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CLOSED-FORM SOLUTION FOR THE DISTRIBUTION OF THE TOTAL TIME SPENT IN A SUBSET OF STATES OF A HOMOGENEOUS MARKOV PROCESS DURING A FINITE OBSERVATION PERIOD

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Abstract

Markov process are widely used to model computer systems. De Souza e Silva and Gail [3] calculated numerically the distribution of the cumulative operational time of repairable computer systems modelled by Markovian processes, that is, the distribution of the total time during which the system was in operation over a finite observation period. An extension of their approach is presented here. A closed-form solution is obtained for the distribution of the total time spent in a subset of states of a homogeneous Markov process during a finite observation period, which is theoretically and numerically interesting. We also give an application of this result to a fault-tolerant system.

UNIFORMIZATION TECHNIQUE; CUMULATIVE OPERATIONAL TIME; REPAIRABLE COMPUTER SYSTEMS

1. Introduction

Consider a continuous-time homogeneous Markov process, say $X = (X_t)_{t \geq 0}$, over a finite state space denoted by E. The states of E are divided into two disjoint subsets. In practice, one subset may represent the states for which the system is up (the operational states) and the other the states for which the system is down (the failed states). This division is supposed to be made by some criteria specified by the user. Let L, $1 \leqq L < M$, be the number of operational states, and define

$$B = \{1, \cdots, L\} \text{ (set of operational states)},$$

$$B^c = \{L+1, \cdots, M\} \text{ (set of failed states)},$$

$$E = \{1, \cdots, M\} = B \cup B^c.$$

We assume that the system modelled by such a process is used during a finite interval of time denoted by $(0, t)$. The random variable of interest is the total time spent by the Markov process X into the subset of states B during $(0, t)$. Denoting by W_t this random variable, we have

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$$W_t = \int_0^t 1_{(X_s \in B)} ds$$

where 1 denotes the indicator function. If B is a subset of transient states of E, the distribution of W_∞ can be found in [1] for semi-Markov reward processes. Assume that the process X is given by its infinitesimal generator A and by its initial probability distribution α. Let $A(i,j)$ denote the (i,j) entry of the matrix A and let us define

$$\lambda(i) = -A(i,i) = \sum_{j \neq i} A(i,j)$$

which represents the output rate from state i.

To obtain the distribution of the random variable W_t, we first recall in the following section the result obtained in [3] and then we show how a closed-form solution for the distribution of W_t can be obtained using renewal equations and a matrix representation. Section 3 gives an application of this result to a fault-tolerant system.

2. Distribution of W_t

In [3] the authors first show, using the uniformization technique (see [3] or [2] for more details of this technique) that, for $s < t$, (and so $k \leqq n$)

$$\mathbb{P}\left[W_t \leqq s \;\middle|\; \begin{array}{l} n \text{ transitions in } (0,t) \\ k \text{ visits to the states of } B \end{array}\right] = \sum_{i=k}^{n} C_n^i \left(\frac{s}{t}\right)^i \left(1 - \frac{s}{t}\right)^{n-i}.$$

Letting $\Omega(n,k)$, $0 \leqq k \leqq n+1$, be the probability that the uniformized Markov chain visits k states of B during n transitions and unconditioning on the number of visits to the states of B, they write

$$\text{(1)} \qquad \mathbb{P}(W_t \leqq s \mid n \text{ transitions on } (0,t)) = \sum_{k=0}^{n} \Omega(n,k) \sum_{i=k}^{n} C_n^i \left(\frac{s}{t}\right)^i \left(1 - \frac{s}{t}\right)^{n-i}.$$

Finally, unconditioning on the number of transitions in $(0,t)$, they obtain the following relation (for $s < t$):

$$\mathbb{P}(W_t \leqq s) = \sum_{n=0}^{+\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \Omega(n,k) \sum_{i=k}^{n} C_n^i \left(\frac{s}{t}\right)^i \left(1 - \frac{s}{t}\right)^{n-i}$$

where λ is a positive real such that $\lambda \geqq \max(\lambda(i), i \in E)$. For $s \geqq t$, it is easy to verify that $\mathbb{P}(W_t \leqq s) = 1$. In the following, we focus on giving a closed-form solution for $\Omega(n,k)$ and so a closed-form expression for the distribution of W_t.

If P denotes the transition probability matrix of the uniformized Markov chain ($P = I + A/\lambda$), we decompose P and the initial probability vector α with respect to the partition $\{B, B^c\}$ of E as follows:

$$P = \begin{pmatrix} P_B & P_{BB^c} \\ P_{B^cB} & P_{B^c} \end{pmatrix}; \qquad \alpha = (\alpha_B, \alpha_{B^c}).$$

We now introduce some useful notation. We denote by T the transpose operator and for convenience the following quantities are set to 0 when $k < 0$ or $n < 0$:

$$\Omega_i(n,k) \stackrel{\text{def}}{=} \mathbb{P}(k \text{ visits to the states of } B \mid n \text{ transitions}, X_0 = i)$$

$$V_B(n,k) \stackrel{\text{def}}{=} (\Omega_1(n,k),\cdots,\Omega_L(n,k))^{\mathrm{T}}$$

$$V_{B^c}(n,k) \stackrel{\text{def}}{=} (\Omega_{L+1}(n,k),\cdots,\Omega_M(n,k))^{\mathrm{T}}$$

$$U_B(n,k) \stackrel{\text{def}}{=} (V_B^{\mathrm{T}}(n,k), V_B^{\mathrm{T}}(n-1,k),\cdots, V_B^{\mathrm{T}}(0,k))^{\mathrm{T}}$$

$$U_{B^c}(n,k) \stackrel{\text{def}}{=} (V_{B^c}^{\mathrm{T}}(n,k), V_{B^c}^{\mathrm{T}}(n-1,k),\cdots, V_{B^c}^{\mathrm{T}}(0,k))^{\mathrm{T}}$$

$$I_B(n) \stackrel{\text{def}}{=} \text{the } ((n+1)L,(n+1)L) \text{ identity matrix}$$

$$I_{B^c}(n) \stackrel{\text{def}}{=} \text{the } ((n+1)(M-L),(n+1)(M-L)) \text{ identity matrix}$$

$$\mathbb{1}_B(n) \stackrel{\text{def}}{=} (1,1,\cdots,1)^{\mathrm{T}} \qquad (n+1)L \text{ entries}$$

$$\mathbb{1}_{B^c}(n) \stackrel{\text{def}}{=} (1,1,\cdots,1)^{\mathrm{T}} \qquad (n+1)(M-L) \text{ entries}$$

$$e_B(n) \stackrel{\text{def}}{=} (0,\cdots,0,\mathbb{1}_B^{\mathrm{T}}(0))^{\mathrm{T}} \qquad (n+1)L \text{ entries}$$

$$e_{B^c}(n) \stackrel{\text{def}}{=} (0,\cdots,0,\mathbb{1}_{B^c}^{\mathrm{T}}(0))^{\mathrm{T}} \qquad (n+1)(M-L) \text{ entries}$$

$$\alpha_B(n) \stackrel{\text{def}}{=} (\alpha_B, 0,\cdots,0) \qquad (n+1)L \text{ entries}$$

$$\alpha_{B^c}(n) \stackrel{\text{def}}{=} (\alpha_{B^c}, 0,\cdots,0) \qquad (n+1)(M-L) \text{ entries}$$

$$\mathbf{1}_{(C)} \stackrel{\text{def}}{=} 1 \quad \text{if } C \text{ is true and 0 otherwise.}$$

Definition 2.1. *If G is a (r,s) matrix then for $n \geqq 1$, $G^{[n]}$ is the $((n+1)r,(n+1)s)$ matrix with its (i,j) block $(1 \leqq i,j \leqq n+1)$ equal to G if $j = i+1$ and equal to 0 otherwise. Moreover, $G^{[0]} = 0$.*

Let us now write the renewal equations.

For every $i \in B$,

$$\Omega_i(n,k) = \sum_{j\in E} P(i,j)\Omega_j(n-1,k-1) + \mathbf{1}_{(n=0,k=1)}.$$

For every $i \in B^c$,

$$\Omega_i(n,k) = \sum_{j\in E} P(i,j)\Omega_j(n-1,k) + \mathbf{1}_{(n=0,k=0)}.$$

This gives in matrix notation

$$V_B(n,k) = P_B V_B(n-1,k-1) + P_{BB^c} V_{B^c}(n-1,k-1) + \mathbb{1}_B(0)\,1_{(n=0,k=1)}$$

$$V_{B^c}(n,k) = P_{B^cB} V_B(n-1,k) + P_{B^c} V_{B^c}(n-1,k) + \mathbb{1}_{B^c}(0)\,1_{(n=0,k=0)}$$

and in block matrix notation, using Definition 2.1,

$$U_B(n,k) = P_B^{[n]} U_B(n,k-1) + P_{BB^c}^{[n]} U_{B^c}(n,k-1) + e_B(n)\,1_{(k=1)}, \tag{2}$$

$$U_{B^c}(n,k) = P_{B^cB}^{[n]} U_B(n,k) + P_{B^c}^{[n]} U_{B^c}(n,k) + e_{B^c}(n)\,1_{(k=0)}. \tag{3}$$

Relation (3) gives

$$U_{B^c}(n,k) = (I_{B^c}(n) - P_{B^c}^{[n]})^{-1}(P_{B^cB}^{[n]} U_B(n,k) + e_{B^c}(n)\,1_{(k=0)}). \tag{4}$$

Replacing now $U_{B^c}(n,k-1)$ into (2) we get

$$\begin{aligned} U_B(n,k) = {} & (P_B^{[n]} + P_{BB^c}^{[n]}(I_{B^c}(n) - P_{B^c}^{[n]})^{-1}P_{B^cB}^{[n]})U_B(n,k-1) \\ & + (P_{BB^c}^{[n]}(I_{B^c}(n) - P_{B^c}^{[n]})^{-1}e_{B^c}(n) + e_B(n))\,1_{(k=1)}. \end{aligned} \tag{5}$$

Remarking that

$$P_B^{[n]}\mathbb{1}_B(n) + P_{BB^c}^{[n]}\mathbb{1}_{B^c}(n) + e_B(n) = \mathbb{1}_B(n) \tag{6}$$

$$P_{B^cB}^{[n]}\mathbb{1}_B(n) + P_{B^c}^{[n]}\mathbb{1}_{B^c}(n) + e_{B^c}(n) = \mathbb{1}_{B^c}(n), \tag{7}$$

relation (5) can be written as

$$U_B(n,k) = H(n)U_B(n,k-1) + (I_B(n) - H(n))\mathbb{1}_B(n)\,1_{(k=1)}$$

where $H(n) = P_B^{[n]} + P_{BB^c}^{[n]}(I_{B^c}(n) - P_{B^c}^{[n]})^{-1}P_{B^cB}^{[n]}$.

That is, with the convention $H(0)^0 = I_B(0)$ (see that $H(0) = 0$),

$$\begin{aligned} & U_B(n,0) = 0 \\ & U_B(n,k) = H(n)^{k-1}(I_B(n) - H(n))\mathbb{1}_B(n) \quad \text{for } k \geqq 1. \end{aligned} \tag{8}$$

Let us now define $\beta(n) \stackrel{\text{def}}{=} \alpha_B(n) + \alpha_{B^c}(n)\,(I_{B^c}(n) - P_{B^c}^{[n]})^{-1}P_{B^cB}^{[n]}$. The vector $\beta(n)$ and the matrix $H(n)$ can also be written as follows:

$$\beta(n) = (\alpha_B, \alpha_{B^c}P_{B^cB}, \alpha_{B^c}P_{B^c}P_{B^cB}, \cdots, \alpha_{B^c}P_{B^c}^{n-1}P_{B^cB}); \quad \beta(0) = \alpha_B,$$

$$H(n) = \begin{bmatrix} 0 & P_B & U_0 & U_1 & U_2 & \cdots & U_{n-3} & U_{n-2} \\ 0 & 0 & P_B & U_0 & U_1 & \cdots & U_{n-4} & U_{n-3} \\ 0 & 0 & 0 & P_B & U_0 & \cdots & U_{n-5} & U_{n-4} \\ 0 & 0 & 0 & 0 & P_B & \cdots & U_{n-6} & U_{n-5} \\ \vdots & & & & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & P_B & U_0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & P_B \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}; \quad H(0) = 0,$$

where for $i \geqq 0$, $U_i \stackrel{\text{def}}{=} P_{BB^c}P_{B^c}^i P_{B^cB}$.

Remark that the matrix $H(n)$ verifies $H(n)^{n+1} = 0$ and $\beta(n)H(n)^n\mathbb{1}_B(n) = \alpha_B P_B^n \mathbb{1}_B(0)$.

Now, since $\Omega(n,k)=\alpha_B(n)U_B(n,k)+\alpha_{B^c}(n)U_{B^c}(n,k)$, combining relations (4), (8) and (7), we obtain

$$\Omega(n,k)=\begin{cases}1-\beta(n)\mathbb{1}_B(n) & \text{if } k=0\\ \beta(n)H(n)^{k-1}(I_B(n)-H(n))\mathbb{1}_B(n) & \text{if } 1\leqq k\leqq n+1\\ 0 & \text{if } k\geqq n+2.\end{cases}$$

We are now able to give a closed-form expression for the distribution of the random variable W_t. Writing relation (1) as

$$\mathbb{P}(W_t\leqq s\mid n \text{ transitions on } (0,t))=\sum_{i=0}^{n} C_n^i\left(\frac{s}{t}\right)^i\left(1-\frac{s}{t}\right)^{n-i}\sum_{k=0}^{i}\Omega(n,k)$$

and since

$$\sum_{k=0}^{i}\Omega(n,k)=1-\beta(n)H(n)^i\mathbb{1}_B(n)$$

the following relation holds:

$$\mathbb{P}(W_t\leqq s\mid n \text{ transitions on } (0,t))=1-\beta(n)\left[I_B(n)-\frac{s}{t}(I_B(n)-H(n))\right]^n\mathbb{1}_B(n).$$

Finally, for $s<t$, we obtain the following expression:

$$\mathbb{P}(W_t\leqq s)=1-\sum_{n=0}^{+\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\beta(n)\left[I_B(n)-\frac{s}{t}(I_B(n)-H(n))\right]^n\mathbb{1}_B(n)$$

which can be also written

$$\mathbb{P}(W_t\leqq s)=1-\sum_{n=0}^{+\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{i=0}^{n}C_n^i\left(\frac{s}{t}\right)^i\left(1-\frac{s}{t}\right)^{n-i}\beta(n)H(n)^i\mathbb{1}_B(n). \tag{9}$$

3. Application to a fault-tolerant system

Consider a system with three identical and independent components. Only one of these components performs tasks. Each component fails with a constant failure rate ϕ. When the performing component fails, a recovery process tries to reconfigure the system by replacing the failed component by one (either) of the two remaining components. The expected time to execute the recovery process is equal to $1/\rho$. The success probability of the recovery process is constant d (failure means the crash of the system). The recovery rate ρ, of course, is assumed to be larger than the failure rate ϕ. The Markov process associated to this system is shown in Figure 1. States D_1, D_2 are the recovery states and state 0 is the down state (in these states the system is not operational). States 1, 2 and 3 are the operational ones. The initial state is state 3 with probability 1. We have $B=\{3,2,1\}$ and $B^c=\{D_2,D_1,0\}$. The initial probability distribution is $\alpha_B=\{1,0,0\}$ and $\alpha_{B^c}=\{0,0,0\}$. The transition rate matrix A of the obtained Markov process can easily be constructed from Figure 1.

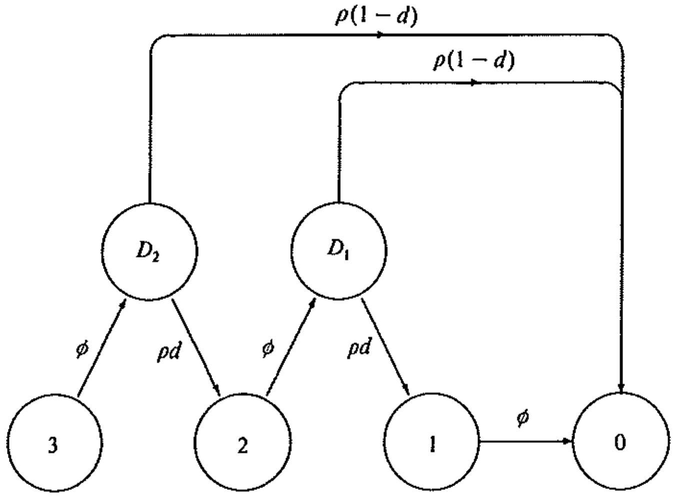

Figure 1. A fault-tolerant three-component system

Due to the above assumptions, we can choose $\lambda = \max(\lambda(i), i \in E) = \rho$. The transition probability matrix P of the uniformized Markov chain has the following properties:

$$\forall i \geqq 1, \quad U_i = P_{BB^c} P_{B^c}^i P_{B^cB} = 0; \quad U_0^3 = 0; \quad P_B = (1 - \phi/\rho) I_B(0).$$

Recall that $U_0 = P_{BB^c} P_{B^cB}$. Using the properties on U_i, $i \geqq 1$, we obtain for every $n \in \mathbb{N}$

$$\beta(n) H(n)^i \mathbb{1}_B(n) = \begin{cases} \alpha_B (P_B + U_0)^i \mathbb{1}_B(0), & i = 0, 1, \cdots, n-2 \\ \alpha_B (P_B^{n-1} + (n-1) P_B^{n-2} U_0) \mathbb{1}_B(0), & i = n-1 \\ \alpha_B P_B^n \mathbb{1}_B(0), & i = n. \end{cases}$$

This gives, using relation (9) of the previous section and after some algebraic manipulation,

$$\mathbb{P}(W_t \leqq s) = 1 - \alpha_B (\exp(-\rho s) - \rho(t-s)\exp(-\rho t) - \exp(-\rho t)) \exp(\rho s (P_B + U_0)) \mathbb{1}_B(0)$$
$$- \alpha_B (\rho(t-s) I_B + \rho^2 s (t-s) U_0 + I_B) \exp(-\rho t) \exp(-\rho s P_B) \mathbb{1}_B(0).$$

Using now that $U_0^3 = 0$ and $P_B = (1 - \phi/\rho) I_B(0)$, it is easy to obtain the following expression:

$$\mathbb{P}(W_t \leqq s) = 1 - e^{-\phi s} \Big[1 + (1 - \exp(-\rho(t-s))) d\phi s$$
$$+ (1 - \exp(-\rho(t-s)) - \rho(t-s)\exp(-\rho(t-s))) \frac{d^2 \phi^2 s^2}{2} \Big].$$

4. Conclusion

The main contributions of this paper are the closed-form solutions given for two probability distributions concerning homogeneous Markov processes. The first one is the distribution of the total number of visits to the states of a subset during the n first transitions of a Markov chain. It is obtained using Markov renewal theory and a matrix representation. Using this result, the second expression obtained is the distribution of the total time spent in a subset of states of a Markov process.

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