Methodol Comput Appl Probab
DOI 10.1007/s11009-008-9075-5

Moments' Analysis in Homogeneous Markov Reward Models

F. Castella · G. Dujardin · B. Sericola

Received: 25 October 2007 / Revised: 14 February 2008 /
Accepted: 17 March 2008
© Springer Science + Business Media, LLC 2008

Abstract We analyze the moments of the accumulated reward over the interval $(0, t)$ in a continuous-time Markov chain. We develop a numerical procedure to compute efficiently the normalized moments using the uniformization technique. Our algorithm involves auxiliary quantities whose convergence is analyzed, and for which we provide a probabilistic interpretation.

Keywords Markov models · Accumulated reward · Performability · Uniformization

AMS 2000 Subject Classification Primary 60J22; Secondary 60J27

1 Introduction

In the dependability analysis of repairable computing systems, there is an increasing interest in evaluating transient measures, and in particular, the accumulated reward over a given period. This measure is also known as a performability measure that takes into account both the performance of a system and its reliability. The state of the system is modeled by an irreducible continuous-time homogeneous Markov chain $X = \{X_t, t \geq 0\}$, over a finite state space S. To each state $i \in S$, is associated a reward rate $f(i)$, which is assumed to be a nonnegative real number, without any loss of generality. The accumulated reward $Y(t)$ over interval $(0, t)$ averaged over t and normalized by the maximum reward rate $f = \max\{f(j), j \in S\}$ is defined, for every $t > 0$, by

$$Y(t) = \frac{1}{ft}\int_0^t f(X_s)ds.$$

F. Castella · G. Dujardin
Université de Rennes I, Campus de Beaulieu, 35042 Rennes Cedex, France

B. Sericola(✉)
INRIA Rennes - Bretagne Atlantique, Campus de Beaulieu, 35042 Rennes Cedex, France
e-mail: Bruno.Sericola@irisa.fr

The distribution of accumulated reward has been considered in several papers in the case of homogeneous Markov chains (Meyer 1980, 1982; Iyer et al. 1986; Kulkarni et al. 1986; Ciardo et al. 1990; Smith et al. 1988; Nabli and Sericola 1996; de Souza e Silva and Gail 1998; Donatiello and Grassi 1991; Cloth and Haverkort 2006). The case of non-homogeneous Markov reward models has been studied in Telek et al. (2004) and Stenberg et al. (2007) deals with non-homogeneous semi-Markov reward models applied to insurance problems. This paper deals with the study and the evaluation of the moments of accumulated reward. As we shall see there is no need to evaluate the distribution if only some of the moments are required. Their evaluation has been studied in Rácz and Telek (1999) where an algorithm has been proposed to compute them. In paper Rácz and Telek (1999) the moments considered are not averaged over $(0, t)$ so that they increase to infinity when t goes to infinity. The first moment of $Y(t)$ has been analyzed in Sericola (1999), where a stationary regime detection has been developed to improve its computational time.

In this paper, we consider the normalized and averaged accumulated reward $Y(t)$ and we are interested in the numerical computation of its moments. As we show, the normalization by ft in the definition of $Y(t)$ above, allows us to resolve the calculation of the moments into the calculation of some coefficients defined as $U_i(n, r)$ in the sequel. In Section 2, we provide an analysis of the moments of the accumulated reward and we introduce the terms $U_i(n, r)$. Taking advantage of the fact that the computation of the moments of $Y(t)$ reduces to that of the coefficients $U_i(n, r)$, we present a first algorithm for the numerical computation of these moments called the *uniformization method*. An important feature of the $U_i(n, r)$'s is that they all belong to the interval $[0, 1]$. In Section 3, we next analyze the convergence of the sequences of the $U_i(n, r)$'s as n goes to infinity. Using these convergence properties, we derive a new, improved procedure for the numerical computation of the first moments of accumulated reward. This new procedure is based on the uniformization method which we adapt to take into account the stationarity detection. This is the key result of the present paper. Section 4 then deals with some probabilitic characterizations of the $U_i(n, r)$'s through Theorems 5, 6 and 7. In Section 5, we provide a numerical example to compare the two algorithms for the computation of the first moments of accumulated reward. This example shows that the algorithm with the stationarity detection can considerably reduce the computational time.

2 Moments analysis

We denote respectively by α and Q the initial distribution and the infinitesimal generator of the irreducible continuous-time homogeneous Markov process $X = \{X_t, t \geq 0\}$ over the finite state space S. For $i \in S$ and $r \geq 0$, we denote by $m_i^{(r)}(t)$ the r-th moment of the accumulated reward $Y(t)$ over $(0, t)$, given that the initial state of X is equal to i, i.e.

$$m_i^{(r)}(t) = E(Y(t)^r | X_0 = i).$$

We denote by $m^{(r)}(t)$ the column vector with the i-th element equal to $m_i^{(r)}(t)$. Clearly, we have $m^{(0)}(t) = \mathbb{1}$, where $\mathbb{1}$ is the column vector with all entries equal to 1. The r-th moment of the accumulated reward over interval $(0, t)$ is thus given, for $r \geq 0$, by

$$E(Y(t)^r) = \alpha m^{(r)}(t).$$

We denote by D the diagonal matrix defined by $D = \text{diag}(d(i))$, where $d(i) = f(i)/f$. The column vectors $m^{(r)}(t)$ can be obtained recursively from the following result.

Theorem 1 *For every $r \geq 1$ and $t > 0$, we have*

$$m^{(r)}(t) = \frac{r}{t^r} \int_0^t u^{r-1} e^{Q(t-u)} D m^{(r-1)}(u) du. \tag{1}$$

Proof Let $g_r(s)$ be the random variable defined, for $r \geq 0$ and $0 \leq s \leq t$, by

$$g_r(s) = \left(\int_s^t f(X_u) du \right)^r.$$

Differentiating with respect to s, we get

$$g_r'(s) = -r f(X_s) \left(\int_s^t f(X_u) du \right)^{r-1} = -r f(X_s) g_{r-1}(s),$$

which is equivalent to

$$g_r(s) = r \int_s^t f(X_u) g_{r-1}(u) du. \tag{2}$$

Note that from the homogeneity of the Markov chain X, we have, for $r \geq 0$ and $0 \leq s \leq t$,

$$\begin{aligned} E(g_r(s) | X_s = j) &= E\left(\left(\int_s^t f(X_u) du \right)^r \Big| X_s = j \right) \\ &= E\left(\left(\int_0^{t-s} f(X_u) du \right)^r \Big| X_0 = j \right) \\ &= f^r (t-s)^r m_j^{(r)}(t-s). \end{aligned} \tag{3}$$

Taking the expectation in Eq. 2, we get, using the Fubini theorem

$$\begin{aligned} E(g_r(s) | X_0 = i) &= r \int_s^t E(f(X_u) g_{r-1}(u) | X_0 = i) du \\ &= r \int_s^t \sum_{j \in S} E(f(X_u) g_{r-1}(u) 1_{\{X_u = j\}} | X_0 = i) du \\ &= r \int_s^t \sum_{j \in S} f(j) E(g_{r-1}(u) | X_u = j, X_0 = i) \Pr\{X_u = j | X_0 = i\} du \\ &= r \int_s^t \sum_{j \in S} f(j) E(g_{r-1}(u) | X_u = j) \Pr\{X_u = j | X_0 = i\} du \\ &= r f^{r-1} \int_s^t \sum_{j \in S} f(j) (t-u)^{r-1} m_j^{(r-1)}(t-u) (e^{Qu})_{i,j} du, \end{aligned}$$

where the fourth equality follows from the Markov property and the fifth is due to homogeneity property Eq. 3. Taking $s = 0$, we obtain

$$m_i^{(r)}(t) = \frac{r}{t^r}\int_0^t \sum_{j\in S} d(j)(t-u)^{r-1} m_j^{(r-1)}(t-u)(e^{Qu})_{i,j}du,$$

which can be written, in matrix notation

$$m^{(r)}(t) = \frac{r}{t^r}\int_0^t (t-u)^{r-1} e^{Qu} D m^{(r-1)}(t-u)du,$$

or, by a variable change,

$$m^{(r)}(t) = \frac{r}{t^r}\int_0^t u^{r-1} e^{Q(t-u)} D m^{(r-1)}(u)du,$$

which completes the proof. □

Differentiating Eq. 1 with respect to t, we get

$$rm^{(r)}(t) + t\frac{dm^{(r)}}{dt}(t) = tQm^{(r)}(t) + rDm^{(r-1)}(t). \tag{4}$$

We now make use of the uniformization technique (Ross 1983). We introduce the uniformized Markov chain $Z = \{Z_n, n \geq 0\}$ associated to the Markov chain X, which is characterized by its uniformization rate ν and by its transition probability matrix P_ν. The uniformization rate ν verifies $\nu \geq \max\{-Q(i,i), i \in S\}$ and matrix P_ν is related to the infinitesimal generator Q by $P_\nu = I + Q/\nu$, where I denotes the identity matrix. To simplify notation, we write in the sequel P for P_ν. The number of transitions during the interval $(0, t)$, which we denote by N_t, is a Poisson process with rate ν. Since $Q = -\nu(I - P)$, Eq. 4 can be written as

$$rm^{(r)}(t) + t\frac{dm^{(r)}}{dt}(t) = -\nu t m^{(r)}(t) + \nu t P m^{(r)}(t) + rDm^{(r-1)}(t). \tag{5}$$

In the following theorem, we determine the sequence of column vectors $U(n, r)$ so that the solution to Eq. 5 has the form of the series in Eq. 6.

Theorem 2 *For every $t \geq 0$, we have*

$$m^{(r)}(t) = \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!} U(n, r), \tag{6}$$

where the column vectors $U(n, r) = (U_i(n, r), i \in S)$ are given by

$$U(0, r) = D^r \mathbb{1} \text{ for } r \geq 0, \tag{7}$$

$$U(n, 0) = \mathbb{1} \text{ for } n \geq 0, \tag{8}$$

and, for $n, r \geq 1$,

$$U(n, r) = \frac{n}{n+r} PU(n-1, r) + \frac{r}{n+r} DU(n, r-1). \tag{9}$$

Proof Differentiating Eq. 6 with respect to t, we get

$$t\frac{dm^{(r)}}{dt}(t) = -\nu t m^{(r)}(t) + \nu t \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!}U(n+1,r)$$
$$= -\nu t m^{(r)}(t) + \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!} n U(n,r).$$

It is worth noticing that the above term-by-term differentiation is perfectly licit since, as we check *a posteriori* (see Section 2.1 below), the $U(n,r)$'s all belong to the segment [0, 1], and all considered series expansion therefore converge thanks to the factors $1/n!$.

Replacing this expression together with Eq. 6 in Eq. 5, we obtain, for every $t \geq 0$,

$$\sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!}(n+r)U(n,r) = \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^{n+1}}{n!} PU(n,r) + \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!} rDU(n,r-1)$$
$$= \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!}(nPU(n-1,r) + rDU(n,r-1)).$$

Note that for $n = 0$, it is not necessary to define $U(n-1,r)$ since, in that case, we have $nPU(n-1,r) = 0$. The equality holds for every $t \geq 0$, so we get, for every $n \geq 0$,

$$U(n,r) = \frac{n}{n+r}PU(n-1,r) + \frac{r}{n+r}DU(n,r-1).$$

The initial condition easily follows from the fact that the paths of the Markov chain X are supposed right continuous at $t = 0$. We thus have, for every $r \geq 0$

$$\lim_{t\longrightarrow 0} m^{(r)}(t) = D^r \mathbb{1}.$$

Taking $t = 0$ in Eq. 6, we get

$$U(0,r) = D^r \mathbb{1}.$$

Since $m^{(0)}(t) = \mathbb{1}$, we obtain

$$U(n,0) = \mathbb{1}, \text{ for every } n \geq 0,$$

which completes the proof. □

2.1 The uniformization method

The computation of the moments of the normalized accumulated reward $E(Y(t)^r)$ is based on the following relation, which can be easily obtained from Eq. 6,

$$E(Y(t)^r) = \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!}u(n,r),$$

where $u(n,r) = \alpha U(n,r)$. Note that since matrix P is stochastic and since diagonal matrix D has its entries in the interval [0, 1], we easily obtain from Eqs. 7, 8 and 9 that

$$0 \leq U_i(n,r) \leq 1, \text{ and thus } 0 \leq u(n,r) \leq 1.$$

Let $\varepsilon > 0$ be a given specified error tolerance and N be defined as

$$N = \min \left\{ n \in \mathbb{N} \;\middle|\; \sum_{j=0}^{n} e^{-\nu t} \frac{(\nu t)^j}{j!} \geq 1 - \varepsilon \right\}. \tag{10}$$

Then we obtain

$$E(Y(t)^r) = \sum_{n=0}^{N} e^{-\nu t} \frac{(\nu t)^n}{n!} u(n,r) + e(N),$$

where the remainder of the series $e(N)$ verifies

$$0 \leq e(N) = \sum_{n=N+1}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} u(n,r) \leq \sum_{n=N+1}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} = 1 - \sum_{n=0}^{N} e^{-\nu t} \frac{(\nu t)^n}{n!} \leq \varepsilon.$$

The computation of integer N can be made without any numerical problems even for large values of νt by using the method described in Bowerman et al. (1990).

The truncation level N is in fact a function of t, say $N(t)$. For a fixed value of ε, $N(t)$ is an increasing function of t. It follows that if we want to compute the R first moments $E(Y(t)^r)$, $r = 1, \ldots, R$, for J distinct values of t, denoted by $t_1 < \cdots < t_J$, we only need to compute $u(n,r)$ for $n = 1, \ldots, N(t_J)$ since the values of $u(n,r)$ are independent of the parameter t.

The pseudo-code of the uniformization method is presented in Algorithm 1.

Uniformization is a widely used technique in performance and dependability analysis of systems modelled by Markov chains, see for instance de Souza e Silva and Gail (2000). This technique has several nice properties. The operations generally involve only elements in [0, 1], as in Eq. 6. As a consequence, the numerical robustness of the method is very good. Furthermore, the recursion to calculate the vectors $U(n,r)$ preserves the structure of the matrix P. This is very important when the state space is large since, in practice, matrix Q and thus P are sparse matrices. Another advantage is that the error is bounded by the tail of the Poisson distribution which can be computed a priori. As mentioned in de Souza e Silva and Gail (2000) several other approaches exist, based on Laplace transform or ODE solvers, but they are costly and prone to numericals problems. A comparison of several algorithms is provided in Cloth and Haverkort (2006). The main problem concerning uniformization is that it is costly when the product νt is large because the truncation step N given in Eq. 10 is increasing with the product νt. This problem is considered in the next section where we use the stationarity detection, i.e. the convergence of the coefficents $U_i(n,r)$, to avoid unnecessary numerical computation.

Algorithm 1 Algorithm for the computation of $E(Y(t_j)^r)$, $j = 1, \ldots, J$ and $r = 1, \ldots, R$

input : $\varepsilon, t_1 < \cdots < t_J, R$
output : $E(Y(t_j)^r)$ for $j = 1, \ldots, J$ and $r = 1, \ldots, R$
Compute N from Eq. 10 with $t = t_J$
for $r = 0$ **to** R **do**
  $U(0, r) = D^r \mathbb{1}$
  $u(0, r) = \alpha U(0, r)$
endfor
for $n = 1$ **to** N **do**
  $U(n, 0) = \mathbb{1}$
  for $r = 1$ **to** R **do**
    $U(n, r) = \dfrac{n}{n+r} PU(n-1, r) + \dfrac{r}{n+r} DU(n, r-1)$
    $u(n, r) = \alpha U(n, r)$
  endfor
endfor
for $j = 1$ **to** J **do**
  for $r = 1$ **to** R **do**
    $E(Y(t_j)^r) = \displaystyle\sum_{n=0}^{N} e^{-\nu t_j} \frac{(\nu t_j)^n}{n!} u(n, r)$
  endfor
endfor

3 Stationarity detection

We consider in this section the sequence of column vectors $U(n, r)$ and we show that it converges when n tends to infinity. This allows us to stop the computation of the $U(n, r)$ as soon as they are close enough to their limit. In the following theorem, we express the vectors $U(n, r)$ recursively over index r. This recursive expression will be used in Theorem 4 to prove the convergence of the sequence $U(n, r)$.

Theorem 3 *For every $n \geq 0$ and $r \geq 1$,*

$$U(n, r) = \frac{1}{\binom{n+r}{r}} \sum_{\ell=0}^{n} \binom{\ell + r - 1}{r - 1} P^{n-\ell} DU(\ell, r-1). \tag{11}$$

Proof We prove this relation by recurrence over index n. For $n = 0$, this relation gives $U(0, r) = DU(0, r-1)$. This leads to $U(0, r) = D^r U(0, 0) = D^r \mathbb{1}$, which is Eq. 7.

Suppose that Eq. 11 is true for integer n. We have to show that

$$U(n+1, r) = \frac{1}{\binom{n+1+r}{r}} \sum_{\ell=0}^{n+1} \binom{\ell + r - 1}{r - 1} P^{n+1-\ell} DU(\ell, r-1).$$

From Eq. 9 and using the recurrence hypothesis, we have

$$\begin{aligned}
U(n+1,r) &= \frac{n+1}{n+1+r} PU(n,r) + \frac{r}{n+1+r} DU(n+1,r-1) \\
&= \frac{n+1}{(n+1+r)\binom{n+r}{r}} \sum_{\ell=0}^{n} \binom{\ell+r-1}{r-1} P^{n+1-\ell} DU(\ell,r-1) \\
&\quad + \frac{r}{n+1+r} DU(n+1,r-1) \\
&= \frac{1}{\binom{n+1+r}{r}} \sum_{\ell=0}^{n} \binom{\ell+r-1}{r-1} P^{n+1-\ell} DU(\ell,r-1) \\
&\quad + \frac{r}{n+1+r} DU(n+1,r-1) \\
&= \frac{1}{\binom{n+1+r}{r}} \sum_{\ell=0}^{n+1} \binom{\ell+r-1}{r-1} P^{n+1-\ell} DU(\ell,r-1),
\end{aligned}$$

which is the desired result. □

We denote by π the stationary probability distribution of the Markov chain X. This row vector satisfies $\pi Q = 0$ or equivalently $\pi = \pi P$.

Theorem 4 *For every $r \geq 0$, we have*

$$\lim_{n \longrightarrow \infty} U(n,r) = (\pi D \mathbb{1})^r \mathbb{1}. \tag{12}$$

Proof We proceed by recurrence over integer r. The result is true for $r = 0$, since we have $U(n,0) = \mathbb{1}$.

Suppose the result is true for integer $r-1$, i.e. suppose that $\lim_{n \longrightarrow \infty} U(n,r-1) = (\pi D \mathbb{1})^{r-1} \mathbb{1}$.

Let us define

$$\beta(n,\ell) = \frac{\binom{\ell+r-1}{r-1}}{\binom{n+r}{r}}, \quad H(\ell) = P^\ell D, \quad U(\ell) = U(\ell,r-1), \text{ and } V(n) = U(n,r).$$

We have $\beta(n,\ell) \geq 0$ and

$$\sum_{\ell=0}^{n} \beta(n,\ell) = 1,$$

and, since the sequence $\beta(n,\ell)$ is increasing with ℓ,

$$\lim_{n \longrightarrow \infty} \sup_{0 \leq \ell \leq n} \beta(n,\ell) = \lim_{n \longrightarrow \infty} \beta(n,n) = \lim_{n \longrightarrow \infty} \frac{r+1}{n+r+1} = 0.$$

To ensure the convergence of the sequence of matrices P^n, we require either that the output rates $-Q(i,i)$, $i \in S$, of the Markov chain X are not all equal or that the uniformization rate ν is such that $\nu > \max\{-Q(i,i), i \in S\}$. This guarantees that the transition probability matrix P is aperiodic which means that it is ergodic and thus, we have

$$H = \lim_{\ell \longrightarrow \infty} H(\ell) = \Pi D,$$

where Π is the matrix with all its lines equal to π. By using the recurrence hypothesis, we get

$$U = \lim_{\ell \longrightarrow \infty} U(\ell) = (\pi D \mathbb{1})^{r-1} \mathbb{1} = (\Pi D)^{r-1} \mathbb{1}.$$

From Eq. 11 and using Lemma 3, we get

$$\lim_{n \longrightarrow \infty} U(n,r) = \lim_{n \longrightarrow \infty} V(n) = \Pi D (\Pi D)^{r-1} \mathbb{1} = (\Pi D)^r \mathbb{1} = (\pi D \mathbb{1})^r \mathbb{1},$$

which completes the proof. □

Using this result, Algorithm 1 can be modified as follows to take into account the convergence of the sequence $U(n,r)$ and thus of the sequence $u(n,r)$. In order to do that we make the following assumption, which is satisfied in practice:

if $\exists\, K < N$ s. t. $\max_{r \le R} |u(K,r) - (\pi D \mathbb{1})^r| \le \varepsilon$ then $\forall n \ge K$, $\max_{r \le R} |u(n,r) - (\pi D \mathbb{1})^r| \le \varepsilon$.

If such a K does not exist, there is no stationarity detection and we come back to the previous algorithm. Such a situation means that the value of N is not large and neither is the value of t_J. If such a K exists, we have

$$E(Y(t)^r) = \sum_{n=0}^{K} e^{-\nu t} \frac{(\nu t)^n}{n!} u(n,r) + (\pi D \mathbb{1})^r \left(1 - \sum_{n=0}^{K} e^{-\nu t} \frac{(\nu t)^n}{n!}\right) + e_1(K),$$

where the remainder of the series $e_1(K)$ verifies

$$\begin{aligned} |e_1(K)| &= \left| \sum_{n=K+1}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} \left(u(n,r) - (\pi D \mathbb{1})^r\right) \right| \\ &\le \sum_{n=K+1}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} \left| u(n,r) - (\pi D \mathbb{1})^r \right| \le \varepsilon. \end{aligned}$$

We obtain Algorithm 2, in which we suppose that $\pi D \mathbb{1}$ has been computed beforehand.

The following section is devoted to probabilistic interpretations of the sequence $U(n,r)$.

4 Probabilistic interpretation

In this section, we provide three explicit formulae expressing the value of the $U(n,r)$'s for any value of n and r. The corresponding results are formulated in

Algorithm 2 Algorithm for the computation of $E(Y(t)^r)$ using stationarity detection

input : $\varepsilon, t_1 < \cdots < t_J, R$
output : $E(Y(t_j)^r)$ for $j = 1, \ldots, J$ and $r = 1, \ldots, R$
Compute N from Eq. 10 with $t = t_J$
for $r = 0$ **to** R **do**
  $U(0, r) = D^r \mathbb{1}$
  $u(0, r) = \alpha U(0, r)$
endfor
$K = N$
for $n = 1$ **to** N **do**
  $U(n, 0) = \mathbb{1}$
  for $r = 1$ **to** R **do**
    $U(n, r) = \frac{n}{n+r} PU(n-1, r) + \frac{r}{n+r} DU(n, r-1)$
    $u(n, r) = \alpha U(n, r)$
  endfor
  if $|u(n, r) - (\pi D\mathbb{1})^r| \leq \varepsilon$ **for every** $r = 1, \ldots, R$ **then**
    K = n
    break
  endif
endfor
for $j = 1$ **to** J **do**
  if $K = N$ **then**
    for $r = 1$ **to** R **do**
$$E(Y(t_j)^r) = \sum_{n=0}^{N} e^{-\nu t_j} \frac{(\nu t_j)^n}{n!} u(n, r)$$
    endfor
  endif
  if $K < N$ **then**
    for $r = 1$ **to** R **do**
$$E(Y(t_j)^r) = \sum_{n=0}^{K} e^{-\nu t_j} \frac{(\nu t_j)^n}{n!} u(n, r) + (\pi D\mathbb{1})^r \left(1 - \sum_{n=0}^{K} e^{-\nu t_j} \frac{(\nu t_j)^n}{n!}\right)$$
    endfor
  endif
endfor

Theorems 5, 6 and 7. Naturally, these expressions are not to be used to *actually compute* the $U(n, r)$'s: the recursion formulae obtained in Theorem 2 are much more appropriate from the computational point of view. The point here is to relate the $U(n, r)$'s with other known quantities describing the process X or its uniformized Markov chain Z.

Our results are the following. In Theorem 5, we show that $U(n, r)$ is completely determined as an explicit function of the transition matrix P, and of the matrix of rewards D, the latter dependence being naturally homogeneous of degree r in the matrix D. Theorem 6 on the other hand provides a probabilistic interpretation of $U(n,r)$ as a function of the process Z itself: we show that $U(n,r)$ can be explicitly determined as the expectation of a homogeneous function of degree r in the random rewards $d(Z_i)$'s $(i = 1, \ldots, n)$. In other words, $U(n,r)$ is shown to be the r-th moment of an accumulated reward during the n first transitions. Last, in Theorem 7, we express $U(n,r)$ as a function of the original process X, instead of Z, and we establish the formula $U_i(n,r) = E\left(Y(t)^r \middle| N_t = n, X_0 = i\right)$. Again, this formula allows to interpret $U(n,r)$ as the r-th moment of an accumulated reward during the n first transitions. However, in our opinion, the important point here is that the quantity $E\left(Y(t)^r \middle| N_t = n, X_0 = i\right)$, which a priori depends on time t, is here shown to be a universal, time independent function of the original Markov chain, the two time dependences of $Y(t)$ and N_t in t somehow cancelling out in this case.

Let us come to the technical statements.

Theorem 5 *For every $n \geq 0$ and $r \geq 1$, we have*

$$U(n,r) = \frac{1}{\binom{n+r}{r}} \sum_{\ell_1=0}^{n} P^{n-\ell_1} D \sum_{\ell_2=0}^{\ell_1} P^{\ell_1-\ell_2} D \cdots \sum_{\ell_{r-1}=0}^{\ell_{r-2}} P^{\ell_{r-2}-\ell_{r-1}} D \sum_{\ell_r=0}^{\ell_{r-1}} P^{\ell_r} D \mathbb{1}. \quad (13)$$

Proof For $r = 1$, Eq. 13 reads

$$U(n,1) = \frac{1}{n+1} \sum_{\ell_1=0}^{n} P^{\ell_1} D \mathbb{1},$$

which is obtained by Eq. 11 and Eq. 8.

Suppose that Eq. 13 is true for integer $r-1$, i.e. suppose that, for every $\ell_1 \geq 0$,

$$U(\ell_1, r-1) = \frac{1}{\binom{\ell_1+r-1}{\ell_1-1}} \sum_{\ell_2=0}^{\ell_1} P^{\ell_1-\ell_2} D \sum_{\ell_3=0}^{\ell_2} P^{\ell_2-\ell_3} D \cdots \sum_{\ell_{r-1}=0}^{\ell_{r-2}} P^{\ell_{r-2}-\ell_{r-1}} D \sum_{\ell_r=0}^{\ell_{r-1}} P^{\ell_r} D \mathbb{1}.$$

Using Eq. 11, we obtain directly Eq. 13. □

Theorem 6 *For every $n \geq 0$, $r \geq 1$ and $i \in S$, we have*

$$U_i(n,r) = \frac{1}{\binom{n+r}{r}} E\left(\sum_{\ell_1=0}^{n} d(Z_{\ell_1}) \sum_{\ell_2=\ell_1}^{n} d(Z_{\ell_2}) \cdots \sum_{\ell_{r-1}=\ell_{r-2}}^{n} d(Z_{\ell_{r-1}}) \sum_{\ell_r=\ell_{r-1}}^{n} d(Z_{\ell_r}) \middle| Z_0 = i \right). \quad (14)$$

Proof The relation is true for $r = 1$ since the right hand side becomes

$$\frac{1}{n+1} E\left(\sum_{\ell_1=0}^{n} d(Z_{\ell_1}) \Big| Z_0 = i\right) = \frac{1}{n+1} \sum_{\ell_1=0}^{n} E\left(d(Z_{\ell_1})|Z_0 = i\right)$$
$$= \frac{1}{n+1} \sum_{\ell_1=0}^{n} \sum_{j\in S} (P^{\ell_1})_{i,j} d(j)$$
$$= \frac{1}{n+1} \sum_{\ell_1=0}^{n} (P^{\ell_1} D \mathbb{1})_i$$
$$= U_i(n, 1).$$

Suppose that Eq. 14 is true for integer $r-1$, i.e.

$$U_i(n, r-1) = \frac{E\left(\sum_{\ell_1=0}^{n} d(Z_{\ell_1}) \sum_{\ell_2=\ell_1}^{n} d(Z_{\ell_2}) \cdots \sum_{\ell_{r-2}=\ell_{r-3}}^{n} d(Z_{\ell_{r-2}}) \sum_{\ell_{r-1}=\ell_{r-2}}^{n} d(Z_{\ell_{r-1}}) \Big| Z_0 = i\right)}{\binom{n+r-1}{r-1}}.$$

We then have

$$E\left(\sum_{\ell_1=0}^{n} d(Z_{\ell_1}) \sum_{\ell_2=\ell_1}^{n} d(Z_{\ell_2}) \cdots \sum_{\ell_{r-1}=\ell_{r-2}}^{n} d(Z_{\ell_{r-1}}) \sum_{\ell_r=\ell_{r-1}}^{n} d(Z_{\ell_r}) \Big| Z_0 = i\right)$$
$$= \sum_{j\in S} E\left(\sum_{\ell_1=0}^{n} d(Z_{\ell_1}) \sum_{\ell_2=\ell_1}^{n} d(Z_{\ell_2}) \cdots \sum_{\ell_{r-1}=\ell_{r-2}}^{n} d(Z_{\ell_{r-1}}) \sum_{\ell_r=\ell_{r-1}}^{n} d(Z_{\ell_r}) \Big| Z_{\ell_1} = j, Z_0 = i\right) (P^{\ell_1})_{i,j}$$
$$= \sum_{\ell_1=0}^{n} \sum_{j\in S} (P^{\ell_1})_{i,j} d(j) E\left(\sum_{\ell_2=\ell_1}^{n} d(Z_{\ell_2}) \cdots \sum_{\ell_{r-1}=\ell_{r-2}}^{n} d(Z_{\ell_{r-1}}) \sum_{\ell_r=\ell_{r-1}}^{n} d(Z_{\ell_r}) \Big| Z_{\ell_1} = j, Z_0 = i\right)$$
$$= \sum_{\ell_1=0}^{n} \sum_{j\in S} (P^{\ell_1})_{i,j} d(j) E\left(\sum_{\ell_2=\ell_1}^{n} d(Z_{\ell_2}) \cdots \sum_{\ell_{r-1}=\ell_{r-2}}^{n} d(Z_{\ell_{r-1}}) \sum_{\ell_r=\ell_{r-1}}^{n} d(Z_{\ell_r}) \Big| Z_{\ell_1} = j\right)$$
$$= \sum_{\ell_1=0}^{n} \sum_{j\in S} (P^{\ell_1})_{i,j} d(j) E\left(\sum_{\ell_2=0}^{n-\ell_1} d(Z_{\ell_2}) \cdots \sum_{\ell_{r-1}=\ell_{r-2}}^{n-\ell_1} d(Z_{\ell_{r-1}}) \sum_{\ell_r=\ell_{r-1}}^{n-\ell_1} d(Z_{\ell_r}) \Big| Z_0 = j\right)$$
$$= \sum_{\ell_1=0}^{n} \sum_{j\in S} (P^{\ell_1})_{i,j} d(j) \binom{n-\ell_1+r-1}{r-1} U_j(n-\ell_1, r-1)$$
$$= \sum_{\ell_1=0}^{n} \binom{\ell_1+r-1}{r-1} \sum_{j\in S} (P^{n-\ell_1})_{i,j} d(j) U_j(\ell_1, r-1)$$

$$= \sum_{\ell_1=0}^{n} \binom{\ell_1 + r - 1}{r - 1} (P^{n-\ell_1} DU(\ell_1, r - 1))_i$$

$$= \binom{n + r}{r} U_i(n, r),$$

which completes the proof. Note that the third equality follows from the Markov property, the fourth equality follows from the homogeneity of the Markov chain Z, the fifth one follows from the recurrence hypothesis and the last one is Eq. 11. □

The two following lemmas will be used to prove Theorem 7.

Lemma 1 *For every $t \geq 0$, we have*

$$\int_0^t (t - u)^\ell u^n du = \binom{n + \ell}{\ell} \frac{t^{n+\ell+1}}{(n + \ell + 1)}.$$

Proof Let $I(n, \ell)$ denote the integral in the left hand side. Using an integration by parts, we obtain

$$I(n, \ell) = \frac{n}{\ell + 1} I(n - 1, \ell + 1).$$

This leads to

$$I(n, \ell) = \binom{n + \ell}{\ell} I(0, n + \ell) = \binom{n + \ell}{\ell} \frac{t^{n+\ell+1}}{n + \ell + 1},$$

which completes the proof. □

Let us define $G_i(t, n, r) = E(Y(t)^r | N_t = n, X_0 = i)$.

Lemma 2 *For every $t \geq 0$, $n \geq 1$, $r \geq 0$ and $i \in S$, we have*

$$G_i(t, n, r) = \frac{n}{t^{n+r}} \sum_{\ell=0}^{r} \binom{r}{\ell} d(i)^\ell \sum_{j \in S} P_{i,j} \int_0^t (t - u)^\ell u^{n+r-\ell-1} G_j(u, n - 1, r - \ell) du.$$

Proof Let us define the quantity $V_i(t,n,r) = E(Y(t)^r 1_{\{N_t=n\}}|X_0 = i)$. Let T_1 be the sojourn time in the initial state. Using a renewal argument, we have

$$
\begin{aligned}
&V_i(t,n,r)\\
&\quad = E(Y(t)^r 1_{\{N_t=n\}}|X_0 = i)\\
&\quad = \int_0^t \sum_{j\in S} P_{i,j} E(Y(t)^r 1_{\{N_t=n\}}|X_u = j,\, T_1 = u,\, X_0 = i)\nu e^{-\nu u} du\\
&\quad = \int_0^t \sum_{j\in S} P_{i,j} E\left(\left[f_i u + \int_u^t f(X_s)ds\right]^r 1_{\{N_t-N_u=n-1\}}|X_u = j,\, T_1 = u,\, X_0 = i\right)\nu e^{-\nu u} du\\
&\quad = \int_0^t \sum_{j\in S} P_{i,j} E\left(\left[f_i u + \int_u^t f(X_s)ds\right]^r 1_{\{N_t-N_u=n-1\}}\middle|X_u = j\right)\nu e^{-\nu u} du\\
&\quad = \int_0^t \sum_{j\in S} P_{i,j} E\left(\left[f_i u + Y(t-u)\right]^r 1_{\{N_t-N_u=n-1\}}|X_u = j\right)\nu e^{-\nu u} du\\
&\quad = \int_0^t \sum_{j\in S} P_{i,j} E\left(\left[f_i u + Y(t-u)\right]^r 1_{\{N_{t-u}=n-1\}}|X_0 = j\right)\nu e^{-\nu u} du\\
&\quad = \sum_{\ell=0}^r \binom{r}{\ell} f(i)^\ell \sum_{j\in S} P_{i,j} \int_0^t u^\ell E\left(Y(t-u)^{r-\ell} 1_{\{N_{t-u}=n-1\}}|X_0 = j\right)\nu e^{-\nu u} du\\
&\quad = \sum_{\ell=0}^r \binom{r}{\ell} f(i)^\ell \sum_{j\in S} P_{i,j} \int_0^t u^\ell V_j(t-u, n-1, r-\ell)\nu e^{-\nu u} du.
\end{aligned}
$$

Unconditioning on the number of transitions during $(0,t)$, we get

$$G_i(t,n,r) = \frac{n! V_i(t,n,r)}{f^r t^r e^{-\nu t}(\nu t)^n},$$

and thus

$$
\begin{aligned}
G_i(t,n,r) &= \frac{n}{t^{n+r}} \sum_{\ell=0}^r \binom{r}{\ell} d(i)^\ell \sum_{j\in S} P_{i,j} \int_0^t u^\ell (t-u)^{n+r-\ell-1} G_j(t-u, n-1, r-\ell) du\\
&= \frac{n}{t^{n+r}} \sum_{\ell=0}^r \binom{r}{\ell} d(i)^\ell \sum_{j\in S} P_{i,j} \int_0^t (t-u)^\ell u^{n+r-\ell-1} G_j(u, n-1, r-\ell) du,
\end{aligned}
$$

which completes the proof. □

Theorem 7 *For every $n \geq 0$, $r \geq 0$, $i \in S$ and $t \geq 0$, we have*

$$U_i(n,r) = E(Y(t)^r | N_t = n,\, X_0 = i). \tag{15}$$

Proof Conditioning on N_t, we have

$$E(Y(t)^r | X_0 = i) = \sum_{n=0}^\infty e^{-\nu t} \frac{(\nu t)^n}{n!} E(Y(t)^r | N_t = n,\, X_0 = i),$$

and from Eq. 6, we have

$$E(Y(t)^r|X_0 = i) = \sum_{n=0}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} U_i(n, r).$$

So, to prove the theorem, we just have to show that $E(Y(t)^r|N_t = n, X_0 = i)$ is independent of t. We proceed by recurrence over index n.

For $n = 0$, we have $E(Y(t)^r|N_t = 0, X_0 = i) = d(i)^r$ since $Y(t) = f(i)t$ when $N_t = 0$ and $X_0 = i$.

Suppose that $U_i(n-1, r) = E(Y(t)^r|N_t = n-1, X_0 = i)$. Using this hypothesis and Lemma 2, we obtain

$$G_i(t, n, r) = \frac{n}{t^{n+r}} \sum_{\ell=0}^{r} \binom{r}{\ell} d(i)^{\ell} \sum_{j \in S} P_{i,j} U_j(n-1, r-\ell) \int_0^t (t-u)^{\ell} u^{n+r-\ell-1} du.$$

Using now Lemma 1, we obtain

$$\begin{aligned} G_i(t, n, r) &= \frac{n}{t^{n+r}} \sum_{\ell=0}^{r} \binom{r}{\ell} d(i)^{\ell} \sum_{j \in S} P_{i,j} U_j(n-1, r-\ell) \int_0^t (t-u)^{\ell} u^{n+r-\ell-1} du \\ &= \frac{n}{n+r} \sum_{\ell=0}^{r} \binom{n-1+r}{\ell} \binom{r}{\ell} d(i)^{\ell} \sum_{j \in S} P_{i,j} U_j(n-1, r-\ell), \end{aligned}$$

which is independent of t. □

In the next section, we present some numerical experiments to show the importance of the stationarity detection in the reduction of the computational time of the moments $E(Y(t)^r)$.

5 Numerical example

We consider a fault-tolerant multiprocessor system which consists of n identical processors and b buffer stages. Processors fail independently at rate λ and are repaired singly with rate μ. Buffers stages fail independently at rate γ and are repaired with rate τ. Processor failures cause a graceful degradation of the system and the number of operational processors is decreased by one. The system is in a failed state when all the processors have failed or any of the buffer stages has failed. No additional processor failures are assumed to occur when the system is in a failed state. The model is represented by a Markov process with state transition diagram shown in Fig. 1. The state space of the system is $S = \{(i, j); 0 \le i \le n, \ j = 0, 1\}$. The component i of a state (i, j) means that there are i operational processors and the component j is zero if any of the buffer stages has failed, otherwise it is one. It follows that the set U of operational states is $U = \{(i, 1); 1 \le i \le n\}$. The reward structure we choose here is given by $f(i) = 1$ if $i \in U$ and $f(i) = 0$ otherwise. We suppose that the

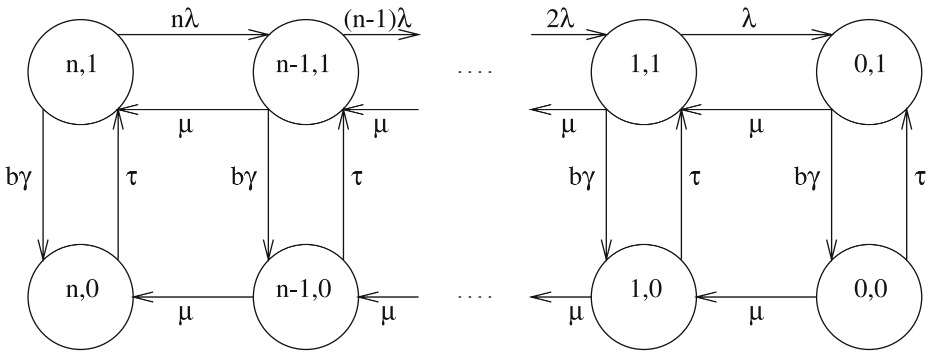

Fig. 1 State-transition diagram for a n-processor system

initial state of the system is state $(n, 1)$. The number of processors is fixed to 16, each with a failure rate $\lambda = 0.00006$ per hour and a repair rate $\mu = 0.1666$ per hour. The number of buffer stages is fixed to 1024, each with a failure rate $\gamma = 0.00131$ per hour and a repair rate $\tau = 0.1666$ per hour. The error tolerance is $\varepsilon = 0.00001$.

In Fig. 2, we plot the first five moments $Y(t)$, as a function of t.

Let us now consider higher values of t. Solving the linear system $\pi Q = 0$ with $\pi \mathbb{1} = 1$, we obtain $\pi D \mathbb{1} = 0.110475$. The complexities of Algorithm 1 and Algorithm 2 are mainly due to the products of matrix P by vector $U(n-1, r)$. The number of such products is equal to NR for Algorithm 1 and it is equal to KR, with $K \leq N$, for Algorithm 2. We show in Fig. 3, the values of K and N obtained when $R = 5$, for different values of $t = t_J$. As expected the value of $N = N(t_J)$ increases

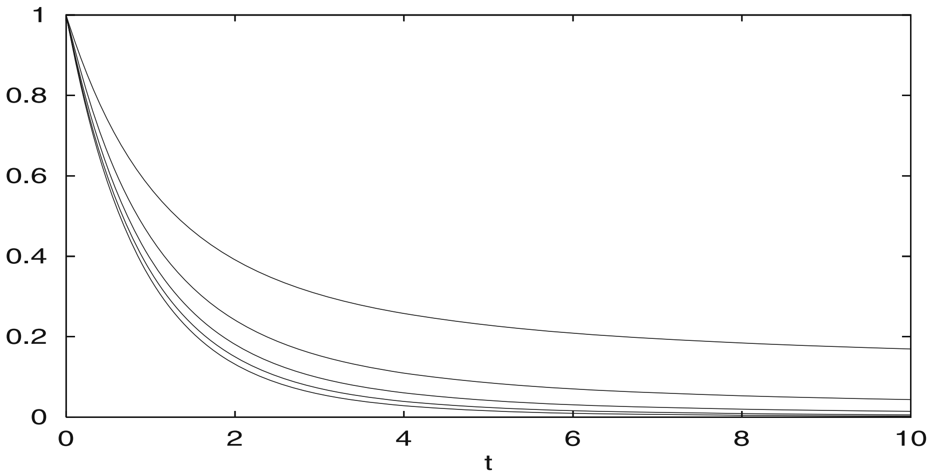

Fig. 2 (Top to bottom) $E(Y(t)^r)$ for $r = 1, 2, 3, 4, 5$

t	50000	60000	70000	80000	90000	100000
N	76621	91823	107015	122200	137379	152154
K	76621	84955	84955	84955	84955	84955

Fig. 3 Stationarity detection for different values of t, $R = 5$

with t_J and the value of K constant after the first instant where $K < N$. This instant, which we denote by T_ε, does depend on ε, and may be called the time to stationarity. It is defined as

$$T_\varepsilon = \inf\{t \geq 0 | K < N(t)\}.$$

In our example, it is between 50000 and 60000. A more detailed analysis shows that $\lceil T_\varepsilon \rceil = 55482$.

Appendix

Let $\beta(n, \ell)$, for $n \geq 0$ and $0 \leq \ell \leq n$, be real numbers, $H(\ell)$ be a sequence of matrices and $U(\ell)$ be a sequence of column vectors with the same finite dimension. We define the sequence of column vectors $V(n)$ by

$$V(n) = \sum_{\ell=0}^{n} \beta(n, \ell) H(n-\ell) U(\ell).$$

We ave the following result.

Lemma 3 *Assume that*

$$\beta(n, \ell) \geq 0, \sum_{\ell=0}^{n} \beta(n, \ell) = 1, \lim_{n \longrightarrow \infty} \sup_{0 \leq \ell \leq n} \beta(n, \ell) = 0, \lim_{\ell \longrightarrow \infty} H(\ell) = H \text{ and } \lim_{\ell \longrightarrow \infty} U(\ell) = U$$

then

$$\lim_{n \longrightarrow \infty} V(n) = HU.$$

Proof The convergence of the sequences $H(\ell)$ and $U(\ell)$ implies that both sequences are uniformly bounded, i.e. there exists an integer M such that, for every $\ell \geq 0$,

$$\|H(\ell)\| \leq M \text{ and } \|U(\ell)\| \leq M.$$

Moreover, their convergence implies that for any $\varepsilon > 0$, there exists L such that for any $\ell, m \geq L$, we have

$$\|H(m) - H\| + \|U(\ell) - U\| \leq \frac{\varepsilon}{2M}.$$

As a consequence, for $n \geq 2L$, we recover

$$\begin{aligned}
\|V(n) - HU\| &= \left\| \sum_{\ell=0}^{n} \beta(n,\ell) H(n-\ell) U(\ell) - HU \right\| \\
&= \left\| \sum_{\ell=0}^{n} \beta(n,\ell) [H(n-\ell) U(\ell) - HU] \right\| \\
&\leq \left\| \sum_{\ell=0}^{L-1} \beta(n,\ell) [H(n-\ell) U(\ell) - HU] \right\| \\
&\quad + \left\| \sum_{\ell=L}^{n-L} \beta(n,\ell) [H(n-\ell) U(\ell) - HU] \right\| \\
&\quad + \left\| \sum_{\ell=n-L+1}^{n} \beta(n,\ell) [H(n-\ell) U(\ell) - HU] \right\|.
\end{aligned}$$

We start with the second term. Since $L \leq \ell \leq n - L$, we have $\ell \geq L$ and $n - \ell \geq L$, and we can write

$$\begin{aligned}
\|H(n-\ell)U(\ell) - HU\| &= \|[H(n-\ell) - H]U(\ell) + H[U(\ell) - U]\| \\
&\leq \|H(n-\ell) - H\| \, \|U(\ell)\| + \|H\| \, \|U(\ell) - U\| \\
&\leq M\left(\|H(n-\ell) - H\| + \|U(\ell) - U\|\right) \leq \frac{\varepsilon}{2}.
\end{aligned}$$

We thus have

$$\begin{aligned}
\left\| \sum_{\ell=L}^{n-L} \beta(n,\ell)[H(n-\ell)U(\ell) - HU] \right\| &\leq \sum_{\ell=L}^{n-L} \beta(n,\ell) \, \|H(n-\ell)U(\ell) - HU\| \\
&\leq \frac{\varepsilon}{2} \sum_{\ell=L}^{n-L} \beta(n,\ell) \quad \leq \quad \frac{\varepsilon}{2}.
\end{aligned}$$

Concerning the first and the third terms, we use the fact that, for every $0 \leq \ell \leq n$, we have

$$\begin{aligned}
\|H(n-\ell)U(\ell) - HU\| &= \|H(n-\ell)U(\ell)\| + \|HU\| \\
&\leq \|H(n-\ell)\| \, \|U(\ell)\| + \|H\| \, \|U\| \leq 2M^2.
\end{aligned}$$

Let us define the sequence $\beta(n) = \sup_{0 \leq \ell \leq n} \beta(n,\ell)$. Since, by hypothesis, $\beta(n)$ converges to 0 when n goes towards infinity and since L is fixed, we can determine N such that for any $n \geq N$, we have $\beta(n) \leq \varepsilon/(8M^2 L)$.

We then have

$$\left\| \sum_{\ell=0}^{L-1} \beta(n,\ell)[H(n-\ell)U(\ell) - HU] \right\| + \left\| \sum_{\ell=n-L+1}^{n} \beta(n,\ell)[H(n-\ell)U(\ell) - HU] \right\|$$

$$\leq \sum_{\ell=0}^{L-1} \beta(n,\ell) \, \|H(n-\ell)U(\ell) - HU]\| + \sum_{\ell=n-L+1}^{n} \beta(n,\ell) \, \|H(n-\ell)U(\ell) - HU]\|$$

$$\leq 2M^2 \left(\sum_{\ell=0}^{L-1} \beta(n,\ell) + \sum_{\ell=n-L+1}^{n} \beta(n,\ell) \right) \leq 4M^2 L\beta(n) \leq \frac{\varepsilon}{2}.$$

Using the previous results, we obtain that for every $n \geq \max(2L, N)$,

$$\|V(n) - HU\| \leq \varepsilon,$$

which completes the proof. □

References

Bowerman PN, Nolty RG, Scheuer EM (1990) Calculation of the Poisson cumulative distribution function. IEEE Trans Reliab 39:158–161

Ciardo G, Marie R, Sericola B, Trivedi KS (1990) Performability analysis using semi-Markov reward processes. IEEE Trans Comput C-39:1251–1264, October

Cloth L, Haverkort BR (2006) Five performability algorithms: a comparison. In: Markov anniversary meeting, Charleston, SC, USA, Boson Books, pp 39–54

de Souza e Silva E, Gail HR (1998) An algorithm to calculate transient distributions of cumulative rate and impluse based reward. Commun Stat, Stoch Models 14(3)

de Souza e Silva E, Gail HR (2000) Transient solutions for Markov chains. In: Grassmann WK (ed) Computational probability, chapter 3. Kluwer

Donatiello L, Grassi V (1991) On evaluating the cumulative performance distribution of fault-tolerant computer systems. IEEE Trans Comput 45(11)

Iyer BR, Donatiello L, Heidelberger P (1986) Analysis of performability for stochastic models of fault-tolerant systems. IEEE Trans Comput, C-35:902–907, October

Kulkarni VG, Nicola VF, Smith RM, Trivedi KS (1986) Numerical evaluation of performability and job completion time in repairable fault-tolerant systems. In: Proceedings IEEE 16-th fault-tolerant computing symposium, Vienna, Austria, pp 252–257, July

Meyer JF (1980) On evaluating the performability of degradable computing systems. IEEE Trans Comput C-29:720–731, August

Meyer JF (1982) Closed-form solutions for performability. IEEE Trans Comput C-31:648–657, July

Nabli H, Sericola B (1996) Performability analysis: a new algorithm. IEEE Trans Comput C-45:491–494, April

Rácz S, Telek M (1999) Numerical analysis of large Markovian reward models. Perform Eval 36–37

Ross SM (1983) Stochastic processes. John Wiley and Sons

Sericola B (1999) Availability analysis of repairable computer systems and stationarity detection. IEEE Trans Comput C-48:1166–1172, November

Smith RM, Trivedi KS, Ramesh AV (1988) Performability analysis: measures, an algorithm, and a case study. IEEE Trans Comput C-37:406–417, April

Stenberg F, Manca R, Silvestrov D (2007) An algorithmic approach to discrete time non-homogeneous backward semi-Markov reward processes with an application to disability insurance. Methodol Comput Appl Probab 9(4):497–519

Telek M, Horváth A, Horváth G (2004) Analysis of inhomogeneous Markov reward models. Linear Algebra Appl 386:383–405