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STOCHASTIC ANALYSIS OF AVERAGE BASED DISTRIBUTED ALGORITHMS

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Abstract

We analyse average-based distributed algorithms relying on simple and pairwise random interactions among a large and unknown number of anonymous agents. This allows the characterization of global properties emerging from these local interactions. Agents start with an initial integer value, and at each interaction keep the average integer part of both values as their new value. The convergence occurs when, with high probability, all the agents possess the same value which means that they all know a property of the global system. Using a well chosen stochastic coupling, we improve upon existing results by providing explicit and tight bounds of the convergence time. We apply these general results to both the proportion problem and the system size problem.

Keywords: Averaging stochastic process; Interacting particle systems; Markov chain; Coupling; Distributed algorithms

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1. Introduction

This paper focuses on the deep analysis of a particular type of averaging stochastic processes. In the averaging process, as introduced in [1], the n agents start independently from each other with an initial integer value, interact randomly by pairs,

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and at each interaction, keep the average of both values as their new value. This type of processes belongs to the general category of stochastic interacting particle systems [9], that are applied in many fields (biology, computer science, physics, etc) to characterize global properties emerging from local interactions. For instance, in [12], the authors used this model to analyze the rumor spreading time which is the number of interactions needed for all the agents of the network to learn a rumor initially known by only one agent. In [7], the authors analyze biological agents (gene or infectious disease) spreading (mean spreading time and stable gene distribution) for diverse type of networks and considering pairwise interactions. In [8], the author considers a voter model variant in which agents have preferences over a set of songs and upon meeting update their own preferences incrementally towards those of the other agents they meet. This is a continuous time model, in which a Poisson process is associated between every pair of agents and whose times correspond to their meetings. Using the spectral gap of an associated Markov chain, the author gives a geometry dependent result on the asymptotic consensus time of the model. A similar continuous model, the compulsive gambler process, is studied in [2] where agents meet pairwise at random times according to Poisson processes and, upon meeting, play an instantaneous fair game in which one wins the other's money. This process behaves like a reversal of our process since interactions between agents end up with their values as different as possible, rather than as equal as possible.

In our context of large-scale distributed algorithms, where interaction-based algorithms are represented by the population protocol model, agents have little computational power, limited size memory, are indistinguishable from one another and are unaware of the population size n of the system [4]. It follows, in particular, that each agent can only be in a finite number of states. The key is to propose efficient algorithms that make agents cooperate to perform computational tasks such as determining the proportion of agents that started their computation with a given integer value, or computing the population size. Both problems (proportion of agents that start with some initial value A and population size) can be solved by relying on average-based population protocols. In a previous work [11], we analyzed the convergence time (also called the mixing time) at which all the n agents of the distributed system are able to determine the proportion of them that started with value A. This work has been used in [6], where the authors tackle a consensus problem derived from the proportion one.

We introduce the discrete-time stochastic process $C := \{C_t,\ t \geq 0\}$, where the random vector C_t is defined by $C_t := (C_t^{(1)}, \ldots, C_t^{(n)})$, to represent the evolution of the agent values. For all agents $i = 1, \ldots, n$, $C_t^{(i)}$ represents its value at time $t \geq 0$. Since each agent only uses a finite number of states, we assume that the $C_t^{(i)}$ are integers, for all t and all i. The average based technique then results that, when two agents interact, one gets the floor of their mean value and the other the ceiling of this value, as formally defined in Relation (1). It follows that process C does not converge to a unique absorbing state, as in [1, 10] when dealing with real numbers, but to an absorbing class of states for which each entry (or each agent value) is either equal to $\lfloor \ell \rfloor$ or equal to $\lceil \ell \rceil$, where $\ell := \sum_{i=1}^{n} C_0^{(i)}/n$ is the initial average value. This class has radius less than 1 in the infinity norm. A similar absorbing behavior is described in [5], where the authors consider a random graph representing the evolution of relationships in a fixed size population. At each instant, a link between two individuals is either created, removed or unchanged, at random. They show, using Markov chains, that the population evolves to a closed class and they give a method for finding the stationary distribution over this class.

The discrepancy of the system, that is the difference between the maximum and minimum value among all nodes, defined by $\max_{1 \leq i \leq n} C_t^{(i)} - \min_{1 \leq i \leq n} C_t^{(i)}$, has been studied in [13], where the authors show that its expectation is in $O(1)$ for regular graphs. Moreover they note that in many applications, agent values cannot be divided arbitrarily, and we need to deal with the discrete case where the values of each node can only be integers. This discretization entails a non-linear behavior due to its rounding errors, which makes this analysis much harder than in the continuous case. In [3] the authors propose a two phases algorithm called average-and-conquer to solve the majority problem in population protocols. The first phase of their algorithm is based on the average of the agents' values and a second phase called conquer is needed for propagating the majority value to all the agents of the networks. It is important to be aware of the fact that, as for most of the papers on the subject (e.g. [1, 3, 6, 13]), the complexities are always of the $O(n \log n)$ type, without any study of the constants arising in these complexities. These analyses are interesting but not sufficiently precise because if the constants occurring in a $O(n \log n)$ complexity are large, they totally annihilate the effect of the logarithm. Indeed, in practice the number n of agents involved in large scale systems is always bounded. For instance, in the IoT (Internet of Things) infrastructure the numbers of nodes never exceeds 10^9 nodes or agents, which logarithm is about 20.72. That is why we focus in this work on precise values of the complexities, i.e. of the type $n(a \ln(n) + b)$, where a and b are constants. This is the main objective of this paper which necessitates a quite detailed mathematical analysis of the behavior of the system.

We provide in this paper a rigorous theoretical analysis of the behavior of average based distributed algorithms. The main contributions of this paper are the following.

- Theorem 2 shows that if the mean initial value, ℓ, is as near as possible to a half-integer then the process converges in linear time to a position where only the two integers closest to ℓ appear.
- Theorem 3 shows that for arbitrary ℓ the process converges in linearithmic time to a position where only the *three* integers closest to ℓ appear. More precisely, it is proved in Theorem 3 that for all $\delta \in (0,1)$, and for all $t \geq (n-1)\left(2\ln\left(K+\sqrt{n}\right) - \ln\delta - \ln 2\right)$, we have

$$\mathbb{P}\left\{\|C_t - L\|_\infty \geq 3/2\right\} \leq \delta,$$

  where K is a constant depending only on the initial vector condition C_0. The proof of this result is based on a quite novel coupling technique for which the coupling process is called the shadow process of C.

  Note that this result is best possible, since for example if the starting configuration has one agent of value 2, two of value 0 and $n-3$ of value 1, then it takes quadratic time to stabilise on two values (since this will only happen when the values of the selected agents are 0 and 2, which has probability $4/(n(n-1))$ to occur at each step). Using this result, it is shown in Corollary 1 that the discrepancy of the system is equal to 0 or 1 with arbitrarily high probability, i.e. that

$$\mathbb{P}\left\{\max_{1\leq i\leq n} C_t^{(i)} - \min_{1\leq i\leq n} C_t^{(i)} > 2\right\} \leq \delta.$$

- First, we apply our result to the proportion problem (see Theorem 4), and show that it significantly improves the one obtained in [11] (see Figure 1). We also apply our result to another problem, the system size problem (see Lemma 7).

The remaining of the paper is orchestrated as follows. Section 2 presents the mathematical model which is based on random interactions between the agents. Section 3 details the analysis of the convergence. The main contribution is the use of the shadow process, a novel stochastic coupling technique. In Section 4, we apply our results to both the proportion and the system size problems. Section 5 concludes the paper.

2. The model

We denote by X_t the random pair of distinct nodes chosen at time t to interact and for every $i, j = 1, \dots, n$, with $i \neq j$, we define

$$p_{i,j}(t) = \mathbb{P}\{X_t = (i, j)\}.$$

The time unit is discrete and corresponds to a single interaction. At each discrete instant t, two distinct indices i and j are chosen among $1, \dots, n$ with probability $p_{i,j}(t)$. Once chosen, the pair of agents (i, j) interacts, and both agents update their respective local value $C_t^{(i)}$ and $C_t^{(j)}$ by taking the mean value of their values prior to this interaction. This average-based technique leads to

$$\left(C_{t+1}^{(i)}, C_{t+1}^{(j)}\right) = \left(\left\lfloor \frac{C_t^{(i)} + C_t^{(j)}}{2} \right\rfloor, \left\lceil \frac{C_t^{(i)} + C_t^{(j)}}{2} \right\rceil\right) \text{ and } C_{t+1}^{(r)} = C_t^{(r)} \text{ for } r \neq i, j. \quad (1)$$

We suppose that the sequence $\{X_t,\ t \geq 0\}$ is a sequence of independent and identically distributed random variables. Since C_t is entirely determined by the values of $C_0, X_0, X_1, \dots, X_{t-1}$, this means in particular that the random variables X_t and C_t are independent and that the stochastic process $C = \{C_t,\ t \geq 0\}$ is a discrete-time homogeneous Markov chain. Classically, we suppose that X_t is uniformly distributed, that is,

$$p_{i,j}(t) = \frac{1_{\{i \neq j\}}}{n(n-1)},$$

where 1_A denotes the indicator function, which is equal to 1 if condition A is true and 0 otherwise.

3. Convergence time of average-based algorithms

We will use the Euclidean norm denoted simply by $\|.\|$ and the infinity norm denoted by $\|.\|_\infty$ and defined for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ by

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \text{ and } \|x\|_\infty = \max_{i=1,\dots,n} |x_i|.$$

We recall the following invariant result of average-based population protocols.

Lemma 1. *For every $t \geq 0$, we have*

$$\sum_{i=1}^{n} C_t^{(i)} = \sum_{i=1}^{n} C_0^{(i)}.$$

Proof. For all integers k, we have $k = \lfloor k/2 \rfloor + \lceil k/2 \rceil$, so the transformation from C_t to C_{t+1} described in Relation (1) does not change the sum of the entries of C_{t+1}. □

We denote by ℓ the mean value of the entries of C_t and by L the row vector of $\mathbb{R}^n$ with all its entries equal to ℓ, that is

$$\ell := \frac{1}{n} \sum_{i=1}^{n} C_t^{(i)} \text{ and } L := (\ell, \dots, \ell). \tag{2}$$

Remark that C has a finite value space composed of a set of transient vectors and an absorbing class of vectors whose entries are equal to $\lfloor \ell \rfloor$ or $\lceil \ell \rceil$. This absorbing class is reduced to a single absorbing vector when ℓ is an integer.

We first bound the decay of the expected value $\mathbb{E}\left(\|C_t - L\|^2\right)$.

Theorem 1. *For every $t \geq 0$, we have*

$$\mathbb{E}\left(\|C_t - L\|^2\right) \leq \left(1 - \frac{1}{n-1}\right)^t \mathbb{E}\left(\|C_0 - L\|^2\right) + \frac{n}{4} - \frac{1_{\{n \text{ odd}\}}}{4n}. \tag{3}$$

Proof. In order to simplify the writing we use the notation $Y_t := \|C_t - L\|^2$. One can deduce from Relation (1) that

$$Y_{t+1} = Y_t - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\left(C_t^{(i)} - C_t^{(j)}\right)^2 - 1_{\{C_t^{(i)} + C_t^{(j)} \text{ odd}\}}\right] 1_{\{X_t = (i,j)\}}.$$

We recall that X_t and C_t are independent and that $p_{i,j}(t) = 1/(n(n-1))$. Conditioning first by C_t, then taking the expectations, we get

$$\begin{aligned}\mathbb{E}\left(Y_{t+1}|C_t\right) &= Y_t - \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(C_t^{(i)} - C_t^{(j)}\right)^2 - 1_{\{C_t^{(i)}+C_t^{(j)} \text{ odd}\}}\right]\right) p_{i,j}(t) \\ &= Y_t - \frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(C_t^{(i)} - C_t^{(j)}\right)^2 - 1_{\{C_t^{(i)}+C_t^{(j)} \text{ odd}\}}\right].\end{aligned}$$

Using that (see [10])

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(C_t^{(i)} - C_t^{(j)}\right)^2 = 2nY_t \quad \text{and} \quad \sum_{i=1}^{n}\sum_{j=1}^{n} 1_{\{C_t^{(i)}+C_t^{(j)} \text{ odd}\}} = 2q_t(n-q_t),$$

where integer q_t is the number of odd entries of vector C_t, we deduce that

$$\mathbb{E}\left(Y_{t+1} \mid C_t\right) = \left(1 - \frac{1}{n-1}\right) Y_t + \frac{q_t(n-q_t)}{n(n-1)}. \tag{4}$$

Since $q_t \in \{0, 1, \ldots, n\}$, the function g defined, for $x \in [0, n]$, by $g(x) = x(n-x)$ has its maximum at point $x = n/2$, so we have $0 \leq g(x) \leq n^2/4$. Thus $g(q_t) = q_t(n-q_t) \leq n^2/4$. If n is even then q_t can be equal to $n/2$ which means that the best upper bound of $g(q_t)$ is $n^2/4$. If n is odd then q_t cannot be equal to $n/2$. The maximum of $g_t(q_t)$ is then reached either at point $q_t = (n-1)/2$ or at point $q_t = (n+1)/2$. For both points, we have $g(q_t) \leq (n-1)(n+1)/4 = n^2/4 - 1/4$, so the best upper bound of $g(q_t)$ is $n^2/4 - 1/4$. Putting together the two cases, we obtain

$$q_t(n-q_t) \leq \frac{n^2}{4} - \frac{1_{\{n \text{ odd}\}}}{4}.$$

Using this inequality in Relation (4), we get

$$\mathbb{E}\left(Y_{t+1} \mid C_t\right) \leq \left(1 - \frac{1}{n-1}\right) Y_t + \frac{n}{4(n-1)} - \frac{1_{\{n \text{ odd}\}}}{4n(n-1)}.$$

Taking the expectation in both sides, we obtain

$$\mathbb{E}\left(Y_{t+1}\right) \leq \left(1 - \frac{1}{n-1}\right) \mathbb{E}\left(Y_t\right) + \frac{n}{4(n-1)} - \frac{1_{\{n \text{ odd}\}}}{4n(n-1)}.$$

Solving this recurrence leads to Relation (3). □

3.1. A first bound on the convergence time

We introduce λ, the distance between ℓ and its nearest integer, that is

$$\lambda := \min\left\{\ell - \lfloor \ell \rfloor, \lceil \ell \rceil - \ell\right\} = \min\left\{\ell - \lfloor \ell \rfloor, 1 - (\ell - \lfloor \ell \rfloor)\right\}.$$

It is easily checked that we have $0 \le \lambda \le 1/2$. In Theorem 4 of [11], we dealt with the case where λ is equal to 1/2. In the following, we extend that analysis first to the case where $\lambda = \left(n - 1_{\{n \text{ odd}\}}\right)/(2n)$ (see Theorem 2) and then, to all $\lambda \in [0, 1/2]$ (see Section 3.2). We start by the following two lemmas.

Lemma 2. *Let* $h = \lfloor \ell \rfloor + 1/2$ *and* $H = (h, h, \dots, h) \in \mathbb{R}^n$. *If* $\lambda = \left(n - 1_{\{n\ odd\}}\right)/(2n)$, *then*

$$\|C_t - L\|^2 = \|C_t - H\|^2 - \frac{1_{\{n\ odd\}}}{4n} \ge \frac{n}{4} - \frac{1_{\{n\ odd\}}}{4n}. \tag{5}$$

Proof. Vector $C_t - L$ is orthogonal to vector e, with all entries equal to 1. Indeed,

$$\langle C_t - L, e \rangle = \sum_{i=1}^{n} \left(C_t^{(i)} - \ell \right) = n\ell - n\ell = 0.$$

Hence, since $L - H = (\ell - h)e$, we deduce that $C_t - L$ and $L - H$ are orthogonal too. Applying Pythagoras' Theorem, we obtain

$$\|C_t - L\|^2 = \|C_t - H\|^2 - \|L - H\|^2. \tag{6}$$

We moreover have $\|L - H\|^2 = n(\ell - h)^2 = n\left(1/2 - (\ell - \lfloor \ell \rfloor)\right)^2$. By definition of λ and since $\lambda = (n - 1_{\{n \text{ odd}\}})/(2n)$, we have either $\ell - \lfloor \ell \rfloor = (n - 1_{\{n \text{ odd}\}})/2n$ or $\ell - \lfloor \ell \rfloor = (n + 1_{\{n \text{ odd}\}})/2n$. In both cases, we get

$$\|L - H\|^2 = \frac{1_{\{n \text{ odd}\}}}{4n}. \tag{7}$$

Observe that

$$\|C_t - H\|^2 \ge n \min_{1 \le i \le n} \left| C_t^{(i)} - (\lfloor \ell \rfloor + 1/2) \right|^2 \ge n|1/2|^2 = n/4. \tag{8}$$

Injecting (7) in Relation (6), and applying Inequality (8), we get Inequality (5). □

Lemma 3. *If* $\lambda = \left(n - 1_{\{n\ odd\}}\right)/(2n)$ *then we have*

$$\max_{1 \le i \le n} C_t^{(i)} - \min_{1 \le i \le n} C_t^{(i)} > 1 \Longleftrightarrow \|C_t - L\|_\infty > 1 - \lambda = \frac{n + 1_{\{n\ odd\}}}{2n}. \tag{9}$$

Proof. Since $\lambda > 0$, ℓ is not an integer, so $\max_{1 \le i \le n} C_t^{(i)} \ge \lceil \ell \rceil$ and $\min_{1 \le i \le n} C_t^{(i)} \le \lfloor \ell \rfloor$, which implies that $\|C_t - L\|_\infty \ge 1 - \lambda$. Thus $\max_{1 \le i \le n} C_t^{(i)} - \min_{1 \le i \le n} C_t^{(i)} = 1$ implies that $\max_{1 \le i \le n} C_t^{(i)} = \lceil \ell \rceil$ and $\min_{1 \le i \le n} C_t^{(i)} = \lfloor \ell \rfloor$, which means that $\|C_t - L\|_\infty = 1 - \lambda$.

Conversely, if $\|C_t - L\|_\infty = 1 - \lambda$ then we have $\ell - 1 < C_t^{(i)} < \ell + 1$, which means that $\max_{1 \le i \le n} C_t^{(i)} = \lceil \ell \rceil$ and $\min_{1 \le i \le n} C_t^{(i)} = \lfloor \ell \rfloor$, that is $\max_{1 \le i \le n} C_t^{(i)} - \min_{1 \le i \le n} C_t^{(i)} = 1$. □

We can now prove the following theorem.

Theorem 2. *For all* $\delta \in (0,1)$*, if* $\lambda = \left(n - 1_{\{n \text{ odd}\}}\right)/(2n)$ *and if there exists a constant* K *such that* $\|C_0 - L\| \leq K$ *then, for every* $t \geq (n-1)(2\ln K - \ln\delta - \ln 2)$*, we have*

$$\mathbb{P}\left\{\|C_t - L\|_\infty > \frac{n + 1_{\{n \text{ odd}\}}}{2n}\right\} = \mathbb{P}\left\{\max_{1\leq i\leq n} C_t^{(i)} - \min_{1\leq i\leq n} C_t^{(i)} > 1\right\} \leq \delta, \tag{10}$$

or equivalently,

$$\mathbb{P}\left\{\max_{1\leq i\leq n} C_t^{(i)} = \min_{1\leq i\leq n} C_t^{(i)} + 1\right\} \geq 1 - \delta. \tag{11}$$

Proof. We first show that

$$\max_{1\leq i\leq n} C_t^{(i)} - \min_{1\leq i\leq n} C_t^{(i)} > 1 \Longrightarrow \|C_t - L\|^2 \geq \frac{n}{4} - \frac{1_{\{n \text{ odd}\}}}{4n} + 2. \tag{12}$$

Let $h = \lfloor \ell \rfloor + 1/2$ and $H = (h, h, \ldots, h) \in \mathbb{R}^n$. In the same way, if $\max_{1\leq i\leq n} C_t^{(i)} - \min_{1\leq i\leq n} C_t^{(i)} > 1$, then there exists an agent i such that $|C_t^{(i)} - h| \geq 3/2$, and for all $j \in \{1, 2, \ldots, n\} \setminus \{i\}$, $|C_t^{(j)} - h| \geq 1/2$. We can thus write

$$\max_{1\leq i\leq n} C_t^{(i)} - \min_{1\leq i\leq n} C_t^{(i)} > 1 \implies \|C_t - H\|^2 \geq \frac{n-1}{4} + \left(\frac{3}{2}\right)^2 = \frac{n}{4} + 2.$$

Applying Lemma 2, we thus obtain Relation (12), and deduce that

$$\mathbb{P}\left\{\max_{1\leq i\leq n} C_t^{(i)} - \min_{1\leq i\leq n} C_t^{(i)} > 1\right\} \leq \mathbb{P}\left\{\|C_t - L\|^2 \geq \frac{n}{4} - \frac{1_{\{n \text{ odd}\}}}{4n} + 2\right\}. \tag{13}$$

Then, from Relation (3) of Theorem 1, we obtain

$$\mathbb{E}\left(\|C_t - L\|^2 - \frac{n}{4} + \frac{1_{\{n \text{ odd}\}}}{4n}\right) \leq \left(1 - \frac{1}{n-1}\right)^t \mathbb{E}(\|C_0 - L\|^2).$$

Let $\tau = (n-1)(2\ln K - \ln\delta - \ln 2)$. For $t \geq \tau$, we have

$$\left(1 - \frac{1}{n-1}\right)^t \leq e^{-t/(n-1)} \leq e^{-\tau/(n-1)} = \frac{2\delta}{K^2}.$$

Moreover, since $\|C_0 - L\| \leq K$, we get $\mathbb{E}(\|C_0 - L\|^2) \leq K^2$ and thus

$$\mathbb{E}\left(\|C_t - L\|^2 - \frac{n}{4} + \frac{1_{\{n \text{ odd}\}}}{4n}\right) \leq 2\delta.$$

Using the Markov inequality (Lemma 2 ensures that we take the expectation of a non negative random variable), we obtain for $t \geq \tau$,

$$\mathbb{P}\left\{\|C_t - L\|^2 - \frac{n}{4} + \frac{1_{\{n \text{ odd}\}}}{4n} \geq 2\right\} \leq \delta.$$

Hence, we deduce from Relation (13) that, for $t \geq \tau$,

$$\mathbb{P}\left\{\max_{1\leq i\leq n} C_t^{(i)} - \min_{1\leq i\leq n} C_t^{(i)} > 1\right\} \leq \delta.$$

Remark that $\max_{1\leq i\leq n} C_t^{(i)}$ cannot be equal to $\min_{1\leq i\leq n} C_t^{(i)}$ here. Indeed, if so, then vector C_t is equal to vector L, implying that ℓ is an integer. In such a case we have $\lambda = 0$, which is impossible since $n \geq 2$. Hence,

$$\mathbb{P}\left\{\max_{1\leq i\leq n} C_t^{(i)} - \min_{1\leq i\leq n} C_t^{(i)} \leq 1\right\} = \mathbb{P}\left\{\max_{1\leq i\leq n} C_t^{(i)} = \min_{1\leq i\leq n} C_t^{(i)} + 1\right\}$$

and we directly obtain Relation (11). Finally, applying Lemma 3, we deduce that

$$\mathbb{P}\left\{\max_{1\leq i\leq n} C_t^{(i)} - \min_{1\leq i\leq n} C_t^{(i)} > 1\right\} = \mathbb{P}\left\{\|C_t - L\|_\infty > \frac{n + 1_{\{n \text{ odd}\}}}{2n}\right\},$$

which ends the proof. □

Note that $\max_{1\leq i\leq n} C_t^{(i)} = \min_{1\leq i\leq n} C_t^{(i)} + 1$ implies that $\min_{1\leq i\leq n} C_t^{(i)} = \lfloor \ell \rfloor$ and $\max_{1\leq i\leq n} C_t^{(i)} = \lceil \ell \rceil$. Hence, Theorem 2 assures us that if $\lambda = \left(n - 1_{\{n \text{ odd}\}}\right)/(2n)$, then the protocol converges after at least $(n-1)\left(2\ln K - \ln\delta - \ln 2\right)$ interactions, with arbitrarily high probability $1 - \delta$ towards a class of absorbing states which are vectors with entries equal to either $\lfloor \ell \rfloor$ or $\lceil \ell \rceil$.

3.2. The shadow process and the main result

The goal of this section is to obtain a result identical to the one of Theorem 2, but without any assumption on λ. This is done by using a stochastic coupling technique in which the process coupled with process C is called the shadow process of C.

The shadow process associated with process C is denoted by $D := \{D_t,\ t \geq 0\}$ and defined at time $t = 0$ by $D_0^{(i)} = C_0^{(i)} + 1_{\{i\in B_0\}}$, where B_0 is any fixed non empty subset of b agents with $b \leq n - 1$, i.e. $B_0 \subset \{1, \ldots, n\}$ and $|B_0| = b$.

For every $t \geq 1$, the random vector D_t is defined as C_t, that is, when the couple (i, j) is chosen to interact at time t, i.e. when $X_t = (i, j)$, the vector D_{t+1} is given by

$$\left(D_{t+1}^{(i)}, D_{t+1}^{(j)}\right) = \left(\left\lfloor \frac{D_t^{(i)} + D_t^{(j)}}{2} \right\rfloor, \left\lceil \frac{D_t^{(i)} + D_t^{(j)}}{2} \right\rceil\right) \quad \text{and} \quad D_{t+1}^{(r)} = D_t^{(r)} \text{ for } r \neq i, j.$$

In other words, processes C_t and D_t are coupled by process X_t: they behave identically in the sense that at each time, the same two agents are chosen for the interaction. The only difference lies in their initial values. Lemma 4 shows that, if at time $t = 0$, D_0 is initially in the shadow of C_0 then at any time $t \geq 0$, D_t remains in the shadow of C_t.

Lemma 4. *For all $t \geq 0$, there exists a non empty set B_t of b agents, i.e. $B_t \subset \{1, \ldots, n\}$ and $|B_t| = b$, such that for all $i \in \{1, 2, \ldots, n\}$, we have*

$$D_t^{(i)} = C_t^{(i)} + 1_{\{i \in B_t\}}. \tag{14}$$

Proof. The proof is made by induction. Relation (14) is clearly true for $t = 0$ by definition of D_0. Suppose that at time $t \geq 0$, there exists a set $B_t \subset \{1, 2, \ldots, n\}$ with $|B_t| = b$, satisfying Relation (14). Let i and j be the two agents interacting at time t, i.e. let $X_t = (i, j)$, for both processes C_t and D_t. We distinguish the following cases.

- **Case 1:** $i, j \in B_t$. In this case, we have
$$D_{t+1}^{(i)} = \left\lfloor \frac{D_t^{(i)} + D_t^{(j)}}{2} \right\rfloor = \left\lfloor \frac{C_t^{(i)} + C_t^{(j)} + 2}{2} \right\rfloor = C_{t+1}^{(i)} + 1.$$
In the same way, we have $D_{t+1}^{(j)} = C_{t+1}^{(j)} + 1$, which means that $i, j \in B_{t+1}$. The other entries being invariant, we have $B_{t+1} = B_t$.
- **Case 2:** $i, j \notin B_t$. In this case, we have $D_{t+1}^{(i)} = C_{t+1}^{(i)}$ and $D_{t+1}^{(j)} = C_{t+1}^{(j)}$ which means that $i, j \notin B_{t+1}$. The other entries being invariant, we have $B_{t+1} = B_t$.
- **Case 3.1:** $i \in B_t$ and $j \notin B_t$ and $C_t^{(i)} + C_t^{(j)}$ is even. In this case, we have
$$D_{t+1}^{(i)} = \left\lfloor \frac{D_t^{(i)} + D_t^{(j)}}{2} \right\rfloor = \left\lfloor \frac{C_t^{(i)} + 1 + C_t^{(j)}}{2} \right\rfloor = \left\lfloor \frac{C_t^{(i)} + C_t^{(j)}}{2} \right\rfloor = C_{t+1}^{(i)}.$$
In the same way, we have $D_{t+1}^{(j)} = C_{t+1}^{(j)} + 1$, which means that $i \notin B_{t+1}$ and $j \in B_{t+1}$. We thus have $B_{t+1} = (B_t \setminus \{i\}) \cup \{j\}$ and so $|B_{t+1}| = |B_t| = b$.
- **Case 3.2:** $i \in B_t$ and $j \notin B_t$ and $C_t^{(i)} + C_t^{(j)}$ is odd. In a similar way to the case 3.1, we have $D_{t+1}^{(i)} = C_{t+1}^{(i)} + 1$ and $D_{t+1}^{(j)} = C_{t+1}^{(j)}$, which means that $i \in B_{t+1}$ and $j \notin B_{t+1}$ and so $B_{t+1} = B_t$.
- **Case 4.1:** $i \notin B_t$ and $j \in B_t$ and $C_t^{(i)} + C_t^{(j)}$ is even. In a similar way to the case 3.2, we have $D_{t+1}^{(i)} = C_{t+1}^{(i)}$ and $D_{t+1}^{(j)} = C_{t+1}^{(j)} + 1$, which means that $i \notin B_{t+1}$ and $j \in B_{t+1}$ and so $B_{t+1} = B_t$.
- **Case 4.2:** $i \notin B_t$ and $j \in B_t$ and $C_t^{(i)} + C_t^{(j)}$ is odd. In a similar way to the case 3.1, we have $D_{t+1}^{(i)} = C_{t+1}^{(i)} + 1$ and $D_{t+1}^{(j)} = C_{t+1}^{(j)}$, which means that $i \in B_{t+1}$ and $j \notin B_{t+1}$. We thus have $B_{t+1} = (B_t \setminus \{j\}) \cup \{i\}$ and so $|B_{t+1}| = |B_t| = b$.

In all cases, we have shown that $B_{t+1} \subset \{1, 2, \dots, n\}$, that $|B_{t+1}| = |B_t| = b$ and that (14) is true at time $t+1$, which completes the proof. □

As we did for process C, we denote by ℓ_D the mean value of the entries of D_t and by L_D the row vector of $\mathbb{R}^n$ with all its entries equal to ℓ_D, that is

$$\ell_D := \frac{1}{n}\sum_{i=1}^{n} D_t^{(i)} \text{ and } L_D := (\ell_D, \dots, \ell_D).$$

We use the shadow process D to extend the results of Theorem 2 for any value of λ. We first show that for any process C, we can construct a shadow process D verifying the condition of Theorem 2, that is a shadow process D such that the fractional part λ_D of ℓ_D, defined by $\lambda_D := \min\{\ell_D - \lfloor \ell_D \rfloor, \lceil \ell_D \rceil - \ell_D\} = \min\{\ell_D - \lfloor \ell_D \rfloor, 1 - (\ell_D - \lfloor \ell_D \rfloor)\}$ verifies

$$\lambda_D = \frac{n - 1_{\{n \text{ odd}\}}}{2n}.$$

Recall, from Relation (2) and Lemma 1, that $n\ell = \sum_{i=1}^{n} C_0^{(i)}$ is an integer.

Lemma 5. *For any process C, there exists a shadow process D with parameter b such that $\ell_D - \lfloor \ell_D \rfloor = (n - 1_{\{n \text{ odd}\}})/(2n)$. More precisely, let $d \geq 0$ be the smallest integer such that n divides $n\ell + d$. Then*

- *If $0 \leq d \leq n/2$, then $b = d + (n - 1_{\{n \text{ odd}\}})/2$,*
- *If $n/2 < d < n$, then $b = d - (n + 1_{\{n \text{ odd}\}})/2$.*

Proof. Let B_0 be a set of b agents with $b \in \{1, \cdots, n-1\}$ and D_t be the corresponding shadow process, associated with process C_t, defined in Section 3.2. By definition of ℓ_D, we have, from (14), $\ell_D = \ell + b/n$ which gives $\ell_D - \lfloor \ell_D \rfloor = \ell + b/n - \lfloor \ell + b/n \rfloor$. Let d be the smallest integer such that n divides $n\ell + d$. Integer d thus belongs to $\{0, \cdots, n-1\}$ and we have $\ell_D - \lfloor \ell_D \rfloor = (b-d)/n - \lfloor (b-d)/n \rfloor$.

- **Case 1:** if $0 \leq d \leq n/2$ then, by taking $b = d + (n - 1_{\{n \text{ odd}\}})/2$, we check that $b \in \{1, \cdots, n-1\}$. Since $0 < (n - 1_{\{n \text{ odd}\}})/(2n) < 1$, we have
$$\ell_D - \lfloor \ell_D \rfloor = \frac{n - 1_{\{n \text{ odd}\}}}{2n} - \left\lfloor \frac{n - 1_{\{n \text{ odd}\}}}{2n} \right\rfloor = \frac{n - 1_{\{n \text{ odd}\}}}{2n}.$$
- **Case 2:** if $n/2 < d < n$ then, by taking $b = d - (n + 1_{\{n \text{ odd}\}})/2$, we also check that $b \in \{1, \cdots, n-1\}$. Since $-1 < -(n + 1_{\{n \text{ odd}\}})/(2n) < 0$, we have
$$\ell_D - \lfloor \ell_D \rfloor = -\frac{n + 1_{\{n \text{ odd}\}}}{2n} - \left\lfloor -\frac{n + 1_{\{n \text{ odd}\}}}{2n} \right\rfloor = \frac{n - 1_{\{n \text{ odd}\}}}{2n}.$$

Hence, $\ell_D - \lfloor \ell_D \rfloor = (n - 1_{\{n \text{ odd}\}})/(2n)$, which ends the proof. □

The shadow process D, associated with process C, is thus constructed from the rest of the Euclidean division of $n\ell$ by n. Taking the complement of this rest to n, we deduce the value of parameter b of the shadow process D. In order to prove the main theorem of this paper, we still need the following technical result.

Lemma 6. *For all $t \geq 0$, we have*

$$\|C_t - L\|_\infty - \|D_t - L_D\|_\infty \leq \frac{n-1}{n} \quad \textit{and} \quad \|D_t - L_D\| - \|C_t - L\| < \sqrt{n}.$$

Proof. From Lemma 4, we easily get

$$\ell_D = \frac{1}{n}\sum_{i=1}^{n} D_t^{(i)} = \ell + \frac{|B_t|}{n} = \ell + \frac{b}{n}.$$

Observing that

$$\|D_t - L_D\|_\infty = \max\{\ell_D - \min_{1\leq i\leq n} D_t^{(i)}, \max_{1\leq i\leq n} D_t^{(i)} - \ell_D\},$$
$$\|C_t - L\|_\infty = \max\{\ell - \min_{1\leq i\leq n} C_t^{(i)}, \max_{1\leq i\leq n} C_t^{(i)} - \ell\},$$

we first deduce that

$$\|D_t - L_D\|_\infty \geq \left(\ell_D - \min_{1\leq i\leq n} D_t^{(i)}\right) \text{ and } \|D_t - L_D\|_\infty \geq \left(\max_{1\leq i\leq n} D_t^{(i)} - \ell_D\right). \quad (15)$$

We distinguish the following two cases.

If $\|C_t - L\|_\infty = \ell - \min_{1\leq i\leq n} C_t^{(i)}$ then, applying Relation (15), we obtain

$$\|C_t - L\|_\infty - \|D_t - L_D\|_\infty \leq \left(\ell - \min_{1\leq i\leq n} C_t^{(i)}\right) - \left(\ell_D - \min_{1\leq i\leq n} D_t^{(i)}\right),$$

and since, from Lemma 4, we have $\min_{1\leq i\leq n} D_t^{(i)} \leq \min_{1\leq i\leq n} C_t^{(i)} + 1$, we deduce

$$\|C_t - L\|_\infty - \|D_t - L_D\|_\infty \leq \ell - \ell_D + 1 = 1 - \frac{b}{n} \leq \frac{n-1}{n}.$$

If $\|C_t - L\|_\infty = \max_{1\leq i\leq n} C_t^{(i)} - \ell$ then, applying Relation (15), we obtain

$$\|C_t - L\|_\infty - \|D_t - L_D\|_\infty \leq \left(\max_{1\leq i\leq n} C_t^{(i)} - \ell\right) - \left(\max_{1\leq i\leq n} D_t^{(i)} - \ell_D\right),$$

and since, from Lemma 4, we have $\max_{1\leq i\leq n} C_t^{(i)} \leq \max_{1\leq i\leq n} D_t^{(i)}$, we deduce again

$$\|C_t - L\|_\infty - \|D_t - L_D\|_\infty \leq \ell - \ell_D = \frac{b}{n} \leq \frac{n-1}{n},$$

which completes the proof of the first inequality.

To prove the second one, note that $D_t - L_D$ is orthogonal to unit vector e. Indeed

$$\langle D_t - L_D, e\rangle = \sum_{i=1}^{n} \left(D_t^{(i)} - \ell_D\right) = n\ell_D - n\ell_D = 0.$$

Hence, since $L_D - L = (\ell_D - \ell)e$, we deduce that $D_t - L_D$ and $L_D - L$ are orthogonal too. The Pythagoras' Theorem then gives $\|D_t - L\|^2 = \|D_t - L_D\|^2 + \|L_D - L\|^2$, which implies that $\|D_t - L_D\| \le \|D_t - L\|$.

From Relation (14), we have $D_t^{(i)} - C_t^{(i)} = 1_{\{i \in B_t\}}$ for every $i = 1, \dots, n$. Since $|B_t| = b$, this leads to $\|D_t - C_t\| = \sqrt{b}$. From the triangle inequality and since $b < n$, we get $\|D_t - L\| \le \|D_t - C_t\| + \|C_t - L\| = \|C_t - L\| + \sqrt{b} \le \|C_t - L\| + \sqrt{n}$. □

The following theorem is the main result of this paper.

Theorem 3. *For all $\delta \in (0,1)$, if there exists a constant K such that $\|C_0 - L\| \le K$, then, for all $t \ge (n-1)\left(2\ln\left(K + \sqrt{n}\right) - \ln\delta - \ln 2\right)$, we have*

$$\mathbb{P}\left\{\|C_t - L\|_\infty \ge 3/2\right\} \le \delta. \tag{16}$$

Proof. Let d be the smallest integer such that n divides $n\ell + d$. From Lemma 5, there exists a shadow process D associated with a set B_0 of b agents, such that

$$\ell_D - \lfloor \ell_D \rfloor = \frac{b-d}{n} - \left\lfloor \ell + \frac{b-d}{n} \right\rfloor = \frac{n - 1_{\{n \text{ odd}\}}}{2n}.$$

Hence,

$$\lambda_D = \min\left(\frac{n - 1_{\{n \text{ odd}\}}}{2n}, 1 - \frac{n - 1_{\{n \text{ odd}\}}}{2n}\right) = \frac{n - 1_{\{n \text{ odd}\}}}{2n}.$$

Moreover, combining the hypothesis $\|C_0 - L\| \le K$ and Lemma 6, we get $\|D_0 - L_D\| \le \|C_0 - L\| + \sqrt{n} \le K + \sqrt{n}$. We can thus apply Theorem 2 to process D with λ_D, L_D and $K + \sqrt{n}$ in place of λ, L and K respectively. We thus obtain, for all $\delta \in (0,1)$ and for every $t \ge (n-1)\left(2\ln(K + \sqrt{n}) - \ln\delta - \ln 2\right)$,

$$\mathbb{P}\left\{\|D_t - L_D\|_\infty > \frac{n + 1_{\{n \text{ odd}\}}}{2n}\right\} = \mathbb{P}\left\{\max_{1 \le i \le n} D_t^{(i)} - \min_{1 \le i \le n} D_t^{(i)} > 1\right\} \le \delta.$$

From Lemma 6, we get $\|C_t - L\|_\infty \le \|D_t - L_D\|_\infty + \frac{n-1}{n}$. This inequality allows us to write

$$\|D_t - L_D\|_\infty \le \frac{n + 1_{\{n \text{ odd}\}}}{2n} \implies \|C_t - L\|_\infty \le \frac{n + 1_{\{n \text{ odd}\}}}{2n} + \frac{n-1}{n} < \frac{3}{2},$$

thus

$$\mathbb{P}\left\{\|C_t - L\|_\infty < \frac{3}{2}\right\} \geq \mathbb{P}\left\{\|D_t - L_D\|_\infty \leq \frac{n + 1_{\{n \text{ odd}\}}}{2n}\right\},$$

or equivalently

$$\mathbb{P}\left\{\|C_t - L\|_\infty \geq \frac{3}{2}\right\} \leq \mathbb{P}\left\{\|D_t - L_D\|_\infty > \frac{n + 1_{\{n \text{ odd}\}}}{2n}\right\} \leq \delta,$$

which completes the proof. □

Theorem 3 thus extends the results of Theorem 6 of [11] to the case of any value for λ. For any λ value, process C_t belongs to the open ball of radius $3/2$ and center L, with arbitrarily high probability in the infinity norm, after no more than $O(n \ln(K + \sqrt{n}))$ time or $O(\ln(K + \sqrt{n}))$ parallel time as shown in Relation (16). Note that in Relation (16), we give explicitly the constant arising in this complexity. This constant depends on the upper bound K of $\|C_0 - L\|$ and the initial vector C_0 is given by the application the user wants to deal with. In the next section, we calculate the upper bound K for two different types of applications : the proportion problem and the system size problem. We conclude this section with the following corollary which shows that under the condition of Theorem 3, the greatest difference among the entries of vector C_t, which represents the values of the agents at time t, is less than or equal to 2 with arbitrarily high probability.

Corollary 1. *For all $\delta \in (0, 1)$, if there exists a constant K such that $\|C_0 - L\| \leq K$, then for all $t \geq (n-1)\left(2 \ln\left(K + \sqrt{n}\right) - \ln \delta - \ln 2\right)$, we have*

$$\mathbb{P}\left\{\max_{1 \leq i \leq n} C_t^{(i)} - \min_{1 \leq i \leq n} C_t^{(i)} > 2\right\} \leq \delta \tag{17}$$

Proof. Observe that

$$\begin{aligned}\|C_t - L\|_\infty < 3/2 &\iff \max_{1 \leq i \leq n} C_t^{(i)} - \ell < 3/2 \text{ and } \ell - \min_{1 \leq i \leq n} C_t^{(i)} < 3/2 \\ &\implies \max_{1 \leq i \leq n} C_t^{(i)} - \min_{1 \leq i \leq n} C_t^{(i)} < 3.\end{aligned}$$

This leads to $\mathbb{P}\{\|C_t - L\|_\infty < 3/2\} \leq \mathbb{P}\{\max_{1 \leq i \leq n} C_t^{(i)} - \min_{1 \leq i \leq n} C_t^{(i)} \leq 2\}$, since the $C_t^{(i)}$ are integers and we conclude by applying Theorem 3. □

4. Applications

In this section, we apply the average-based distributed algorithm studied above to derive, from local average-based interactions, two global properties of our system: first the proportion of agents whose initial value is equal to A and second the number of agents n of the system.

We suppose that agents initially start their execution either with the initial value A or B. Let n_A be the number of agents starting with value A. If agent i starts with value A, we set $C_0^{(i)} = m$ and if he starts with value B, we set $C_0^{(i)} = 0$, where m is an integer, known by all the agents, which will be determined later. We thus have

$$\|C_0 - L\|^2 = n_A \left(m - \frac{n_A m}{n}\right)^2 + (n - n_A)\left(\frac{n_A m}{n}\right)^2 = m^2 n_A \left(1 - \frac{n_A}{n}\right). \tag{18}$$

4.1. Solving the Proportion Problem

The proportion problem consists for each agent to compute the proportion γ_A of agents that initially started the average-based algorithm with the initial value A. We have $\gamma_A = n_A/n$. Recall that the number n of agents in the system is not known to the agents. Relation (18) gives a function of n_A which reaches its maximum for $n_A = n/2$. At that value we obtain $\|C_0 - L\|^2 \le m^2 n/4$, that is

$$\|C_0 - L\| \le m\sqrt{n}/2. \tag{19}$$

Recall that $C_t^{(i)}$ represents the local value of agent i at discrete time t. We show that the local estimation of the proportion γ_A is given by the quantity $C_t^{(i)}/m$. More precisely, the following theorem gives an evaluation of the first instant t from which the distance between $C_t^{(i)}/m$ and γ_A, for all the agents, is less than a fixed ε with arbitrarily high probability $1 - \delta$, the integer value m being determined by the threshold ε.

Theorem 4. *For all $\delta \in (0, 1)$ and for all $\varepsilon \in (0, 1)$, by taking $m = \lceil 3/(2\varepsilon) \rceil$, we have, for all $t \ge (n-1)(\ln n - \ln \delta + 2\ln(2 + 1/\varepsilon) + \ln(9/32))$,*

$$\mathbb{P}\left\{\left|C_t^{(i)}/m - \gamma_A\right| < \varepsilon, \ \text{for all } i = 1, \dots, n\right\} \ge 1 - \delta.$$

Proof. Since $m = \lceil 3/(2\varepsilon) \rceil$, we have, from Relation (19),

$$\|C_0 - L\| \le \frac{m\sqrt{n}}{2} = \left\lceil \frac{3}{2\varepsilon} \right\rceil \frac{\sqrt{n}}{2} \le \left(\frac{3}{2\varepsilon} + 1\right)\frac{\sqrt{n}}{2} = \left(\frac{3 + 2\varepsilon}{4\varepsilon}\right)\sqrt{n}.$$

By choosing $K = (3 + 2\varepsilon)\sqrt{n}/(2\varepsilon)$, we obtain

$$2\ln(K + \sqrt{n}) = 2\ln\left[\left(\frac{3 + 6\varepsilon}{4\varepsilon}\right)\sqrt{n}\right] = \ln n + 2\ln(3/4) + 2\ln(2 + 1/\varepsilon).$$

We are now able to apply Theorem 3, which leads, for all $\delta \in (0, 1)$ and for all $t \geq (n-1)(\ln n - \ln\delta + 2\ln(2 + 1/\varepsilon) + \ln(9/32))$, to

$$\mathbb{P}\{\|C_t - L\|_\infty \geq 3/2\} \leq \delta$$

or equivalently, since $\ell = \gamma_A m$, to

$$\mathbb{P}\left\{\left|C_t^{(i)}/m - \gamma_A\right| < 3/(2m), \text{ for all } i = 1, \dots, n\right\} \geq 1 - \delta.$$

The fact that $m \geq 3/(2\varepsilon)$ completes the proof. □

In Figure 1, we compare our new bound of the convergence time obtained in Theorem 4 for the proportion problem with the one previously obtained in [11]. One can observe that we have considerably improved it. As usual, the parallel convergence time is the convergence time divided by the number n of agents.

4.2. Solving the System Size Problem

We now address the system size problem and suppose that each agent knows n_A. We prove that each agent is able to determine either the exact value of the number n of agents or an approximation of this number, depending on the initial input value m with arbitrarily high probability.

We introduce the following two functions, $\omega_{\min}$ and $\omega_{\max}$, which will be used by each node to get the lower and the upper bound of n, respectively. They are defined, for all integers k, by

$$\omega_{\min}(k) = \left\lceil \frac{2n_A m}{2k+3} \right\rceil \quad \text{and} \quad \omega_{\max}(k) = \begin{cases} +\infty & \text{if } k \leq 1 \\ \left\lfloor \frac{2n_A m}{2k-3} \right\rfloor & \text{if } k \geq 2. \end{cases}$$

We first start by a general result on the convergence time for the system size problem.

Lemma 7. *For all $\delta \in (0, 1)$, for all positive integers m and n_A, and for all $t \geq (n-1)\left(\ln\left(m\sqrt{n_A(1 - n_A/n)} + \sqrt{n}\right) - \ln\delta - \ln 2\right)$, we have*

$$\mathbb{P}\left\{\omega_{\min}\left(C_t^{(i)}\right) \leq n \leq \omega_{\max}\left(C_t^{(i)}\right), \text{ for all } i = 1, \dots, n\right\} \geq 1 - \delta.$$

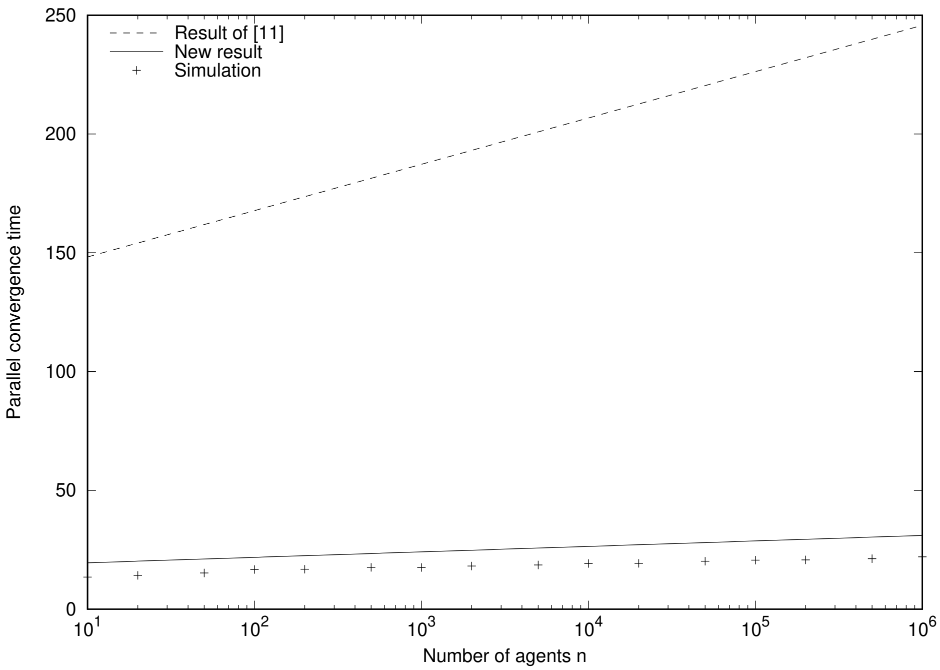

FIGURE 1: Bounds comparison of the parallel convergence time for the proportion problem. For each n, we simulate the parallel convergence time (star points) using 10^5 experiments. We also compute the two upper bounds of the parallel convergence time obtained in [11] (dashed line) and in Theorem 4 (plain line). For each experiment the initial proportion of agents starting with A is a uniform random number in [0,1] and we have taken $\delta = 10^{-4}$ and $\varepsilon = 10^{-2}$, which gives $m = 150$. Note the logarithmic scale of the x-axis.

Proof. From (18), we obtain $\|C_0 - L\| \leq m\sqrt{n_A(1 - n_A/n)}$. Applying Theorem 3 with $K = m\sqrt{n_A(1 - n_A/n)}$, we get

$$\mathbb{P}\left\{\|C_t - L\|_\infty < 3/2\right\} \geq 1 - \delta, \tag{20}$$

for all $t \geq (n-1)\left(\ln(m\sqrt{n_A(1-n_A/n)} + \sqrt{n}) - \ln\delta - \ln 2\right)$. Then, recalling that $\ell = n_A m/n$ and using the fact that

$$\|C_t - L\|_\infty < 3/2 \iff \text{for all } i = 1, \ldots, n,\ C_t^{(i)} - 3/2 < n_A m/n < C_t^{(i)} + 3/2, \tag{21}$$

we deduce first that, for all $i = 1, \ldots, n$, $n_A m/(C_t^{(i)} + 3/2) < n$, which implies that $\omega_{\min}\left(C_t^{(i)}\right) = \left\lceil n_A m/(C_t^{(i)} + 3/2)\right\rceil \leq n$.
By definition of $\omega_{\max}$, if $C_t^{(i)} \leq 1$, then obviously in that case $n \leq \omega_{\max}(C_t^{(i)})$. If $C_t^{(i)} \geq 2$, we deduce from Relation (21) that $n < n_A m/C_t^{(i)} - 3/2$, which means

that $n \leq \left\lfloor n_A m/(C_t^{(i)} - 3/2) \right\rfloor = \omega_{\max}(C_t^{(i)})$. We thus have shown that, for all $t \geq (n-1)\left(\ln(m\sqrt{n_A(1-n_A/n)} + \sqrt{n}) - \ln\delta - \ln 2\right)$, we have

$$\|C_t - L\|_\infty < 3/2 \Longrightarrow \omega_{\min}\left(C_t^{(i)}\right) \leq n \leq \omega_{\max}\left(C_t^{(i)}\right), \text{ for all } i = 1, \ldots, n. \tag{22}$$

The use of Relation (20) completes the proof. □

Suppose that an upper bound N of n is known. Then we prove that with arbitrarily high probability, after a given number of interactions (computed below), any agent i can locally compute the exact system size n, i.e. $\omega_{\min}(C_t^{(i)}) = \omega_{\max}(C_t^{(i)}) = n$.

Theorem 5. *For all $\delta \in (0,1)$, for all positive integers n_A and N with $n \leq N$, by taking $m \geq 3N(N+1)/n_A$, we have, for all $t \geq (n-1)(\ln n_A + 2\ln m - \ln\delta)$,*

$$\mathbb{P}\left\{\omega_{\min}\left(C_t^{(i)}\right) = \omega_{\max}\left(C_t^{(i)}\right) = n, \text{ for all } i = 1, \ldots, n\right\} \geq 1 - \delta. \tag{23}$$

Proof. Since $n \leq N$, the condition on m gives $3n(n+1) \leq n_A m$ or equivalently to $3n^2 \leq n_A m - 3n$. Multiplying each side of this inequality by $4n_A m/n^2$, we obtain

$$12 n_A m \leq (2n_A m/n)^2 - 12 n_A m/n = 4(n_A m/n - 3/2)^2 - 9. \tag{24}$$

On the other hand, from (18), we have $\|C_0 - L\| \leq m\sqrt{n_A(1-n_A/n)}$. Using the fact that $\sqrt{1-x} \leq 1 - x/2$, for all $x \leq 1$, we get

$$\|C_0 - L\| \leq \sqrt{n_A} m \left(1 - \frac{n_A}{2n}\right).$$

Denoting by K this upper bound of $\|C_0 - L\|$ and using successively the condition $mn_A \geq 3n(n+1)$ and the fact that $n_A \geq 1$, we obtain

$$K + \sqrt{n} = \sqrt{n_A} m + \sqrt{n} - \frac{mn_A\sqrt{n_A}}{2n} \leq \sqrt{n_A} m + \sqrt{n} - \frac{3(n+1)}{2} \leq \sqrt{n_A} m.$$

Using this inequality in Theorem 3, we get, for all $t \geq (n-1)(\ln n_A + 2\ln m - \ln\delta - \ln 2)$, $\mathbb{P}\{\|C_t - L\|_\infty < 3/2\} \geq 1 - \delta$. Observe now that $\|C_t - L\|_\infty < 3/2$ implies that $n_A m/n - 3/2 < C_t^{(i)}$, for all i, which in turn implies, from (24), that

$$0 \leq 12 n_A m \leq 4(n_A m/n - 3/2)^2 - 9 < 4(C_t^{(i)})^2 - 9,$$

in which we used the condition $m \geq 3n(n+1)/n_A$ to ensure that $n_A m/n - 3/2 > 0$. Combining these two results and using the definitions of the integer functions $\omega_{\min}$ and

$\omega_{\max}$, we obtain

$$\|C_t - L\|_\infty < 3/2 \Longrightarrow \frac{12n_A m}{4(C_t^{(i)})^2 - 9} < 1 \Longrightarrow \frac{2n_A m}{2C_t^{(i)} - 3} - \frac{2n_A m}{2C_t^{(i)} + 3} < 1$$
$$\Longrightarrow \omega_{\max}\left(C_t^{(i)}\right) - \omega_{\min}\left(C_t^{(i)}\right) < 1 \Longrightarrow \omega_{\max}\left(C_t^{(i)}\right) = \omega_{\min}\left(C_t^{(i)}\right).$$

From (22) in Lemma 7, we also have

$$\|C_t - L\|_\infty < 3/2 \Longrightarrow \omega_{\min}\left(C_t^{(i)}\right) \le n \le \omega_{\max}\left(C_t^{(i)}\right), \text{ for all } i = 1, \ldots, n.$$

Thus, $1 - \delta \le \mathbb{P}\left\{\|C_t - L\|_\infty < 3/2\right\} \le \mathbb{P}\left\{\omega_{\min}\left(C_t^{(i)}\right) = \omega_{\max}\left(C_t^{(i)}\right) = n\right\}$. □

5. Conclusion

In this paper we have presented a thorough analysis of the bound of the convergence time of average-based population protocols, and applied it to both the proportion problem and the system size one. Thanks to a well chosen stochastic coupling, we have considerably improved existing results by providing explicit and tight bounds of the time required to converge to the solution of these problems. Numerical simulations illustrate the tightness of our bounds of convergence times.

A possible extension of this work would be to consider more general graphs for the interactions between agents, instead of the complete graph used in this paper. Another future direction would be to deal with a continuous time model in which interactions occur at the transitions' instants of a Poisson process or more generally at the transitions' instants of a Phase-type renewal process which preserves the Markov property.

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