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Interval availability analysis using operational periods

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Abstract

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Consider a *repairable* computer system alternating periods during which it delivers useful service (*operational* periods) with periods in which it is being repared (*unoperational* periods). Interval availability is a dependability measure of such a system defined by the fraction of time during which a system is in operation over a finite observation period. In general, to evaluate the interval availability distribution of repairable computer systems analysts use Markov models. In this paper, we first deal with semi-Markov processes and under some conditions we develop a method to compute this distribution. It is based on the analysis of the length of the successive operational and unoperational periods. Particular attention is then devoted to the Markov case leading to a specific algorithm. As a byproduct, we obtain the distribution of the cumulative operational time up to the nth operational period.

Keywords: Repairable computer systems, semi-Markov processes, Markov processes, interval availability, cumulative operational time, operational periods.

1. Introduction

Let us call *repairable* a computer system that is able to put itself (or to be put) back in operation after a fault in its hardware or its software, owing to some kind of redundancy in its conception. We consider such a system modeled by a stochastic process X in continuous time. The state space S is assumed to be finite and partitionned into two classes: the subset U of *operational* states and the subset D of *unoperational* states. When the system is in an operational state (we say also that the system is *up*), it is considered to give some kind of service to the user while in an unoperational one (or *down*), no useful service is delivered and only system operations are performed to restore an operational situation. The criteria defining each state as up or down is specified by the designer of the model according to the objectives of the modeling task. To avoid terminology discussions, we will call *failure* a transition of the process from the subset U to the subset D and only those, and *repair* a transition from D to U and also only those transitions.

The states of process X are indexed in the following way:

$$\text{card}(S) = M \text{ and } S = \{1, 2, \ldots, M\},$$

$$\text{card}(U) = L \text{ and } U = \{1, 2, \ldots, L\},$$

$$\text{card}(D) = M - L \text{ and } D = \{L+1, L+2, \ldots, M\}.$$

The following assumptions will be made throughout the paper. The process X is irreducible in the following sense:

$$\text{for all } i, j \in S \text{ there exists } t \text{ such that } \Pr(X_t = j \mid X_0 = i) > 0.$$

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Moreover, to simplify the developments, we consider that the system starts always in an operational state, that is,

$$X_0 \in U.$$

The *interval availability* in $[0, T]$, where T is a given positive constant, is the proportion of $[0, T]$ in which the system has been in operation. Its distribution is an important measure, especially for dependable computer systems. This random variable (r.v.) can be written as the ratio

$$\frac{C_T}{T}$$

where C_T is the *cumulative operational time* up to the instant T, defined in turn by

$$C_T \stackrel{\text{def}}{=} \int_0^T 1_{(X_s \in U)} \, \mathrm{d}s$$

($1_{(E)}$ denotes the indicator of the event E).

Since T is a constant (usually called the *mission time*), the analysis of the stochastic properties of the interval availability can be performed by analyzing those of C_T, avoiding the T denominator. In the sequel, we will only consider this last r.v. Previous work can be found in [5] where the problem is stated in terms of a linear hyperbolic system of partial differential equations and it is solved by explicit finite-difference methods in [4]. The uniformization method has been used in [2] for Markov processes, bounding errors caused by truncation of an infinite series. This work has been extended in [9] to obtain a closed form expression for the distribution of C_T, which is given in terms of growing size matrices.

In this paper, we describe the behaviour of the r.v. C_T, for semi Markov processes, using the operational and unoperational periods durations. The distribution of these durations can be found in [7] in the Markov case and in [8] for semi Markov reward processes. The expression of the distribution of C_T that we present here needs, to be tractable, that the sequences of operational and unoperational periods are independent. This property is analysed using the canonical embedded Markov chain of the process X

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and lead, in the Markovian case, to a specific algorithm obtained by truncating infinite series. We show that very simple conditions are sufficient to have the independence property. These conditions trivially hold at least in some particular and usual cases. The interest of our algorithm is that it performs much better than the general one given in [2].

In the next section, we relate the r.v. C_T and the sojourn times of the process in the subsets U and D. We follow the approach of [3] where the case $\text{card}(U) = \text{card}(D) = 1$ is considered in an introduction to renewal theory. We analyze some conditions allowing to apply it to our more general case. Since this approach involves the computation of sums of sojourn times in the subset U and sums of sojourn times in D, these random variables are analyzed in Section 3 in the Markovian case. We give the form of their distributions and an algorithm to compute them. In the same section, we give an algorithm to compute the distribution of the cumulative operational time up to time T under the conditions of Section 2. The last section is devoted to some conclusions.

2. Operational and unoperational periods

Let us consider the stochastic process defined at any instant t by the r.v. $1_{(X_t \in U)}$. Its trajectories alternates between values 1 and 0, starting from value 1 since we assume that $X_0 \in U$. The intervals in which the value is 1, that is, in which the system is in operation, are called *operational periods* and their lengths will be denoted by $U_1, U_2 \cdots$ In the same way, the remaining intervals are the *unoperational periods* whose lengths are $D_1, D_2 \cdots$ Of course, we will only consider regular processes so that there will be no explosive behaviour: for any r.v. U_i and D_j we assume that

$$\Pr(0 < U_i < +\infty) = \Pr(0 < D_j < +\infty) = 1. \tag{1}$$

The distributions of these r.v. can be found in [7] in the case of an irreducible Markov model and in [8] for semi-Markov reward models with absorbing states.

Following [3], we denote by τ_t the r.v.

$$\tau_t \stackrel{\text{def}}{=} \inf\{s \mid C_s \geqslant t\}.$$

See that $C_{\tau_t} = t$. Since for all t we have $C_t \leqslant t$, we also have that

$$\text{for all } t,\ \tau_t \geqslant t$$

and it follows that

$$\text{for all } t < T,\ \{C_T < t\} = \{\tau_t > T\}. \tag{2}$$

Let us denote by F_t the number of failures appearing strictly before t and by R_t the number of repairs before t. That is, if F^i is the instant of the ith failure and R^j is the instant of the jth repair, $i \geqslant 1$ and $j \geqslant 1$, we have

$$F_t = \text{card}(\{i \geqslant 1 \mid F^i < t\}),$$

$$R_t = \text{card}(\{j \geqslant 1 \mid R^j \leqslant t\}).$$

Note that in the definition of the counter F_t we do not include an eventual failure at time t. This assumption is only made to simplify the two next lemmas and it is particularly without repercussion if the event "there is a failure exactly at time t" is of measure 0. We then have trivially that

$$F_{\tau_t} = R_{\tau_t}.$$

To see this, just observe that the random instant τ_t belongs almost surely (a.s.) to an interval of the form $]R^k, F^{k+1}]$ with the additionnal convention $R^0 \stackrel{def}{=} 0$.

Let us consider the r.v. "cumulative operational time up to the mth operational period", denoted by TU_m, and "cumulative unoperational time up to the nth unoperational period", denoted by TD_n, that is,

$$TU_m \stackrel{\text{def}}{=} U_1 + U_2 + \cdots + U_m,$$

$$TD_n \stackrel{\text{def}}{=} D_1 + D_2 + \cdots + D_n.$$

For convenience, we set $TU_0 \stackrel{def}{=} 0$ and $TD_0 \stackrel{def}{=} 0$. We construct two auxiliary counting processes $\tilde{F} = (\tilde{F}_t)_{t \geqslant 0}$ and $\tilde{R} = (\tilde{R}_t)_{t \geqslant 0}$ by

$$\tilde{F}_t \stackrel{\text{def}}{=} \text{card}(\{m \geqslant 1 \mid TU_m < t\}), \tag{3}$$

$$\tilde{R}_t \stackrel{\text{def}}{=} \text{card}(\{n \geqslant 1 \mid TD_n \leqslant t\}). \tag{4}$$

We can state now the first result of this approach in the following lemma.

Lemma 2.1. *At any instant t we have $\tilde{F}_t = \tilde{R}_{\tau_t - t}$ a.s.*

Proof. Let k be the number of failures appearing strictly before the instant τ_t. Since the cumulative operational time up to τ_t is t, we have that $\tilde{F}_t = k$. The cumulative unoperational time up to τ_t is $\tau_t - t$. This means that the kth point in process $\tilde{R}$ arrives at the instant $\tau_t - t$, which implies that $\tilde{R}_{\tau_t - t} = k = \tilde{F}_t$. □

The basic result which relates the r.v. C_T to the processes $\tilde{F}$ and $\tilde{R}$ is

Lemma 2.2. *For $t < T$, we have $\{\tau_t > T\} = \{\tilde{F}_t > \tilde{R}_{T-t}\}$ a.s.*

Proof. Again as in the previous lemma, we consider the evolution of process X up to the random instant τ_t. If $T < \tau_t$ then $\tilde{R}_{T-t} < \tilde{R}_{\tau_t - t}$ since there is an arrival in $\tilde{R}$ exactly at time $\tau_t - t$ (Lemma 2.1). Reciprocally, when $\tilde{R}_{T-t} < \tilde{F}_t = \tilde{R}_{\tau_t - t}$ then necessarily we have $T - t < \tau_t - t$ since $\tilde{R}_s$ is a non decreasing function of s. □

Remark. In [3] there is a minor error in this statement since it is claimed in Subsection 2.3 (page 115) that $\{\tau_t > T\} = \{\tilde{F}_t \geqslant \tilde{R}_{T-t}\}$. Note that $\tilde{F}_t$ and $\tilde{R}_{\tau_t - t}$ are integer random variables so that the point is relevant.

Consider the distribution of C_T. Observe first that

$$\Pr(C_T = T) = \Pr(U_1 \geqslant T)$$

so that the distribution of C_T has a jump with value $\Pr(U_1 \geqslant T)$ in T. This value is the *reliability* of the system at time T. Putting together (2) and Lemma 2.2, we obtain that if $t < T$,

$$\begin{aligned} \Pr(C_T < t) &= \Pr(\tau_t > T) \\ &= \Pr(\tilde{F}_t > \tilde{R}_{T-t}). \end{aligned} \tag{5}$$

If $\tilde{F}$ and $\tilde{R}$ are independent, we can write for $t < T$,

$$\Pr(C_T < t) = \sum_{n=0}^{+\infty} \Pr(\tilde{F}_t > n) \Pr(\tilde{R}_{T-t} = n) \tag{6}$$

and analyse independently the two process distributions.

This independence condition obviously holds in the simple case of a two-state homogeneous Markov process, with $L = 1$ (one state is operational, the other is unoperational or failed). This model has been extensively analyzed in several works, and in particular closed forms of the distribution of C_T have been obtained. Let us denote by λ the transition rate from the only operational state and by μ the transition rate from the failed state. Note first that

$$\Pr(C_T = T) = \Pr(U_1 \geqslant T) = e^{-\lambda T}.$$

Let us take $t < T$. In [1] the following expression using Bessel functions is derived by specializing previous work by Takács:

$$\Pr(C_T \leqslant t) = 1 - e^{-\lambda t}\left[1 + \sqrt{\lambda\mu t}\int_0^{T-t}\frac{e^{-\mu\xi}}{\sqrt{\xi}} I_1\left(2\sqrt{\lambda\mu t\xi}\right) d\xi\right]$$

where

$$I_1(x) = \sum_{j=0}^{\infty}\left(\frac{x}{2}\right)^{2j+1}\frac{1}{j!(j+1)!}.$$

In the book by S. Ross [6], using uniformization it is derived that

$$\Pr(C_T \leqslant t) = \sum_{n=1}^{\infty} e^{-(\lambda+\mu)T}\frac{(\lambda+\mu)^n T^n}{n!} \times \sum_{k=1}^{n}\binom{n}{k-1}\left(\frac{\mu}{\lambda+\mu}\right)^{k-1}\left(\frac{\lambda}{\lambda+\mu}\right)^{n-k+1}$$
$$\times \sum_{i=k}^{n}\binom{n}{i}\left(\frac{t}{T}\right)^{i}\left(1-\frac{t}{T}\right)^{n-i}.$$

The expression obtained in [3] is

$$\Pr(C_T \leqslant t) = \sum_{n=0}^{\infty}\frac{[\mu(T-t)]^n}{n!} e^{-\mu(T-t)}\sum_{k=n+1}^{\infty}\frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

(in fact, in [3] the error in the statement corresponding to Lemma 2.2 induces a wrong initial value of index k in the imbedded series).

Another interesting simple case is the two-state model with constant repairs and exponential operational times. Let $r > 0$ be the repair time and, as before, denote by λ the fialure rate. In this situation, $\tilde{R}_{T-t} = n$ is equivalent to $\lfloor (T-t)/r \rfloor = n$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the real x.

Then,

$$\Pr\left(\tilde{R}_{T-t} = n\right) = 1_{(n = \lfloor (T-t)/r \rfloor)}$$

and

$$\Pr(C_T < t) = \sum_{n=0}^{+\infty}\Pr\left(\tilde{F}_t > n\right)\Pr\left(\tilde{R}_{T-t} = n\right)$$
$$= \Pr\left(\tilde{F}_t > \lfloor (T-t)/r \rfloor\right)$$
$$= 1 - e^{-\lambda t}\sum_{i=0}^{\lfloor (T-t)/r \rfloor}\frac{(\lambda t)^i}{i!}.$$

This distribution has $\lceil T/r \rceil$ jumps at the points $t = T - kr$ for $k = 0, 1, \ldots, \lceil T/r \rceil - 1$, where $\lceil x \rceil$ denotes the least integer greater than or equal to x. The size of the jump for $t = T - kr$ is

$$e^{-\lambda(T-kr)} \frac{(\lambda(T-kr))^k}{k!}.$$

Consider again the general case and relation (6). Rewriting it in terms of operational and unoperational periods lengths, we have that

$$\Pr(\tilde{F}_t > n) = \Pr(TU_{n+1} < t) \tag{7}$$

and

$$\Pr(\tilde{R}_{T-t} = n) = \Pr(TD_n \leqslant T - t) - \Pr(TD_{n+1} \leqslant T - t), \tag{8}$$

so that we can write

$$\Pr(C_T < t) = \sum_{n=0}^{+\infty} \left[\Pr(TD_n \leqslant T - t) - \Pr(TD_{n+1} \leqslant T - t)\right] \Pr(TU_{n+1} < t). \tag{9}$$

A question is now under which conditions the processes $\tilde{F}$ and $\tilde{R}$ are independent to allow the application of relation (9). We will consider this question under the assumption that X is semi-Markov: The sojourn times of X in each state i, the *holding times* in i, are i.i.d. and they are independent of the holding times in the other states. Moreover, if Y_n is the $(n+1)$th visited state by X, the process $(Y_n)_{n \geqslant 0}$ is a homogeneous Markov chain with transition probability matrix denoted by P and, given Y_n, the r.v. Y_{n+1} does not depend on the holding times of X.

Suppose that there exist two particular states i in U and j in D such that every operational period begins in i and every unoperational period begins in j (this implies in particular that $X_0 = i$). It is then clear that the two sequences $(U_i)_{i \geqslant 1}$ and $(D_j)_{j \geqslant 1}$ are independent. Moreover, they are both i.i.d. sequences of r.v. The next result gives a generalization of this statement.

Let us first define a condition to be checked directly in P. In the sequel, we use the notation

$$\text{for all } i \in S \text{ and } B \subseteq S,\ P(i, B) \stackrel{\text{def}}{=} \sum_{j \in B} P(i, j).$$

Definition 2.3. *The failures (transitions from U to D) are U-independent iff matrix P verifies*

for all $i \in U$ such that $P(i, D) > 0$, we have

for all $j \in D$, $\dfrac{P(i, j)}{P(i, D)}$ does not depend on i.

In the same way, the repairs are D-independent iff

for all $j \in D$ such that $P(j, U) > 0$, we have

for all $i \in U$, $\dfrac{P(j, i)}{P(j, U)}$ does not depend on j.

We have now the following condition for independence between operational and unoperational periods lengths.

Theorem 2.4. *If the failures are U-independent and the repairs are D-independent, then*
(i) *the two sequences $(U_i)_{i \geqslant 1}$ and $(D_j)_{j \geqslant 1}$ are independent,*
(ii) *the sequence $(D_j)_{j \geqslant 1}$ is i.i.d.,*
(iii) *the r.v. $(U_i)_{i \geqslant 1}$ are independent and the r.v. $(U_i)_{i \geqslant 2}$ are identically distributed.*

Proof. Let us denote by

$$U^- = \text{the "input states" of } U$$
$$\stackrel{\text{def}}{=} \{i \in U \mid \text{there exists } j \in D \text{ such that } P(j, i) > 0\},$$
$$U^+ = \text{the "output states" of } U$$
$$\stackrel{\text{def}}{=} \{i \in U \mid \text{there exists } j \in D \text{ such that } P(i, j) > 0\}.$$

In the same way, we define the subsets D^- and D^+ of D. Observe that $P(i, D) = P(i, D^-)$ for all i in U and that $P(j, U) = P(j, U^-)$ for any j in D. Let us denote by IU_m (resp. ID_n) the state in which the mth operational (resp. nth unoperational) period begins and by OU_m (resp. OD_n) the last state of the mth operational (resp. nth unoperational) period. If the failures are U-independent, then for any state j in D^- and for all $m \geq 1$,

$$\begin{aligned}\Pr(ID_m = j) &= \sum_{i \in U^+} \Pr(ID_m = j \mid OU_m = i) \Pr(OU_m = i) \\ &= \sum_{i \in U^+} \frac{P(i, j)}{P(i, D)} \Pr(OU_m = i) \\ &= \frac{P(i, j)}{P(i, D)} \quad \text{for any } i \in U^+. \end{aligned} \tag{10}$$

Symetrically, for any operational state i and for $n \geq 2$, the condition of D-independence on the repairs allows us to write

$$\begin{aligned}\Pr(IU_n) = i) &= \sum_{j \in D^+} \Pr(IU_n) = i \mid OD_{n-1} = j) \Pr(OD_{n-1} = j) \\ &= \sum_{j \in D^+} \frac{P(j, i)}{P(j, U)} \Pr(OD_{n-1} = j) \\ &= \frac{P(j, i)}{P(j, U)} \quad \text{for any } j \in D^+. \end{aligned} \tag{11}$$

Let us define the random vectors $\vec{U}_m$ and $\vec{D}_n$ by

$$\vec{U}_m \stackrel{\text{def}}{=} (U_1, \ldots, U_m), \ \vec{U}_0 \stackrel{\text{def}}{=} 0,$$
$$\vec{D}_n \stackrel{\text{def}}{=} (D_1, \ldots, D_n), \ \vec{D}_0 \stackrel{\text{def}}{=} 0.$$

We also denote by $\vec{t}_m$ and $\vec{s}_n$ the non negative real vectors $(t_1, \ldots, t_m)$ and $(s_1, \ldots, s_n)$, with $\vec{t}_0 = t_0 \geq 0$ and $\vec{s}_0 = s_0 \geq 0$.

For $n \geq 1$, the joint distribution of $(\vec{U}_n, \vec{D}_n)$ can be written as

$$\Pr\left(\vec{U}_n \leq \vec{t}_n, \vec{D}_n \leq \vec{s}_n\right) = \Pr\left(\vec{U}_n \leq \vec{t}_n, \vec{D}_{n-1} \leq \vec{s}_{n-1}\right) \Pr\left(D_n \leq s_n \mid \vec{U}_n \leq \vec{t}_n, \vec{D}_{n-1} \leq \vec{s}_{n-1}\right).$$

Let us rewrite the second factor of the right hand side by conditionning with respect to the first state of the nth unoperational period.

$$\Pr\left(D_n \leq s_n \mid \vec{U}_n \leq \vec{t}_n, \vec{D}_{n-1} \leq \vec{s}_{n-1}\right) =$$
$$\sum_{j \in D^-} \Pr\left(D_n \leq s_n \mid \vec{U}_n \leq \vec{t}_n, \vec{D}_{n-1} \leq \vec{s}_{n-1}, ID_n = j\right) \Pr\left(ID_n = j \mid \vec{U}_n \leq \vec{t}_n, \vec{D}_{n-1} \leq \vec{s}_{n-1}\right).$$

It is clear that

$$\Pr\left(D_n \leqslant s_n \mid \vec{U}_n \leqslant \vec{t}_n,\ \vec{D}_{n-1} \leqslant \vec{s}_{n-1},\ ID_n = j\right) = \Pr(D_n \leqslant s_n \mid ID_n = j)$$

and the second factor of the general term in the sum can be rewritten by conditionning with respect to the last state of the nth operational period using (10) as

$$\begin{aligned}\Pr\left(ID_n = j \mid \vec{U}_n \leqslant \vec{t}_n,\ \vec{D}_{n-1} \leqslant \vec{s}_{n-1}\right) &= \sum_{i \in U^+} \Pr\left(ID_n = j \mid \vec{U}_n \leqslant \vec{t}_n,\ \vec{D}_{n-1} \leqslant \vec{s}_{n-1},\ OU_n = i\right) \\ &\quad \times \Pr\left(OU_n = i \mid \vec{U}_n \leqslant \vec{t}_n,\ \vec{D}_{n-1} \leqslant \vec{s}_{n-1}\right) \\ &= \sum_{i \in U^+} \frac{P(i, j)}{P(i, D)} \Pr\left(OU_n = i \mid \vec{U}_n \leqslant \vec{t}_n,\ \vec{D}_{n-1} \leqslant \vec{s}_{n-1}\right) \\ &= \sum_{i \in U^+} \Pr(ID_n = j) \Pr\left(OU_n = i \mid \vec{U}_n \leqslant \vec{t}_n,\ \vec{D}_{n-1} \leqslant \vec{s}_{n-1}\right) \\ &= \Pr(ID_n = j).\end{aligned}$$

Coming back to the initial relation,

$$\begin{aligned}\Pr\left(\vec{U}_n \leqslant \vec{t}_n,\ \vec{D}_n \leqslant \vec{s}_n\right) &= \Pr\left(\vec{U}_n \leqslant \vec{t}_n,\ \vec{D}_{n-1} \leqslant \vec{s}_{n-1}\right) \sum_{j \in D^-} \Pr(D_n \leqslant s_n \mid ID_n = j) \Pr(ID_n = j) \\ &= \Pr\left(\vec{U}_n \leqslant \vec{t}_n,\ \vec{D}_{n-1} \leqslant \vec{s}_{n-1}\right) \Pr(D_n \leqslant s_n).\end{aligned}$$

This proves by induction that the components of the random vector $(\vec{U}_n, \vec{D}_n)$ are independent. Observe that the proof is valid even if $n = 1$. It remains to verify that the independence also holds when we consider the random vector $(\vec{U}_{n+1}, \vec{D}_n)$. The argument is quite similar using (11) and will not be developped here.

We have shown that any finite sequence of operational and unoperational periods $U_1, D_1, U_2, \cdots$ is composed by independent r.v. In particular, this means that the r.v. $(U_i)_{i \geqslant 1}$ are independent and that the r.v. $(D_j)_{j \geqslant 1}$ are independent. Moreover, since (10) is valid for any $m \geqslant 1$, clearly the distribution of D_m does not depends on m and in the same way, the distribution of U_n does not depend on n if $n \geqslant 2$. The only exception is the distribution of U_1. For instance, if $\text{card}(U^-) > 1$ an if we know that the process starts in, say, state i, then necessarily the distribution of U_1 will be different of the distribution of U_n for all $n \geqslant 2$. □

As we see, in order to apply (6), the previous theorem gives us a condition that can be easily checked on the transition probability matrix. Note that it is always satisfied if

$$\left[\text{card}(U^-) = 1 \text{ or } \text{card}(D^+) = 1\right] \text{ and } \left[\text{card}(U^+) = 1 \text{ or } \text{card}(D^-) = 1\right]. \tag{12}$$

In particular, this is the case if one of the sets U or D is reduced to a single state.

It can be noted that the applicability conditions of the previous theorem (U-independent failures and D-independent repairs) are by no means necessary. Just consider a Markov model with transition rate matrix A satisfying the well known *strong lumpability* conditions with respect to the partition $\{U, D\}$, that is,

$$\text{for all } i \in U,\ \sum_{j \in D} A(i, j) \text{ does not depend on } i,$$

$$\text{for all } j \in D,\ \sum_{i \in U} A(j, i) \text{ does not depend on } j.$$

Under these conditions, the sequences of sojourn times of process X in the subsets U and D are i.i.d. (exponential) random variables and the two sequences are independent. Moreover, this property holds for any initial distribution of X. It is possible (and easy to verify) that the strong lumpability conditions can be satisfied while the U-independent failures and D-independent repairs are not and viceversa.

The application of (6) involves the distribution of the r.v. TU_n and TD_n. These r.v. are interesting in themselves since they give supplementary information about the evolution of X with time. Similar r.v. appear when considering the number of failures during the mission time. Recalling that F_t denotes the number of failures appearing in the interval $]0, t[$, we have that

$$\begin{aligned}\Pr(F_t > n) &= \Pr(U_1 + D_1 + U_2 + \cdots + D_{n-1} + U_n < t)\\ &= \Pr(TU_n + TD_{n-1} < t)\end{aligned}$$

and thus, we find again the distribution of a sum of period lengths, here operational and unoperational. In the next section, we limit ourselves to the Markovian case to analyse in detail the distributions of TU_n and TD_n.

3. The Markovian case

In this section, we consider the case where X is a continuous time Markov process. We first recall some recent work about the distribution of the nth operational period length. We then derive an expression for the cumulative operational time up to the nth operational period. This result will be used to obtain a formula to compute the cumulative operational time up to time T, under the hypothesis of Theorem 2.4.

The Markov process $X = (X_t)_{t \geq 0}$ is supposed to be homogeneous and irreducible. It is given by its transition rate matrix $A = (A(i, j), i, j \in S)$ (its infinitesimal generator) where

$$A(i, i) = -\sum_{j \neq i} A(i, j).$$

The initial distribution is concentrated on the operational states ($X_0 \in U$) and we denote it by the L dimensioned row vector α_U. We denote by 1^* the column vector with all its coordinates equal to 1 (we always use row vectors and $(.)^*$ denotes the transpose operator) and by I the identity matrix, their dimensions being defined by the context.

We give in the following subsection an algorithm to compute the distribution of TU_n, the cumulative operational time up to the nth operational period. In the same way, we shall obtain the distribution of TD_n, the cumulative unoperational time up to the nth unoperational period.

3.1. Calculating the distributions of TU_n and TD_n

Let P be the transition probability matrix of the embedded Markov chain of X at the transition instants and Λ the $M \times M$ diagonal matrix whose ith entry, λ_i, is the output rate from state i.

The matrices A and P are related by

$$P = I + \Lambda^{-1} A.$$

We decompose the matrices A, P and Λ into submatrices as follows:

$$A = \begin{pmatrix} A_U & A_{UD} \\ A_{DU} & A_D \end{pmatrix}, \quad P = \begin{pmatrix} P_U & P_{UD} \\ P_{DU} & P_D \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \Lambda_U & 0 \\ 0 & \Lambda_D \end{pmatrix}.$$

The distribution of U_n can be found in [7]. It is given by

$$\Pr(U_n \leq t) = 1 - \alpha_U G_U^{n-1} \, e^{A_U t} 1^* \tag{13}$$

where $G_U = A_U^{-1} A_{UD} A_D^{-1} A_{DU} = (I - P_U)^{-1} P_{UD} (I - P_D)^{-1} P_{DU}$. A study on the properties of matrix G_U has been done in [7].

Let us now analyse the cumulative operational time up to the nth operational period. Define the row vectors

$$z_U(n, t) \stackrel{\text{def}}{=} (\Pr(TU_n > t \mid X_0 = i),\ i \in U),$$

$$z_D(n, t) \stackrel{\text{def}}{=} (\Pr(TU_n > t \mid X_0 = i),\ i \in D)$$

and observe that if $X_0 \in U$,

$$\Pr(TU_n > t) = \alpha_U z_U^*(n, t).$$

To derive the distribution of TU_n, we will use the following lemma.

Lemma 3.1. *For all $n \geqslant 1$,*

$$z_D^*(n, t) = (I - P_D)^{-1} P_{DU} z_U^*(n, t).$$

Proof. For every $i \in D$,

$$\Pr(TU_n > t \mid X_0 = i) = \sum_{j \in U} P(i, j) \Pr(TU_n > t \mid X_0 = j) + \sum_{j \in D} P(i, j) \Pr(TU_n > t \mid X_0 = j).$$

This gives in matrix notation

$$z_D^*(n, t) = P_{DU} z_U^*(n, t) + P_D z_D^*(n, t).$$

Solving this last equation for z_D concludes the proof. □

The following result gives the distribution of TU_n.

Theorem 3.2. *For all $n \geqslant 1$, the distribution of TU_n is given by*

$$\Pr(TU_n \leqslant t) = 1 - \alpha_{U,n} e^{Q_{U,n} t} 1^*$$

where $\alpha_{U,n} = (\alpha_U, 0, \ldots, 0)$ is the nL dimensioned row vector in which each 0 represents the L dimensioned null vector and $Q_{U,n}$ is the $nL \times nL$ matrix

$$Q_{U,n} = \begin{pmatrix} Q'_U & Q''_U & 0 & 0 & & \cdots & & & 0 & 0 \\ 0 & Q'_U & Q''_U & 0 & 0 & & \cdots & & 0 & 0 \\ 0 & 0 & Q'_U & Q''_U & 0 & 0 & & \cdots & 0 & 0 \\ \vdots & & & & \ddots & \ddots & & & \vdots & \vdots \\ \vdots & & & & & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & & & \cdots & & & 0 & Q'_U & Q''_U \\ 0 & 0 & & & \cdots & & & & 0 & Q'_U \end{pmatrix}$$

with $Q_{U,1} = Q'_U \stackrel{\text{def}}{=} A_U = -\Lambda_U(I - P_U)$ and $Q''_U \stackrel{\text{def}}{=} -A_{UD} A_D^{-1} A_{DU} = \Lambda_U P_{UD} (I - P_D)^{-1} P_{DU}$. In matrix $Q_{U,n}$, each 0 represents a $L \times L$ null matrix.

Proof. For $n = 1$, the result is immediat since $TU_1 = U_1$ (see eq. (13)).

Let $n \geqslant 2$ and $i \in U$. Writing the renewal equations for the Markov process X we obtain

$$\Pr(TU_n > t \mid X_0 = i) = \sum_{j \in U} P(i, j) \int_0^t \Pr(TU_n > t - s \mid X_0 = j) \lambda_i \, e^{-\lambda_i s} \, ds$$

$$+ \sum_{j \in D} P(i, j) \int_0^t \Pr(TU_{n-1} > t - s \mid X_0 = j) \lambda_i \, e^{-\lambda_i s} \, ds.$$

This gives after a variable change

$$\Pr(TU_n > t \mid X_0 = i) = \sum_{j \in U} \lambda_i \, \mathrm{e}^{-\lambda_i t} \int_0^t \mathrm{e}^{\lambda_i s} P(i, j) \Pr(TU_n > s \mid X_0 = j) \, \mathrm{d}s$$

$$+ \sum_{j \in D} \lambda_i \, \mathrm{e}^{-\lambda_i t} \int_0^t \mathrm{e}^{\lambda_i s} P(i, j) \Pr(TU_{n-1} > s \mid X_0 = j) \, \mathrm{d}s.$$

In matrix notation we have

$$z_U^*(n, t) = \Lambda_U \, \mathrm{e}^{-\Lambda_U t} \left[\int_0^t \mathrm{e}^{\Lambda_U s} P_U z_U^*(n, s) \, \mathrm{d}s + \int_0^t \mathrm{e}^{\Lambda_U s} P_{UD} z_D^*(n-1, s) \, \mathrm{d}s \right]$$

and using Lemma 3.1

$$z_U^*(n, t) = \Lambda_U \, \mathrm{e}^{-\Lambda_U t} \left[\int_0^t \mathrm{e}^{\Lambda_U s} P_U z_U^*(n, s) \, \mathrm{d}s + \int_0^t \mathrm{e}^{\Lambda_U s} P_{UD} (I - P_D)^{-1} P_{DU} z_U^*(n-1, s) \, \mathrm{d}s \right].$$

Taking now the derivative with respect to the variable t, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} z_U^*(n, t) = -\Lambda_U z_U^*(n, t) + \Lambda_U \left[P_U z_U^*(n, t) + P_{UD} (I - P_D)^{-1} P_{DU} z_U^*(n-1, t) \right]$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t} z_U^*(n, t) = -\Lambda_U (I - P_U) z_U^*(n, t) + \Lambda_U P_{UD} (I - P_D)^{-1} P_{DU} z_U^*(n-1, t)$$

$$= Q'_U z_U^*(n, t) + Q''_U z_U^*(n-1, t).$$

If $l_U(n, t)$ denotes the row vector $(z_U(n, t), z_U(n-1, t), \ldots, z_U(1, t))$ which dimension is nL, we obtain that, for every $n \geq 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t} l_U^*(n, t) = Q_{U,n} l_U^*(n, t)$$

that is, since $l_U^*(n, 0) = 1^*$ for every $n \geq 1$,

$$l_U^*(n, t) = \mathrm{e}^{Q_{U,n} t} 1^*.$$

Finally,

$$\Pr(TU_n > t) = \alpha_U z_U^*(n, t) = \alpha_{U,n} l_U^*(n, t) = \alpha_{U,n} \, \mathrm{e}^{Q_{U,n} t} 1^*$$

that is, $\Pr(TU_n \leq t) = 1 - \alpha_{U,n} \, \mathrm{e}^{Q_{U,n} t} 1^*$. □

In order to compute the distribution of TU_n for a fixed $n \geq 2$, one can proceed as follows. Let λ_U be the positive real number defined by $\lambda_U \stackrel{\text{def}}{=} \max(-Q'_U(i, i), i \in U)$, which represents the greatest output rate from the states of U since $Q'_U = \Lambda_U$. We denote by $P_{U,n}$ the $nL \times nL$ matrix

$$P_{U,n} \stackrel{\text{def}}{=} I + Q_{U,n} / \lambda_U.$$

Matrix $P_{U,n}$ is a function of the $L \times L$ matrices $P'_U \stackrel{\text{def}}{=} \mathrm{I} + \mathrm{Q}'_U / \lambda_U$ and $P''_U \stackrel{\text{def}}{=} \mathrm{Q}''_U / \lambda_U$. Remark that, since $(Q'_U + Q''_U) 1^* = 0$, the matrix $(P'_U + \mathrm{P}''_U)$ is a $L \times L$ stochastic matrix. It follows that $P_{U,n}^h 1^* = 1^*$ for every $h \leq n-1$. With this notation, we obtain for every $n \geq 1$

$$\Pr(TU_n \leq t) = \sum_{h=n}^{+\infty} \mathrm{e}^{-\lambda_U t} \frac{(\lambda_U t)^h}{h!} \left(1 - \alpha_{U,n} P_{U,n}^h 1^*\right).$$

Denoting now by $x_U^*(n, h)$ the vector composed by the L first entries of $P_{U,n}^h 1^*$, it follows that

$$\Pr(TU_n \leqslant t) = \sum_{h=n}^{\infty} e^{-\lambda_U t} \frac{(\lambda_U t)^h}{h!} (1 - \alpha_U x_U^*(n, h))$$

$$= \sum_{h=n}^{H} e^{-\lambda_U t} \frac{(\lambda_U t)^h}{h!} (1 - \alpha_U x_U^*(n, h)) + e_U(H, t, n)$$

where $e_U(H, t, n)$ is the rest of the series when truncating to step H, the sum from n to H being equal to 0 if $n > H$. It is easy to verify that

$$e_U(H, t, n) = \sum_{h=H+1}^{\infty} e^{-\lambda_U t} \frac{(\lambda_U t)^h}{h!} (1 - \alpha_U x_U^*(n, h)) \leqslant 1 - \sum_{h=0}^{H} e^{-\lambda_U t} \frac{(\lambda_U t)^h}{h!}$$

and so H can be evaluated beforehand for a given error tolerance. Since $P_{U,n}^h$ is a block upper triangular matrix, we can decompose it into four submatrices

$$P_{U,n}^h = \begin{pmatrix} P_U'^h & W_{n-1,h} \\ 0 & P_{U,n-1}^h \end{pmatrix}$$

where $W_{n-1,1}$ is the $L \times (n-1)L$ matrix $(P_U'', 0, \ldots, 0)$. These considerations lead to recursive relations:

$$W_{n-1,h} = P_U' W_{n-1,h-1} + W_{n-1,1} P_{U,n-1}^{h-1}$$

and

$$\begin{aligned} x_U^*(n, h) &= P_U'^h 1^* + W_{n-1,h} 1^* \\ &= P_U'^h 1^* + P_U' W_{n-1,h-1} 1^* + W_{n-1,1} P_{U,n-1}^{h-1} 1^* \\ &= P_U'^h 1^* + P_U' W_{n-1,h-1} 1^* + P_U'' x_U^*(n-1, h-1) \\ &= P_U' \left(P_U'^{h-1} 1^* + W_{n-1,h-1} 1^* \right) + P_U'' x_U^*(n-1, h-1) \\ &= P_U' x_U^*(n, h-1) + P_U'' x_U^*(n-1, h-1). \end{aligned}$$

This last relation induces a simple algorithm to evaluate $x_U^*(n, h)$ with initial values $x_U^*(0, j) = 0$ for every $j \geqslant 0$ and $x_U^*(j, 0) = 1^*$, for every $j \geqslant 1$.

In the same way, concerning the cumulative unoperational time up to the nth unoperational period TD_n, we have the same result by replacing the subset U by the subset D and taking $\beta_D = -\alpha_U A_U^{-1} A_{UD} = \alpha_U (I - P_U)^{-1} P_{UD}$ in place of α_U. The matrices used for the computation of the distribution of TD_n are now

$$Q_D' = A_D, \; Q_D'' = -A_{DU} A_U^{-1} A_{UD}$$

and

$$P_D' = I + Q_D' / \lambda_D, \; P_D'' = Q_D'' / \lambda_D$$

where $\lambda_D = \max(-Q_D'(i, i), i \in D)$. That is,

$$\Pr(TD_n \leqslant t) = \sum_{k=n}^{K} e^{-\lambda_D t} \frac{(\lambda_D t)^k}{k!} (1 - \beta_D x_D^*(n, k)) + e_D(K, t, n).$$

The sum is equal to 0 if $n > K$. The vectors $x_D^*(n, k)$ verify the following recursion

$$x_D^*(n, k) = P_D' x_D^*(n, k-1) + P_D'' x_D^*(n-1, k-1)$$

with initial values $x_D^*(0, j) = 0$ for every $j \geqslant 0$ and $x_D^*(j, 0) = 1^*$, for every $j \geqslant 1$. The rest $e_D(K, t, n)$ verifies

$$e_D(K, t, n) \leqslant 1 - \sum_{k=0}^{K} e^{-\lambda_D t} \frac{(\lambda_D t)^k}{k!}.$$

3.2. Calculating the distribution of C_T

In this subsection we develop an algorithm to compute the distribution of C_T when failures are U-independent and repairs are D-independent, applying relation (9).

It is easy to verify that the condition "failures are U-independent" is equivalent to the condition $P_{UD} = \theta_U^* g_D$ where $\theta_U^* = P_{UD} 1^*$. Observe that $g_D 1^* = 1$. In the same way, "repairs are D-independent" is equivalent to the condition $P_{DU} = \theta_D^* g_U$. It follows that matrix P_U'' has a more convenient form in which there is no need to deal with inverse matrices. We have

$$\begin{aligned} P_U'' &= \Lambda_U P_{UD}(I - P_D)^{-1} P_{DU}/\lambda_U \\ &= \Lambda_U \theta_U^* g_D (I - P_D)^{-1} \theta_D^* g_U/\lambda_U \\ &= \Lambda_U \theta_U^* g_U/\lambda_U \\ &= A_{UD} 1^* g_U/\lambda_U \end{aligned}$$

and symetrically,

$$P_D'' = A_{DU} 1^* g_D/\lambda_D.$$

Let ϵ be a given error tolerance specified by the user to compute $\Pr(C_T \leqslant t)$ when $t < T$. Let H and K be the smallest integers verifying

$$1 - \sum_{h=0}^{H} e^{-\lambda_U t} \frac{(\lambda_U t)^h}{h!} \leqslant \frac{\epsilon}{3},$$

$$1 - \sum_{k=0}^{K} e^{-\lambda_D (T-t)} \frac{(\lambda_D (T-t))^k}{k!} \leqslant \frac{\epsilon}{3}$$

and let N be defined by

$$N \stackrel{\text{def}}{=} \min(K, H - 1).$$

The distribution of C_T, under the two independence conditions, can be written as

$$\begin{aligned} \Pr(C_T \leqslant t) = &\sum_{n=0}^{N} \left[\Pr(TD_n \leqslant T - t) - \Pr(TD_{n+1} \leqslant T - t)\right] \Pr(TU_{n+1} \leqslant t) \\ &+ \sum_{n=N+1}^{\infty} \left[\Pr(TD_n \leqslant T - t) - \Pr(TD_{n+1} \leqslant T - t)\right] \Pr(TU_{n+1} \leqslant t). \end{aligned}$$

The first sum is denoted by Σ and will be decomposed into three different sums, so we denote the second sum in the previous formula by Σ_4. This second sum, Σ_4, verify

$$\begin{aligned} \Sigma_4 &\leqslant \Pr(TU_{N+2} \leqslant t) \sum_{n=N+1}^{\infty} \left[\Pr(TD_n \leqslant T - t) - \Pr(TD_{n+1} \leqslant T - t)\right] \\ &= \Pr(TU_{N+2} \leqslant t) \Pr(TD_{N+1} \leqslant T - t) \\ &\leqslant \max(\Pr(TU_{H+1} \leqslant t), \Pr(TD_{K+1} \leqslant T - t)) \\ &\leqslant \frac{\epsilon}{3}. \end{aligned}$$

Using the results of the previous subsection, the first sum, Σ, can be decomposed in the following way:

$$\Sigma = \sum_{n=0}^{N} \left(\sum_{k=n}^{K} \mathrm{e}^{-\lambda_D(T-t)} \frac{(\lambda_D(T-t))^k}{k!} \beta_D(x_D^*(n+1, k) - x_D^*(n, k)) \right)$$
$$\times \left(\sum_{h=n+1}^{H} \mathrm{e}^{-\lambda_U t} \frac{(\lambda_U t)^h}{h!} (1 - \alpha_U x_U^*(n+1, h)) \right)$$
$$+ \sum_{n=0}^{N} \sum_{k=n}^{K} \mathrm{e}^{-\lambda_D(T-t)} \frac{(\lambda_D(T-t))^k}{k!} \beta_D(x_D^*(n+1, k) - x_D^*(n, k)) e_U(H, t, n+1)$$
$$+ \sum_{n=0}^{N} \Pr(TU_{n+1} \leqslant t)[e_D(K, T-t, n) - e_D(K, T-t, n+1)].$$

These three terms are respectively denoted by Σ_1, Σ_2 and Σ_3. We then have

$$\Sigma_2 = \sum_{n=0}^{N} \sum_{k=n}^{K} \mathrm{e}^{-\lambda_D(T-t)} \frac{(\lambda_D(T-t))^k}{k!} \beta_D(x_D^*(n+1, k) - x_D^*(n, k)) e_U(H, t, n+1)$$
$$\leqslant \frac{\epsilon}{3} \sum_{n=0}^{N} \sum_{k=n}^{K} \mathrm{e}^{-\lambda_D(T-t)} \frac{(\lambda_D(T-t))^k}{k!} \beta_D(x_D^*(n+1, k) - x_D^*(n, k))$$
$$= \frac{\epsilon}{3} \sum_{k=0}^{N} \mathrm{e}^{-\lambda_D(T-t)} \frac{(\lambda_D(T-t))^k}{k!} \sum_{n=0}^{k} \beta_D(x_D^*(n+1, k) - x_D^*(n, k))$$
$$+ \frac{\epsilon}{3} \sum_{k=N+1}^{K} \mathrm{e}^{-\lambda_D(T-t)} \frac{(\lambda_D(T-t))^k}{k!} \sum_{n=0}^{N} \beta_D(x_D^*(n+1, k) - x_D^*(n, k))$$
$$= \frac{\epsilon}{3} \sum_{k=0}^{N} \mathrm{e}^{-\lambda_D(T-t)} \frac{(\lambda_D(T-t))^k}{k!} \beta_D x_D^*(k+1, k)$$
$$+ \frac{\epsilon}{3} \sum_{k=N+1}^{K} \mathrm{e}^{-\lambda_D(T-t)} \frac{(\lambda_D(T-t))^k}{k!} \beta_D x_D^*(N+1, k)$$
$$\leqslant \frac{\epsilon}{3} \sum_{k=0}^{K} \mathrm{e}^{-\lambda_D(T-t)} \frac{(\lambda_D(T-t))^k}{k!}$$
$$\leqslant \frac{\epsilon}{3}$$

and

$$\Sigma_3 = \sum_{n=0}^{N} \Pr(TU_{n+1} \leqslant t)[e_D(K, T-t, n) - e_D(K, T-t, n+1)]$$
$$\leqslant \sum_{n=0}^{N} [e_D(K, T-t, n) - e_D(K, T-t, n+1)]$$
$$= e_D(K, T-t, 0) - e_D(K, T-t, N+1)$$
$$\leqslant e_D(K, T-t, 0)$$

$$= \sum_{k=K+1}^{\infty} e^{-\lambda_D(T-t)} \frac{(\lambda_D(T-t))^k}{k!} (1 - \beta_D x_D^*(0, k))$$

$$= \sum_{k=K+1}^{\infty} e^{-\lambda_D(T-t)} \frac{(\lambda_D(T-t))^k}{k!}$$

$$\leqslant \frac{\epsilon}{3}.$$

Finally, we have the following result, which leads easily to an algorithm.

$$0 \leqslant \Pr(C_T \leqslant t) - \Sigma_1 \leqslant \epsilon$$

where

$$\Sigma_1 = \sum_{n=0}^{\min(K,H-1)} \left(\sum_{k=n}^{K} e^{-\lambda_D(T-t)} \frac{(\lambda_D(T-t))^k}{k!} \beta_D (x_D^*(n+1, k) - x_D^*(n, k)) \right)$$
$$\times \left(\sum_{h=n+1}^{H} e^{-\lambda_U t} \frac{(\lambda_U t)^h}{h!} (1 - \alpha_U x_U^*(n+1, h)) \right).$$

This formula can be seen as the application of a double uniformization. In practice, highly dependable computing systems lead to models in which failure rates are very small with respect to repair rates, so that λ_U will be very small with respect to λ_D. For such systems, the interesting values of t are near of the mission time T. Now, it is well known that H is $O(\lambda_U t)$ and K is $O(\lambda_D(T-t))$. If the value of t is near of T, the truncation steps K and H are both small since λ_U and $T-t$ are small. These numbers K and H must be compared with the truncation step corresponding to the more general algorithm given in [2] whose value is $O(\lambda_D T)$. To be more specific, just consider the storage necessary to perform the recursions in [2] which is equal to $O(\lambda_D TM)$, for any value of t (recall that M is the size of the state space). In our case, we need to store $O(NL)$ numbers for the recursions concerning the vectors x_U^* and $O(N(M-L))$ for the recursions concerning the vectors x_D^* (L is the number of operational states, $M-L$ is the number of unoperational states and $N = \min(K, H-1)$). So, we need $O(N \max(L, M-L))$ cells. As N can be several orders of magnitude less than $\lambda_D T$, specially when the Markov model is stiff, the gain in storage (and so, in time) can be considerable. For instance, assume that $M = 2L$, $\lambda_U = 10^{-6}$, $\lambda_D = 1$ and $T = 10^8$. The storage needed in [2] is about $O(10^8 M)$. If $t = 10^8 - 10^7$, N is about 90 and thus, the storage needed by the algorithm presented here is about $O(45 M)$.

4. Conclusion

In this paper we give a new method to obtain the interval availability distribution when some conditions are satisfied by the transition probability matrix of the model. A first interest in this approach is that it allows to consider semi-Markov processes. The conditions lead basically to operational periods independent and identically distributed and unoperational periods also i.i.d. Moreover, the two sequences are independent of each other, which is the key property to be able to apply the technique. The conditions trivially hold when the subset of operational states (or the subset of unoperational states) has only one state. They also trivially hold, for instance, when the subset of operational states (or the subset of unoperational states) has only one entry state and only one exit state.

We analyse in detail the Markovian case and we give an algorithm to compute the distribution of the cumulative operational time over a finite observation period, which is equivalent to give the interval availability distribution. The algorithm includes as input data an error tolerance given by the user. Its main interest lies in the fact that it performs much better than a general one since it takes into account the particular structure of the model. As a byproduct, we analyse the cumulative operational time up to the nth operational period in the Markovian case, obtaining its distribution and an algorithm to compute it. This is done in the general case, that is, without the previous considered independence conditions. We also note that a similar method can be applied to obtain the distribution of the number of failures (or the number of repairs) during the mission time.

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