Stochastic Models, 21:599–613, 2005

ISSN: 1532-6349 print/1532-4214 online
DOI: 10.1081/STM-200056225

TRANSIENT ANALYSIS OF AVERAGED QUEUE LENGTH IN MARKOVIAN QUEUES

Landy Rabehasaina □ *LMC-IMAG, rue des Mathématiques, Grenoble, France*

Bruno Sericola □ *IRISA-INRIA, Campus de Beaulieu, Rennes, France*

□ *We consider a Markovian queue and its associated exponentially averaged length. The set of partial differential equations satisfied by the joint distribution of the queue and the averaged queue length is given. We obtain a recursive expression for the moments of the averaged queue length, and develop a stable algorithm to compute them. These results are illustrated through numerical examples.*

Keywords Exponential averaging; Fluid queue; Markov chain; RED.

Mathematics Subject Classification Primary 60J25, 60K20; Secondary 90B20.

1. INTRODUCTION

One of the important issues in teletraffic research is congestion control and quality of service assurance. Different mechanisms were proposed to tackle these problems and one of these mechanisms is the random early detection (RED). Suggested by Floyd and Jacobson[2], it was first aimed at preventing buffer overflows in networks as well as smoothing TCP traffic arriving at a queue. Indeed in RED, packets are randomly dropped according to a probability that depends on an average of the queue length over time: this allows TCP sources not to increase or halve their rates all at the same time. Other advantages of RED, developed in Floyd[2], include less sensitivity to bursty traffic, as opposed to other congestion avoidance mechanisms, such as Drop Tail. Several modifications of RED have been proposed in the literature (see for instance Bodin[1] and Lin[6]) to cope in addition with different types of traffics.

Received September 2004; Accepted February 2005
Address correspondence to Bruno Sericola, IRISA-INRIA, Campus de Beaulieu, 35042 Rennes Cedex, France; E-mail: bruno.sericola@irisa.fr

In this paper, we study the distribution and the moments of the averaged length of a general Markovian queue, which is a first step towards the analysis of the RED mechanism itself. Indeed, in that more difficult case, we shall be able to tell, after evaluating the distribution or the moments of the averaged queue, whether the RED protocol performs well or not, since the difference between the averaged queue in our present paper and the averaged queue in RED gives the averaged number of customers rejected up to time t. This will be particularly interesting if we want to meet constraints, say on the amount of rejected customers at a fixed time (finite horizon constraint) or in the long run (infinite horizon constraint).

The model we consider here generalizes the one studied by Kuumola et al.[4,5], where the authors studied the averaged length of M/M/1/K queue. More specifically, we consider a Markovian queue and its exponentially averaged queue length between $[0, t)$. As underlined in Kuumola[4], the exponentially averaged queue length is taken with respect to time, as opposed to normal RED, where this average is computed at the arrival time of each new packet. Time average may in fact have been better suited than average at each arrival, as was pointed out by the authors (Kuumola[4,5]). Our paper generalizes some results from Kuumola[4,5], where analytical results were obtained in the case of a finite M/M/1 queue of size 2 or 3. We are particularly interested in a general Markovian queue, which includes some classical queues such as MMPP/PH/1 or BMAP/PH/1 queues. The Batch Markovian Arrival Process (BMAP) was first introduced by Neuts[8] as the versatile Markovian point process. It generalizes the Markovian Arrival Process (MAP) introduced by Lucantoni et al.[7].

This work is also related to Kella[3] and Rabehasaina[9,10] in which the authors study a Markov modulated fluid queue with a service rate depending linearly on the fluid level in the queue. However, the difference with those latter papers is the interpretation of the model: the averaged queue length and the instantaneous queue considered here are seen respectively as the fluid queue level and the Markov chain driving the fluid queue in Kella[3] and Rabehasaina[9,10]. We believe that these different interpretations, as well as the techniques related to each of these two points of view, can be exploited in future works. Stationary regime of the queue (i.e. of the exponentially averaged queue length in our context) is studied in Kella[3], in the case where the driving Markov chain has two states, and in Rabehasaina[9], where the model in addition features a white noise factor. Attention is given to the transient distribution in Rabehasaina[10]. As in Rabehasaina[9,10], we focus in the present paper on expressing recursively the moments of the exponentially averaged queue length and on obtaining them easily from an algorithmic point of view.

The paper is organized as follows. In section 2 we describe the model and set the differential equation satisfied by the distribution function of the exponentially averaged queue length by using a standard first passage argument on the Markov process formed by the queue length at time t and its exponentially averaged queue length over $[0, t)$. In section 3 we compute all moments of this averaged queue length recursively and we give a numerically stable algorithm to compute these moments, which requires only additions and multiplications of non-negative quantities. These results are illustrated through numerical examples.

2. MODEL DESCRIPTION

We consider a Markovian queue with either a finite or infinite capacity waiting room. The states of this queue are represented by the regular continuous time Markov chain $X = \{X_t, t \in \mathbb{R}^+\}$ given by $X_t = (L_t, \Phi_{L_t})$ where L_t represents the instantaneous queue length at time t and Φ_{L_t} represents the environment, also called the phase process, associated with L_t needed to ensure that X is a Markov chain. Note that the phase process at time t may depend on the queue length L_t at time t. For example, if the Markovian queue is the BMAP/PH/1 queue, then Φ_{L_t} is the phase of the arrival process if $L_t = 0$ and is a couple of which the first entry is the phase of the arrival process and the second entry is the phase of the server at time t, if $L_t \geq 1$. That is why we shall sometimes call Φ_{L_t} the phase process. Of course, if the Markovian queue is the M/M/1 queue then there is no need of the process Φ_{L_t}, since $X_t = L_t$ in that case.

The state space of the Markov chain X is denoted by E given by

$$E = \bigcup_{j \in \mathbb{K}} \{j\} \times \mathscr{E}_j,$$

where $\mathbb{K}$ is the set $\{0, \ldots, K\}$ if the capacity of the queue is finite and equal to K and is the set $\mathbb{N}$ if the capacity is infinite. $\mathscr{E}_j$ is the set of values taken by the phase process Φ_{L_t} when $L_t = j$. We suppose that for every $j \in \mathbb{K}$, $\mathscr{E}_j$ is finite. We denote by Q the infinitesimal generator of the Markov chain X and by $q_{j,\phi}$ the output rate of state (j, ϕ), where of course $\phi \in \mathscr{E}_j$. The initial distribution of X is denoted by the row vector $\pi(0)$.

The continuous version of the RED mechanism is based on the exponentially averaged queue length S_t defined, for small values of h, by $S_0 = 0$ and

$$S_{t+h} = e^{-wh} S_t + (1 - e^{-wh}) L_t,$$

where $w \geq 0$ is a weighting or averaging parameter. By subtracting S_t from each side, dividing by h, and letting h tend to 0, we obtain

$$\frac{d}{dt} S_t = -w(S_t - L_t). \tag{1}$$

This equation leads to the following expressions of S_t

$$S_t = \int_0^t L_{t-u} w e^{-wu}\, du = e^{-wt} \int_0^t L_u w e^{wu}\, du. \tag{2}$$

We first study in this paper the joint distribution of the process (S_t, L_t). To do that we consider the process (S_t, X_t) which is a Markov process with state space

$$\mathcal{S} = \{(x, (j, \phi)) \mid x \in \mathbb{R}^+, j \in \mathbb{K}, \phi \in \mathcal{E}_j\}.$$

Note that if the capacity is finite and equal to K, we can restrict the state space by taking $x \le K$.

For every $(x, (j, \phi)) \in \mathcal{S}$ and $t \in \mathbb{R}^+$, we denote by $F_{j,\phi}(t, x)$ the joint distribution of (S_t, X_t), that is

$$F_{j,\phi}(t, x) = \Pr\{S_t \le x, X_t = (j, \phi)\}.$$

It is easy to check that for a fixed $t > 0$, the distribution of S_t has jumps which correspond to the fact that the number of customers in the queue remains the same during the whole interval $[0, t)$. More precisely, we denote by B_j, for $j \in \mathbb{K}$, the subset of states of the state space E corresponding to a queue length equal to j, i.e.,

$$B_j = \{(l, \phi) \in E \mid l = j, \phi \in \mathcal{E}_j\}.$$

For $j \in \mathbb{K}$ and $t > 0$, we then have from either Relation (1) or Relation (2)

$$S_t = j(1 - e^{-wt}) \iff L_s = j, \quad \forall s \in [0, t) \iff X_s \in B_j, \quad \forall s \in [0, t).$$

It follows that

$$\Pr\{S_t = j(1 - e^{-wt})\} = \pi_{B_j}(0) e^{Q_{B_j B_j} t} \mathbb{1},$$

where $Q_{B_j B_j}$ is the sub-infinitesimal generator of dimension $|B_j|$ obtained from Q by considering only the internal transitions of the subset B_j and $\pi_{B_j}(0)$ is the subvector of dimension $|B_j|$ obtained from the initial probability distribution $\pi(0)$ by considering only the initial probabilities of states of B_j. The vector $\mathbb{1}$ is the column vector with all its entries equal to 1, its dimension being given by the context, which is $|B_j|$ here. In particular, when the capacity of the queue is equal to K, we have $S_t \le K(1 - e^{-wt})$.

Theorem 2.1. *For every $t > 0$ and $x \ne j(1 - e^{-wt})$ for all $j \in \mathbb{K}$, we have*

$$\frac{\partial F_{j,\phi}(t, x)}{\partial t} = w(x - j) \frac{\partial F_{j,\phi}(t, x)}{\partial x} + \sum_{(k,\theta) \in E} F_{k,\theta}(t, x) Q_{(k,\theta),(j,\phi)}. \tag{3}$$

Proof. By conditioning on the number $N_{t,t+s}$ of transitions of the Markov chain X in $[t, t+s)$, we have

$$\begin{aligned}\Pr\{S_{t+s} \le x, X_{t+s} = (j,\phi)\} = {}& \Pr\{S_{t+s} \le x, X_{t+s} = (j,\phi), N_{t,t+s} = 0\} \\ &+ \Pr\{S_{t+s} \le x, X_{t+s} = (j,\phi), N_{t,t+s} = 1\} \\ &+ \Pr\{S_{t+s} \le x, X_{t+s} = (j,\phi), N_{t,t+s} \ge 2\}.\end{aligned}$$

We separately consider these three terms. For the first term, since $X_{t+s} = (j,\phi)$ is equivalent to $X_t = (j,\phi)$ when $N_{t,t+s} = 0$, we have

$$\begin{aligned}&\Pr\{S_{t+s} \le x, X_{t+s} = (j,\phi), N_{t,t+s} = 0\} \\ &= \Pr\{S_{t+s} \le x, X_t = (j,\phi), N_{t,t+s} = 0\} \\ &= \Pr\{S_{t+s} \le x \mid X_t = (j,\phi), N_{t,t+s} = 0\}\Pr\{X_t = (j,\phi), N_{t,t+s} = 0\} \\ &= \Pr\{S_t \le (x-j)e^{ws} + j \mid X_t = (j,\phi), N_{t,t+s} = 0\}\Pr\{X_t = (j,\phi), N_{t,t+s} = 0\} \\ &= \Pr\{S_t \le (x-j)e^{ws} + j \mid X_t = (j,\phi)\}\Pr\{X_t = (j,\phi), N_{t,t+s} = 0\} \\ &= \Pr\{S_t \le (x-j)e^{ws} + j, X_t = (j,\phi)\}\Pr\{N_{t,t+s} = 0 \mid X_t = (j,\phi)\} \\ &= \Pr\{N_{t,t+s} = 0 \mid X_t = (j,\phi)\}F_{j,\phi}(t, (x-j)e^{ws} + j) \\ &= e^{-q_{j,\phi}s}F_{j,\phi}(t, (x-j)e^{ws} + j) \\ &= (1 - q_{j,\phi}s)F_{j,\phi}(t, (x-j)e^{ws} + j) + o(s).\end{aligned}$$

The third equality follows from the fact that, if $X_t = (j,\phi)$, which means that $L_t = j$ and $N_{t,t+s} = 0$, we have $S_{t+s} = e^{-ws}S_t + j(1 - e^{-ws})$. The fourth and the seventh equalities follow from the Markov property.

For the second term, which we denote by $G(s)$, we define

$$G_{(k,\theta)}(s) = \Pr\{S_{t+s} \le x \mid X_t = (k,\theta), X_{t+s} = (j,\phi), N_{t,t+s} = 1\}.$$

We then have

$$\begin{aligned}G(s) &= \Pr\{S_{t+s} \le x, X_{t+s} = (j,\phi), N_{t,t+s} = 1\} \\ &= \sum_{(k,\theta)\in E} G_{(k,\theta)}(s)\Pr\{X_t = (k,\theta)\}\Pr\{X_{t+s} = (j,\phi), N_{t,t+s} = 1 \mid X_t = (k,\theta)\} \\ &= \sum_{(k,\theta)\in E} G_{(k,\theta)}(s)\Pr\{X_t = (k,\theta)\}H_{(k,\theta),(j,\phi)}q_{k,\theta}s + o(s), \qquad (4)\end{aligned}$$

where $H = (H_{(k,\theta),(j,\phi)})$ is the transition probability matrix of the embedded Markov chain at the instants of state changes of X. It is given by the relation

$$H = I + \Lambda^{-1} Q, \tag{5}$$

where I is the identity matrix whose dimension is given by the context and Λ is the diagonal matrix containing the output rates $q_{j,\phi}$ of the states of X.

When $X_t = (k, \theta)$, $X_{t+s} = (j, \phi)$ and $N_{t,t+s} = 1$, we have

$$e^{-ws} S_t + m(1 - e^{-ws}) \le S_{t+s} \le e^{-ws} S_t + M(1 - e^{-ws}),$$

where $m = \min\{k, j\}$ and $M = \max\{k, j\}$. We thus have, using the Markov property,

$$\begin{aligned} &\Pr\{S_t \le (x - M)e^{ws} + M \mid X_t = (k, \theta)\} \\ &\quad \le G_{k,\theta}(s) \le \Pr\{S_t \le (x - m)e^{ws} + m \mid X_t = (k, \theta)\}. \end{aligned}$$

We then obtain from (4)

$$\sum_{(k,\theta)\in E} F_{k,\theta}(t, (x - M)e^{ws} + M) H_{(k,\theta),(j,\phi)} q_{k,\theta} s + o(s) \le G(s),$$

and

$$G(s) \le \sum_{(k,\theta)\in E} F_{k,\theta}(t, (x - m)e^{ws} + m) H_{(k,\theta),(j,\phi)} q_{k,\theta} s + o(s).$$

By dividing by s and taking the limit when s tends to 0, we get

$$\lim_{s\to 0} \frac{G(s)}{s} = \sum_{(k,\theta)\in E} F_{k,\theta}(t, x) H_{(k,\theta),(j,\phi)} q_{k,\theta}.$$

For the third term, we have

$$\Pr\{S_{t+s} \le x, X_{t+s} = (j, \phi), N_{t,t+s} \ge 2\} \le \Pr\{N_{t,t+s} \ge 2\} = o(s).$$

Combining the three terms, we obtain

$$\begin{aligned} &\frac{F_{j,\phi}(t + s, x) - F_{j,\phi}(t, x)}{s} \\ &\quad = \frac{(1 - q_{j,\phi} s) F_{j,\phi}(t, (x - j)e^{ws} + j) - F_{j,\phi}(t, x)}{s} + \frac{G(s)}{s} + \frac{o(s)}{s} \\ &\quad = \frac{F_{j,\phi}(t, (x - j)e^{ws} + j) F_{j,\phi}(t, x)}{s} - q_{j,\phi} F_{j,\phi}(t, (x - j)e^{ws} + j) + \frac{G(s)}{s} + \frac{o(s)}{s} \end{aligned}$$

If now s tends to 0, we get

$$\frac{\partial F_{j,\phi}(t,x)}{\partial t} = w(x-j)\frac{\partial F_{j,\phi}(t,x)}{\partial x} - q_{j,\phi}F_{j,\phi}(t,x) + \sum_{(k,\theta)\in E} F_{k,\theta}(t,x)H_{(k,\theta),(j,\phi)}q_{k,\theta}.$$

Because from (5), $Q = -\Lambda + \Lambda H$, we obtain that

$$\frac{\partial F_{j,\phi}(t,x)}{\partial t} = w(x-j)\frac{\partial F_{j,\phi}(t,x)}{\partial x} + \sum_{(k,\theta)\in E} F_{k,\theta}(t,x)Q_{(k,\theta),(j,\phi)},$$

which completes the proof.

We denote by $F(t,x)$ the row vector $(F_j(t,x))_{j\in\mathbb{K}}$ where each $F_j(t,x)$ is itself a row vector of dimension $|B_j|$ equal to $(F_{j,\phi}(t,x))_{\phi\in\mathscr{E}_j}$ and by $D_j(x)$ the diagonal matrix $D_j(x) = w(x-j)I$ of dimension $|B_j|$. Relation (3) can then be written as

$$\frac{\partial F_j(t,x)}{\partial t} = \frac{\partial F_j(t,x)}{\partial x}D_j(x) + \sum_{k\in\mathbb{K}} F_k(t,x)Q_{B_kB_j},$$

where $Q_{B_kB_j}$ is the sub-matrix of Q containing the transitions from states of B_k to states of B_j. It can be also written as

$$\frac{\partial F(t,x)}{\partial t} = \frac{\partial F(t,x)}{\partial x}D(x) + F(t,x)Q,$$

where $D(x)$ is the block diagonal matrix whose blocks are the diagonal matrices $D_j(x)$.

3. MOMENTS EVALUATION

We introduce the transient state probabilities of the Markov chain X defined, for every $(j,\phi)\in E$, by $\pi_{j,\phi}(t) = \Pr\{X_t = (j,\phi)\}$. We denote by $\pi(t)$ the row vector containing all the $\pi_{j,\phi}(t)$. This probability distribution is given by

$$\pi(t) = \pi(0)e^{Qt}.$$

For every $(j,\phi)\in E$, $r\geq 0$ and $t\geq 0$, we denote by $V_{j,\phi}(t,r)$ the rth moment of S_t when $X_t = (j,\phi)$, that is

$$V_{j,\phi}(t,r) = E(S_t^r 1_{\{X_t=(j,\phi)\}}) = \int_0^\infty x^r\, dF_{j,\phi}(t,x),$$

where $1_{\{c\}}$ equals 1 if condition c holds and 0 otherwise. For $t \geq 0$ and $r = 0$, we have $V_{j,\phi}(t, 0) = \pi_{j,\phi}(t)$ and for $t = 0$ we have $V_{j,\phi}(0, r) = \pi_{j,\phi}(0)1_{\{r=0\}}$.

We denote by $V(t, r)$ the row vector containing the $V_{j,\phi}(t, r)$, for $(j, \phi) \in E$. We first make sure that all these moments exist, i.e., that $E(S_t^r) < +\infty$ for all $r > 0$ and $t \geq 0$. This can be done as follows. From Relation (2), we have

$$\begin{aligned} S_t^r &= \left(\int_0^t L_{t-u} w e^{-wu} du \right)^r \\ &\leq \left(\int_0^t w L_{t-u} du \right)^r = \left(\int_0^t w L_u du \right)^r \\ &= t^r w^r \left(\frac{1}{t} \int_0^t L_u du \right)^r \\ &\leq t^{r-1} w^r \int_0^t L_u^r du \quad \text{from Jensen's inequality.} \end{aligned}$$

Taking expectations on both sides, we get

$$E(S_t^r) \leq t^{r-1} w^r \int_0^t E(L_u^r) du.$$

The Markov chain X being supposed to be regular, the moments $E(L_u^r)$ are finite for all $u \geq 0$ and $r \in \mathbb{N}$, and so the moments $E(S_t^r)$ are also finite for all $t \geq 0$ and $r \in \mathbb{N}$.

The following theorem gives a recursive expression for the moments of S_t. We denote by D the block diagonal matrix whose jth block D_j is of dimension $|B_j|$ and is given by $D_j = jI$, for $j \in \mathbb{K}$.

Theorem 3.1. *For every $r \geq 1$ and $t \geq 0$, we have*

$$V(t, r) = wr \int_0^t V(t - u, r - 1) D e^{(Q - wrI)u} du. \tag{6}$$

Proof. From Relation (1), we easily get, for $r \geq 1$,

$$\frac{d}{dt} S_t^r = r S_t^{r-1} \frac{d}{dt} S_t = -wr S_t^r + wr S_t^{r-1} L_t.$$

For $r \geq 1$, we have $S_0^r = 0$, so the solution to this equation is

$$S_t^r = \int_0^t wr e^{-wru} S_{t-u}^{r-1} L_{t-u} du.$$

Multiplying both sides by $1_{\{X_t=(j,\phi)\}}$ and taking expectation, we get, for all $(j,\phi)\in E$,

$$
\begin{aligned}
V_{j,\phi}(t,r) &= E\left(\int_0^t wre^{-wru}S_{t-u}^{r-1}L_{t-u}du\ 1_{\{X_t=(j,\phi)\}}\right)\\
&= \int_0^t wre^{-wru}E(S_{t-u}^{r-1}L_{t-u}1_{\{X_t=(j,\phi)\}})du\\
&= \int_0^t wre^{-wru}\sum_{(k,\theta)\in E}E(S_{t-u}^{r-1}L_{t-u}1_{\{X_{t-u}=(k,\theta)\}}1_{\{X_t=(j,\phi)\}})du\\
&= \int_0^t wre^{-wru}\sum_{(k,\theta)\in E}kE(S_{t-u}^{r-1}1_{\{X_{t-u}=(k,\theta)\}}1_{\{X_t=(j,\phi)\}})du. \qquad (7)
\end{aligned}
$$

The expectation on the right hand side is given by

$$
\begin{aligned}
&E(S_{t-u}^{r-1}1_{\{X_{t-u}=(k,\theta)\}}1_{\{X_t=(j,\phi)\}})\\
&\quad = E(S_{t-u}^{r-1}\mid X_{t-u}=(k,\theta),X_t=(j,\phi))\Pr\{X_{t-u}=(k,\theta),X_t=(j,\phi)\}\\
&\quad = E(S_{t-u}^{r-1}\mid X_{t-u}=(k,\theta))\Pr\{X_{t-u}=(k,\theta),X_t=(j,\phi)\}\\
&\quad = E(S_{t-u}^{r-1}1_{\{X_{t-u}=(k,\theta)\}})\Pr\{X_t=(j,\phi)\mid X_{t-u}=(k,\theta)\}\\
&\quad = V_{k,\theta}(t-u,r-1)(e^{Qu})_{(k,\theta),(j,\phi)},
\end{aligned}
$$

where the second equality follows from the Markov property. Replacing this result in Relation (7), we obtain

$$
\begin{aligned}
V_{j,\phi}(t,r) &= \int_0^t wre^{-wru}\sum_{(k,\theta)\in E}kV_{k,\theta}(t-u,r-1)(e^{Qu})_{(k,\theta),(j,\phi)}du\\
&= wr\int_0^t \sum_{(k,\theta)\in E}kV_{k,\theta}(t-u,r-1)(e^{(Q-wr)u})_{(k,\theta),(j,\phi)}du.
\end{aligned}
$$

This gives in matrix notation

$$
V(t,r) = wr\int_0^t V(t-u,r-1)De^{(Q-wrI)u}du,
$$

which completes the proof.

As $V(t, 0) = \pi(t)$ for every $t \geq 0$, we have from (6)

$$\begin{aligned} V(t, 1) &= w \int_0^t \pi(t-u) D e^{(Q-wI)u} du \\ &= w\pi(0) \int_0^t e^{Q(t-u)} D e^{(Q-wI)u} du \\ &= w e^{-wt} \pi(0) \int_0^t e^{(Q+wI)u} D e^{Q(t-u)} du. \end{aligned}$$

This gives, since $e^{Q(t-u)}\mathbb{1} = \mathbb{1}$,

$$E(S_t) = V(t, 1)\mathbb{1} = w e^{-wt} \pi(0) \int_0^t e^{(Q+wI)u} du D\mathbb{1}. \tag{8}$$

Note that for $w > 0$, the matrix $Q + wI$ is in general a singular matrix.

3.1. Moments Computation

We suppose in this subsection that the Markov chain X is uniform which means that $v = \max\{q_{j,\phi}, (j, \phi) \in E\}$ is finite. We denote by P the transition probability matrix of the uniformized discrete-time Markov chain, with respect to v, associated to X. The matrices P and Q are related by $P = I + Q/v$. We then have the following theorem.

Theorem 3.1.1. *For every $t \geq 0$, $r \geq 0$, and for every $\theta \geq v$, we have*

$$V(t, r) = \sum_{n=0}^{\infty} e^{-\theta t} \frac{(\theta t)^n}{n!} U(n, r), \tag{9}$$

where the row vectors $U(n, r)$ are given by

$$U(n, 0) = \pi(0) \left(\frac{\theta - v}{\theta} I + \frac{v}{\theta} P \right)^n \quad \textit{for } n \geq 0 \tag{10}$$

$$U(0, r) = 0 \quad \textit{for } r \geq 1 \tag{11}$$

$$\begin{aligned} U(n, r) = &\frac{wr}{\theta} U(n-1, r-1) D + \left(1 - \frac{v + wr}{\theta} \right) U(n-1, r) \\ &+ \frac{v}{\theta} U(n-1, r) P \quad \textit{for } n \geq 1 \textit{ and } r \geq 1 \end{aligned} \tag{12}$$

Proof. When $r = 0$, we have $V(t, 0) = \pi(t) = \pi(0)e^{Qt}$ and

$$Q = -v(I - P) = -\theta \left[I - \left(\frac{\theta - v}{\theta} I + \frac{v}{\theta} P \right) \right],$$

so, we easily get

$$V(t,0) = \sum_{n=0}^{\infty} e^{-\theta t} \frac{(\theta t)^n}{n!} \pi(0) \left(\frac{\theta - v}{\theta} I + \frac{v}{\theta} P \right)^n.$$

In order to satisfy Relation (9) for $r = 0$, we take

$$U(n,0) = \pi(0) \left(\frac{\theta - v}{\theta} I + \frac{v}{\theta} P \right)^n.$$

For $r \geq 1$ and $t = 0$, we have $V(0,r) = 0$ and Relation (9) gives $V(0,r) = U(0,r)$, so we take $U(0,r) = 0$ for $r \geq 1$.

Consider Relation (6). Using a change of variable, it can be written, for $t > 0$ and $r \geq 1$, as

$$V(t,r) = wr \int_0^t V(u, r-1) D e^{-(Q-wrI)u} du e^{(Q-wrI)t}.$$

Differentiating with respect to t, we obtain

$$\frac{dV(t,r)}{dt} = wrV(t,r-1)D + V(t,r)(Q - wrI). \tag{13}$$

We write the solution to that differential equation as (9) and we determine the relations that must be satisfied by the row vectors $U(n,r)$. From Relation (9), we have

$$\frac{dV(t,r)}{dt} = \sum_{n=0}^{\infty} e^{-\theta t} \frac{(\theta t)^n}{n!} \theta(U(n+1,r) - U(n,r)),$$

$$wrV(t,r-1)D = \sum_{n=0}^{\infty} e^{-\theta t} \frac{(\theta t)^n}{n!} wrU(n,r-1)D,$$

and, using also the relation $Q = -v(I - P)$,

$$\begin{aligned} V(t,r)(Q - wrI) &= \sum_{n=0}^{\infty} e^{-\theta t} \frac{(\theta t)^n}{n!} U(n,r)(Q - wrI) \\ &= \sum_{n=0}^{\infty} e^{-\theta t} \frac{(\theta t)^n}{n!} U(n,r)(-(v + wr)I + vP). \end{aligned}$$

It follows that if the $U(n,r)$ are such that

$$\theta(U(n+1,r) - U(n,r)) = wrU(n,r-1)D - (v + wr)U(n,r) + vU(n,r)P,$$

then Equation (13) is satisfied. This relation can be rewritten, for $n \geq 1$ and $r \geq 1$, as

$$U(n,r) = \frac{wr}{\theta} U(n-1, r-1)D + \left(1 - \frac{v + wr}{\theta}\right) U(n-1, r) + \frac{v}{\theta} U(n-1, r)P.$$

3.2. Algorithmic Aspects

We suppose in this subsection that the Markovian queue has a finite capacity that we denote by K. We consider the computation of the R first moments $E(S_t), E(S_t^2), \ldots, E(S_t^R)$ for ℓ different values of t, say $t_1 < \cdots < t_\ell$.

By taking $\theta = v + wR$, Relation (12) becomes a convex combination of three vectors. It is then easy to check from Relations (10), (11) and (12) that, for every $n \geq 0$ and $r \geq 0$, the vectors $U(n, r)$ have non-negative entries and that we have

$$U(n,r)\mathbb{1} \leq K^r.$$

Also from Relations (11) and (12), it can be shown by recurrence that $U(n, r) = 0$ for $r \geq n + 1$. These considerations yield a computational method that avoids numerical problems since all the computed quantities are bounded and require only additions and multiplications of nonnegative numbers.

Let ε be the desired precision for the computation of all these moments. For every $t > 0$, we define the integer $N(t)$ by

$$N(t) = \min\left\{ n \geq 0 \;\middle|\; \sum_{i=0}^{n} e^{-\theta t} \frac{(\theta t)^i}{i!} \geq 1 - \frac{\varepsilon}{K^R} \right\}. \tag{14}$$

We thus have

$$E(S_t^r) = V(t,r)\mathbb{1} = \sum_{n=0}^{N(t)} e^{-\theta t} \frac{(\theta t)^n}{n!} U(n,r)\mathbb{1} + e(N(t)),$$

where, by definition of $N(t)$, the remainder of the series $e(N(t))$ satisfies

$$\begin{aligned} e(N(t)) &= \sum_{n=N(t)+1}^{\infty} e^{-\theta t} \frac{(\theta t)^n}{n!} U(n,r)\mathbb{1} \leq K^r \sum_{n=N(t)+1}^{\infty} e^{-\theta t} \frac{(\theta t)^n}{n!} \\ &\leq K^R \left(1 - \sum_{n=0}^{N(t)} e^{-\theta t} \frac{(\theta t)^n}{n!}\right) \leq \varepsilon. \end{aligned}$$

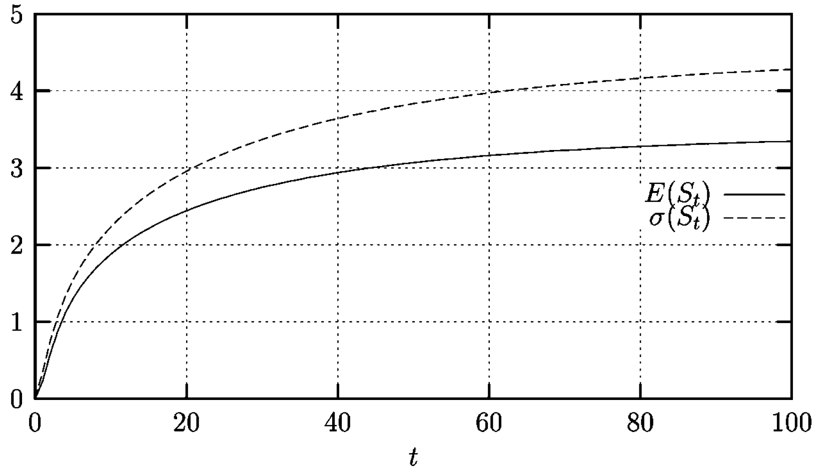

FIGURE 1 Mean value and standard deviation of S_t as a function of t for $w = 1$.

In order to compute the R first moments of S_t at the instants $t_1 < \cdots < t_\ell$, we need only to evaluate the vectors $U(n, r)$ for $n = 0, 1, \ldots, N(t_\ell)$, as these vectors do not depend on the values of $t_1 < \cdots < t_\ell$ and as the function $N(t)$ is an increasing function of t.

3.3. Numerical Examples

Consider the M/Cox$_2$/1/K queue with finite capacity equal to K, arrival rate equal to λ and with a two phases Coxian service time distribution given

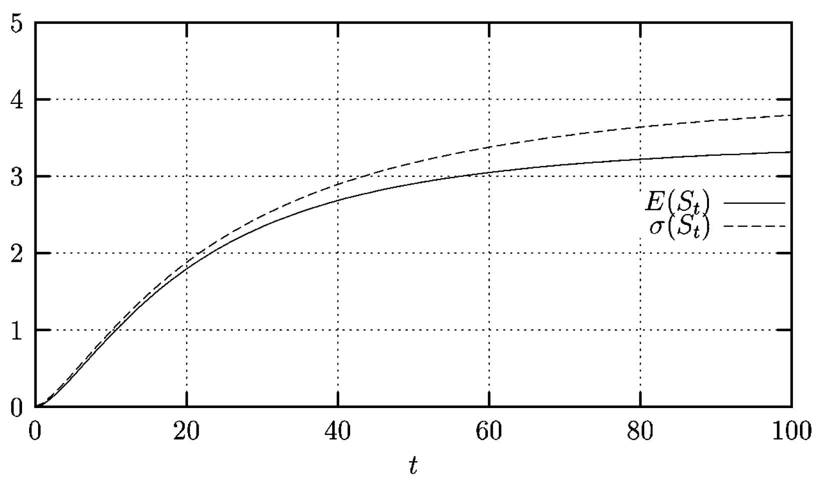

FIGURE 2 Mean value and standard deviation of S_t as a function of t for $w = 0.1$.

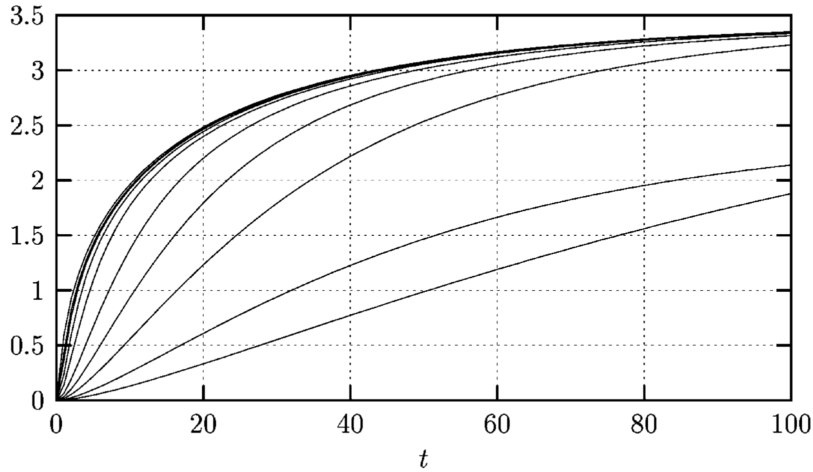

FIGURE 3 From top to the bottom : Mean values of L_t and S_t as a function of t for $w = 3, 2, 1, 0.5, 0.2, 0.1, 0.05, 0.03, 0.01$ respectively.

by the classical representation (β, T) of phase-type distributions, where

$$\beta = (1, 0) \quad \text{and} \quad T = \begin{pmatrix} -3/2 & 1 \\ 0 & -2 \end{pmatrix}.$$

The mean service time $-\beta T^{-1}\mathbb{1}$ is thus equal to 1. The queue is supposed to be initially empty and we take $\lambda = 0.8$ and $K = 50$. The precision is specified as $\varepsilon = 10^{-5}$.

Figure 1 and Figure 2 show the mean value $E(S_t)$ and the standard deviation $\sigma(S_t) = [E(S_t^2) - E(S_t)^2]^{1/2}$ as a function of t when the weighting parameter w is equal to 1 and 0.1 respectively.

We observed that when the weighting parameter w is greater than 1, the results obtained for the mean value $E(S_t)$ and the standard deviation $\sigma(S_t)$ are very close to those obtained in the case where $w = 1$. This behavior is illustrated in Figure 3 which shows the convergence of $E(S_t)$ to $E(L_t)$ when w tends to infinity.

REFERENCES

1. Bodin, U.; Schelén, O.; Pink, S. Load-tolerant differentiation with active queue management. ACM Computer Communications Review **2000**, *30* (4), http://www.acm.org/sigcomm/ccr/archive/2000/july00/bodin.pdf.
2. Floyd, S.; Jacobson, V. Random early detection gateways for congestion avoidance. IEEE/ACM Transactions on Networking **1993**, *1* (4), 397–413.
3. Kella, O.; Stadje, W. Exact Results for a fluid model with state-dependent flow rates. Probability in the Engineering and Informational Sciences **2002**, *16* (4), 389–402.
4. Kuumola, E.; Resing, J.; Virtamo, J. Analytical Results on the Stochastic Behaviour of an Averaged Queue Length. In *Proceedings of the Sixteenth Nordic Teletraffic Seminar, NTS 16*, 2002; 88–99.

5. Kuumola, E.; Resing, J.; Virtamo, J. Joint Distribution of Instantaneous and Averaged Queue Length in an M/M/1/K System. In *Proceedings of the 15th Teletraffic Congress Specialist Seminar "Internet Traffic Engineering and Traffic Management"*; Wurzburg, Germany, 2002; 58–67.
6. Lin, D.; Morris, R. Dynamics of random early detection. In *Proceedings of ACM SIGCOMM 97;* Cannes, France, 1997; 127–137.
7. Lucantoni, D.M.; Meier-Hellstern, K.S.; Neuts, M.F. A single server queue with server vacations and a class of non-renewal arrival processes. Advances in Applied Probability **1990**, *22*, 676–705.
8. Neuts, M.F. A versatile Markovian point process. Journal of Applied Probability **1979**, *16*, 764–779.
9. Rabehasaina, L.; Sericola, B. A second order Markov modulated fluid queue with linear service rate. Journal of Applied Probability **2004**, *41* (3), 758–777.
10. Rabehasaina, L.; Sericola, B. Transient Analysis of a Markov Modulated Fluid Queue with Linear Service Rate. In *Proceedings of the 10th Conference in Analytical and Stochastic Modelling Techniques and Applications ASMTA'03*, 2003; 234–239.