Adv. Appl. Prob. **25**, 702–713 (1993)
Printed in N. Ireland

TRANSIENT ANALYSIS OF THE $M/M/1$ QUEUE

P. LEGUESDRON,* *INSA, Rennes*
J. PELLAUMAIL,* *INSA, Rennes*
G. RUBINO,** *IRISA-INRIA, Rennes*
B. SERICOLA,** *IRISA-INRIA, Rennes*

Abstract

A new approach is used to obtain the transient probabilities of the $M/M/1$ queueing system. The first step of this approach deals with the generating function of the transient probabilities of the uniformized Markov chain associated with this queue. The second step consists of the inversion of this generating function. A new analytical expression of the transient probabilities of the $M/M/1$ queue is then obtained.

UNIFORMIZATION; GENERATING FUNCTION

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60K25
SECONDARY 60J27; 60J35

1. Introduction

Several methods have been proposed for the study of the transient behaviour of the $M/M/1$ queueing system. In most cases the derived expressions are complicated by the fact that they often refer to Bessel functions. These expressions are generally obtained by a combined use of generating functions and Laplace transforms [1], [3], [6], [11], [14]. The use of Bessel functions leads to complex numerical solutions. An overview of these methods can be found in [2]. Sourouri and Krinik [13] recently proposed an approach using only Taylor series, but it leads also to numerical problems since the coefficients of these series are alternately positive and negative.

In this paper, we propose a new method based on the uniformization technique and on generating functions. In the following section, we give an expression of the generating function of the transient probabilities of the uniformized Markov chain associated with the Markov process describing the $M/M/1$ queue. In Section 3 this generating function is inverted and we obtain an analytical expression of the transient probabilities of the uniformized Markov chain which leads to an analytical expression for the transient probabilities of the $M/M/1$ queue. This expression is quite simple in the case where the queue is initially empty. When the queue is not initially empty, the expression obtained is reduced to the previous one (Theorem 1(b) of [4]) by adding a term corresponding to the transient probabilities of the

Received 8 July 1992; revision received 30 October 1992.
* Postal address: INSA, 20 avenue des Buttes de Coësmes, 35043 Rennes Cédex, France.
** Postal address: IRISA-INRIA, Campus Universitaire de Beaulieu, 35042 Rennes Cédex, France.

associated absorbing process obtained by making state 0 absorbing. In deriving this last expression, we obtain also a simple expression of the probability to go from state i to state j in n transitions on the absorbing process. The last section is devoted to some conclusions.

2. Generating functions and uniformized Markov chain

Consider the classical $M/M/1$ queue with arrival rate λ and service rate μ. We denote by $X = (X_t)_{t \geqq 0}$ the continuous-time Markov process associated with the state of the queue. The infinitesimal generator of the process X is denoted by A. The non-zero entries of this matrix are

$$A_{0,0} = -\lambda, \quad A_{0,1} = 2 \quad \text{and for any } i \geqq 1, \quad A_{i,i-1} = \mu, \quad A_{i,i} = -(\lambda + \mu), \quad A_{i,i+1} = \lambda.$$

For every $i, j \in \mathbb{N}$ and $t \geqq 0$, we denote by $P_{i,j}(t)$ the probability of having j customers in the queue (including the one being served) at time t given that there are i customers in the queue at time 0. The following classical result holds:

$$P_{i,j}(t) = \mathbb{P}(X_t = j \mid X_0 = i) = (\exp(At))_{i,j}.$$

To compute these transient probabilities, we first consider the uniformized discrete-time Markov chain (see for instance [10]) associated with process X. The transient probability matrix of this Markov chain is denoted by P. We then have the following relation between matrices A and P:

$$P = I + \frac{1}{\lambda + \mu} A,$$

where I denotes the identity matrix, that is $I_{i,j} = 1$ if $i = j$ and 0 otherwise. This relation allows us to write the matrix $\exp(At)$ as

$$\exp(At) = \sum_{k=0}^{+\infty} \exp(-(\lambda + \mu)t) \frac{(\lambda + \mu)^k t^k}{k!} P^k. \tag{1}$$

If p and q are defined by

$$p \stackrel{\text{def}}{=} \frac{\lambda}{\lambda + \mu} \quad \text{and} \quad q \stackrel{\text{def}}{=} \frac{\mu}{\lambda + \mu},$$

the non-zero entries of the matrix P are

$$P_{0,0} = q, \quad \forall i \geqq 1 \quad P_{i,i-1} = q, \quad \text{and} \quad \forall i \geqq 0 \quad P_{i,i+1} = p.$$

In what follows, we focus on the calculation of P^k using generating functions.

Let us consider the functions M defined on $\mathbb{N} \times \mathbb{N}$ with complex values. As usual, we see them as infinite matrices and denote by $M_{i,j}$ the value $M(i, j)$. With each M we associate the value

$$\nu(M) \stackrel{\text{def}}{=} \sup_i \sum_j |M_{ij}|$$

and we denote by $\mathcal{M}$ the set of infinite matrices M such that $\nu(M) < +\infty$. It is well known that ν is a norm on the set $\mathcal{M}$ and that $(\mathcal{M}, \nu)$ is a Banach algebra [7]. The *potential kernel* of the matrix $M \in \mathcal{M}$, denoted by Φ_M, is the complex function defined by

$$\Phi_M(z) \stackrel{\text{def}}{=} \sum_{k=0}^{+\infty} M^k z^k$$

for all z satisfying $|z| < 1/\nu(M)$. It can be seen as the generating function of the sequence of the powers of matrix M. It is well known that Φ_M is the only solution to the matrix equation in X

$$\forall\, |z| < 1/\nu(M), \qquad X(z) = I + zMX(z). \tag{2}$$

More generally, for every matrix H, $H\Phi_M$ is the only solution to the matrix equation in X

$$\forall\, |z| < 1/\nu(M), \qquad X(z) = H + zX(z)M. \tag{3}$$

Another useful property of the potential kernel is given below as a lemma.

Lemma 2.1. For any functions M and N of $\mathcal{M}$ we have

$$\Phi_{M+N}(z) = \Phi_M(z) + z\Phi_M(z)N\Phi_{M+N}(z) \tag{4}$$

$$= \Phi_M(z) + z\Phi_{M+N}(z)N\Phi_M(z) \tag{5}$$

for all z such that $|z| < \inf(1/\nu(M), 1/\nu(M+N))$.

Proof. These relations follow from

$$\forall k \geqq 1 \quad (M+N)^k = M^k + \sum_{j=0}^{k-1} (M+N)^j N M^{k-j-1}$$
$$= M^k + \sum_{j=0}^{k-1} M^j N (M+N)^{k-j-1}$$

which can be easily checked by induction on k.

Consider the potential kernel of P, that is

$$\Phi_P(z) = \sum_{k=0}^{+\infty} P^k z^k.$$

We are going to find an explicit expression of Φ_P and we shall invert it to obtain an explicit expression of P^k. We now need some notation and a lemma. Let V, W and R be the matrices defined by

$$V_{i,j} = I_{i+1,j}, \qquad W_{i,j} = I_{i,j+1}, \qquad R_{i,j} = I_{i,0}I_{0,j} \quad \forall i, j \in \mathbb{N}.$$

Lemma 2.2. Let α and β be any complex numbers. Let X be the infinite matrix defined by $X_{i,j} = \alpha^i \beta^j$ and Y the infinite matrix defined by

$$Y = \sum_{k=0}^{+\infty} W^k X V^k.$$

Then matrix Y satisfies

$$(6) \qquad (\alpha W + \beta V)Y = (1 + \alpha\beta)Y - I.$$

Proof. Observe that Y is well defined for every $i, j \in \mathbb{N}$ as

$$Y_{i,j} = \begin{cases} \beta^{j-i} s_i & \text{if } i \leqq j \\ \alpha^{i-j} s_j & \text{if } i \geqq j \end{cases}$$

where for every $m \geqq 0$,

$$s_m = \sum_{l=0}^{m} (\alpha\beta)^l = \frac{1 - (\alpha\beta)^{m+1}}{1 - \alpha\beta} \quad \text{if} \quad \alpha\beta \neq 1.$$

To prove (6), we denote by L its left-hand side, that is, $L = (\alpha W + \beta V)Y$. For $i > j$, we have

$$\begin{aligned} L_{i,j} &= \alpha Y_{i-1,j} + \beta Y_{i+1,j} \quad (\text{with the convention } Y_{-1,j} = 0) \\ &= \alpha\alpha^{i-j-1} s_j + \beta\alpha^{i-j+1} s_j \\ &= (1 + \alpha\beta)\alpha^{i-j} s_j \\ &= (1 + \alpha\beta) Y_{i,j}. \end{aligned}$$

The case $i < j$ is similar to the case $i > j$. On the diagonal we have

$$\begin{aligned} L_{i,i} &= \alpha Y_{i-1,i} + \beta Y_{i+1,i} \\ &= \alpha\beta s_{i-1} + \alpha\beta s_i \\ &= s_i - 1 + \alpha\beta s_i \\ &= (1 + \alpha\beta) s_i - 1 \\ &= (1 + \alpha\beta) Y_{i,i} - 1. \end{aligned}$$

The next theorem gives an expression of the potential kernel $\Phi_P(z)$ of matrix P. First let us recall that the Catalan numbers are defined for all non-negative integer n by

$$c_n \stackrel{\text{def}}{=} \binom{2n}{n} \frac{1}{n+1}.$$

The generating function of the sequence of Catalan numbers,

$$(7) \qquad C(z) \stackrel{\text{def}}{=} \sum_{n=0}^{+\infty} c_n z^n,$$

converges for $|z| \leq \frac{1}{4}$ and we have

$$C(z) = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

Note that this complex function satisfies

$$C(z) = 1 + zC^2(z). \tag{8}$$

Theorem 2.3. Let $|z|<1$ and $\eta(z) \stackrel{\text{def}}{=} C(pqz^2)$. Let $X(z)$ be the infinite matrix defined by

$$X_{ij}(z) \stackrel{\text{def}}{=} (qz\eta(z))^i (pz\eta(z))^j$$

and $Y(z)$ the infinite matrix defined as in Lemma 2.2 by

$$Y(z) = \sum_{k=0}^{+\infty} W^k X(z) V^k.$$

Then, we have

$$\forall z \text{ such that } |z|<\tfrac{1}{2}, \qquad \Phi_P(z) = \eta(z)Y(z) + \frac{qz\eta(z)}{1-qz\eta(z)}\eta(z)X(z). \tag{9}$$

Proof. Let us denote by Q the matrix $pV+qW$. We have $P=Q+qR$. For $|z|<1$, we have $|pqz^2|<\frac{1}{4}$ (observe that since $p+q=1$, we have $pq \leqq \frac{1}{4}$) and so $\eta(z)$ is well defined and satisfies $|\eta(z)|<2$. Moreover, writing (7) with $z=pq$ and after some algebra, we obtain that $|pz\eta(z)|<1$ and $|qz\eta(z)|<1$, and so

$$\nu(X(z)) = \frac{1}{1-p\,|z|\,|\eta(z)|}.$$

Applying the previous lemma, in which we set $\alpha = qz\eta(z)$ and $\beta = pz\eta(z)$, we obtain

$$z\eta(z)QY(z) = (1+pqz^2\eta^2(z))Y(z) - I.$$

Relation (8) implies that $\eta(z) = 1 + pqz^2\eta^2(z)$, so we have

$$\eta(z)Y(z) = I + zQ\eta(z)Y(z).$$

Now since $\nu(Q)=1$, we can apply (2) to obtain

$$\forall z \text{ such that } |z|<1, \qquad \eta(z)Y(z) = \Phi_Q(z).$$

From (4) in which we set $M=Q$ and $N=qR$, we have

$$\forall z \text{ such that } |z|<1, \qquad \Phi_P(z) = \Phi_Q(z) + qz\Phi_P(z)R\Phi_Q(z),$$

that is, since $\Phi_Q(z) = \eta(z)Y(z)$,

$$\forall z \text{ such that } |z|<1, \qquad \Phi_P(z) = \eta(z)Y(z) + z\Phi_P(z)qR\eta(z)Y(z).$$

To prove (9), we fix z such that $|z|<\frac{1}{2}$ and we apply (3) in which we set $H(z)=\eta(z)Y(z)$ and $M(z) = qR\eta(z)Y(z)$. We must ensure that for $|z|<\frac{1}{2}$, we have $|z|<1/\nu(M(z))$. By definition of $Y(z)$ and since $RW=0$, we have $RY(z)=RX(z)$.

This leads to

$$\forall z \text{ such that } |z| < \tfrac{1}{2}, \qquad \begin{aligned} \nu(M(z)) &= \nu(qR\eta(z)Y(z)) \\ &= q\,|\eta(z)|\,\nu(RY(z)) \\ &= q\,|\eta(z)|\,\nu(RX(z)) \\ &= q\,|\eta(z)|\,\nu(X(z)) \\ &= \frac{q\,|\eta(z)|}{1 - p\,|z|\,|\eta(z)|} \\ &< \frac{2q}{1-p} = 2. \end{aligned}$$

So,

$$|z| < \tfrac{1}{2} \Rightarrow \nu(M(z)) < 2 \Rightarrow |z| < 1/\nu(M(z)).$$

We can now apply (3) to obtain

$$\forall z \text{ such that } |z| < \tfrac{1}{2}, \qquad \Phi_P(z) = \eta(z)Y(z)\sum_{k=0}^{+\infty} (qz\eta(z))^k (RY(z))^k$$

or (since $RY(z) = RX(z)$),

$$\forall z \text{ such that } |z| < \tfrac{1}{2}, \qquad \Phi_P(z) = \eta(z)Y(z)\sum_{k=0}^{+\infty} (qz\eta(z))^k (RX(z))^k.$$

Observing that $RX(z)R = R$ and so that for every $k \geqq 1$, we have $(RX(z))^k = RX(z)$, we can write

$$\forall z \text{ such that } |z| < \tfrac{1}{2}, \qquad \Phi_P(z) = \eta(z)Y(z) + \eta(z)Y(z)\sum_{k=1}^{+\infty} (qz\eta(z))^k RX(z)$$

that is, since $|qz\eta(z)| < 1$,

$$\forall z \text{ such that } |z| < \tfrac{1}{2}, \qquad \Phi_P(z) = \eta(z)Y(z) + \frac{qz\eta(z)}{1 - qz\eta(z)}\,\eta(z)Y(z)RX(z).$$

By definition of $Y(z)$ and since $VR = 0$, we have $Y(z)R = X(z)R$. Furthermore, it is easy to check that $X(z)RX(z) = X(z)$. Finally,

$$\forall z \text{ such that } |z| < \tfrac{1}{2}, \qquad \Phi_P(z) = \eta(z)Y(z) + \frac{qz\eta(z)}{1 - qz\eta(z)}\,\eta(z)X(z)$$

which completes the proof.

3. Transient probabilities

In this section, we focus on (9) to obtain an expression of $(P^n)_{i,j}$, the transient probabilities of the uniformized Markov chain of the $M/M/1$ queue. These discrete transient probabilities will lead by means of (1) to the transient probabilities of the $M/M/1$ queue. In what follows, we always work in the complex domain $|z|<\frac{1}{2}$.

For $i, j \in \mathbb{N}$, $Y_{i,j}(z)$ is given in the proof of Lemma 2.2 in which we set $\alpha = qz\eta(z)$ and $\beta = pz\eta(z)$, that is

$$Y_{i,j}(z) = \begin{cases} (pz\eta(z))^{j-i} \sum_{k=0}^{i} p^k q^k z^{2k} \eta(z)^{2k} & \text{if } i \leqq j \\ (qz\eta(z))^{i-j} \sum_{k=0}^{j} p^k q^k z^{2k} \eta(z)^{2k} & \text{if } i \geqq j. \end{cases}$$

In what follows, we consider only the case $i \leqq j$. The case $i \geqq j$ is similar since, as mentioned in [4], p. 325, we have for every $i, j \geqq 0$,

$$P_{i,j}(t) = \left(\frac{\lambda}{\mu}\right)^{j-i} P_{j,i}(t).$$

So, let $i, j \in \mathbb{N}$ such that $i \leqq j$. By definition of $X(z)$ we obtain

$$(10) \quad (\Phi_P(z))_{i,j} = (pz\eta(z))^{j-i} \sum_{k=0}^{i} p^k q^k z^{2k} \eta(z)^{2k+1} + p^i q^{i+1} z^{i+j+1} \eta(z)^{i+j+1} \frac{\eta(z)}{1 - qz\eta(z)}.$$

In the next subsection, we first consider the case $i = 0$, that is, the case when the queue is initially empty.

3.1. *The queue is initially empty.* The next theorem gives an expression of the transient probabilities of the uniformized Markov chain associated with X, the process describing the behaviour of the $M/M/1$ queue. To prove it we need the following lemma which gives an analytical expression of the powers of $\eta(z)$.

Lemma 3.1. For every $k \geqq 1$ and for every z such that $|z| \leqq \frac{1}{4}$, we have

$$C^k(z) = \sum_{n=0}^{+\infty} s(k, n) z^n,$$

where

$$s(k, n) = k \frac{(2n + k - 1)!}{n!\,(n + k)!}.$$

Proof. See for instance [8], p. 154.

Theorem 3.2. For every $j \geqq 0$,

$$(P^n)_{0,j} = \begin{cases} 0 & \text{if } n < j \\ \dfrac{p^j}{q^j} \displaystyle\sum_{k=0}^{\lfloor (n-j)/2 \rfloor} s(n + 1 - 2k, k) p^k q^{n-k}, & \text{if } n \geqq j, \end{cases}$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than the real number x.

Proof. By setting $i = 0$ in (10), we obtain

$$(\Phi_P(z))_{0,j} = (pz\eta(z))^j \frac{\eta(z)}{1 - qz\eta(z)}. \tag{11}$$

This relation can be written as

$$(\Phi_P(z))_{0,j} = \frac{p^j}{q^{j+1}z} \sum_{k=j+1}^{+\infty} [qz\eta(z)]^k.$$

Using Lemma 3.1, this gives

$$[qz\eta(z)]^k = [qzC(pqz^2)]^k = \sum_{n=0}^{+\infty} s(k, n)p^n q^{n+k} z^{2n+k}.$$

It follows that for every $j \geqq 0$,

$$\begin{aligned}
(\Phi_P(z))_{0,j} &= \frac{p^j}{q^{j+1}z} \sum_{k=j+1}^{+\infty} \sum_{n=0}^{+\infty} s(k, n)p^n q^{n+k} z^{2n+k} \\
&= \frac{p^j}{q^{j+1}z} \sum_{n=0}^{+\infty} \sum_{k=j+1}^{+\infty} s(k, n)p^n q^{n+k} z^{2n+k} \\
&= \frac{p^j}{q^{j+1}z} \sum_{n=0}^{+\infty} \sum_{k=2n+j+1}^{+\infty} s(k-2n, n)p^n q^{k-n} z^k \\
&= \frac{p^j}{q^{j+1}z} \sum_{k=j+1}^{+\infty} \sum_{n=0}^{\lfloor (k-(j+1))/2 \rfloor} s(k-2n, n)p^n q^{k-n} z^k \\
&= \frac{p^j}{q^j} \sum_{k=j}^{+\infty} \sum_{n=0}^{\lfloor (k-j)/2 \rfloor} s(k+1-2n, n)p^n q^{k-n} z^k
\end{aligned}$$

and so we obtain the desired expression for $(P^n)_{0,j}$.

The transient probabilities of the $M/M/1$ queue given that the queue is empty at time $t = 0$ are then

$$P_{0,j}(t) = \frac{p^j}{q^j} \sum_{n=j}^{+\infty} \exp(-(\lambda + \mu)t) \frac{(\lambda + \mu)^n t^n}{n!} \sum_{k=0}^{\lfloor (n-j)/2 \rfloor} s(n+1-2k, k)p^k q^{n-k} \tag{12}$$

that is,

$$P_{0,j}(t) = \frac{p^j}{q^j} \sum_{n=j}^{+\infty} \exp(-(\lambda + \mu)t) \frac{(\lambda + \mu)^n t^n}{n!} \sum_{k=0}^{\lfloor (n-j)/2 \rfloor} \frac{n+1-2k}{n+1} \binom{n+1}{k} p^k q^{n-k}. \tag{13}$$

If $j = 0$, we obtain from Theorem 3.2,

$$(P^n)_{0,0} = \sum_{k=0}^{\lfloor n/2 \rfloor} s(n+1-2k, k)p^k q^{n-k},$$

but a simpler expression can be obtained using the expression of $(\Phi_P(z))_{0,0}$ as follows. If we set $j = 0$ in (11), we have

$$(\Phi_P(z))_{0,0} = \frac{\eta(z)}{1 - qz\eta(z)}$$

and the relation $\eta(z) = 1 + pqz^2\eta^2(z)$ leads to

$$(\Phi_P(z))_{0,0} = \frac{1 - pz\eta(z)}{1 - z}$$

which gives

$$(\Phi_P(z))_{0,0} = \sum_{n=0}^{+\infty} \left(1 - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{2k}{k} \frac{p^{k+1}q^k}{k+1}\right) z^n$$

with the convention that when an index decreases in a sum the value of that sum is 0. With this convention, we have

$$(P^n)_{0,0} = 1 - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{2k}{k} \frac{p^{k+1}q^k}{k+1}.$$

An interpretation of this result is the following. Let NBP denote the number of customers served during a busy period. Using the distribution of NBP (see for instance [9], p. 65), that is, for $k \geqq 1$

$$\mathbb{P}(NBP = k) = \binom{2k-2}{k-1} \frac{p^{k-1}q^k}{k},$$

we obtain

$$(P^n)_{0,0} = 1 - \frac{p}{q} \mathbb{P}(NBP \leqq n).$$

It is shown in [12] that the right-hand side of this last relation is the probability that the nth customer will find the queue empty on its arrival.

Using now the expression of $(P^n)_{0,0}$, we get

$$P_{0,0}(t) = \sum_{n=0}^{+\infty} \exp(-(\lambda+\mu)t) \frac{(\lambda+\mu)^n t^n}{n!} \left(1 - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{2k}{k} \frac{p^{k+1}q^k}{k+1}\right). \tag{14}$$

If BP denotes the busy period of the $M/M/1$ queueing system, it is known that

$$P_{0,0}(t) = 1 - \frac{\lambda}{\mu} \mathbb{P}(BP \leqq t)$$

(see [1], p. 153.) Relation (14) gives then an expression of the busy period distribution:

$$\mathbb{P}(BP \leqq t) = \sum_{n=1}^{+\infty} \exp(-(\lambda+\mu)t) \frac{(\lambda+\mu)^n t^n}{n!} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{2k}{k} \frac{p^k q^{k+1}}{k+1}.$$

This expression was also obtained and studied in [5].

3.1.1. *Limiting behaviour of the transient probabilities.* We can verify that the well-known limiting behaviour of the $M/M/1$ queue is obtained when t tends to infinity. Let ρ be defined as $\rho = \lambda/\mu = p/q$ if $\lambda < \mu$ and $\rho \equiv 1$ otherwise.

Using Theorem 3.2 for $j = 0$ and the previous relation giving $(P^n)_{0,0}$ we obtain the following relation:

$$1 - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{2k}{k} \frac{p^{k+1}q^k}{k+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} s(n+1-2k, k)p^k q^{n-k}.$$

If $\pi_{0,j}$ denotes the limit of $P_{0,j}(t)$ when t grows to infinity, we have

$$\begin{aligned}
\pi_{0,j} &= \lim_{t \to +\infty} P_{0,j}(t) = \lim_{n \to +\infty} (P^n)_{0,j} \\
&= \frac{p^j}{q^j} \lim_{n \to +\infty} \sum_{k=0}^{\lfloor (n-j)/2 \rfloor} s(n+1-2k, k)p^k q^{n-k} \\
&= \frac{p^j}{q^j} \lim_{n \to +\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} s(n+1-2k, k)p^k q^{n-k} \\
&= \frac{p^j}{q^j} \lim_{n \to +\infty} \left[1 - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{2k}{k} \frac{p^{k+1}q^k}{k+1} \right] \\
&= \frac{p^j}{q^j} (1 - pC(pq)) \\
&= \rho^j (1-\rho)
\end{aligned}$$

which is the well-known formula for the stationary behaviour of the $M/M/1$ queue.

3.2. *The queue is not initially empty.* Using some results obtained in [4] (Equations (3), (6), (10) and Theorem 1(b)), we can write $P_{i,j}(t)$, for $i \leqq j$, as follows:

$$(15)\quad P_{i,j}(t) = \exp(-(\lambda+\mu)t) \left(\frac{\lambda}{\mu}\right)^{(j-i)/2} [I_{j-i}(2t\sqrt{\lambda\mu}) - I_{i+j+2}(2t\sqrt{\lambda\mu})] + \left(\frac{\mu}{\lambda}\right)^{i+1} P_{0,i+j+1}(t)$$

where I_k denotes the modified Bessel function of the first kind, that is

$$I_k(x) = \sum_{m=0}^{+\infty} \frac{(x/2)^{2m+k}}{m!\,(k+m)!}.$$

So, the general case reduces to the case of the queue initially empty by means of (15). However, it may be interesting to show how to derive (15) from the generating function expression. Combining (10) and (11), we can write for all $0 \leqq i \leqq j$,

$$(16)\qquad (\Phi_P(z))_{i,j} = (\Phi_Q(z))_{i,j} + \frac{q^{i+1}}{p^{i+1}} (\Phi_P(z))_{0,i+j+1}$$

where Q is the matrix defined in the proof of Theorem 2.3 by $Q = pV + qW$. To obtain an expression of $P_{i,j}(t)$ in the general case, it remains to perform the inversion of Φ_Q. Let us consider the so-called absorbing process associated with X, obtained by making state 0 absorbing. For every i, j, $(Q^n)_{i,j}$ is the probability to go from state $i+1$ to state $j+1$ in n transitions on the absorbing process. The potential kernel, $\Phi_Q(z)$, of Q is equal to $\eta(z)Y(z)$ (see Theorem 2.3). It follows that, for $i \leqq j$,

$$(\Phi_Q(z))_{i,j} = \sum_{k=0}^{i} p^{k+j-i} q^k z^{2k+j-i} \eta(z)^{2k+j-i+1} \tag{17}$$

which, using Lemma 3.1, can be written

$$(\Phi_Q(z))_{i,j} = \sum_{n=0}^{+\infty} \sum_{k=0}^{i} s(2k+j-i+1, n) p^{n+k+j-i} q^{n+k} z^{2n+2k+j-i}.$$

In the last sum, the variable change $n+k \to k$ leads to

$$(\Phi_Q(z))_{i,j} = \sum_{n=0}^{+\infty} \sum_{k=n}^{n+i} s(2k-2n+j-i+1, n) p^{k+j-i} q^k z^{2k+j-i}.$$

Exchanging first the order of summations and exchanging then the indexes n and k, we obtain

$$(\Phi_Q(z))_{i,j} = \sum_{n=0}^{i-1} \sum_{k=0}^{n} s(2n-2k+j-i+1, k) p^{n+j-i} q^n z^{2n+j-i} + \sum_{n=i}^{+\infty} \sum_{k=n-i}^{n} s(2n-2k+j-i+1, k) p^{n+j-i} q^n z^{2n+j-i}.$$

It follows that the only non-zero values of the sequence $(Q^n)_{i,j}$ are the following:

$$(Q^{2n+j-i})_{i,j} = \begin{cases} p^{n+j-i} q^n \sum_{k=0}^{n} s(2n-2k+j-i+1, k) & \text{if } n \leqq i \\ p^{n+j-i} q^n \sum_{k=n-i}^{n} s(2n-2k+j-i+1, k) & \text{if } n \geqq i+1. \end{cases}$$

In order to simplify this expression, we use the following result which is immediate to prove:

$$\sum_{k=0}^{l} s(h-2k+1, k) = \binom{h}{l} \quad \text{for every } l \geqq 0 \text{ and } h \geqq 2l.$$

We obtain that

$$(Q^{2n+j-i})_{i,j} = \begin{cases} p^{n+j-i} q^n \binom{2n+j-i}{n} & \text{if } n \leqq i \\ p^{n+j-i} q^n \left[\binom{2n+j-i}{n} - \binom{2n+j-i}{n-i-1} \right] & \text{if } n \geqq i+1. \end{cases}$$

Putting these results into (16) we obtain, after some algebra, the general expression (15).

4. Conclusion

In this paper we have derived analytical formulae giving the transient behaviour of the $M/M/1$ queueing system. These formulae are quite simple, especially when the queue is initially empty. Moreover, we get a simple expression of the probability to go from state i to state j in n transitions on the absorbing process obtained by making state 0 absorbing. Further work could be the extension of this method to more general queueing systems such as the $M/M/c$ queue.

Acknowledgement

The authors would like to thank the referee for his helpful comments concerning (16) and the interpretation of the probability $(P^n)_{0,0}$.

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