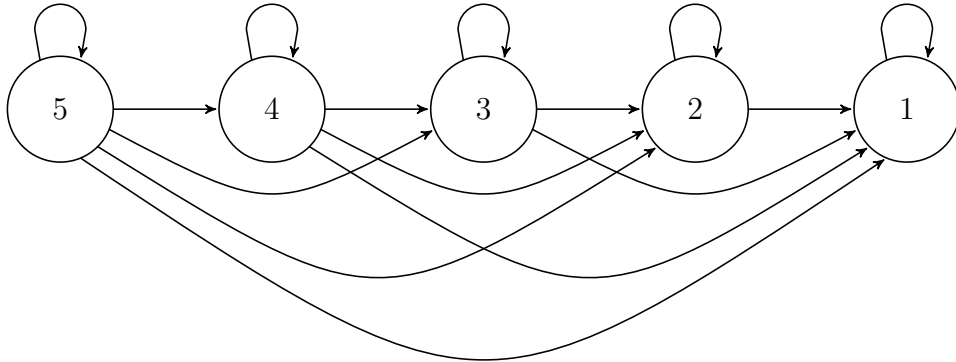
Figure 1: Transition graph of Markov chain X when $n = 5$

value and that this group of j processes is selected to continue the competition. We thus have, for $1 \leq j < i \leq n$,

$$P_{i,j} = \frac{1}{2}\left[\frac{1}{2^i}\binom{i}{j} + \frac{1}{2^i}\binom{i}{j-i}\right] = \frac{1}{2^i}\binom{i}{j}.$$

The states $2, 3, \ldots, n$ are thus transient states and state 1 is absorbing since $P_{1,1} = 1$. Figure 1 shows the graph of the Markov chain X in the case where $n = 5$.

When n processors are initially competing, the number T_n of steps needed by the Test&Set protocol is the hitting time of state 1 by Markov chain X, that is

$$T_n = \inf\{\ell \geq 0 \mid X_\ell = 1\}.$$

It is well-known, see for instance [7], that the distribution of T_n is given, for $k \geq 0$, by

$$\mathbb{P}\{T_n > k\} = \alpha Q^k \mathbb{1},$$

where Q is the matrix of dimension $n-1$ obtained from P by deleting the row and the column corresponding to absorbing state 1, α is the row vector containing the initial probabilities of the transient states, that is $\alpha_n = 1$ and $\alpha_i = 0$ for $i = 2, \ldots, n-1$, and $\mathbb{1}$ is the column vector of dimension $n - 1$ with all its entries equal to 1. In the next section, we analyze the execution time of the algorithm, that is the number of steps needed to reach state 1.

3 Execution time analysis

The expected value of T_n is given by

$$\mathbb{E}\{T_n\} = \sum_{k=0}^{\infty} \mathbb{P}\{T_n > k\} = \alpha(I - Q)^{-1}\mathbb{1},$$

It can also be evaluated using the well-known recurrence relation, see for instance [7],

$$\mathbb{E}\{T_n \mid X_0 = n\} = 1 + \sum_{k=2}^{n} P_{n,k}\mathbb{E}\{T_k \mid X_0 = k\}. \tag{1}$$

A first result of this paper is given by the following theorem. We denote by log the logarithm function to the base 2.

Theorem 1 *The expected time $\mathbb{E}\{T_n \mid X_0 = n\}$ needed to terminate the Test&Set protocol when n processes are initially competing satisfies*

$$\mathbb{E}\{T_n \mid X_0 = n\} = \Theta(\log(n)).$$

More precisely, for all $\varepsilon > 0$, there exists a positive integer n_0 such that, for every $n \geq n_0$, we have

$$\left(\frac{1}{2} - \varepsilon\right)\log(n) \leq \mathbb{E}\{T_n \mid X_0 = n\} \leq (2 + \varepsilon)\log(n).$$

Proof. See Appendix A. ∎

We now give an explicit expression of the distribution of the execution time T_n As seen previously, we have, for every $n \geq 2$ and $k \geq 0$,

$$\mathbb{P}\{T_n > k\} = \alpha Q^k \mathbb{1}.$$

An expression of matrix Q^k is given by the next theorem.

Theorem 2 *For every $n \geq 2$ and $k \geq 0$ and $i, j \in \{2, \ldots, n\}$, we have*

$$\left(Q^k\right)_{i,j} = \begin{cases} 0 & if \quad i < j \\[2mm] \dfrac{1}{2^{(i-1)k}} & if \quad i = j \\[4mm] \dfrac{1}{2^{ik}}\binom{i}{j}\sum_{\ell=1}^{2^k-1}\ell^{i-j} & if \quad i > j. \end{cases}$$

Proof. The result is clearly true for $k = 0$ since Q^0 is the identity matrix and since an empty sum is equal to 0 by convention. For $k = 1$ we see that this expression reduces to the definition of matrix Q given previously. Let $k \geq 2$. Since matrix Q is triangular with $Q_{i,j} = 0$ for $i < j$, we also have $\left(Q^k\right)_{i,j} = 0$. Moreover, the diagonal entries are given by

$$\left(Q^k\right)_{i,i} = (Q_{i,i})^k = \frac{1}{2^{(i-1)k}}.$$

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Consider now the case where $i > j$. We proceed by recurrence. Suppose that the results is true for integer $k - 1$, *i.e.* suppose that we have

$$\left(Q^{k-1}\right)_{i,j} = \frac{1}{2^{i(k-1)}} \binom{i}{j} \sum_{\ell=1}^{2^{k-1}-1} \ell^{i-j}.$$

We then recover, for $1 \le j \le i - 1$,

$$\left(Q^{k}\right)_{i,j} = \sum_{m=j}^{i} \left(Q^{k-1}\right)_{i,m} Q_{m,j}$$

$$= \left(Q^{k-1}\right)_{i,j} Q_{j,j} + \left(Q^{k-1}\right)_{i,i} Q_{i,j} + \sum_{m=j+1}^{i-1} \left(Q^{k-1}\right)_{i,m} Q_{m,j}$$

$$= \frac{1}{2^{i(k-1)+j-1}} \binom{i}{j} \sum_{\ell=1}^{2^{k-1}-1} \ell^{i-j} + \frac{1}{2^{(i-1)(k-1)}} \frac{1}{2^{i}} \binom{i}{j}$$

$$+ \frac{1}{2^{i(k-1)}} \sum_{m=j+1}^{i-1} \frac{1}{2^{m}} \binom{i}{m} \binom{m}{j} \sum_{\ell=1}^{2^{k-1}-1} \ell^{i-m}$$

$$= \frac{1}{2^{i(k-1)+j}} \binom{i}{j} \sum_{\ell=1}^{2^{k-1}-1} \ell^{i-j} + \frac{1}{2^{ik}} \binom{i}{j} + \frac{1}{2^{i(k-1)}} \sum_{m=j}^{i} \frac{1}{2^{m}} \binom{i}{m} \binom{m}{j} \sum_{\ell=1}^{2^{k-1}-1} \ell^{i-m}.$$

Exchanging the order of the sums in the third term and using the variable change $m := m+j$, we get

$$\left(Q^{k}\right)_{i,j} = \frac{1}{2^{ik}} \binom{i}{j} \left(1 + \sum_{\ell=1}^{2^{k-1}-1} (2\ell)^{i-j} + \sum_{\ell=1}^{2^{k-1}-1} \sum_{m=j}^{i} \binom{i-j}{m-j} (2\ell)^{i-m}\right)$$

$$= \frac{1}{2^{ik}} \binom{i}{j} \left(1 + \sum_{\ell=1}^{2^{k-1}-1} (2\ell)^{i-j} + \sum_{\ell=1}^{2^{k-1}-1} \sum_{m=0}^{i-j} \binom{i-j}{m} (2\ell)^{i-j-m}\right)$$

$$= \frac{1}{2^{ik}} \binom{i}{j} \left(1 + \sum_{\ell=1}^{2^{k-1}-1} (2\ell)^{i-j} + \sum_{\ell=1}^{2^{k-1}-1} (2\ell + 1)^{i-j-m}\right)$$

$$= \frac{1}{2^{ik}} \binom{i}{j} \sum_{\ell=1}^{2^{k}-1} \ell^{i-j},$$

which completes the proof. ∎

This theorem allows us to get a simple expression of the distribution of T_n.

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Corollary 3 *For every $n \geq 2$ and $k \geq 0$, we have*

$$\mathbb{P}\{T_n > k\} = 1 - \frac{n}{2^k} \sum_{\ell=1}^{2^k-1} \left(\frac{\ell}{2^k}\right)^{n-1}. \tag{2}$$

Proof. Since

$$\mathbb{P}\{T_n > k\} = \alpha Q^k \mathbb{1} = \sum_{j=2}^{n} \left(Q^k\right)_{n,j} = \frac{1}{2^{(n-1)k}} + \sum_{j=2}^{n-1} \left(Q^k\right)_{n,j},$$

we obtain, using Theorem 2,

$$
\begin{aligned}
\mathbb{P}\{T_n > k\} &= \frac{1}{2^{(n-1)k}} + \frac{1}{2^{nk}} \sum_{j=2}^{n-1} \binom{n}{j} \sum_{\ell=1}^{2^k-1} \ell^{n-j} \\
&= \frac{1}{2^{(n-1)k}} + \frac{1}{2^{nk}} \sum_{\ell=1}^{2^k-1} \ell^n \sum_{j=2}^{n-1} \binom{n}{j} \left(\frac{1}{\ell}\right)^j \\
&= \frac{1}{2^{(n-1)k}} + \frac{1}{2^{nk}} \sum_{\ell=1}^{2^k-1} \ell^n \left(\left(1 + \frac{1}{\ell}\right)^n - 1 - \frac{n}{\ell} - \frac{1}{\ell^n}\right) \\
&= \frac{1}{2^{(n-1)k}} + \frac{1}{2^{nk}} \left(\sum_{\ell=1}^{2^k-1} [(\ell+1)^n - \ell^n] - n \sum_{\ell=1}^{2^k-1} \ell^{n-1} - 2^k + 1 \right) \\
&= 1 - \frac{n}{2^{nk}} \sum_{\ell=1}^{2^k-1} \ell^{n-1},
\end{aligned}
$$

which completes the proof. ∎

Using this result, we obtain an explicit expression of the expectation of T_n which is

$$\mathbb{E}\{T_n\} = \sum_{k=0}^{\infty} \mathbb{P}\{T_n > k\} = \sum_{k=0}^{\infty} \alpha Q^k \mathbb{1} = 1 + \sum_{k=1}^{\infty} \left(1 - \frac{n}{2^k} \sum_{\ell=1}^{2^k-1} \left(\frac{\ell}{2^k}\right)^{n-1}\right).$$

Figure 2 shows the complementary distribution of T_n for several values of n.

For instance, for fixed n and for small values of ε, it is interesting to evaluate the first value of k such that $\mathbb{P}\{T_n > k\} \leq \varepsilon$. Table 1 shows this value k^* defined by

$$k^* = \inf\{k \geq 0 \mid \mathbb{P}\{T_n > k\} \leq \varepsilon\},$$
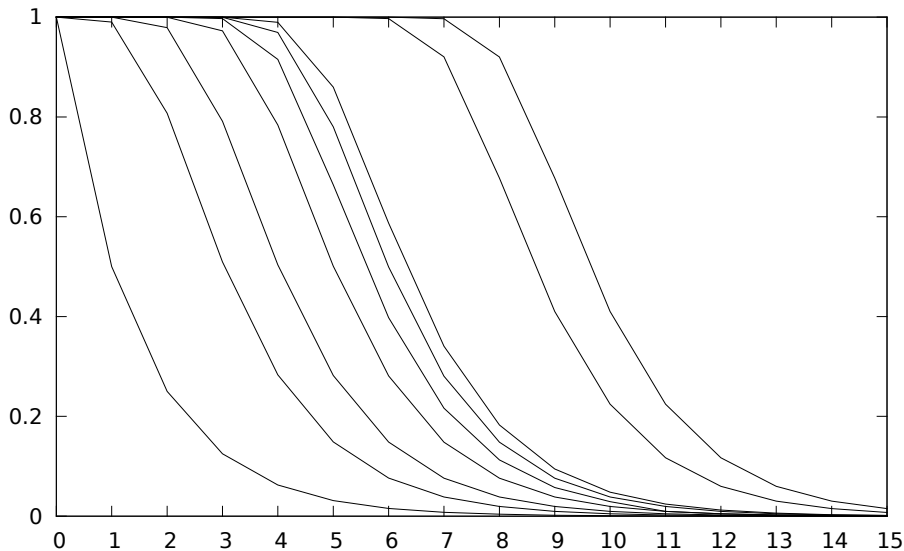
for several values of n and ε.

Figure 2: From top to the bottom: $\mathbb{P}\{T_n > k\}$ for $n = 2, 10, 20, 40, 60, 80, 100, 500, 1000$, as functions of k.

	$n = 2$	$n = 50$	$n = 100$	$n = 500$	$n = 1000$
$\varepsilon = 10^{-2}$	7	12	13	15	16
$\varepsilon = 10^{-3}$	10	15	16	18	19
$\varepsilon = 10^{-4}$	14	18	19	22	23

Table 1: Values of k^* for different values of n and ε.

This result shows, for instance, that for $n = 1000$ initially competing processes, the number of steps needed to determine the winning process is less than or equal to 23 with probability 0.9999.

4   Total contention analysis

We consider now the total contention before termination. For $\ell \geq 0$, we denote by $W_\ell(n)$ the number of processes that executed step ℓ of the protocol when n processes are initially competing. This random variable, also called the contention of the Test&Set at step ℓ, is defined, for $n \geq 2$, by

$$W_\ell(n) = \sum_{i=2}^{n} i 1_{\{X_\ell = i\}}.$$

Since the initial state is state n, we have $W_0(n) = n$ with probability 1. The total contention before termination is denoted by $N(n)$ and given by

$$N(n) = \sum_{\ell=0}^{\infty} W_\ell(n).$$

The next theorem gives the expectation of $N(n)$.

Theorem 4 *For every $n \geq 2$ and $\ell \geq 0$, we have*

$$\mathbb{E}\{W_\ell(n)\} = \frac{n}{2^\ell} \ \text{and} \ \mathbb{E}\{N(n)\} = 2n.$$

Proof. Since the initial state is state n, we have, for $\ell \geq 0$,

$$\mathbb{E}\{W_\ell(n)\} = \sum_{i=2}^{n} i \mathbb{P}\{X_\ell = i \mid X_0 = n\} = \sum_{i=2}^{n} i \left(Q^\ell\right)_{n,i}.$$

For $\ell = 0$, we have $\mathbb{E}\{W_0(n)\} = n$. For $\ell \geq 1$, we write

$$\mathbb{E}\{W_\ell(n)\} = \sum_{i=2}^{n} i \sum_{j=i}^{n} Q_{n,j} \left(Q^{\ell-1}\right)_{j,i}$$

$$= \sum_{j=2}^{n} Q_{n,j} \sum_{i=2}^{j} i \left(Q^{\ell-1}\right)_{j,i}$$

$$= \sum_{j=2}^{n} Q_{n,j} \mathbb{E}\{W_{\ell-1}(j)\}.$$

We pursue by recurrence over index ℓ. The result being true for $\ell = 0$, suppose that for every $j \geq 2$, we have $\mathbb{E}\{W_{\ell-1}(j)\} = j/2^{\ell-1}$. We then get, for every $n \geq 2$,

$$\mathbb{E}\{W_\ell(n)\} = Q_{n,n} \mathbb{E}\{W_{\ell-1}(n)\} + \sum_{j=2}^{n-1} Q_{n,j} \mathbb{E}\{W_{\ell-1}(j)\}$$

$$= \frac{1}{2^{n-1}} \frac{n}{2^{\ell-1}} + \frac{1}{2^n} \sum_{j=2}^{n-1} \binom{n}{j} \frac{j}{2^{\ell-1}}$$

$$= \frac{n}{2^n 2^{\ell-1}} + \frac{1}{2^n} \sum_{j=2}^{n} \binom{n}{j} \frac{j}{2^{\ell-1}}$$

$$= \frac{1}{2^n 2^{\ell-1}} \sum_{j=1}^{n} j \binom{n}{j}$$

$$= \frac{n}{2^n 2^{\ell-1}} \sum_{j=1}^{n} \binom{n-1}{j-1}$$

$$= \frac{n}{2^\ell}.$$

We then have

$$\mathbb{E}\{N(n)\} = \sum_{\ell=0}^{\infty} \mathbb{E}\{W_\ell(n)\} = 2n,$$

which completes the proof. ∎

This theorem shows that the expected total contention experienced by each process, which is $\mathbb{E}\{N(n)\}/n$, is equal to 2.

Let us now consider the distribution of $N(n)$ for every $n \geq 2$. Denoting by V_i the total number of visits of Markov chain X to state i, $2 \leq i \leq n$, we have

$$V_i = \sum_{\ell=0}^{\infty} 1_{\{X_\ell=i\}}.$$

With this notation and by definition of $N(n)$, we have, for $n \geq 2$,

$$N(n) = \sum_{i=2}^{n} iV_i.$$

We set $N(0) = N(1) = 0$ and for every $n \geq 0$, we define for every $n, k \geq 0$, $F_n(k) = \mathbb{P}\{N(n) > k \mid X_0 = n\}$. We then have the following result.

Theorem 5 *For every $k \geq 0$, we have $F_0(k) = F_1(k) = 0$. For $n \geq 2$ and $k = 0, \ldots, n-1$, we have $F_n(k) = 1$. For every $n \geq 2$ and $k \geq n$, we have*

$$F_n(k) = \frac{1}{2^{n-1}} F_n(k-n) + \frac{1}{2^n} \sum_{j=2}^{n-1} \binom{n}{j} F_j(k-n).$$

Proof. Since $N(0) = N(1) = 0$ we clearly have $F_0(k) = F_1(k) = 0$ for every $k \geq 0$. We consider now the case $n \geq 2$. Since $X_0 = n$, we have $N(n) > n-1$ with probability 1, which means that we have $F_n(k) = 1$, for $k = 0, \ldots, n-1$. For $k \geq n$, conditionning on the state reached after the first transition, we get

$$\mathbb{P}\{N(n) > k \mid X_0 = n\} = \sum_{j=1}^{n} \mathbb{P}\{N(n) > k \mid X_1 = j, X_0 = n\} P_{n,j}.$$

If $X_1 = 1$, we have, since state 1 is absorbing, $N(n) = n$ and thus $\mathbb{P}\{N(n) > k \mid X_1 = 1, X_0 = n\} = 0$. We thus obtain, using the Markov and the homogeneity properties,

$$
\begin{aligned}
\mathbb{P}\{N(n) > k \mid X_0 = n\} &= \sum_{j=2}^{n} \mathbb{P}\{N(n) > k \mid X_1 = j, X_0 = n\} P_{n,j} \\
&= \sum_{j=2}^{n} \mathbb{P}\{N(j) + n > k \mid X_1 = j, X_0 = n\} P_{n,j} \\
&= \sum_{j=2}^{n} \mathbb{P}\{N(j) > k - n \mid X_0 = j\} P_{n,j},
\end{aligned}
$$

which can be written as

$$F_n(k) = \sum_{j=2}^{n} F_j(k-n) P_{n,j}$$

or, by definition of $P_{n,j}$,

$$F_n(k) = \frac{1}{2^{n-1}} F_n(k-n) + \frac{1}{2^n} \sum_{j=2}^{n-1} \binom{n}{j} F_j(k-n).$$

This completes the proof. ∎

For $n \geq 2$, it is easily checked that $F_n(n) = 1 - n/2^n$.

Figure 3 shows the complementary distribution of the total contention for several values of n, the number of initially competing processes.

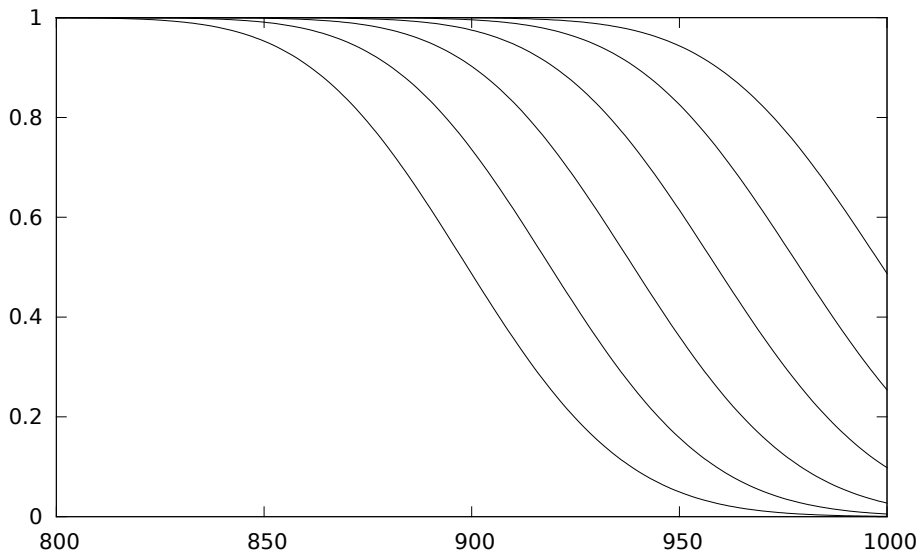Figure 3: From bottom to the top: $\mathbb{P}\{N(n) > k\}$ for $n = 450, 460, 470, 480, 490, 500$, as functions of k.

These results are quite interesting since, for instance, for $n = 470$ initially competing processes, the total contention is greater than 950 with probability 0.36102 whereas for $n = 480$ this probability becomes 0.61436 just by adding a very small number (10 over 470) of initially competing processes. This suggests further work on a deeper analysis of the distribution of the total contention for large n and k.

5   Conclusion

We have proved that the expected step complexity of our synchronization solution is logarithmic in the number of involved processes and we obtained the distribution of the number of

steps needed to solve the Test&Set problem. Moreover we have shown that the expected total contention experienced by all the n involved process is equal to $2n$ and we have obtained the distribution of the total contention. Further work on this subject concerns the asymptotic behaviour of the distributions of the number of steps needed to solve the Test&Set problem and of the total contention around their respective expected values.

A Proof of Theorem 1

For $n \geq 2$, we introduce the notation $u_n = \mathbb{E}\{T_n \mid X_0 = n\}$. Using this notation and the expression of the transition probability matrix P, Relation (1) can be written as

$$
\begin{cases}
u_2 & = \quad 2 \\
\\
u_n & = \quad \dfrac{2^{n-1}}{2^{n-1} - 1} \left(1 + \dfrac{1}{2^n} \sum_{k=2}^{n-1} \binom{n}{k} u_k \right), \quad \text{for } n \geq 3.
\end{cases}
\tag{3}
$$

We are interested in the behavior of this sequence when n tends to infinity. In order to get this behavior, we need the following lemma.

We first introduce the notation $\gamma_{n,k} = \dfrac{1}{2^n} \binom{n}{k}$.

Lemma 6 *For every $\alpha \in (0, 1/2)$, there exists an integer n_1 such that for every $n \geq n_1$ and for the integers $k \in E_n = [0, n/2 - n^{\alpha+1/2}) \cup (n/2 + n^{\alpha+1/2}, n]$, we have*

$$
\gamma_{n,k} \leq e\sqrt{2} \exp(-2n^{2\alpha}).
$$

Proof. Using the Stirling formula, we have for all $n \geq 1$,

$$
\sqrt{2\pi}\sqrt{n}n^n e^{-n} \leq n! \leq e\sqrt{n}n^n e^{-n}.
$$

We then have, for $1 \leq k \leq n-1$,

$$
\gamma_{n,k} = 2^{-n} \binom{n}{k} \leq e2^{-n} \frac{\sqrt{n}n^n e^{-n}}{\sqrt{k}k^k e^{-k}\sqrt{n-k}(n-k)^{(n-k)}e^{-(n-k)}} = e\delta_{n,k},
\tag{4}
$$

where

$$
\begin{aligned}
\delta_{n,k} &= \frac{\sqrt{n}}{\sqrt{k(n-k)}} 2^{-n} n^n k^{-k}(n-k)^{-(n-k)} \\
&= \frac{\sqrt{n}}{\sqrt{k(n-k)}} \exp\left(n\ln(n/2) - k\ln(k) - (n-k)\ln(n-k) \right).
\end{aligned}
$$

Now, for every $n \geq 3$, the two functions ϕ_n and ψ_n, defined by

$$
\begin{aligned}
x \in [1, n-1] &\longmapsto \phi_n(x) = \sqrt{x(n-x)}, \\
x \in [1, n-1] &\longmapsto \psi_n(x) = n\ln(n/2) - x\ln(x) - (n-x)\ln(n-x)
\end{aligned}
$$

both are increasing on $[1, n/2]$ and decreasing on $[n/2, n-1]$ (this is obvious for function ϕ_n, while the derivative of ψ_n with respect to x is $\psi_n'(x) = -\ln(x) + \ln(n - x) = -\ln(x/(n - x))$ which is ≥ 0 when $1 \leq x \leq n/2$, and ≤ 0 when $n/2 \leq x \leq n - 1$).

We now take a small α, $0 < \alpha < 1/2$, and we estimate $\gamma_{n,k}$ for the k's belonging to the set $E_n \cap \{1, \ldots, n-1\}$. Taking $0 < \alpha < 1/2$, there exists an integer n_2 such that for $n \geq n_2$, the two intervals forming E_n are non empty. For this α and $n \geq n_2$, we have for all $x \in E_n$,

$$\phi_n(x) \geq \sqrt{n-1} \text{ and } \psi_n(x) \leq \psi_n\left(n/2 - n^{\alpha+1/2}\right) = \psi_n\left(n/2 + n^{\alpha+1/2}\right),$$

with

$$\begin{aligned}
\psi_n\left(n/2 + n^{\alpha+1/2}\right) &= n \ln(n/2) - (n/2 - n^{\alpha+1/2}) \ln(n/2 - n^{\alpha+1/2}) \\
&\quad - (n/2 + n^{\alpha+1/2}) \ln(n/2 + n^{\alpha+1/2}) \\
&= -(n/2 - n^{\alpha+1/2}) \ln(1 - 2n^{\alpha-1/2}) \\
&\quad - (n/2 + n^{\alpha+1/2}) \ln(1 + 2n^{\alpha-1/2}).
\end{aligned}$$

Using the Taylor-Lagrange formula, we get

$$-\ln(1 + 2n^{\alpha-1/2}) = -2n^{\alpha-1/2} + K_1\left(2n^{\alpha-1/2}\right)^2 \text{ with } 0 \leq K_1 \leq 1/2 \text{ for } n \geq n_3.$$

$$-\ln(1 - 2n^{\alpha-1/2}) = 2n^{\alpha-1/2} + K_2\left(2n^{\alpha-1/2}\right)^2 \text{ with } 0 \leq K_2 \leq 1 \text{ for } n \geq n_4,$$

This leads, for $n \geq n_1 = \max(n_2, n_3, n_4)$, to

$$\begin{aligned}
\psi_n\left(n/2 + n^{\alpha+1/2}\right) &= (n/2 - n^{\alpha+1/2})\left(2n^{\alpha-1/2} + 4K_2 n^{2\alpha-1}\right) \\
&\quad - (n/2 + n^{\alpha+1/2})\left(2n^{\alpha-1/2} + 4K_1 n^{2\alpha-1}\right) \\
&= -4n^{2\alpha} + 2(K_2 - K_1)n^{2\alpha} - (K_2 + K_1)n^{3\alpha-1} \\
&\leq -4n^{2\alpha} + 2(K_2 - K_1)n^{2\alpha} \\
&\leq -2n^{2\alpha}.
\end{aligned}$$

We thus obtain from (4), for all $k \in E_n \cap \{1, \ldots, n-1\}$ and $n \geq n_1$,

$$\gamma_{n,k} \leq \frac{e\sqrt{n}\exp(\psi_n(k))}{\phi_n(k)} \leq e\frac{\sqrt{n}}{\sqrt{n-1}}\exp(-2n^{2\alpha}) \leq e\sqrt{2}\exp(-2n^{2\alpha}). \tag{5}$$

For $k = 0$ and $k = n$, we have $\gamma_{n,0} = \gamma_{n,n} \leq \gamma_{n,1}$, which completes the proof. $\blacksquare$

We are now ready to prove Theorem 1.

Using Lemma (6) and introducing the bound $M_n = \max_{2 \leq k \leq n} u_k$, we obtain from Relation (3) and for $\alpha \in (0, 1/2)$ and $n \geq n_1$,

$$u_n \leq \frac{1}{1 - 2^{-(n-1)}}\left(1 + e\sqrt{2}ne^{-2n^{2\alpha}}M_n + \sum_{k \in F_n \cap \mathbb{N}} \gamma_{n,k}u_k\right),$$

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where $F_n = [n/2 - n^{\alpha+1/2}, n/2 + n^{\alpha+1/2}]$.

Since $1/(1-x) \le 1 + 2x$, for $0 < x \le 1/2$ and $e\sqrt{2}(1 + 2^{-(n-2)})ne^{-n^{2\alpha}} \le 1$, for n large enough (*i.e.* $n \ge n_2$ for some n_2 whose precise value is irrelevant), we obtain for $n \ge \max(n_1, n_2)$,

$$u_n \le \left(1 + 2^{-(n-2)}\right)\left(1 + \sum_{k \in F_n \cap \mathbb{N}} \gamma_{n,k} u_k\right) + e^{-n^{\alpha}} M_n. \tag{6}$$

We introduce a dyadic partition of the indices n, and set, for any $j \ge 1$, the notation $U_j = M_{2^j}$, that is

$$U_j = \max_{2 \le k \le 2^j} u_k.$$

Now for every $j \ge 1$, we now estimate U_{j+1} as a function of U_j. To do so, we take n such that $2^j < n \le 2^{j+1}$ and we write

$$\sum_{k \in F_n \cap \mathbb{N}} \gamma_{n,k} u_k = \sum_{k \in F_n \cap \mathbb{N}, \, 0 \le k \le 2^j} \gamma_{n,k} u_k + \sum_{k \in F_n \cap \mathbb{N}, \, 2^j < k < n} \gamma_{n,k} u_k$$

$$\le U_j \left(\sum_{k=0}^{2^j} \gamma_{n,k}\right) + U_{j+1}\left(\sum_{k=2^j+1}^{n} \gamma_{n,k}\right)$$

$$= U_j s_{n,j} + U_{j+1}(1 - s_{n,j}), \tag{7}$$

where $s_{n,j}$ is defined by

$$s_{n,j} = \sum_{k=0}^{2^j} \gamma_{n,k},$$

and we have used the fact that $\sum_{k=0}^{n} \gamma_{n,k} = 1$. Note the obvious estimate $0 \le s_{n,j} \le 1$ and note also that, since $n \in (2^j, 2^{j+1}]$, we have

$$s_{n,j} = \sum_{k=0}^{2^j} \gamma_{n,k} \ge \sum_{0 \le k \le n/2} \gamma_{n,k},$$

while, since $\gamma_{n,k} = \gamma_{n,n-k}$,

$$1 = \sum_{k=0}^{n} \gamma_{n,k} = \sum_{0 \le k \le n/2} \gamma_{n,k} + \sum_{n/2 < k \le n} \gamma_{n,k}$$

$$= \sum_{0 \le k \le n/2} \gamma_{n,k} + \sum_{0 \le k < n/2} \gamma_{n,k} \le 2 \sum_{0 \le k \le n/2} \gamma_{n,k},$$

from which it comes

$$s_{n,j} \ge 1/2.$$

Relations (6) and (7) eventually provide

$$U_{j+1} \leq \left(1 + 2^{-(n-2)}\right)\left(1 + U_j s_{n,j} + U_{j+1}(1 - s_{n,j})\right) + e^{-n^\alpha} M_n$$
$$\leq \left(1 + 2^{-(n-2)}\right)\left(1 + \frac{U_j + U_{j+1}}{2}\right) + e^{-n^\alpha} M_n,$$

where we have used $s_{n,j} \geq 1/2$ together with $U_j \leq U_{j+1}$. In other words, and taking into account $n \in (2^j, 2^{j+1}]$, we have

$$U_{j+1} \leq \left(1 + 2^{-(2^j-2)}\right)\left(1 + \frac{U_j + U_{j+1}}{2}\right) + e^{-2^{j\alpha}} U_{j+1}.$$

Hence we arrive at

$$U_{j+1} \leq \frac{1 + 2^{-(2^j-2)}}{1 - 2^{-(2^j-2)} - 2e^{-2^{j\alpha}}} \left(2 + U_j\right). \tag{8}$$

We rewrite (8) in a more convenient form. To do so, we fix some β such that $0 < \beta < \alpha$ and we introduce

$$\beta_j = e^{-2^{j\beta}}.$$

It is clear that for j large enough, *i.e.* $j \geq j_1$ for some j_1, Relation (8) implies, for $j \geq j_1$, that

$$U_{j+1} \leq (1 + \beta_j)(2 + U_j). \tag{9}$$

We are now in position to conclude. Formula (9) implies that

$$U_{j+1} \leq 2\left[(1 + \beta_j) + (1 + \beta_j)(1 + \beta_{j-1}) + \cdots + \prod_{k=j_1}^{j}(1 + \beta_k)\right] + U_{j_1}\prod_{k=j_1}^{j}(1 + \beta_k).$$

Introducing the products P_j defined by

$$P_j = \prod_{k=j_1-1}^{j}(1 + \beta_k),$$

the above bound rewrites

$$U_{j+1} \leq 2\left[\frac{P_j}{P_{j-1}} + \frac{P_j}{P_{j-2}} + \cdots + \frac{P_j}{P_{j_1-1}}\right] + \frac{P_j}{P_{j_1-1}}U_{j_1}.$$

Hence, since the infinite product $\prod_{k>j_1}(1 + \beta_k)$ clearly converges, we have

$$\lim_{j \to \infty} P_j = P > 0$$

and we may write,

$$U_{j+1} \leq 2P_j \sum_{\ell=j_1-1}^{j-1} \frac{1}{P_\ell} + \frac{P_j}{P_{j_1-1}}U_{j_1}.$$

We denote by $\widetilde{U}_j$ the right hand side of this last inequality. It is clear, using a standard fact about Cesàro means, that

$$P_j \sum_{\ell=j_1-1}^{j-1} \frac{1}{P_\ell} \underset{j\to\infty}{\sim} P \sum_{\ell=j_1-1}^{j-1} \frac{1}{P} \underset{j\to\infty}{\sim} j \text{ and } \frac{P_j}{P_{j_1-1}} U_{j_1} \underset{j\to\infty}{\sim} \frac{P}{P_{j_1-1}} U_{j_1}.$$

We deduce that

$$U_{j+1} \leq \widetilde{U}_j \text{ with } \widetilde{U}_j \underset{j\to\infty}{\sim} 2j.$$

Defining $\widetilde{u}_n = \widetilde{U}_j$, for $2^j < n \leq 2^{j+1}$ with $n \geq \max(n_1, n_2)$ and $j \geq j_1$, we get

$$\frac{\widetilde{U}_j}{2(j+1)} \leq \frac{\widetilde{U}_j}{2\log_2(n)} = \frac{\widetilde{u}_n}{2\log_2(n)} < \frac{\widetilde{U}_j}{2j}.$$

We deduce that

$$u_n \leq U_{j+1} \leq \widetilde{U}_j = \widetilde{u}_n \text{ with } \widetilde{u}_n \underset{n\to\infty}{\sim} 2\log_2(n).$$

This proves the second inequality.

Symmetrically, starting from (3), we write

$$u_n \geq 1 + \sum_{k\in F_n\cap\mathbb{N}} \gamma_{n,k} u_k.$$

We introduce the following quantities, for $j \geq 2$,

$$V_j = \min_{2^{j-1}<k\leq 2^j} u_k \text{ and } W_j = \min(V_j, V_{j-1}).$$

For any n such that $2^j < n \leq 2^{j+1}$, we observe that the set F_n satisfies $F_n \subset [2^{j-2}, 2^{j+1}]$, for j large enough. Besides, whenever n is such that $2^j < n \leq 2^{j+1} - 2^{(j+1)(\alpha+1/2)+1}$, we have $F_n \subset [2^{j-2}, 2^j]$. Indeed, for such n, we have

$$\frac{n}{2} + n^{\alpha+1/2} = \frac{n}{2}(1 + 2n^{\alpha-1/2}) \leq \left(2^j - 2^{(j+1)(\alpha+1/2)}\right)\left(1 + 2^{(j+1)(\alpha-1/2)+1}\right)$$
$$= 2^j\left(1 - 2^{(j+1)(\alpha-1/2)+1}\right)\left(1 + 2^{(j+1)(\alpha-1/2)+1}\right)$$
$$\leq 2^j.$$

Hence, for n such that $2^j < n \leq 2^{j+1} - 2^{(j+1)(\alpha+1/2)+1}$, we may write using (5)

$$u_n \geq 1 + \sum_{k\in F_n\cap\mathbb{N}} \gamma_{n,k} u_k \geq 1 + W_j \sum_{k\in F_n\cap\mathbb{N}} \gamma_{n,k} = 1 + W_j\left(1 - \sum_{k\in E_n\cap\mathbb{N}} \gamma_{n,k}\right).$$

Using Lemma 6, we get

$$u_n \geq 1 + W_j(1 - e\sqrt{2}(n+1)e^{-2n^{2\alpha}}) \geq 1 + W_j(1 - e^{-2^{\alpha j}}), \text{ for } j \text{ large enough }.$$

In particular, since $W_j \leq U_j \leq \widetilde{U}_j \sim 2j$, we deduce that for j large enough we also have $u_n \geq W_j$ under these circumstances. Indeed, for all $\varepsilon > 0$, we have $W_j \leq (2 + \varepsilon)j$, which gives, for j large enough,

$$u_n \geq 1 + W_j(1 - e^{-2^{\alpha j}}) = W_j + (1 - W_j e^{-2^{\alpha j}}) \geq W_j + 1 - (2 + \varepsilon)j e^{-2^{\alpha j}} \geq W_j.$$

For the remaining n, which belong to $(2^{j+1} - 2^{(j+1)(\alpha+1/2)+1}, 2^{j+1}]$, we observe that $F_n \subset [2^{j-1}, 2^{j+1} - 2^{(j+1)(\alpha+1/2)+1}]$. Indeed, for such n, we have

$$
\begin{aligned}
\frac{n}{2} + n^{\alpha+1/2} &\leq 2^j + 2^{(j+1)(\alpha+1/2)} \\
&= 2^{j+1} - 2^{(j+1)(\alpha+1/2)+1} + \underbrace{\left(2^j - 2^{j+1} + 3 \times 2^{(j+1)(\alpha+1/2)}\right)}_{\leq 0 \text{ for } j \text{ large enough}},
\end{aligned}
$$

together with

$$\frac{n}{2} - n^{\alpha+1/2} \geq 2^j - 2^{(j+1)(\alpha+1/2)} - 2^{(j+1)(\alpha+1/2)} \geq 2^{j-1}, \text{ for } j \text{ large enough }.$$

We write, as a consequence, using the now proved fact that $u_k \geq W_j$ whenever $k \in (2^{j-1}, 2^{j+1} - 2^{(j+1)(\alpha+1/2+)+1}]$,

$$u_n \geq 1 + \sum_{k \in F_n \cap \mathbb{N}} \gamma_{n,k} u_k \geq 1 + W_j \sum_{k \in F_n \cap \mathbb{N}} \gamma_{n,k} \geq 1 + W_j(1 - e^{-2^{\alpha j}}).$$

Eventually we have proved that, for $2^j < n \leq 2^{j+1}$,

$$V_{j+1} \geq 1 + W_j(1 - e^{-2^{\alpha j}}) \geq W_j.$$

Re-iterating the argument establishes as well, using $V_{j+1} \geq W_j$ and $V_j \geq W_j$, which implies that $W_{j+1} \geq W_j$,

$$V_{j+2} \geq 1 + W_{j+1}(1 - e^{-2^{\alpha(j+1)}}) \geq 1 + W_j(1 - e^{-2^{\alpha j}}) \geq W_j.$$

As a consequence, we have

$$W_{j+2} \geq 1 + W_j(1 - e^{-2^{\alpha j}}).$$

Introducing the notation $\alpha_j = e^{-2^{\alpha j}}$, we infer for j large enough, *i.e.* $j \geq j_2$ for some j_2,

$$
\begin{aligned}
W_j &\geq 1 + (1 - \alpha_{j-2})W_{j-2} \\
&\geq 1 + (1 - \alpha_{j-2}) + (1 - \alpha_{j-2})(1 - \alpha_{j-4})W_{j-4} \geq \cdots \\
&\geq 1 + (1 - \alpha_{j-2}) + (1 - \alpha_{j-2})(1 - \alpha_{j-4}) + \cdots + \prod_{k=0}^{\lfloor (j-j_2)/2 \rfloor - 2} (1 - \alpha_{j-2-2k}) \\
&\quad + W_{j'} \prod_{k=0}^{\lfloor (j-j_2)/2 \rfloor - 1} (1 - \alpha_{j-2-2k}).
\end{aligned}
$$

Introducing the products P_j defined by

$$P_j = \prod_{k=0}^{\lfloor (j-j_2)/2 \rfloor} (1 - \alpha_{j-2-2k}),$$

the above bound rewrites, using the notation $j' = j - 2\lfloor (j - j_2)/2 \rfloor$,

$$W_j \geq \left[\frac{P_j}{P_j} + \frac{P_j}{P_{j-2}} + \frac{P_j}{P_{j-4}} + \cdots + \frac{P_j}{P_{j'+2}} \right] + \frac{P_j}{P_{j'}} W_{j'}.$$

Hence, since the infinite product $\displaystyle\prod_{k \geq \lfloor (j_2-2)/2 \rfloor} (1 - \alpha_{2k})$ clearly converges, we have

$$\lim_{j \to \infty} P_j = P > 0$$

and we may write,

$$W_j \geq P_j \sum_{\ell=0}^{\lfloor (j-j_2)/2 \rfloor} \frac{1}{P_{j'+2\ell}} + \frac{P_j}{P_{j'}} W_{j'}.$$

We denote by $\widetilde{W}_j$ the right hand side of this last inequality. It is clear, using a standard fact about Cesàro means, that

$$P_j \sum_{\ell=0}^{\lfloor (j-j_2)/2 \rfloor} \frac{1}{P_{j'+2\ell}} \underset{j \to \infty}{\sim} P \sum_{\ell=0}^{\lfloor (j-j_2)/2 \rfloor} \frac{1}{P} \underset{j \to \infty}{\sim} \frac{j}{2} \text{ and } \frac{P_j}{P_{j'}} W_{j'} \underset{j \to \infty}{\sim} \frac{P}{P_{j_2}} W_{j_2}.$$

We deduce that

$$W_j \geq \widetilde{W}_j \text{ with } \widetilde{W}_j \underset{j \to \infty}{\sim} \frac{j}{2}.$$

Defining $\widetilde{\widetilde{u}}_n = \widetilde{W}_j$, for $2^j < n \leq 2^{j+1}$ with n and j large enough, we get

$$\frac{\widetilde{W}_j}{j/2} > \frac{\widetilde{W}_j}{\log_2(n)/2} = \frac{\widetilde{\widetilde{u}}_n}{2\log_2(n)} \geq \frac{\widetilde{W}_j}{(j+1)/2}.$$

We deduce that

$$u_n \geq W_j \geq \widetilde{W}_j = \widetilde{\widetilde{u}}_n \text{ with } \widetilde{u}_n \underset{n \to \infty}{\sim} \frac{\log_2(n)}{2}.$$

This proves the first inequality and completes the proof of Theorem 1.

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