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Sojourn Times in the $M/PH/1$ Processor Sharing Queue

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Abstract. We give in this paper an algorithm to compute the sojourn time distribution in the processor sharing, single server queue with Poisson arrivals and phase type distributed service times. In a first step, we establish the differential system governing the conditional sojourn times probability distributions in this queue, given the number of customers in the different phases of the PH distribution at the arrival instant of a customer. This differential system is then solved by using a uniformization procedure and an exponential of matrix. The proposed algorithm precisely consists of computing this exponential with a controlled accuracy. This algorithm is then used in practical cases to investigate the impact of the variability of service times on sojourn times and the validity of the so-called reduced service rate (RSR) approximation, when service times in the different phases are highly dissymmetrical. For two-stage PH distributions, we give conjectures on the limiting behavior in terms of an $M/M/1$ PS queue and provide numerical illustrative examples.

Keywords: phase type distribution, processor sharing discipline, sojourn time, asymptotic estimates

AMS subject classification: 60K25, 68M20

1. Introduction

Over the past few years, the study of processor sharing disciplines has gained renewed interest in relation with the problematic of bandwidth sharing of elastic flows in packet telecommunication networks [20]. As a matter of fact, the processor sharing discipline, which has been studied for decades in the queueing literature [14], ideally represents, at the expense of several simplifying assumptions (no latency in rate adaptation, same round trip times, etc.), how bandwidth is shared among flows controlled by TCP (Transmission Control Protocol). Almost 90% of the total volume of data transmitted through the Internet are nowadays controlled by this transport protocol.

A flow may actually correspond to a single TCP connection (micro-flow) or to a group of TCP connections (macro-flow) having some characteristics in common. In the latter case, the different TCP connections of a flow may belong to the same session or have some common addressing information (for instance the same prefix in the destination address information element in packet headers). Moreover, several studies have shown that when observing a transmission link, flows may reasonably be assumed to arrive as

Poisson processes. An empirical study of the flow arrival process on an Internet backbone link can be found in [4]. In [5], further arguments are given for justifying the Poisson assumption for the arrival process of flows. Hence, the $M/G/1$ Processor Sharing (PS) queue may be used to study how bandwidth is shared among the different flows crossing a link; in this model, customers correspond to flows introduced above and service times correspond to the amounts of data in flows.

When customers have exponentially distributed service times, the model for representing bandwidth sharing reduces to the well-known $M/M/1$ PS queue, which has been extensively studied in the technical literature. In particular, the differential system satisfied by the vector composed of the complementary distributions of the sojourn time of a customer conditioned on the number of customers in the queue upon his arrival, was established by Coffman et al. in the early 1970 [8] and an iterative algorithm can be used to compute the sojourn time distribution (see Asmussen [3]).

An explicit closed form expression for the sojourn time distribution in an $M/M/1$ PS has been obtained by Morrison [15] via Laplace transform inversion. More recently, Borst et al. [6] obtained the asymptotic behavior of the sojourn time distribution by using the correspondence between the $M/M/1$ PS queue and the $M/M/1$ queue with the random order service (ROS) discipline and a result by Flatto [9]; this asymptotic result was first established by Pollaczek [18] (see the book by Riordan [19]). In [10], it is shown that the $M/M/1$ PS queue has an underlying orthogonal structure related to Pollaczek orthogonal polynomials and their associated weight function.

When service times are not exponentially but simply independent and identically distributed (i.e., in the $M/GI/1$ PS case), the Laplace transform of the sojourn time of a customer conditionally upon its requested service time has been established by Yashkov [24] and Ott [17] (see also Kitaev [13] for the transient behavior). On the basis of the Laplace transform, numerical procedures can then be used to compute the probability distribution function of the sojourn time, such as those designed by Abate and Whitt (see [1] for instance). The major difficulty in this approach, however, consists of computing with high accuracy the Laplace transform at some points in order to run a numerical Laplace transform inversion algorithm.

In this paper, we consider that service times are phase type distributed. The basic motivation for studying the $M/PH/1$ PS is that PH distributions can approximate any probability distribution function, including heavy tailed distribution (see Starobinski and Sidi [22]). Moreover, the PH distribution offers the possibility of modeling several classes of customers, i.e., customers with different types of service time distributions. This can be done by adequately choosing the parameters of the PH distribution, namely the vector, which ith component is the probability that a customer initiates its service in phase i, and the transition matrix of the underlying Markov chain. The major advantage of the $M/PH/1$ PS queue is that by exploiting the Markovian structure of the system, it is possible to design a numerical algorithm for directly computing the probability distribution function of the sojourn time without handling Laplace transforms and then running a Laplace transform inversion procedure.

By parameterizing the PH distribution so as to obtain different customer classes, the numerical algorithm is then used to investigate the so-called reduced service rate (RSR) approximation. When service times are heavy-tailed (in particular Pareto distributed), a RSR approximation (also called Reduced Load Equivalence in the technical literature, see Agrawal et al. [2]) has been shown to hold under mild assumptions; see Zwart and Boxma [25], Núñez-Queija [16], and Jelenković and Momčilović [12]. In all these references, sufficient conditions are given for the following tail-equivalence between the distributions of the sojourn time V and of the service time B of a customer:

$$\mathbf{P}\{V > x\} \sim \mathbf{P}\{B > x(1-\rho)\}. \tag{1}$$

The equivalence indicates that for a customer with a large sojourn time everything happens as if he were served alone with a reduced service rate $1-\rho$. In other words, the service rate 1 is reduced by the load ρ offered by other customers. The RSR approximation also holds in more complex situations [11].

The contribution of this paper is twofold: First, we establish the differential system governing the conditional sojourn time distributions, which is equivalent to the system of Coffman et al. [8] for the $M/M/1$ PS queue, and secondly, we propose an algorithm for solving the system by means of matrix techniques. The algorithm is then used to investigate different RSR approximations when there are several classes of customers with highly dissymmetrical mean service times. In each case, a rationale is given for intuitively supporting the RSR approximation, which is then formulated in terms of a conjecture with a numerical application.

The organization of this paper is as follows: In Section 2, the notation is introduced and the differential system associated with the $M/PH/1$ PS queue is established. The method of solving this system by means of matrix techniques is described and the resolution algorithm is given in Section 3. Some numerical examples and conjectures are presented in Section 4, where the RSR property is investigated for different values of the parameters. Some concluding remarks are presented in Section 5.

2. The $M/PH/1$ queue and the associated differential system

2.1. Notation

Throughout this paper, we consider an $M/PH/1$ queueing system, where customers arrive according to a Poisson process with rate λ and require PH (Phase-Type) distributed service times. The service time distribution is characterized by the infinitesimal generator

$$\begin{pmatrix} T & v \\ 0 & 0 \end{pmatrix},$$

where T is a $m \times m$ matrix whose entries are denoted by $\mu_{i,j}$ and v is a column vector with dimension m whose entries are denoted by v_i.

We introduce the row vector β whose i-th entry is the probability that a service begins in phase i, for $i = 1, \ldots, m$. The transition rates matrix T satisfies $\mu_{i,i} < 0$ for

$i = 1, \ldots, m$ and $\mu_{i,j} \geq 0$, for $i \neq j$. Also $T\mathbf{1} + v = 0$ and $\beta\mathbf{1} = 1$, where $\mathbf{1}$ is the column vector with all its entries equal to 1, its dimension being given by the context. We thus represent a phase-type distribution by the couple (β, T).

The output rate from state j is denoted by μ_j, i.e. $\mu_j = -\mu_{j,j}$. We assume that the states $1, \ldots, m$ are all transient so that absorption from any initial state is certain. It is equivalent to say that the matrix T is non singular. The mean service rate μ is then given by

$$\mu = \frac{1}{-\beta T^{-1}\mathbf{1}},$$

We suppose throughout the paper that the stability condition $\rho = \lambda/\mu < 1$ holds so that a stationary regime exists for the system.

For every $j = 1, \ldots, m$, we denote by $\mathbf{1}_j$ the unit row vector with all components equal to 0 except the j-th one equal to one. For every $n \geq 0$, we introduce the set $\mathcal{S}_n$ of row vectors $\boldsymbol{n}$ defined by

$$\mathcal{S}_n = \left\{ \boldsymbol{n} = (n_1, \ldots, n_m) \in \mathbf{N}^m \;\middle|\; \sum_{j=1}^{m} n_j = n \right\}.$$

For every $i = 1, \ldots, m$, $n \geq 0$, and $\boldsymbol{n} \in \mathcal{S}_n$, we denote by $W(i, \boldsymbol{n})$ the sojourn time of a customer which initiates his service in phase i and finds at his arrival, n_j customers already in the service phase j for $j = 1, \ldots, m$.

2.2. Differential system governing the sojourn time distribution

By setting $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise, and by denoting by $\mathcal{E}(x)$ a random variable exponentially distributed with mean $1/x$, we have the following lemma for the conditional sojourn time $W(i, \boldsymbol{n})$ of a customer arriving at the queue, while there are n_j customers in phase j, $j = 1, \ldots, m$, so that $\boldsymbol{n} = (n_1, \ldots, n_j, \ldots, n_m)$; the proof of the lemma relies on the lack of memory of the exponential distribution and is omitted.

Lemma 1. For all $i = 1, \ldots, m$, $n \in \mathbf{N}$ and $\boldsymbol{n} \in \mathcal{S}_n$, we have the following equality in distribution

$$W(i, \boldsymbol{n}) \overset{d}{=} \begin{cases} \mathcal{E}(\Lambda(i, \boldsymbol{n})) & w.p.\ \dfrac{v_i}{(n+1)\Lambda(i, \boldsymbol{n})}, \\ \mathcal{E}(\Lambda(i, \boldsymbol{n})) + W(i, \boldsymbol{n} - \mathbf{1}_j + \mathbf{1}_k) & w.p.\ \dfrac{n_j \mu_{j,k}}{(n+1)\Lambda(i, \boldsymbol{n})}\ (j \neq k), \\ \mathcal{E}(\Lambda(i, \boldsymbol{n})) + W(j, \boldsymbol{n}) & w.p.\ \dfrac{\mu_{i,j}}{(n+1)\Lambda(i, \boldsymbol{n})}\ (j \neq i), \\ \mathcal{E}(\Lambda(i, \boldsymbol{n})) + W(i, \boldsymbol{n} - \mathbf{1}_j) & w.p.\ \dfrac{n_j v_j}{(n+1)\Lambda(i, \boldsymbol{n})}, \\ \mathcal{E}(\Lambda(i, \boldsymbol{n})) + W(i, \boldsymbol{n} + \mathbf{1}_j) & w.p.\ \dfrac{\lambda \beta_j}{\Lambda(i, \boldsymbol{n})}, \end{cases} \tag{2}$$

where

$$\Lambda(i, \boldsymbol{n}) = \lambda + \sum_j \frac{(n_j + \delta_{i,j})v_j}{n+1} + \sum_j \sum_{k \neq j} \frac{(n_j + \delta_{i,j})\mu_{j,k}}{n+1}$$
$$= \lambda + \frac{1}{n+1} \sum_{j=1}^{m} (n_j + \delta_{i,j})\mu_j.$$

Equation (2) is obtained by using the following arguments. For the first line on the right hand side of equation (2), given that the new (tagged) customer enters the system in phase i while there are n_j customers in phase j, $j = 1, \ldots, m$, because of the memoryless property of the exponential distribution, the next event (arrival, departure or phase change) takes place after a time exponentially distributed with parameter $\Lambda(i, \boldsymbol{n})$. The tagged customer leaves the system at this time (i.e., its sojourn time is then $\mathcal{E}(\Lambda(i, \boldsymbol{n}))$) if a departure occurs before an arrival and a phase change, and the tagged customer completes its service; this event has probability $v_i/((n+1)\Lambda(i, \boldsymbol{n}))$. For the second line, if a phase change (from phase j to phase k) occurs before an arrival or a departure, owing to the memoryless property of the exponential distribution, everything happens as if the tagged customer had spent an exponential service time with parameter $\Lambda(i, \boldsymbol{n})$ plus the sojourn time spent when he enters the systems with $n_j - 1$ customers in phase j and $n_k + 1$ customers in phase k; this event occurs with probability $n_j \mu_{j,k}/((n+1)\Lambda(i, \boldsymbol{n}))$. The other lines of equation (2) are obtained by invoking similar arguments.

We denote by S the phase in which an arriving customer initiates his service, by X the stationary state of the queue at arrival instants of customers, which is also the stationary distribution at an arbitrary instant by PASTA property, and by W the stationary sojourn time of a customer in the queue. We then define the conditional complementary probability distribution of the sojourn time as

$$K(y \mid i, \boldsymbol{n}) \stackrel{def.}{=} \mathbf{P}\{W(i, \boldsymbol{n}) > y\} = \mathbf{P}\{W > y \mid S = i, X = \boldsymbol{n}\}.$$

From equation (2), it is easily checked that the conditional probability functions $K(y \mid i, \boldsymbol{n})$ satisfy the differential system as stated in the following proposition.

Proposition 1. The conditional complementary probability distribution functions $K(y \mid i, \boldsymbol{n})$ satisfy the differential system

$$\frac{\partial}{\partial y} K(y \mid i, \boldsymbol{n}) = -\Lambda(i, \boldsymbol{n}) K(y \mid i, \boldsymbol{n}) + \lambda \sum_j \beta_j K(y \mid i, \boldsymbol{n} + \mathbf{1}_j)$$
$$+ \sum_j \frac{n_j v_j}{n+1} K(y \mid i, \boldsymbol{n} - \mathbf{1}_j) + \sum_{j \neq i} \frac{\mu_{i,j}}{n+1} K(y \mid j, \boldsymbol{n})$$
$$+ \sum_j \sum_{k \neq j} \frac{n_j \mu_{j,k}}{n+1} K(y \mid i, \boldsymbol{n} - \mathbf{1}_j + \mathbf{1}_k),$$

The above differential system generalizes the one obtained by Coffman et al. [8] for the $M/M/1$ PS queue.

Let us define the state space $\mathcal{T} = \{1, \dots, m\} \times \mathbf{N}^m$ and the subspaces $\mathcal{T}_n$ by $\mathcal{T}_n = \{1, \dots, m\} \times \mathcal{S}_n$. Note that the cardinality $|\mathcal{T}_n|$ of $\mathcal{T}_n$ is given by

$$|\mathcal{T}_n| = m\binom{m+n-1}{n}.$$

By using the partition of the state space $\mathcal{T}$ induced by the subspaces $\mathcal{T}_n$, the above differential system can be written in matrix form as

$$\frac{\partial K(y)}{\partial y} = AK(y), \quad K(0) = \mathbf{1} \tag{3}$$

where K is the infinite column vector, whose n-th component, denoted by K_n, is itself a column vector with $|\mathcal{T}_n|$ entries. The matrix A is a tridiagonal matrix that can be written as

$$A = \begin{pmatrix} C_0 & A_0 & & & \\ B_1 & C_1 & A_1 & & \\ & B_2 & C_2 & A_2 & \\ & & \cdot & \cdot & \cdot \end{pmatrix}.$$

where the matrices A_i, B_i and C_i are defined as follows: For $i \in \{1, \dots, m\}$, $n \geq 0$ and $\boldsymbol{n} = (n_1, \dots, n_m) \in \mathcal{S}_n$, the matrices A_n are defined by

$$a((i, \boldsymbol{n}), (i, \boldsymbol{n} + \mathbf{1}_j)) = \lambda\beta_j, \quad j = 1, \dots, m,$$

and the other elements equal to 0. Similarly, the non null elements of matrices B_n are the elements

$$b((i, \boldsymbol{n}),\ (i, \boldsymbol{n} - \mathbf{1}_j)) = \frac{n_j v_j}{n+1}, \quad n_j > 0,\ j = 1, \dots, m.$$

Finally, the non null elements of the matrices C_n are

$$\begin{cases} c((i, \boldsymbol{n}), (j, \boldsymbol{n})) = \dfrac{\mu_{i,j}}{n+1}, & j = 1, \dots, m, \quad j \neq i, \\ c((i, \boldsymbol{n}), (i, \boldsymbol{n} - \mathbf{1}_j + \mathbf{1}_k)) = \dfrac{n_j \mu_{j,k}}{n+1}, & j, k = 1, \dots, m, \quad j \neq k,\ n_j > 0, \\ c((i, \boldsymbol{n}), (i, \boldsymbol{n})) = -\left(\lambda + \dfrac{1}{n+1}\displaystyle\sum_{j=1}^{m}(n_j + \delta_{i,j})\mu_j\right). & \end{cases}$$

The objective of the next section is to describe how the differential system (3) can be solved by means of matrix analysis methods and in particular, by using a uniformization procedure.

3. Resolution of the differential system

3.1. Solution

For every $n \geq 0$ and $\boldsymbol{n} \in \mathcal{S}_n$, we denote by $p(\boldsymbol{n})$ the stationary probability to have n_j customers in the service phase j for $j = 1, \ldots, m$. It is well-known that under the stability condition $\rho = \lambda/\mu < 1$, which ensures that the stationary regime exists for the queue, we have

$$p(\boldsymbol{n}) = (1-\rho)n!\lambda^n \prod_{j=1}^{m} \frac{(-\beta T^{-1}\mathbf{1}_j)^{n_j}}{n_j!} = (1-\rho)n! \prod_{j=1}^{m} \frac{\rho_j^{n_j}}{n_j!},$$

where $\rho_j = \lambda(-\beta T^{-1}\mathbf{1}_j)$. Note that we have $\rho = \rho_1 + \cdots + \rho_m$.

It is also well-known from PASTA property, that, for every $n \geq 0$ and $\boldsymbol{n} \in \mathcal{S}_n$, we have

$$\mathbf{P}\{X = \boldsymbol{n}\} = p(\boldsymbol{n}).$$

Thus the stationary probability $\pi(i, \boldsymbol{n})$ that an arriving customer initiates his service in phase i and finds n_1 customers in the service phase 1, ..., and n_m customers in the service phase m is given by

$$\pi(i, \boldsymbol{n}) = \mathbf{P}\{S = i, X = \boldsymbol{n}\} = \beta_i p(\boldsymbol{n}),$$

since S and X are independent. The distribution of the stationary sojourn time W of a customer in the queue is given from equation (3) by means of an exponential of matrix as follows.

Lemma 2. The complementary probability distribution function of the sojourn time W is given by

$$\mathbf{P}\{W > y\} = \pi K(y) = \pi e^{Ay}\mathbf{1}, \tag{4}$$

where π is the infinite row vector whose n-th component is denoted by π_n, which is itself a row vector with $|\mathcal{T}_n|$ entries containing the $\pi(i, \boldsymbol{n})$.

For every $j = 1, \ldots, m$, we denote by W_j the sojourn time of a customer who initiates his service in phase j. The probability distribution of W_j is given by the following result.

Proposition 2. The complementary distribution function of the random variable W_j, which is the sojourn time of a customer starting his service in phase j, is given by

$$\mathbf{P}\{W_j > y\} = \pi^{(j)} e^{Ay}\mathbf{1}, \tag{5}$$

where $\pi^{(j)}$ is the infinite row vector whose n-th component is denoted by $\pi_n^{(j)}$, which is itself a row vector with $|\mathcal{T}_n|$ entries denoted by $\pi^{(j)}(i, \boldsymbol{n})$ and defined by

$$\pi^{(j)}(i, \boldsymbol{n}) = \begin{cases} p(\boldsymbol{n}) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have

$$\begin{aligned}
\mathbf{P}\{W_j > y\} &= \mathbf{P}\{W > y \mid S = j\} \\
&= \sum_{n=0}^{\infty} \sum_{\boldsymbol{n} \in \mathcal{S}_n} \mathbf{P}\{X = \boldsymbol{n}\}\mathbf{P}\{W > y \mid S = j, X = \boldsymbol{n}\} \\
&= \sum_{n=0}^{\infty} \sum_{\boldsymbol{n} \in \mathcal{S}_n} p(\boldsymbol{n}) K(y \mid j, \boldsymbol{n}) = \pi^{(j)} K(y) = \pi^{(j)} e^{Ay} \mathbf{1},
\end{aligned}$$

and equation (5) follows. □

To conclude this section, let us simply mention that the distribution of the sojourn time W of an arbitrary customer can be written as a weighted sum of the distributions of the conditional sojourn times W_j:

$$\mathbf{P}\{W > y\} = \sum_{j=1}^{m} \beta_j \mathbf{P}\{W_j > y\}. \tag{6}$$

3.2. *The M/PH/1 PS multiclass queue*

We show in this subsection that the sojourn times in the M/PH/1 PS multiclass queue can be seen as a particular case of the single class queue.

Consider the M/PH/1 PS multiclass queue with J classes such that customers of class j arrive according to a Poisson process with rate λ_j and are served according to the PH distribution represented by the couple $(\beta^{(j)}, T^{(j)})$. The set of states on which vector $\beta^{(j)}$ and matrix $T^{(j)}$ operate is denoted by L_j, for $j = 1, \ldots, J$. We denote by $W^{(j)}$ the sojourn time of a class j customer in the queue. The distribution of $W^{(j)}$ is given by the following result.

Proposition 3. The complementary probability distribution function of the sojourn time $W^{(j)}$ of a class j customer is given by

$$\mathbf{P}\left\{W^{(j)} > y\right\} = \pi^{(L_j)} e^{Ay} \mathbf{1}, \tag{7}$$

where $\pi^{(L_j)}$ is the infinite row vector defined by

$$\pi^{(L_j)}(i, \boldsymbol{n}) = \begin{cases} \beta_i^{(j)} p(\boldsymbol{n}) & \text{if } i \in L_j \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Proof. From the multi-class queue, we construct the M/PH/1 PS monoclass queue in which customers arrive according to a Poisson process with rate $\lambda = \lambda_1 + \cdots + \lambda_J$ and are served according to the PH distribution represented by the couple (β, T) given by

$$\beta = \left(\frac{\lambda_1}{\lambda}\beta^{(1)}, \ldots, \frac{\lambda_J}{\lambda}\beta^{(J)}\right)$$

and T is the block-diagonal matrix containing the matrices $T^{(j)}$, that is

$$T = \begin{pmatrix} T^{(1)} & 0 & \cdots & 0 \\ 0 & T^{(2)} & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & T^{(J)} \end{pmatrix}.$$

The dimension of vector β and matrix T is $m = |L_1| + \cdots + |L_J|$.

We recall that for $\ell = 1, \ldots, m$, W_ℓ is the sojourn time of a customer, who initiates his service in phase ℓ, and that S is the state in which a customer initiates his service. We then have

$$\begin{aligned} \mathbf{P}\{W^{(j)} > y\} &= \mathbf{P}\{W > y \mid S \in L_j\} \\ &= \frac{1}{\mathbf{P}\{S \in L_j\}} \sum_{\ell \in L_j} \mathbf{P}\{S = \ell\}\mathbf{P}\{W > y \mid S = \ell\} \\ &= \frac{1}{\mathbf{P}\{S \in L_j\}} \sum_{\ell \in L_j} \mathbf{P}\{S = \ell\}\mathbf{P}\{W_\ell > y\}. \end{aligned}$$

For $\ell \in L_j$, we have $\mathbf{P}\{S = \ell\} = \lambda_j \beta_\ell^{(j)}/\lambda$ and $\beta^{(j)}\mathbf{1} = 1$, so that

$$\mathbf{P}\big\{W^{(j)} > y\big\} = \sum_{\ell \in L_j} \beta_\ell^{(j)} \mathbf{P}\{W_\ell > y\} = \sum_{\ell \in L_j} \beta_\ell^{(j)} \pi^{(\ell)} e^{Ay}\mathbf{1} = \pi^{(L_j)} e^{Ay}\mathbf{1},$$

where $\pi^{(L_j)}$ is the infinite row vector defined by

$$\pi^{(L_j)} = \sum_{\ell \in L_j} \beta_\ell^{(j)} \pi^{(\ell)}.$$

By using the definition of vector $\pi^{(\ell)}$, the entries of vector $\pi^{(L_j)}$ are given by equation (8). This completes the proof of equation (7). $\square$

From the above result, we can easily deduce the complementary probability distribution function of the sojourn time W of an arbitrary customer.

Corollary 1. The complementary probability distribution function of the sojourn time W of a customer in the multi-class $M/PH/1$ PS queue can be expressed by means of the distributions of the sojourn times $W^{(j)}$ of class j customers, $j = 1, \dots, J$ as

$$\mathbf{P}\{W > y\} = \sum_{j=1}^{J} \frac{\lambda_j}{\lambda} \mathbf{P}\big\{W^{(j)} > y\big\}. \tag{9}$$

Proof. By definition, we have

$$\begin{aligned}\mathbf{P}\{W > y\} &= \sum_{j=1}^{J} \mathbf{P}\{W > y \mid S \in L_j\} \mathbf{P}\{S \in L_j\} \\ &= \sum_{j=1}^{J} \mathbf{P}\big\{W^{(j)} > y\big\} \mathbf{P}\{S \in L_j\} = \sum_{j=1}^{J} \frac{\lambda_j}{\lambda} \mathbf{P}\big\{W^{(j)} > y\big\}\end{aligned}$$

and equation (9) follows. □

3.3. Resolution method

Let ν denote the uniformization rate associated with matrix A, defined by

$$\nu \geq \theta = \sup_{(i,\boldsymbol{n})} |A((i,\boldsymbol{n}),(i,\boldsymbol{n}))|.$$

It is easy to check that we have

$$\theta = \sup_{(i,\boldsymbol{n})} \{-c((i,\boldsymbol{n}),(i,\boldsymbol{n}))\} \leq \lambda + \max_i \{\mu_i\}.$$

We thus choose as uniformization rate $\nu = \lambda + \max_i\{\mu_i\}$ and we define the matrix P as $P = I + A/\nu$, where I is the identity matrix whose dimension is given by the context. The matrix A being a sub-infinitesimal generator, P is a substochastic matrix which can be written as

$$P = \begin{pmatrix} R_0 & P_0 & & & \\ Q_1 & R_1 & P_1 & & \\ & Q_2 & R_2 & P_2 & \\ & & \cdot & \cdot & \cdot \end{pmatrix},$$

where the matrices P_n, Q_n and R_n are given by

$$R_n = I + C_n/\nu, \quad P_n = A_n/\nu \quad \text{and} \quad Q_n = B_n/\nu.$$

By conditioning on the number of events in the interval $(0, y)$ (or equivalently by using equation (4)), we have

$$\mathbf{P}\{W > y\} = \sum_{h=0}^{\infty} e^{-\nu y} \frac{(\nu y)^h}{h!} \pi P^h \mathbf{1}. \tag{10}$$

This relation is an infinite series involving vectors and matrices of infinite dimension, so we have to perform truncations in both the dimension and the number of terms in the series.

For what concerns the dimension, for every $n \geq 0$, we have

$$\sum_{i=1}^{m} \sum_{\boldsymbol{n} \in \mathcal{S}_n} \pi(i, \boldsymbol{n}) = (1 - \rho)\rho^n,$$

which can also be written as $\pi_n \mathbf{1} = (1 - \rho)\rho^n$. Let ε be the desired precision for the computation of $\mathbf{P}\{W > y\}$. For $\varepsilon < 1$, we define the integer M by

$$M = \left\lfloor \frac{\ln(\varepsilon/2)}{\ln(\rho)} \right\rfloor, \tag{11}$$

so that, $\rho^{M+1} \leq \varepsilon/2$. We denote respectively by π^- and π^+ the infinite row vectors given by

$$\pi^- = (\pi_0, \ldots, \pi_M, 0, 0, \ldots) \quad \text{and} \quad \pi^+ = (0, \ldots, 0, \pi_{M+1}, \pi_{M+2}, \ldots),$$

so that we have $\pi = \pi^- + \pi^+$. We thus have from relation (10)

$$\mathbf{P}\{W > y\} = \sum_{h=0}^{\infty} e^{-\nu y} \frac{(\nu y)^h}{h!} \pi^- P^h \mathbf{1} + e_1(M),$$

where $e_1(M)$ satisfies

$$\begin{aligned} e_1(M) &= \sum_{h=0}^{\infty} e^{-\nu y} \frac{(\nu y)^h}{h!} \pi^+ P^h \mathbf{1} \leq \sum_{h=0}^{\infty} e^{-\nu y} \frac{(\nu y)^h}{h!} \pi^+ \mathbf{1} \\ &= \pi^+ \mathbf{1} = \sum_{n=M+1}^{\infty} \pi_n \mathbf{1} = \rho^{M+1} \leq \varepsilon/2. \end{aligned}$$

For what concerns the truncation of the series in relation (10), we define the integer H by

$$H = \min\left\{ n \geq 0 \;\middle|\; \sum_{h=0}^{n} e^{-\nu y} \frac{(\nu y)^h}{h!} \geq 1 - \varepsilon/2 \right\}. \tag{12}$$

We thus have

$$\mathbf{P}\{W > y\} = \sum_{h=0}^{H} e^{-\nu y} \frac{(\nu y)^h}{h!} \pi^- P^h \mathbf{1} + e_1(M) + e_2(H),$$

where $e_2(H)$ satisfies

$$e_2(H) = \sum_{h=H+1}^{\infty} e^{-\nu y} \frac{(\nu y)^h}{h!} \pi^- P^h \mathbf{1} \leq \sum_{h=H+1}^{\infty} e^{-\nu y} \frac{(\nu y)^h}{h!} = 1 - \sum_{h=0}^{H} e^{-\nu y} \frac{(\nu y)^h}{h!} \leq \varepsilon/2.$$

Let us define $F(y)$ by

$$F(y) = \sum_{h=0}^{H} e^{-\nu y} \frac{(\nu y)^h}{h!} \pi^- P^h \mathbf{1}.$$

We have

$$0 \leq \mathbf{P}\{W > y\} - F(y) \leq \varepsilon.$$

Let us define, for every $h \geq 0$, the sequence $v(h) = \pi^- P^h \mathbf{1}$. The computation of $v(h)$ can then be done as follows. We define the infinite row vector $V(h) = \pi^- P^h$, so that $v(h) = V(h)\mathbf{1}$. We then have $V(0) = \pi^-$ and for $h \geq 1$, $V(h) = V(h-1)P$. As we did for vector π, we decompose the vector $V(h)$, for every h, in subvectors $V_n(h)$; $V_n(h)$ being itself a row vector with $|\mathcal{T}_n|$ entries. It is easily checked that, since $V(0)$ has a finite number of nonzero entries and since P is a block-tridiagonal matrix, we have

$$V_{M+r}(h) = 0, \text{ for every } h \geq 0 \quad \text{and} \quad r \geq h+1.$$

The relation $V(h) = V(h-1)P$ can then be expressed, for every $h \geq 1$, as

$$\begin{cases} V_0(h) = V_0(h-1)R_0 + V_1(h-1)Q_1, \\ V_n(h) = V_{n-1}(h-1)P_{n-1} + V_n(h-1)R_n + V_{n+1}(h-1)Q_{n+1}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad 1 \leq n \leq M+h-2, \\ V_{M+h-1}(h) = V_{M+h-2}(h-1)P_{M+h-2} + V_{M+h-1}(h-1)R_{M+h-1}, \\ V_{M+h}(h) = V_{M+h-1}(h-1)P_{M+h-1}. \end{cases}$$

3.4. *Algorithmic aspects*

The truncation level H is in fact a function of y, say H_y. For a fixed value of ε, H_y is an increasing function of y. It follows that if we want to compute $F(y)$ for L distinct values of y, denoted by $y_1 < \cdots < y_L$, we only need to compute $v(h)$ for $n = 1, \ldots, H_{y_L}$ since the values of $v(h)$ are independent of the parameter y.

The algorithm for the computation of the function $F(y)$ is given in Table 1. This algorithm is used in the next section to investigate the behavior of the $M/PH/1$ PS queue.

The same algorithm can be used for the computation of $\mathbf{P}\{W_j > y\}$. It suffices to replace the instruction $[V_n = \pi_n]$ by the instruction $[V_n = \pi_n^{(j)}]$.

To conclude this section, it is worth noting that the complexity of the algorithm given in Table 1 lies essentially in the computation of the vectors W_n, each having $|\mathcal{T}_n|$ entries. Except for the first and last columns, the number of non-zero entries in each column of the matrices P_{n-1}, R_n and Q_{n+1} is equal to m, m^2 and m, respectively. Thus the products $V_{n-1}P_{n-1}$, V_nR_n and $V_{n+1}Q_{n+1}$ require $m|\mathcal{T}_n|$, $m^2|\mathcal{T}_n|$ and $m|\mathcal{T}_n|$ multiplications, respectively. The computation of vector W_n thus relies on $m(m+2)|\mathcal{T}_n|$ multiplications. Since this must be done for $h = 1$ to H and for $n = 0$ to $M + h$, we obtain a computational complexity $C(m, M, H)$ given by

$$\begin{aligned} C(m, M, H) &= m^2(m+2)\sum_{h=1}^{H}\sum_{n=0}^{M+h}\binom{m+n-1}{n} \\ &= m^2(m+2)\sum_{h=1}^{H}\binom{m+M+h}{M+h} \\ &= m^2(m+2)\left[\binom{m+M+H+1}{M+H} - \binom{m+M+1}{M}\right]. \end{aligned}$$

Table 1

Algorithm for the computation of $\mathbf{P}\{W > y\}$.

```
input : ε, y_1 < ··· < y_L
output : F(y_1), ..., F(y_L)
M = ⌊ln(ε/2)/ ln(ρ)⌋
H = min{n ≥ 0 | Σ_{h=0}^{n} e^{-ν y_L} (ν y_L)^h / h! ≥ 1 − ε/2}
for n = 0 to M do
    V_n = π_n
endfor
v(0) = 1 − ρ^{M+1}
for h = 1 to H do
    W_0 = V_0 R_0 + V_1 Q_1
    for n = 1 to M + h − 2 do
        W_n = V_{n−1} P_{n−1} + V_n R_n + V_{n+1} Q_{n+1}
    endfor
    W_{M+h−1} = V_{M+h−2} P_{M+h−2} + V_{M+h−1} R_{M+h−1}
    W_{M+h} = V_{M+h−1} P_{M+h−1}
    v(h) = 0
    for n = 0 to M + h do
        V_n = W_n
        v(h) = v(h) + V_n 1
    endfor
endfor
for j = 1 to L do
    F(y_j) = Σ_{h=0}^{H} e^{−ν y_j} (ν y_j)^h / h! v(h)
endfor
```

This complexity is polynomial in m and can be considerable for large values of m, so it must be used for small values of m only. In the examples given in the next section, we have taken $m = 2$ and the execution time for obtaining all the points for a given curve is about a few second long. The main advantage of our algorithm lies in its numerical robustness. In fact, its main loop is based only upon additions and multiplications of numbers belonging to [0, 1]. This prevents stability problems and the result is guaranteed with a precision ε specified in advance. This is very important for testing conjectures.

The methods using the Laplace transform [17,24] are difficult to implement numerically because the Laplace transform to be inverted is expressed as an integral over $\mathbf{R}_+$ which contains itself other integrals of Laplace transforms that must be inverted first. These procedures may raise stability issues due to underflow or overflow and they are subject to several uncontrolled error sources, which means that we are not sure to obtain the correct result. Nevertheless, they are the only way to obtain the result for large values of m. In fact the Laplace transform inversion method and the uniformization technique must be seen as complementary for large and small values of m, respectively.

4. Numerical examples

4.1. Influence of the variability of service times on the sojourn time

In this section, we illustrate the impact of the variability of services times on the sojourn times of customers. For this purpose, we consider an $M/PH/1$ PS queue, where customers arrive according to a Poisson process with rate λ and services times S have a two-stage hyper-exponential distribution given by

$$\mathbf{P}\{S > x\} = \beta e^{-\mu_1 x} + (1 - \beta)e^{-\mu_2 x} \tag{13}$$

for $\beta \in (0, 1)$. A service time is thus exponential with mean $1/\mu_1$ with probability β or with mean $1/\mu_2$ with probability $1 - \beta$.

We parameterize the hyper-exponential distribution by means of the mean service time $1/\mu_1$ in phase 1, the mean global service time $\mathbf{E}[S]$ and the squared coefficient of variation c^2 defined by $c^2 = \mathrm{var}[S]/\mathbf{E}[S]^2$ so that

$$\mu_2 = \frac{2(1 - \mu_1\mathbf{E}[S])}{\mathbf{E}[S](2 - (1 + c^2)\mu_1\mathbf{E}[S])} \quad \text{and} \quad \beta = \frac{(c^2 - 1)\mu_1^2\mathbf{E}[S]^2}{2 - 4\mu_1\mathbf{E}[S] + (1 + c^2)\mu_1^2\mathbf{E}[S]^2}.$$

It is easily checked that the coefficient c^2 must be greater than or equal to one in order to ensure the existence of $\beta \in [0, 1]$. Note that if $\mu_1\mathbf{E}[S] < 1$, then we shall have $1 \le c^2 < 2/(\mu_1\mathbf{E}[S]) - 1$. Moreover, under the same condition, if $c^2 \to 2/(\mu_1\mathbf{E}[S]) - 1$, then μ_2 tends to infinity; in this case we have very dissymmetrical service rates, which case is studied in the next section.

To illustrate the impact of the coefficient of variation on the sojourn time in the $M/PH/1$ PS queue, we consider the case when $\mu_1\mathbf{E}[S] = 1/2$ and the load of the queue

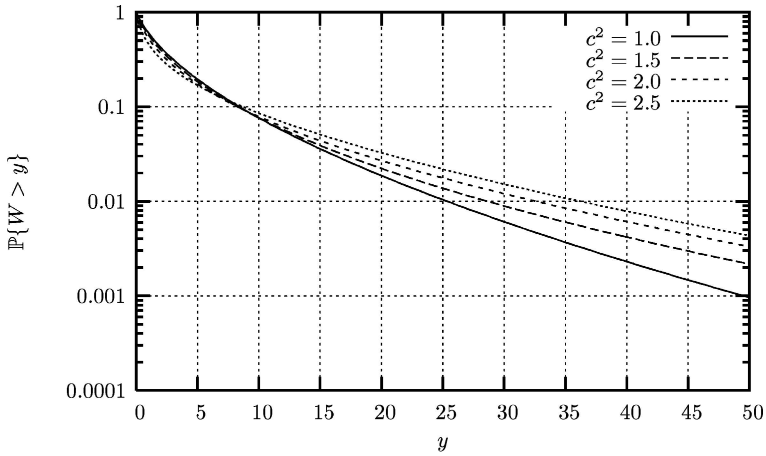

Figure 1. Behavior of the sojourn time in the $M/PH/1$ PS queue for different values of the squared coefficient of variation.

$\rho = \lambda\mathbf{E}[S] = 0.85$. Taking $\mu_1 = 1$, we have $\mathbf{E}[S] = 1/2$ and $\lambda = \rho/\mathbf{E}[S] = 1.7$. The squared coefficient of variation shall then belong to $[1, 3)$. Figure 1 shows the sojourn time distribution for different values of c^2. The service time distribution is more variable than the exponential distribution and we observe that the sojourn time is an increasing function of c^2 for large values of y, in the sense that if $c_2 > c_1$, $\mathbf{P}\{W(c_1) > y\} \leq \mathbf{P}\{W(c_2) > y\}$ for large values of y, the quantity $W(c_i)$ denoting the sojourn time for a squared coefficient of variation c_i^2.

The different above observations are in line with Ross type conjectures [21], which state that more variable arrival processes lead to worse performances, expressed in the case considered in this paper in terms of sojourn times. We specifically observe that more variable traffic yields larger sojourn times. We however note that we do not have a stochastic domination property in the sense that if $c_2 > c_1$, $W(c_1)$ is not stochastically dominated by $W(c_2)$.

4.2. Investigation on the RSR approximation

We investigate in this section under which conditions an RSR approximation may hold for the $M/PH/1$ PS systems, when service times are very dissymmetrical. This is the case for instance when service times can be divided into two classes. The transition rates in one class are much smaller than the transition rates of the other class. In the context of Internet traffic modeling, customers with small mean service times (high transition rates) correspond to short data transfers (referred to as mice in the technical literature), while customers with long mean service times (small transition rates) represent bulk data transfers (known as elephants). By assuming that bandwidth among small and large data transfers is shared according to the processor sharing discipline, the $M/PH/1$ PS queue may serve as a model for studying the integration of both traffic types on a transmission link, in particular the sojourn duration in the system, which corresponds to the time needed to complete a data transfer.

We specifically study the case when service times S are distributed according to a two-stage hyper-exponential distribution as given by equation (13). We moreover assume that the second exponential has a mean much larger than the first one, namely $\mu_2 = \mu_1/N$, where N is a scaling factor, so that the mean service time of a customer in the second stage of the hyper-exponential distribution is N times larger than the mean service time in the first stage.

The load offered by class 1 customers is $\rho_1 = \lambda\beta/\mu_1$ and that by class 2 customers is $\rho_2 = \lambda(1-\beta)/\mu_2 = N\lambda(1-\beta)/\mu_1$. In the following, we assume that the load factors ρ_1 and ρ_2 (with $\rho_1 + \rho_2 < 1$) as well as the service rate for class 1 customers are fixed to $\rho_1 = 0.3$, $\rho_2 = 0.6$ and $\mu_1 = 1$, and we let N vary so that the parameters of the system are defined in terms of these primary variables as

$$\beta = \frac{1}{1 + \frac{\rho_2}{N\rho_1}} \quad \text{and} \quad \lambda = \rho_1\left(1 + \frac{\rho_2}{N\rho_1}\right)\mu_1.$$

Assume that the system is in the stationary regime and let W denote the sojourn time in the system (of an arbitrary customer without taking into account the class). Let W_i denote the sojourn time for class i customers.

In a first step, we consider large values of W. We first note that by equation (6)

$$\mathbf{P}\{W > y\} = \beta\mathbf{P}\{W_1 > y\} + (1-\beta)\mathbf{P}\{W_2 > y\}. \tag{14}$$

Large sojourn times are essentially due to class 2 customers and $\beta\mathbf{P}\{W_1 > y\}$ is negligible when compared with $(1-\beta)\mathbf{P}\{W_2 > y\}$. It then follows that one may reasonably conjecture for $y > 0$ and large values of N

$$\mathbf{P}\{W > Ny\} \sim (1-\beta)\mathbf{P}\{W_2 > Ny\}. \tag{15}$$

Now, for class 2 customers, small service times appear more or less as noise and one may expect that they are seen only through their load. Thus, the sojourn time for class 2 customers should be close to that in an $M/M/1$ PS queue with arrival rate $(1-\beta)\lambda$ and service rate $\mu_2(1-\rho_1)$. Denoting by $V(\lambda, \mu)$ the sojourn time of a customer in an $M/M/1$ PS queue with arrival rate λ and mean service time μ, we have

$$\mu V(\lambda, \mu) \stackrel{d}{=} V(\lambda/\mu, 1). \tag{16}$$

It follows that we should have for $y > 0$ and large N

$$\mathbf{P}\{W_2 > Ny\} \sim \mathbf{P}\left\{V\left(\frac{\rho_2}{1-\rho_1}, 1\right) > \mu_1(1-\rho_1))y\right\}. \tag{17}$$

By using equations (15) and (17), we finally come up with the following conjecture.

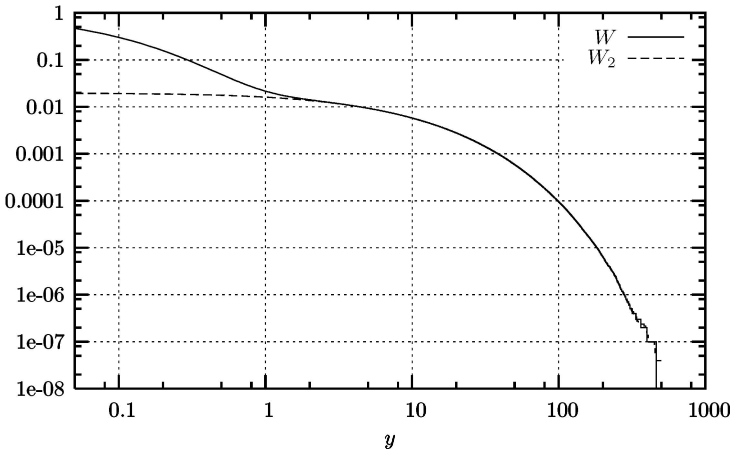

Figure 2. Accuracy of Approximations (15): Comparison between the curves $y \to \mathbf{P}\{W > Ny\}$ and $y \to (1-\beta)\mathbf{P}\{W_2 > Ny\}$ for $N = 100$.

Conjecture 1. For large values of the scaling coefficient N, we have for fixed y

$$\mathbf{P}\{W > Ny\} \sim (1-\beta)\mathbf{P}\left\{V\left(\frac{\rho_2}{1-\rho_1}, 1\right) > \mu_1(1-\rho_1)y\right\}.$$

Approximations (15) and (17) are illustrated in figures 2 and 3 for $N = 2, 5, 100$.

It clearly appears that the function $y \to \mathbf{P}\{(1-\rho_1)W_2/N > y\}$ and $y \to \mathbf{P}\{V(\rho_2/(1-\rho_1), 1) > \mu_1 y\}$ are almost indistinguishable; the function $y \to \mathbf{P}\{W_2 > y\}$ has been obtained by using the resolution algorithm displayed in Table 1 by choosing the adequate initial probabilities as indicated in Section 3.4. Moreover, approximation (15) proves very accurate as soon as the variable y is sufficiently large.

Let us now consider the case when y is small. In equation (14), the term $\mathbf{P}\{W_2 > y\}$ is close to one but the factor $(1-\beta) = O(1/N)$. Hence, when N is large, the dominating term in the right hand side of equation (14) is $\beta\mathbf{P}\{W_1 > y\}$ and we then have the approximation

$$\mathbf{P}\{W > y\} \sim \mathbf{P}\{W_1 > y\}. \tag{18}$$

When N is large, class 2 customers appear as permanent customers for those of class 1. Hence, one may expect that class 2 customers are seen by those of class 1 only through their mean load and that we have the approximation

$$\mathbf{P}\{W_1 > y\} \sim \mathbf{P}\{V(\lambda\beta, \mu_1(1-\rho_2)) > y\}$$

and by using identity (16), we have the following conjecture.

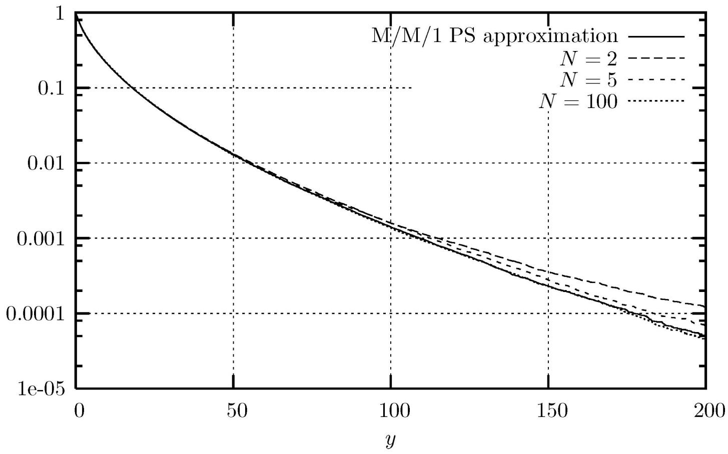

Figure 3. Accuracy of approximation (17): Comparison between the curves $y \to \mathbf{P}\{(1-\rho_1)W_2/N > y\}$ for different values of N and $y \to \mathbf{P}\{V(\rho_2/(1-\rho_1), 1) > \mu_1 y\}$ ($M/M/1$ PS approximation).

Conjecture 2. For large values of the scaling coefficient N, we have

$$\mathbf{P}\{W > y\} \sim \mathbf{P}\{V(\rho_1/(1-\rho_2), 1) > y(1-\rho_2)\mu_1\}. \tag{19}$$

for small and moderate values of y.

Approximations (18) and (19) are illustrated in figure 4, which shows that they prove quite accurate for small values of y. The function $y \to \mathbf{P}\{W_1 > y\}$ has been obtained by using the algorithm displayed in Table 1 with adequate initial probabilities. The graphs of the functions $y \to \mathbf{P}\{W_1 > y\}$ for $N = 100, 1000, 10000$ are indistinguishable one from each other.

Note that the assumption of permanent customers may be valid only for small or moderate values of y. In particular, this assumption cannot be used for computing mean values. This observation readily follows from the fact that the mean sojourn time in an $M/M/1$ PS queue with K permanent customers is equal to $(K+1)$ times the mean sojourn time in a regular $M/M/1$ PS queue (i.e., with no permanent customers) [23]. Hence, by deconditioning upon K, which follows a geometric distribution with parameter ρ_2, we cannot recover via the permanent customer assumption the mean value of the sojourn time in the $M/PH/1$ PS queue (equal to $\mathbf{E}[S]/(1-\rho)$).

4.3. RSR approximation in the case of serial service times

In the previous section, we have considered exponential service times in parallel, that is, an arriving customer has a certain probability of having an exponential service time with a given mean value. The same kind of investigations can be carried out when

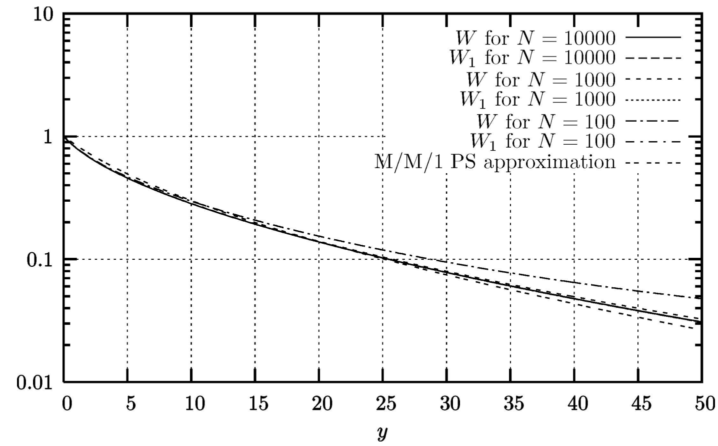

Figure 4. Accuracy of Approximations (18) and (19): Comparison between the curves $y \to \mathbf{P}\{W > y\}$ and $\mathbf{P}\{W_1 > y\}$ for $N = 100$, 1000, 10000, and the curve $y \to \mathbf{P}\{V(\rho_1/(1-\rho_2), 1) > y(1-\rho_2)\mu_1\}$ ($M/M/1$ PS approximation).

service times are in series, i.e., when an arriving customer has an exponential service time with a given mean value, then moves to an other stage of the PH distribution with an exponential service time with another mean value and so on. When the transition rates in the different stages of the PH distribution are very dissymmetrical, one may expect that an RSR approximation pertains so that stages with small mean service times weakly contributes to large sojourn times, which are then essentially due to large service times.

To illustrate the above intuition, we consider a two-stage PH distribution composed of two exponential service times in series. The service rate in the first stage is denoted by μ_1 and that in the second stage by μ_2 such that $\mu_2 = \mu_1/N$. The transition matrix is given by

$$\begin{pmatrix} -\mu_1 & \mu_1 & 0 \\ 0 & -\mu_2 & \mu_2 \\ 0 & 0 & 0 \end{pmatrix}$$

and the initial probabilities by $\beta_1 = 1$ and $\beta_2 = 0$. An arriving customer thus enters the PH distribution through stage 1, remains in stage 1 for an exponential service time with mean $1/\mu_1$ and then moves to state 2 before leaving the system once the service time in the second stage is completed.

When N is large, one may expect that large sojourn times in the system are essentially due to service times in stage 2. Thus, one may reasonably conjecture that large service times can be approximated by sojourn times in an $M/M/1$ PS queue with a mean arrival rate λ and a mean service time $1/\mu_2$. This result should not depend on the initial

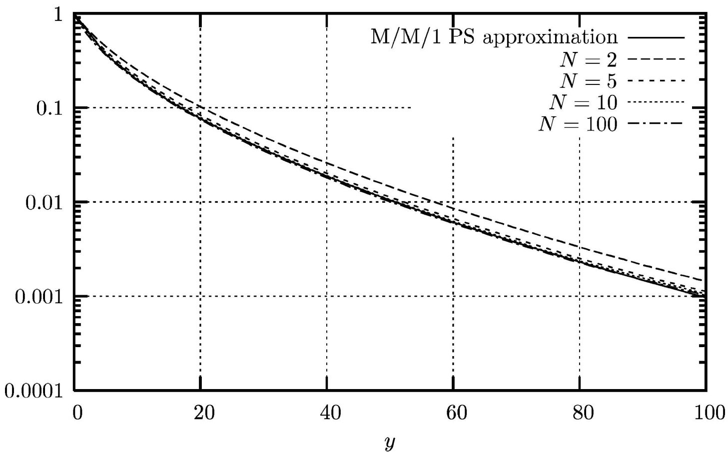

Figure 5. Comparison between the sojourn times in the case of a two dissymmetrical exponential service times in series for different values of N and the sojourn time in the limiting $M/M/1$ PS queue.

distribution and one may take for instance $\beta_1 = \beta_2 = 1/2$. For this choice of the initial probability distribution, figure 5 displays the sojourn time distribution in the $M/PH/1$ PS queue with $\mu_1 = 1, \mu_2 = \mu_1/N, \rho = \lambda\mathbf{E}[S] = 0.85$, so that $\lambda = \rho/(\beta_1/\mu_1 + N/\mu_1)$. As expected, we observe from figure 5 that $\mathbf{P}\{W > yN\} \sim \mathbf{P}\{V(\rho, 1) > \mu_1 y\}$ when N is sufficiently large.

The same phenomenon can be observed if a certain proportion of customers leave the system after their service time in stage 1 has been completed; this amounts to assuming that there exists a leak rate of stage 1 customers. If we denote by v_1 this leak rate, the transition matrix reads

$$\begin{pmatrix} -(v_1 + \mu_1) & \mu_1 & v_1 \\ 0 & -\mu_2 & \mu_2 \\ 0 & 0 & 0 \end{pmatrix}$$

Assume that all customers enter a service through phase 1 ($\beta_1 = 1$). By using the same arguments as above, one may expect that large sojourn times are close to sojourn times in an $M/M/1$ PS queue with a mean arrival rate $\lambda\mu_1/(v_1 + \mu_1)$ and a mean service time $1/\mu_2$. Using the fact that the global sojourn time probability distribution is equal to the weighted sum of the sojourn time probability distributions of those customers with small sojourn times and those with large sojourn times (see equation (6)), we should have

$$\mathbf{P}\{W > yN\} \sim \frac{\mu_1}{\mu_1 + v_1}\mathbf{P}\{V(\rho, 1) > \mu_1 y\}$$

This approximation is illustrated in figure 6 for a load $\rho = 0.85$, $\mu_1 = 1$, $v_1 = \mu_1/2$, $\mu_2 = \mu_1/N$ and different values of the scale parameter N.

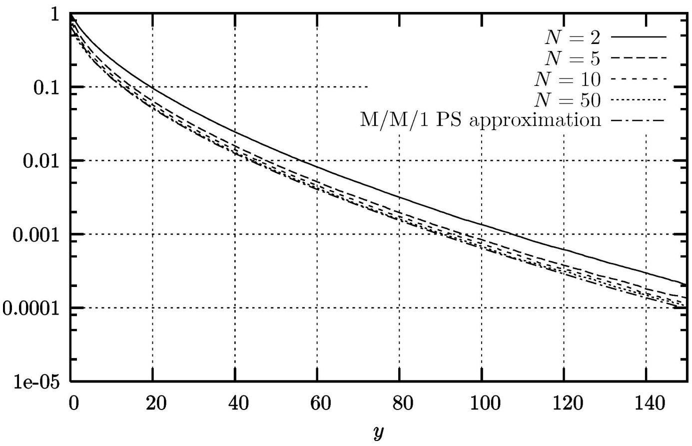

Figure 6. Comparison for different values of the scaling parameter N between the sojourn times in the case of a two dissymmetrical exponential service times in series and a leak rate for stage 1 customers, and the sojourn time in the limiting $M/M/1$ PS queue.

5. Concluding remarks

By exploiting the Markovian nature of the $M/PH/1$ PS system, we have presented in this paper an algorithm for numerically computing the probability distribution function of the sojourn time of an arbitrary customer. The PH distribution can be parameterized so as to model different classes of customers with different service time distributions. In the case of a two-stage PH distribution with highly dissymmetrical service times, we have investigated the validity of the so-called RSR approximation. Such an approximation is very important from an engineering point of view since it allows different classes to be analyzed in isolation. In fact, the RSR approximation is more delicate to show in the case of light tail distributions than in the case of Pareto distributions, where the technonology developped by Zwart, Jelenkovic, Boxma and others, based on ε-breaking, proves very efficient to show asymptotic results for $M/G/1$ PS queues with heavy tailed service times (see [11,12,25]).

It may be possible to rigorously prove the different conjectures presented in this paper by using the expression of the Laplace transform of the sojourn time distribution established by Ott [17] and Yashkov [24]. This point will be addressed in further studies.

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