

A SECOND-ORDER MARKOV-MODULATED FLUID QUEUE WITH LINEAR SERVICE RATE

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Abstract

We consider an infinite-capacity second-order fluid queue governed by a continuous-time Markov chain and with linear service rate. The variability of the traffic is modeled by a Brownian motion and a local variance function modulated by the Markov chain and proportional to the fluid level in the queue. The behavior of this second-order fluid-flow model is described by a linear stochastic differential equation, satisfied by the transient queue level. We study the transient level's convergence in distribution under weak assumptions and we obtain an expression for the stationary queue level. For the first-order case, we give a simple expression of all its moments as well as of its Laplace transform. For the second-order model we compute its first two moments.

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1. Introduction and motivations

We consider in this paper an infinite-capacity fluid queue, the level of which at time t is denoted by $Q(t)$. Fluid arrives into this queue according to a nondecreasing process $A(t)$, and leaves the queue at rate μ_t at time t . We consider a second-order model. This means that the fluid level is not only represented by the incoming fluid and the service rate, but also by a variability factor, which is modeled by a local variance function σ_t and a Brownian motion B_t . In that case, $Q(t)$ satisfies the following stochastic differential equation:

$$dQ(t) = dA(t) - \mu_t dt + \sigma_t dB_t + dL_t,$$

where L_t is a nondecreasing process, interfering only when $Q(t)$ hits 0 and preventing it from being negative.

In practise, we consider the case where $dA(t) = \lambda(X(t), Q(t)) dt$, $\mu_t = \mu(X(t), Q(t))$, and $\sigma_t = \sigma(X(t), Q(t))$, where $X(t)$ is an external stochastic process, called the environment process, which modulates the queue. In that case, $Q(t)$ is a solution to the stochastic differential equation reflected at 0:

$$dQ(t) = (\lambda(X(t), Q(t)) - \mu(X(t), Q(t))) dt + \sigma(X(t), Q(t)) dB_t + dL_t.$$

This model was introduced in [8], and has been studied extensively with different hypotheses on $X(t)$ and on the functions λ and μ , particularly when $X(t)$ is a continuous-time Markov chain:

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see e.g. [4] for a study where λ and μ do not depend on $Q(t)$, or [2] for a set of differential equations satisfied by the density of $Q(t)$. Kella and Stadje [5] obtained the explicit distribution of the steady-state distribution of $Q(t)$ when $X(t)$ is a two-state Markov chain and $\sigma = 0$.

Our main study is focused on the *linear model*. More precisely, the service rate at time t is $\mu(X(t))Q(t)$ (modulated linear release rate), where of course μ is assumed to be nonnegative. We will also suppose that $\sigma(X(t), Q(t)) = \sigma(X(t))Q(t)$. This model has been studied by Asmussen and Kella [1], Kella and Whitt [7], and Kella and Stadje [6] (with a network background in the latter two cases), when A is a Poisson process or a Markov-modulated Lévy process, and when $\sigma = 0$ (first-order model). The authors identify in [6] a functional equation that the steady-state Laplace–Stieltjes transform must satisfy. As pointed out by those authors, even though it is not clear how to solve this functional equation, it can be used to compute the first two moments. We consider in the present paper $dA(t)$ of the form $\lambda(X(t)) dt$, where $\{X(t), t \in \mathbb{R}\}$ is a stationary continuous-time Markov chain, and we obtain explicit analytical results for all order moments as well as for the Laplace transform of the stationary distribution when $\sigma = 0$. When σ is a function not identically equal to 0, it is much harder to get the distribution of the stationary regime: however, we obtain its first two moments.

It is then not difficult to see that $Q(t)$ satisfies the following stochastic differential equation:

$$dQ(t) = \lambda(X(t)) dt - \mu(X(t))Q(t) dt + \sigma(X(t))Q(t) dB_t, \tag{1.1}$$

the term in dL_t having disappeared, 0 then being an impenetrable barrier for $Q(t)$ (see e.g. [1]).

Due to its linearity, this model has some application in finance. Let us see the connection between finance and the model described by (1.1) by considering an insurance risk model, similar to the one in [1]. More precisely, let us consider a Markov chain $\{I(t), t \in \mathbb{R}\}$ and an insurance risk process $\{R(t), t \in \mathbb{R}\}$ with interest rate $\rho(I(t))$ and volatility $s(I(t))$ at time t . Let

$$J(t') - J(t) = \int_t^{t'} v(I(s)) ds$$

be the total claim in the period $[t, t']$, where v is a nonnegative function. The risk $R(t)$ then satisfies the linear stochastic differential equation

$$\begin{aligned} dR(t) &= -dJ(t) + R(t)[\rho(I(t)) dt + s(I(t)) dw(t)] \\ &= -v(I(t)) dt + \rho(I(t))R(t) dt + s(I(t))R(t) dw(t), \end{aligned} \tag{1.2}$$

with the notation traditionally used in finance and where $\{w(t), t \in \mathbb{R}\}$ is a Brownian motion. The Markov chain $\{I(t), t \in \mathbb{R}\}$ typically captures the market behavior: for example, its state space is $\{-1, +1\}$, where $+1$ represents an up-trend of the market, whereas -1 means a down-trend. The state space can be even larger, leading to a more precise description of the market trend (see [13]). Typically, the volatility is small during an up-trend period, as investors are cautious and move slowly, and it increases during a down-trend period, as investors panic and the market becomes more erratic (see again [13]).

Let $R^*(t) := R(-t)$ be the reversed process of R (also called the dual process). From (1.2), R^* satisfies

$$dR^*(t) = v(I^*(t)) dt - \rho(I^*(t))R^*(t) dt + s(I^*(t))R^*(t) \circ dw^*(t), \tag{1.3}$$

where I^* and w^* are, respectively, the reversed Markov chain and Brownian motion defined by $I^*(t) := I(-t)$ and $w^*(t) := w(-t)$. The term $s(I^*(t))R^*(t) \circ dw^*(t)$ is the Stratonovitch

integral, linked to the usual Itô integral via the equality

$$s(I^*(t))R^*(t) \circ dw^*(t) = s(I^*(t))^2 R^*(t) dt + s(I^*(t))R^*(t) dw^*(t)$$

(see e.g. [12]). Thus, (1.3) can be rewritten in the following way:

$$dR^*(t) = v(I^*(t)) dt - [\rho(I^*(t)) - s(I^*(t))^2]R^*(t) dt + s(I^*(t))R^*(t) dw^*(t). \tag{1.4}$$

Note then that, provided that $\rho - s^2 \geq 0$, (1.4) is exactly (1.1) with $\lambda = v$, $\mu = \rho - s^2$, $s = \sigma$, $X = I^*$, $B = w^*$, and $Q = R^*$.

Now, the fundamental duality relation between R^* and R is the following: let us set $\tau(x) := \inf\{t \geq 0 \mid R(t) = 0\}$ to be the ruin time of the process R , where $R(0) = x$, and let W be a random variable towards which $R^*(t)$ converges in distribution as $t \rightarrow +\infty$ (which, with the notation of the present paper, and because of the matching correspondence $Q = R^*$ in (1.4) and (1.1), amounts to $Q(t)$ converging in distribution to W); then the relation between $\tau(x)$ and W is given by (see [1] and [12])

$$P(\tau(x) < \infty) = P(W > x).$$

In other words, there is a close connection between the probability of ruin, $P(\tau(x) < \infty)$, and the distribution of W . This paper is then dedicated to giving some characteristics of W when the volatility $s (= \sigma)$ is equal to 0 or when it is not identically equal to 0.

Throughout this paper, we will suppose that $\{X(t), t \in \mathbb{R}\}$ is a stationary ergodic continuous-time Markov chain on a finite state space $S = \{1, \dots, N\}$ with stationary distribution $\pi = (\pi_1, \dots, \pi_N)$. We will denote by $Q = (q_{ij})_{(i,j) \in S \times S}$ its generator matrix. The environment process $\{X(t), t \in \mathbb{R}\}$ and the Brownian motion $\{B_t, t \in \mathbb{R}\}$ will be taken to be independent. Let us remember that λ is nonnegative. We assume that μ is also nonnegative, and satisfies $\mu(i) > 0$ for some $i \in S$ (the function μ is then not identically equal to 0).

The paper is organized as follows. In Section 2 we show, under suitable assumptions, the convergence in distribution of the queue level. In Section 3 we give an expression for the first moment of the stationary distribution. In Section 4 we study the stationary regime when σ is the function identically equal to 0. We give explicitly the moments of all orders of the stationary distribution as well as of its Laplace transform. To finish, in Section 5 we compute the second moment of the stationary distribution when σ is not necessarily identically equal to 0 under some mild assumption relating μ and σ .

We recall a lemma about exponential martingales (see [11]) that will be used throughout the paper.

Lemma 1.1. *Let $\{Z(t), t \in \mathbb{R}\}$ and $\{w(t), t \in \mathbb{R}\}$ be, respectively, a stationary process and a Brownian motion. Let us suppose that the two processes are independent. Then, for all bounded functions g and all $u \in \mathbb{R}$, the process*

$$\left\{ \exp\left(\int_u^t g(Z(v)) dw(v) - \frac{1}{2} \int_u^t g(Z(v))^2 dv \right), t \geq u \right\}$$

is a martingale given $\{Z(t), t \in \mathbb{R}\}$ adapted to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$ defined by

$$\mathcal{F}_t = \sigma(w(s) - w(u), u \leq s \leq t).$$

Besides, for all $t \geq u$,

$$E\left(\exp\left(\int_u^t g(Z(v)) dw(v) - \frac{1}{2} \int_u^t g(Z(v))^2 dv \right) \middle| Z \right) = 1.$$

2. Stationary regime of the queue

Let us denote by $\{Q_u^y(t), t \geq u\}$ the process solving (1.1) for $t \geq u$ and such that $Q_u^y(u) = y$. It is then standard (see e.g. [11]) that we have an explicit solution for $Q_u^y(t)$: for $t \geq u$,

$$Q_u^y(t) = y \exp\left(-\int_u^t \left[\mu(X(s)) + \frac{\sigma(X(s))^2}{2}\right] ds + \int_u^t \sigma(X(s)) dB_s\right) + \int_u^t \exp\left(-\int_s^t \left[\mu(X(v)) + \frac{\sigma(X(v))^2}{2}\right] dv + \int_s^t \sigma(X(v)) dB_v\right) \lambda(X(s)) ds. \tag{2.1}$$

In addition, the relation between $Q_u^y(t)$ and $Q_u^y(t')$, $t \geq t' \geq u$, is

$$Q_u^y(t) = Q_u^y(t') \exp\left(-\int_{t'}^t \left[\mu(X(s)) + \frac{\sigma(X(s))^2}{2}\right] ds + \int_{t'}^t \sigma(X(s)) dB_s\right) + \int_{t'}^t \exp\left(-\int_s^t \left[\mu(X(v)) + \frac{\sigma(X(v))^2}{2}\right] dv + \int_s^t \sigma(X(v)) dB_v\right) \lambda(X(s)) ds. \tag{2.2}$$

For simplicity's sake, since y will be fixed throughout the paper, we will write

$$Q(t) := Q_0^y(t).$$

We begin by finding a stationary process $W(t)$ solving (1.1).

Theorem 2.1. *Let us set*

$$W(t) = \int_{-\infty}^t \exp\left(-\int_s^t \left[\mu(X(v)) + \frac{\sigma(X(v))^2}{2}\right] dv + \int_s^t \sigma(X(v)) dB_v\right) \lambda(X(s)) ds. \tag{2.3}$$

Then

- (i) $W(t)$ is finite for all t ,
- (ii) $\{W(t), t \in \mathbb{R}\}$ is a stationary process solving (1.1),
- (iii) $Q(t)$ converges in distribution to $W(0)$, independently of the initial condition.

Proof. Let us begin by proving (i) with $t = 0$ without loss of generality. The integral (2.3) lies in $[0, +\infty]$, and we only need to prove that it is finite almost surely. For each $u \leq t$,

$$\begin{aligned} & \mathbb{E}\left(\int_u^0 \exp\left(-\int_s^0 \left[\mu(X(v)) + \frac{\sigma(X(v))^2}{2}\right] dv + \int_s^0 \sigma(X(v)) dB_v\right) \lambda(X(s)) ds\right) \\ &= \int_u^0 \mathbb{E}\left(\exp\left(-\int_s^0 \left[\mu(X(v)) + \frac{\sigma(X(v))^2}{2}\right] dv + \int_s^0 \sigma(X(v)) dB_v\right) \lambda(X(s))\right) ds \\ &\leq \sup_{i \in S} \lambda(i) \int_u^0 \mathbb{E}\left(\exp\left(-\int_s^0 \left[\mu(X(v)) + \frac{\sigma(X(v))^2}{2}\right] dv + \int_s^0 \sigma(X(v)) dB_v\right)\right) ds. \end{aligned} \tag{2.4}$$

Now,

$$\begin{aligned} & \mathbb{E}\left(\exp\left(-\int_s^0\left[\mu(X(v)) + \frac{\sigma(X(v))^2}{2}\right]dv + \int_s^0\sigma(X(v))dB_v\right)\right) \\ &= \mathbb{E}\left(\mathbb{E}\left(\exp\left(-\int_s^0\left[\mu(X(v)) + \frac{\sigma(X(v))^2}{2}\right]dv + \int_s^0\sigma(X(v))dB_v\right)\middle|X\right)\right) \\ &= \mathbb{E}\left(\exp\left(-\int_s^0\mu(X(v))dv\right)\mathbb{E}\left(\exp\left(\int_s^0\sigma(X(v))dB_v - \int_s^0\frac{\sigma(X(v))^2}{2}dv\right)\middle|X\right)\right). \end{aligned}$$

By Lemma 1.1 with $w = B$, $Z = X$, and $g = \sigma$, the process

$$\left\{\exp\left(\int_s^t\sigma(X(v))dB_v - \int_s^t\frac{\sigma(X(v))^2}{2}dv\right), t \geq s\right\}$$

is a martingale given X ; hence,

$$\mathbb{E}\left(\exp\left(\int_s^0\sigma(X(v))dB_v - \int_s^0\frac{\sigma(X(v))^2}{2}dv\right)\middle|X\right) = 1.$$

In order to show that (2.4) admits a finite limit as $u \rightarrow -\infty$, it is then sufficient to show that

$$\alpha = \liminf_{s \rightarrow -\infty} \frac{1}{s} \ln \mathbb{E}\left(\exp\left(-\int_s^0\mu(X(v))dv\right)\right) > 0. \tag{2.5}$$

Since $\mu(X(v)) \geq 0$, we already know that $\alpha \geq 0$. Now, if α were equal to 0 we would have that $\int_s^0\mu(X(v))dv = 0$ almost surely for all $s \leq 0$, which is impossible because $\mu(i) > 0$ for some i in S and the positive recurrence of X . This implies that $\mathbb{E}(\exp(-\int_s^0\mu(X(v))dv)) \leq K \exp(\alpha s)$ for some constant K . Thus, (2.4) admits a finite limit as $u \rightarrow -\infty$. This in particular implies that $W(0) < +\infty$ almost surely.

We prove (ii) in a similar manner to Proposition 3.3 of [9]. Let us first notice that, because X is stationary and B has stationary increments, $Q_u^0(t)$ has the same distribution as $Q_{u-t}^0(0)$ for $t \geq u$. Those two random variables converge respectively to $W(t)$ and $W(0)$ as $u \rightarrow -\infty$. Thus, $W(t)$ and $W(0)$ are identical in distribution. To see that $\{W(t), t \in \mathbb{R}\}$ solves (1.1), we write that, in view of (2.2), for $t \geq 0$,

$$\begin{aligned} Q_u^0(t) &= Q_u^0(0) \exp\left(-\int_0^t\left[\mu(X(s)) + \frac{\sigma(X(s))^2}{2}\right]ds + \int_0^t\sigma(X(s))dB_s\right) \\ &\quad + \int_0^t \exp\left(-\int_0^v\left[\mu(X(v)) + \frac{\sigma(X(v))^2}{2}\right]dv + \int_0^v\sigma(X(v))dB_v\right)\lambda(X(s))ds \end{aligned}$$

and we let $u \rightarrow -\infty$.

Finally, we prove that $\{Q_0^y(t), t \geq 0\}$ converges in distribution to $W(0)$ for all $y \geq 0$ as t tends to infinity. We saw that $Q_{-t}^0(0)$ has the same distribution as $Q_0^0(t)$ for $t \geq 0$; hence, $Q_0^0(t)$ converges in distribution to $W(0)$. To see that $Q_0^y(t)$ converges in distribution to $W(0)$ we write that, thanks to (2.1), for $t \geq 0$,

$$Q_0^y(t) - Q_0^0(t) = y \exp\left(-\int_0^t\left[\mu(X(s)) + \frac{\sigma(X(s))^2}{2}\right]ds + \int_0^t\sigma(X(s))dB_s\right).$$

Then $E(|Q_0^y(t) - Q_0^0(t)|) = E(Q_0^y(t) - Q_0^0(t)) = y E(\exp(-\int_0^t \mu(X(v)) dv))$. As in the proof of (i), this expression tends to 0 as $t \rightarrow +\infty$ exponentially fast because of the positive recurrence of X . Thus, $Q_0^y(t) - Q_0^0(t)$ tends to 0 in $L^1(\Omega)$ as $t \rightarrow +\infty$. The convergence in distribution of $Q_0^y(t)$ towards $W(0)$ follows.

Let us remark that $W(0)$ can also be expressed in the following way:

$$W(0) = \int_0^\infty \exp\left(-\int_0^s \left[\mu(X^*(v)) + \frac{\sigma(X^*(v))^2}{2}\right] dv + \int_0^s \sigma(X^*(v)) dB_v^*\right) \lambda(X^*(s)) ds,$$

where X^* and B^* are the reversed versions of the processes X and B , that is, for all t , $X^*(t) = X((-t)^-)$ and $B_t^* = B_{-t}$. Obviously, B^* is still a Brownian motion. Moreover, it is standard that X^* is a continuous-time Markov chain with transition matrix $Q^* = (q_{ij}^*)_{(i,j) \in S \times S}$, where $q_{ij}^* = q_{ji} \pi_j / \pi_i$.

For clarity, we will often use the notation W instead of $W(0)$.

3. First moment of the stationary distribution

We are interested in this section in the first moment of the queue level in the stationary regime. Note that in Section 4 we get the same expression by another method for the first-order case (i.e. $\sigma = 0$). For some similar results concerning the first moment of the stationary distribution for first-order models, see [1] and [6].

Actually, the result obtained in this section is slightly more general than the computation of the first moment. Moreover, the originality of this section is the martingale approach used to prove the result (as opposed to the method of [1] and [6], where moments are computed by using a Markov renewal equation). In addition, a similar approach to that presented here, although much more subtle, will be used in the computation of the second moment in Section 5.

Let us set, for $t \geq 0$,

$$Y_t = Y_0 - \int_0^t \left[\mu(X^*(s)) + \frac{\sigma(X^*(s))^2}{2} \right] ds + \int_0^t \sigma(X^*(s)) dB_s^*,$$

where Y_0 is a bounded random variable. We then have the following result, which also yields the expression for $E(W)$.

Theorem 3.1. For all functions $h : S \rightarrow \mathbb{R}$,

$$E(Wh(X(0))) = \pi H(D_\mu - Q^*)^{-1} \Lambda \mathbf{1}, \tag{3.1}$$

where $H = \text{diag}(h(1), \dots, h(N))$, $D_\mu = \text{diag}(\mu(1), \dots, \mu(N))$, $\Lambda = \text{diag}(\lambda(1), \dots, \lambda(N))$, and $\mathbf{1} = (1, \dots, 1)^\top$. In particular, if $h(i) = 1$ for all $i \in S$, we get the expression for the first moment of W :

$$E(W) = \pi(D_\mu - Q^*)^{-1} \Lambda \mathbf{1}. \tag{3.2}$$

Proof. Let us set $g(y, i) = \exp(y)f(i)$, $y \in \mathbb{R}$, $i \in S$, where f will be conveniently chosen later on. By Itô's formula,

$$g(Y_t, X^*(t)) - g(Y_0, X^*(0)) - \int_0^t \mathcal{A}g(Y_s, X^*(s)) ds = \int_0^t \sigma(X^*(s)) \partial_y g(Y_s, X^*(s)) dB_s^*,$$

where

$$\begin{aligned} \mathcal{A}g(y, i) &= -(\mu(i) + \frac{1}{2}\sigma(i)^2)\partial_y g(y, i) + \frac{1}{2}\sigma(i)^2\partial_y^2 g(y, i) + \sum_j q_{ij}^* g(y, j) \\ &= -\mu(i) \exp(y) f(i) + \exp(y) \sum_j q_{ij}^* f(j). \end{aligned}$$

Let us then set

$$\begin{aligned} M_t &= g(Y_t, X^*(t)) - g(Y_0, X^*(0)) - \int_0^t \mathcal{A}g(Y_s, X^*(s)) ds \\ &= \int_0^t \sigma(X^*(s))\partial_y g(Y_s, X^*(s)) dB_s^*. \end{aligned}$$

In general, $\{M_t, t \geq 0\}$ is a local martingale given X^* adapted to $\mathcal{F}_t = \sigma(B_s^* - B_0^*, 0 \leq s \leq t)$. In the present case, let us show that M_t is actually a real martingale. Since $\partial_y g(y, i) = \exp(y)f(i)$, we have

$$\begin{aligned} M_t &= \int_0^t \sigma(X^*(s))\partial_y g(Y_s, X^*(s)) dB_s^* \\ &= \int_0^t \sigma(X^*(s)) \exp(Y_s) f(X^*(s)) dB_s^* \end{aligned}$$

with

$$\exp(Y_s) = \exp\left(Y_0 - \int_0^s \left[\mu(X^*(v)) + \frac{\sigma(X^*(v))^2}{2}\right] dv + \int_0^s \sigma(X^*(v)) dB_v^*\right).$$

Since μ, f , and Y_0 are bounded, it is easy to verify that M_t is a martingale if

$$N_t := \exp\left(-\int_0^t \frac{\sigma(X^*(s))^2}{2} ds + \int_0^t \sigma(X^*(s)) dB_s^*\right)$$

admits a finite moment of order 2. It is easy to check that

$$N_t^2 = \exp\left(\int_0^t \sigma(X^*(s))^2 ds\right) P_t,$$

where

$$P_t := \exp\left(-\int_0^t 2\sigma(X^*(s))^2 ds + \int_0^t 2\sigma(X^*(s)) dB_s^*\right).$$

From Lemma 1.1 with $Z = X^*$, $w = B^*$, and $g = 2\sigma$, P_t is a martingale given X^* adapted to \mathcal{F}_t . Thus,

$$\begin{aligned} E(N_t^2) &= E(E(N_t^2 | X^*)) = E\left(E\left(\exp\left(\int_0^t \sigma(X^*(s))^2 ds\right) P_t \mid X^*\right)\right) \\ &= E\left(\exp\left(\int_0^t \sigma(X^*(s))^2 ds\right) E(P_t | X^*)\right) \\ &= E\left(\exp\left(\int_0^t \sigma(X^*(s))^2 ds\right)\right), \end{aligned}$$

which is finite (remember that σ takes its values in the finite set $\{\sigma(1), \dots, \sigma(N)\}$). Then M_t is a real martingale. Now, with the expression for g , we can write $\mathcal{A}g$ in the following way:

$$\mathcal{A}g(y, i) = \exp(y)[(Q^* - D_\mu)\underline{f}](i),$$

where we use the notation $\underline{f} = (f(1), \dots, f(N))^T$ and where, for a column vector $v = (v_1, \dots, v_N)^T$, we set $v(i) := v_i$.

Now, since $(M_t)_{t \geq 0}$ is a martingale given X^* , we have that $E(M_t | X^*(0)) = 0$. Hence,

$$\begin{aligned} & E\left(\int_0^t \exp(Y_s)[(Q^* - D_\mu)\underline{f}](X^*(s)) ds h(X^*(0))\right) \\ &= E\left(E\left(\int_0^t \exp(Y_s)[(Q^* - D_\mu)\underline{f}](X^*(s)) ds h(X^*(0)) \mid X^*(0)\right)\right) \\ &= E\left(E\left(\int_0^t \exp(Y_s)[(Q^* - D_\mu)\underline{f}](X^*(s)) ds \mid X^*(0)\right)h(X^*(0))\right) \\ &= E\left(E\left(\int_0^t \mathcal{A}g(Y_s, X^*(s)) ds \mid X^*(0)\right)h(X^*(0))\right) \\ &= E(-E(\exp(Y_0)f(X^*(0)) | X^*(0))h(X^*(0)) \\ &\quad + E(\exp(Y_t)f(X^*(t)) | X^*(0))h(X^*(0))) \\ &= E(-E(\exp(Y_0)f(X^*(0))h(X^*(0)) | X^*(0)) \\ &\quad + E(\exp(Y_t)f(X^*(t))h(X^*(0)) | X^*(0))) \\ &= -E(\exp(Y_0)f(X^*(0))h(X^*(0))) + E(\exp(Y_t)f(X^*(t))h(X^*(0))). \end{aligned} \tag{3.3}$$

The diagonal matrix D_μ has at least one nonzero entry and the matrix Q^* is irreducible because Q is irreducible by hypothesis, so $Q^* - D_\mu$ is invertible. We now take $Y_0 = 0$ and $\underline{f} = (Q^* - D_\mu)^{-1} \wedge \mathbf{1}$. Then (3.3) reads

$$\begin{aligned} & E\left(\int_0^t \exp\left(-\int_0^s \left[\mu(X^*(v)) + \frac{\sigma(X^*(v))^2}{2}\right] dv + \int_0^s \sigma(X^*(v)) dB_v^*\right)\right. \\ &\quad \left. \times \lambda(X^*(s)) ds h(X^*(0))\right) \\ &= -E([(Q^* - D_\mu)^{-1} \wedge \mathbf{1}](X^*(0))h(X^*(0))) \\ &\quad + E\left(\exp\left(-\int_0^t \left[\mu(X^*(s)) + \frac{\sigma(X^*(s))^2}{2}\right] ds + \int_0^t \sigma(X^*(s)) dB_s^*\right)\right. \\ &\quad \left. \times [(Q^* - D_\mu)^{-1} \wedge \mathbf{1}](X^*(t))h(X^*(0))\right). \end{aligned} \tag{3.4}$$

Using an argument similar to that used to show that (2.4) tends to a finite limit as u tends to $-\infty$, it is easy to show that the last term in the right-hand side of the equality (3.4) converges to 0 as t tends to infinity. Since the left-hand side of (3.4) tends to $E(Wh(X^*(0))) = E(Wh(X(0)))$ as t tends to infinity, by letting $t \rightarrow +\infty$ in (3.4) we then get

$$\begin{aligned} E(Wh(X(0))) &= E([(D_\mu - Q^*)^{-1} \wedge \mathbf{1}](X^*(0))h(X^*(0))) \\ &= E([(D_\mu - Q^*)^{-1} \wedge \mathbf{1}](X(0))h(X(0))), \end{aligned}$$

which can be written in matrix form as (3.1).

4. The case of a first-order model, stationary regime

We study in this section the stationary regime of the queue, still in the first-order model framework. From Theorem 2.1, $Q(t)$ converges in distribution to W . Let us set

$$F(x) := \lim_{t \rightarrow \infty} F(t, x),$$

where $F(t, x)$ is defined by $F(t, x) = (F_1(t, x), \dots, F_N(t, x))^T$, with

$$F_i(t, x) = P(Q(t) \leq x \mid X^*(0) = i).$$

Then $F(x) = (P(W \leq x \mid X^*(0) = i))_{i \in S}$. Letting $t \rightarrow \infty$ in Theorem 3.1 of [10] we get the following result.

Theorem 4.1. *For every $x \geq 0$,*

$$(\Lambda - D_{\mu x})F'(x) = Q^*F(x). \tag{4.1}$$

We will suppose in this section that $\inf_{i \in S} \mu(i) > 0$ (the service rate is then always positive when the queue level is positive), so that W is bounded by $\sup_{i \in S} \lambda(i) / \inf_{i \in S} \mu(i)$ (and thus all its moments exist).

The aim of this section is to find an expression of all moments of the stationary distribution. Let us start by recalling the following lemma.

Lemma 4.1. *Let H be the cumulative distribution function of a nonnegative random variable. For every $r \geq 1$, if the r th-order moment exists, then*

$$\int_0^\infty x^r dH(x) = r \int_0^\infty x^{r-1} (1 - H(x)) dx.$$

Proof. See for instance [3].

Using Lemma 4.1, we can compute the moments of W as well as its Laplace transform. Let us set

$$v_i(k) = E(W^k \mid X(0) = i)$$

and let $V(k)$ be the column vector with the entries $v_i(k)$. An expression for $V(k)$ is given by the following corollary.

Corollary 4.1. *For every $k \geq 1$,*

$$V(k) = \left(D_{\mu} - \frac{Q^*}{k} \right)^{-1} \Lambda V(k - 1).$$

Proof. Since $Q^* \mathbf{1} = 0$, (4.1) can be written as

$$(\Lambda - D_{\mu x})F'(x) = -Q^*(\mathbf{1} - F(x)).$$

Multiplying both sides by x^{k-1} for $k \geq 1$, and after integration, we get

$$\Lambda \int_0^\infty x^{k-1} F'(x) dx - D_{\mu} \int_0^\infty x^k F'(x) dx = -Q^* \int_0^\infty x^{k-1} (\mathbf{1} - F(x)) dx.$$

Using Lemma 4.1, we easily get

$$\Lambda \int_0^\infty x^{k-1} F'(x) dx - D_\mu \int_0^\infty x^k F'(x) dx = \frac{-1}{k} Q^* \int_0^\infty x^k F'(x) dx.$$

Thus,

$$\Lambda V(k-1) - D_\mu V(k) = \frac{-1}{k} Q^* V(k),$$

that is,

$$\left(D_\mu - \frac{Q^*}{k} \right) V(k) = \Lambda V(k-1).$$

The matrix D_μ is diagonal with at least one nonzero entry and so, for $k \geq 1$, the matrix $D_\mu - Q^*/k$ is invertible. This implies that

$$V(k) = \left(D_\mu - \frac{Q^*}{k} \right)^{-1} \Lambda V(k-1),$$

which completes the proof.

Since $V(0) = \mathbf{1}$, we have

$$V(k) = \left(D_\mu - \frac{Q^*}{k} \right)^{-1} \Lambda \left(D_\mu - \frac{Q^*}{k-1} \right)^{-1} \Lambda \cdots \left(D_\mu - \frac{Q^*}{2} \right)^{-1} \Lambda (D_\mu - Q^*)^{-1} \Lambda \mathbf{1}.$$

The k th stationary moment of the queue level is then given by

$$E(W^k) = \pi V(k).$$

Corollary 4.1 shows that the moments of W can be easily evaluated recursively by solving linear systems.

Let us denote by ϕ the Laplace transform of W , that is,

$$\phi(\theta) := E(e^{\theta W}),$$

which is defined for every θ in \mathbb{R} since W is bounded by $\sup_{i \in S} \lambda(i) / \inf_{i \in S} \mu(i)$.

Corollary 4.2. For every θ in \mathbb{R} ,

$$\phi(\theta) = \pi \sum_{k=0}^\infty \frac{V(k)\theta^k}{k!}.$$

Proof. This is done by using the inequality $W \leq \sup_{i \in S} \lambda(i) / \inf_{i \in S} \mu(i)$ and seeing that, since $\phi^{(k)}(0) = \pi V(k)$, for all N ,

$$\begin{aligned} \left| \phi(\theta) - \sum_{k=0}^N \frac{\phi^{(k)}(0)\theta^k}{k!} \right| &= \left| E \left(\exp(\theta W) - \sum_{k=0}^N \frac{W^k \theta^k}{k!} \right) \right| \\ &= \left| E \left(\sum_{k=N+1}^\infty \frac{W^k \theta^k}{k!} \right) \right| \\ &\leq \sum_{k=N+1}^\infty \left(\frac{\sup_{i \in E} \lambda(i)}{\inf_{i \in S} \mu(i)} \right)^k \frac{\theta^k}{k!} \rightarrow 0 \quad \text{as } N \rightarrow \infty. \end{aligned}$$

This completes the proof.

5. Second moment of the stationary distribution for the second-order model

We study the case of the second-order linear model. We impose the following additional condition.

Condition 5.1. *For all $i \in S$, $\mu(i) \geq 4\sigma(i)^2$ and there exists an $i \in S$ such that $\mu(i) > 4\sigma(i)^2$.*

Let us set $z_t = (Q_0^0(t))^2$ for $t \geq 0$. The process z_t then converges in distribution to W^2 . Moreover, Itô's formula applied to $(Q_0^0(t))^2$ yields

$$d(Q_0^0(t))^2 = 2\lambda(X(t))Q_0^0(t) dt + [-2\mu(X(t)) + \sigma(X(t))^2](Q_0^0(t))^2 dt + 2\sigma(X(t))(Q_0^0(t))^2 dB_t,$$

and thus z_t satisfies the following stochastic differential equation:

$$dz_t = 2\lambda(X(t))Q_0^0(t) dt + [-2\mu(X(t)) + \sigma(X(t))^2]z_t dt + 2\sigma(X(t))z_t dB_t \tag{5.1}$$

with the initial condition $z_0 = 0$. In a way, z_t is a second-order linear model with input rate equal to $2\lambda(X(t))Q_0^0(t)$, release rate equal to $2\mu(X(t)) - \sigma(X(t))^2$ (which is assumed negative thanks to the condition $\mu(i) \geq 4\sigma(i)^2$), and local variance $2\sigma(X(t))z_t$. Motivated by this remark, for any $u \in \mathbb{R}$, we set $(Z_t^u)_{t \geq u}$ to be the solution to the following stochastic differential equation:

$$\begin{cases} dZ_t^u = 2\lambda(X(t))W(t) dt + [-2\mu(X(t)) + \sigma(X(t))^2]Z_t^u dt + 2\sigma(X(t))Z_t^u dB_t, \\ Z_u^u = 0. \end{cases} \tag{5.2}$$

Note then that, except for the initial condition, Z_t^u satisfies the same stochastic differential equation as $W(t)^2$ (which again can be easily verified by applying Itô's formula to $W(t)^2$). In fact, we have the following lemma.

Lemma 5.1. *As $t \rightarrow \infty$, Z_t^0 converges in distribution towards*

$$\int_{-\infty}^0 \exp\left(-\int_s^0 [2\mu(X(v)) + \sigma(X(v))^2] dt + \int_s^0 2\sigma(X(v)) dB_v\right) 2\lambda(X(s))W(s) ds. \tag{5.3}$$

In addition, this random variable is equal to W^2 in distribution.

Proof. First note that the solution to (5.2) is (see [11])

$$Z_t^u = \int_u^t \exp\left(-\int_s^t [2\mu(X(v)) + \sigma(X(v))^2] dv + \int_s^t 2\sigma(X(v)) dB_v\right) 2\lambda(X(s))W(s) ds.$$

By an argument similar to that used in Theorem 2.1, we then have that Z_t^0 has the same distribution as Z_0^{-t} . By letting $t \rightarrow \infty$, we then get that Z_t^0 converges in distribution to (5.3).

Let us now prove that (5.3) has the same distribution as W^2 . Since $W(t)^2$ satisfies

$$dW(t)^2 = 2\lambda(X(t))W(t) dt + [-2\mu(X(t)) + \sigma(X(t))^2]W(t) dt + 2\sigma(X(t))W(t) dB_t,$$

which is the same stochastic differential equation as (5.2) except for the initial condition at $t = u$, we have that, for $t = 0$ and $0 \geq u$, similarly to (2.2),

$$\begin{aligned}
 W^2 &= W(u)^2 \exp\left(-\int_u^0 [2\mu(X(s)) + \sigma(X(s))^2] ds + \int_u^0 2\sigma(X(s)) dB_s\right) \\
 &\quad + \int_u^0 \exp\left(-\int_s^0 [2\mu(X(v)) + \sigma(X(v))^2] dv + \int_s^0 2\sigma(X(v)) dB_v\right) 2\lambda(X(s))W(s) ds \\
 &= W(u)^2 M(u) \\
 &\quad + \int_u^0 \exp\left(-\int_s^0 [2\mu(X(v)) + \sigma(X(v))^2] dv + \int_s^0 2\sigma(X(v)) dB_v\right) 2\lambda(X(s))W(s) ds,
 \end{aligned}
 \tag{5.4}$$

where

$$M(u) := \exp\left(-\int_u^0 [2\mu(X(s)) + \sigma(X(s))^2] ds + \int_u^0 2\sigma(X(s)) dB_s\right).$$

For all $x > 0, A > 0, u \leq 0$,

$$\begin{aligned}
 P(W(u)^2 M(u) > x) &= P(W(u)^2 M(u) > x, W(u)^2 > A) \\
 &\quad + P(W(u)^2 M(u) > x, W(u)^2 \leq A) \\
 &\leq P(W(u)^2 > A) + P(AM(u) > x) \\
 &= P(W^2 > A) + P(AM(u) > x),
 \end{aligned}
 \tag{5.5}$$

the equality $P(W(u)^2 > A) = P(W^2 > A)$ holding because of the stationarity of $W(t)$. Now

$$P(AM(u) > x) \leq \frac{A E(M(u))}{x}.$$

Using Lemma 1.1, we can verify that

$$E(M(u)) = E\left(\exp\left(-\int_u^0 (2\mu(X(s)) - \sigma(X(s))^2) ds\right)\right).$$

Condition 5.1 in particular implies that $2\mu(i) - \sigma(i)^2 \geq 0$ for all $i \in S$, and $2\mu(i) - \sigma(i)^2 > 0$ for some $i \in S$. Thus, using an argument similar to that in Theorem 2.1, we can show that

$$E\left(\exp\left(-\int_u^0 (2\mu(X(s)) - \sigma(X(s))^2) ds\right)\right)$$

converges to 0 exponentially fast as $u \rightarrow -\infty$. Hence,

$$P(AM(u) > x) \leq \frac{A E(\exp(-\int_u^0 (2\mu(X(s)) - \sigma(X(s))^2) ds))}{x} \rightarrow 0$$

as $u \rightarrow -\infty$. Taking the lim sup in (5.5), we then get

$$\limsup_{u \rightarrow -\infty} P(W(u)^2 M(u) > x) \leq P(W^2 > A) + 0 \quad \text{for all } A > 0.$$

This inequality holds for all $A > 0$; thus, by letting $A \rightarrow \infty$, we have that

$$\lim_{u \rightarrow -\infty} P(W(u)^2 M(u) > x) = 0$$

for all x . Thus, $W(u)^2 M(u)$ converges to 0 in probability as u tends to $-\infty$. There thus exists a sequence $(u_k)_{k \in \mathbb{N}}$ satisfying $\lim_{k \rightarrow \infty} u_k = -\infty$ such that $W(u_k)^2 M(u_k)$ converges to 0 as $k \rightarrow \infty$.

Letting k tend to $+\infty$ in (5.4) with u replaced by u_k , we conclude that W^2 is equal in distribution to (5.3).

Again by reversing time, we have that (5.3) is equal to

$$\int_0^\infty \exp\left(-\int_0^s [2\mu(X^*(v)) + \sigma(X^*(v))^2] dt + \int_0^s 2\sigma(X^*(v)) dB_v^*\right) 2\lambda(X^*(s)) W^*(s) ds, \tag{5.6}$$

where $W^*(s) := W(-s)$. By reversing time in (2.3), we get

$$W^*(s) = \int_s^\infty \exp\left(-\int_s^r \left[\mu(X^*(h)) + \frac{\sigma(X^*(h))^2}{2}\right] dh + \int_s^r \sigma(X^*(h)) dB_h^*\right) \lambda(X^*(r)) dr.$$

We likewise set

$$M^*(s) := M(-s) = \exp\left(-\int_0^s [2\mu(X^*(v)) + \sigma(X^*(v))^2] dv + \int_0^s 2\sigma(X^*(v)) dB_v^*\right).$$

Then (5.6) can be rewritten as

$$\int_0^\infty M^*(s) 2\lambda(X^*(s)) W^*(s) ds. \tag{5.7}$$

We now show that W and $M^*(t), t \geq 0$, admit moments of order 2.

Lemma 5.2. *We have $E(W^2) < +\infty$.*

Proof. Since W^2 is equal in distribution to (5.7), we have, using Fubini's theorem,

$$\begin{aligned} E(W^2) &= \int_0^\infty E(M^*(s) 2\lambda(X^*(s)) W^*(s)) ds \\ &= \int_0^\infty E(E(M^*(s) 2\lambda(X^*(s)) W^*(s) \mid B_v^*, v \in [0, s], X)) ds. \end{aligned} \tag{5.8}$$

But

$$M^*(s) = \exp\left(-\int_0^s [2\mu(X^*(v)) + \sigma(X^*(v))^2] dv + \int_0^s 2\sigma(X^*(v)) dB_v^*\right)$$

is $B_v^*, v \in [0, s]$, and X^* measurable, and so is $\lambda(X^*(s))$. Thus,

$$\begin{aligned} E(M^*(s) 2\lambda(X^*(s)) W^*(s) \mid B_v^*, v \in [0, s], X^*) \\ = M^*(s) 2\lambda(X^*(s)) E(W^*(s) \mid B_v^*, v \in [0, s], X^*). \end{aligned} \tag{5.9}$$

Recall that

$$W^*(s) = \int_s^\infty \exp\left(-\int_s^r \left[\mu(X^*(h)) + \frac{\sigma(X^*(h))^2}{2}\right] dh + \int_s^r \sigma(X^*(h)) dB_h^*\right) \lambda(X^*(r)) dr;$$

then

$$\begin{aligned} & E(W^*(s) \mid B_v^*, v \in [0, s], X^*) \\ &= E\left(\int_s^\infty \exp\left(-\int_s^r \left[\mu(X^*(h)) + \frac{\sigma(X^*(h))^2}{2}\right] dh + \int_s^r \sigma(X^*(h)) dB_h^*\right) \right. \\ &\quad \left. \times \lambda(X^*(r)) dr \mid B_v^*, v \in [0, s], X^*\right). \end{aligned}$$

The increments of B_h^* for $h \geq s$ are independent of $B_v^*, v \in [0, s]$, so

$$\begin{aligned} & E\left(\int_s^\infty \exp\left(-\int_s^r \left[\mu(X^*(h)) + \frac{\sigma(X^*(h))^2}{2}\right] dh + \int_s^r \sigma(X^*(h)) dB_h^*\right) \right. \\ &\quad \left. \times \lambda(X^*(r)) dr \mid B_v^*, v \in [0, s], X^*\right) \\ &= E\left(\int_s^\infty \exp\left(-\int_s^r \left[\mu(X^*(h)) + \frac{\sigma(X^*(h))^2}{2}\right] dh + \int_s^r \sigma(X^*(h)) dB_h^*\right) \right. \\ &\quad \left. \times \lambda(X^*(r)) dr \mid X^*\right) \\ &= \int_s^\infty E\left(\exp\left(-\int_s^r \left[\mu(X^*(h)) + \frac{\sigma(X^*(h))^2}{2}\right] dh + \int_s^r \sigma(X^*(h)) dB_h^*\right) \right. \\ &\quad \left. \times \lambda(X^*(r)) \mid X^*\right) dr. \end{aligned}$$

Using Lemma 1.1 with $Z = X^*$, $w = B^*$, and $g = \sigma$, we have that

$$\begin{aligned} & E\left(\exp\left(-\int_s^r \left[\mu(X^*(h)) + \frac{\sigma(X^*(h))^2}{2}\right] dh + \int_s^r \sigma(X^*(h)) dB_h^*\right) \lambda(X^*(r)) \mid X^*\right) \\ &= \exp\left(-\int_s^r \mu(X^*(h)) dh\right) \lambda(X^*(r)) \\ &\quad \times E\left(\exp\left(-\int_s^r \frac{\sigma(X^*(h))^2}{2} dh + \int_s^r \sigma(X^*(h)) dB_h^*\right) \mid X^*\right) \\ &= \exp\left(-\int_s^r \mu(X^*(h)) dh\right) \lambda(X^*(r)); \end{aligned}$$

hence,

$$\begin{aligned} & E(W^*(s) \mid B_v^*, v \in [0, s], X) \\ &= \int_s^\infty E\left(\exp\left(-\int_s^r \left[\mu(X^*(h)) + \frac{\sigma(X^*(h))^2}{2}\right] dh + \int_s^r \sigma(X^*(h)) dB_h^*\right) \right. \\ &\quad \left. \times \lambda(X^*(r)) \mid X^*\right) dr \\ &= \int_s^\infty \exp\left(-\int_s^r \mu(X^*(h)) dh\right) \lambda(X^*(r)) dr. \end{aligned} \tag{5.10}$$

Thus, from (5.8), (5.9), and (5.10),

$$\mathbb{E}(W^2) = \int_0^\infty \mathbb{E} \left(M^*(s) 2\lambda(X^*(s)) \int_s^\infty \exp\left(-\int_s^r \mu(X^*(h)) dh\right) \lambda(X^*(r)) dr \right) ds. \quad (5.11)$$

Now, by reconditioning with respect to X^* , we get

$$\begin{aligned} & \mathbb{E} \left(M^*(s) 2\lambda(X^*(s)) \int_s^\infty \exp\left(-\int_s^r \mu(X^*(h)) dh\right) \lambda(X^*(r)) dr \right) \\ &= \mathbb{E} \left(\mathbb{E} \left(M^*(s) 2\lambda(X^*(s)) \int_s^\infty \exp\left(-\int_s^r \mu(X^*(h)) dh\right) \lambda(X^*(r)) dr \mid X^* \right) \right) \\ &= \mathbb{E} \left(2\lambda(X^*(s)) \int_s^\infty \exp\left(-\int_s^r \mu(X^*(h)) dh\right) \lambda(X^*(r)) dr \mathbb{E}(M^*(s) \mid X^*) \right). \end{aligned}$$

We may write $M^*(s)$ in the following way:

$$\begin{aligned} M^*(s) &= \exp\left(-\int_0^s [2\mu(X^*(v)) + \sigma(X^*(v))^2] dv + \int_0^s 2\sigma(X^*(v)) dB_v^*\right) \\ &= \exp\left(-\int_0^s [2\mu(X^*(h)) - \sigma(X^*(h))^2] dh\right) \\ &\quad \times \exp\left(-\int_0^s 2\sigma(X^*(v))^2 dv + \int_0^s 2\sigma(X^*(v)) dB_v^*\right). \end{aligned}$$

Using again Lemma 1.1 with $Z = X^*$, $w = B^*$, and $g = 2\sigma$, we then get easily that

$$\mathbb{E}(M^*(s) \mid X^*) = \exp\left(-\int_0^s [2\mu(X^*(h)) - \sigma(X^*(h))^2] dh\right).$$

Let us now notice that, since $\mu \geq 4\sigma^2$ yields that $\mu \geq \sigma^2$,

$$\begin{aligned} & \int_s^\infty \exp\left(-\int_s^r \mu(X^*(h)) dh\right) \lambda(X^*(r)) dr \mathbb{E}(M^*(s) \mid X^*) \\ &= \int_s^\infty \exp\left(-\int_s^r \mu(X^*(h)) dh\right) \lambda(X^*(r)) dr \exp\left(-\int_0^s [2\mu(X^*(h)) - \sigma(X^*(h))^2] dh\right) \\ &= \int_s^\infty \exp\left(-\int_0^r \mu(X^*(h)) dh\right) \lambda(X^*(r)) dr \exp\left(-\int_0^s [\mu(X^*(h)) - \sigma(X^*(h))^2] dh\right) \\ &\leq \int_s^\infty \exp\left(-\int_0^r \mu(X^*(h)) dh\right) \lambda(X^*(r)) dr. \end{aligned}$$

Thus, we have from (5.11) that

$$\begin{aligned} \mathbb{E}(W^2) &\leq \int_0^\infty \mathbb{E} \left(2\lambda(X^*(s)) \int_s^\infty \exp\left(-\int_0^r \mu(X^*(h)) dh\right) \lambda(X^*(r)) dr \right) ds \\ &\leq 2 \left(\sup_{i \in \mathcal{S}} \lambda(i) \right)^2 \int_0^\infty \mathbb{E} \left(\int_s^\infty \exp\left(-\int_0^r \mu(X^*(h)) dh\right) dr \right) ds \\ &= 2 \left(\sup_{i \in \mathcal{S}} \lambda(i) \right)^2 \int_0^\infty \int_s^\infty \mathbb{E} \left(\exp\left(-\int_0^r \mu(X^*(h)) dh\right) \right) dr ds. \quad (5.12) \end{aligned}$$

Recall from Theorem 2.1 the estimate $E(\exp(-\int_0^r \mu(X^*(h)) dh)) \leq O(\exp(-\alpha r))$ for the constant α defined by (2.5). It follows that

$$\int_s^\infty E\left(\exp\left(-\int_0^r \mu(X^*(h)) dh\right)\right) dr \leq O(\exp(-\alpha s)).$$

Hence,

$$\int_0^\infty \int_s^\infty E\left(\exp\left(-\int_0^r \mu(X^*(h)) dh\right)\right) dr ds$$

is finite. Thus, from (5.12), $E(W^2)$ is finite.

Lemma 5.3. *As $t \rightarrow +\infty$, $E(M^*(t)^2)$ tends to 0 exponentially fast.*

Proof. For all $t \geq 0$,

$$\begin{aligned} M^*(t)^2 &= \exp\left(-\int_0^t [4\mu(X^*(v)) + 2\sigma(X^*(v))^2] dv + \int_0^t 4\sigma(X^*(v)) dB_v^*\right) \\ &= \exp\left(-\int_0^t [4\mu(X^*(v)) - 6\sigma(X^*(v))^2] dv - \int_0^t 8\sigma(X^*(v))^2 dv \right. \\ &\quad \left. + \int_0^t 4\sigma(X^*(v)) dB_v^*\right). \end{aligned}$$

Thus, by the same martingale argument as in the proof of Theorem 2.1, we get that

$$E(M^*(t)^2) = E\left(\exp\left(-\int_0^t [4\mu(X^*(v)) - 6\sigma(X^*(v))^2] dv\right)\right).$$

Condition 5.1 implies that $4\mu(i) - 6\sigma(i)^2 \leq 0$ for all $i \in S$, and that $4\mu(i) - 6\sigma(i)^2 < 0$ for some i . Thus, again using the same argument as in Theorem 2.1, we deduce that $E(M^*(t)^2)$ tends to 0 exponentially fast as $t \rightarrow +\infty$.

We may now state the expression for the second moment of W , which is the main result of this section.

Theorem 5.1. *We define*

$$\begin{aligned} D_{2\mu-\sigma^2} &:= \text{diag}(2\mu(1) - \sigma(1)^2, \dots, 2\mu(N) - \sigma(N)^2), \\ D_{\mu-2\sigma^2} &:= \text{diag}(\mu(1) - 2\sigma(1)^2, \dots, \mu(N) - 2\sigma(N)^2). \end{aligned}$$

Then

$$\begin{aligned} E(W^2) &= 2\pi(D_{2\mu-\sigma^2} - Q^*)^{-1}(Q^* - D_{\mu-4\sigma^2})^{-1}\Lambda^2 \mathbf{1} \\ &\quad - 2\pi(Q^* - D_{\mu-4\sigma^2})^{-1}\Lambda(D_\mu - Q^*)^{-1}\Lambda \mathbf{1}. \end{aligned} \tag{5.13}$$

Proof. As in Theorem 3.1, we write, using Itô's formula,

$$\begin{aligned} g(W^*(t), M^*(t), X^*(t)) - g(W^*(0), M^*(0), X^*(0)) &- \int_0^t \mathcal{A}'g(W^*(s), M^*(s), X^*(s)) ds \\ &= \int_0^t \sigma(X^*(s))W^*(s)\partial_{x_w}g(W^*(s), M^*(s), X^*(s)) dB_s^* \\ &\quad + \int_0^t 2\sigma(X^*(s))M^*(s)\partial_{x_m}g(W^*(s), M^*(s), X^*(s)) dB_s^*, \end{aligned} \tag{5.14}$$

where $\partial_{x_w} g$ and $\partial_{x_m} g$ denote the derivatives of g with respect to the first and second variables respectively and

$$\begin{aligned} \mathcal{A}'g(x_w, x_m, i) &= [(\mu(i) + \sigma(i)^2)x_w - \lambda(i)]\partial_{x_w}g(x_w, x_m, i) \\ &\quad - [2\mu(i) - \sigma(i)^2]x_m\partial_{x_m}g(x_w, x_m, i) \\ &\quad + \frac{1}{2}\sigma(i)^2x_w^2\partial_{x_w}^2g(x_w, x_m, i) + 2\sigma(i)^2x_m^2\partial_{x_m}^2g(x_w, x_m, i) \\ &\quad + 2\sigma(i)^2x_mx_w\partial_{x_m}\partial_{x_w}g(x_w, x_m, i) + \sum_j q_{ij}^*g(x_w, x_m, j). \end{aligned}$$

Given X^* , (5.14) is a local martingale. We now take $g(x_w, x_m, i) = x_w x_m f(i)$, where f will again be chosen later on. By Lemmas 5.2 and 5.3, $W^*(t) \stackrel{D}{=} W$ and $M^*(t)$ are in $L^2(\Omega)$, and thus (5.14) is actually a real martingale.

Taking the expectation in (5.14) with the expressions for g and \mathcal{A}' , we then get, after an easy calculation,

$$\begin{aligned} &E\left(\int_0^t [(\mu(X^*(s)) + \sigma(X^*(s))^2)W^*(s) - \lambda(X^*(s))]M^*(s)f(X^*(s)) \right. \\ &\quad \left. - [2\mu(X^*(s)) - \sigma(X^*(s))^2]M^*(s)W^*(s)f(X^*(s)) \right. \\ &\quad \left. + 2\sigma(X^*(s))^2M^*(s)W^*(s)f(X^*(s)) + \sum_j q_{X^*(s),j}^*W^*(s)M^*(s)f(j) \, ds\right) \\ &= E(W^*(0)M^*(0)f(X^*(0))) - E(W^*(t)M^*(t)f(X^*(t))), \end{aligned}$$

which, with the notation of Theorem 3.1, can be rewritten as

$$\begin{aligned} &E\left(\int_0^t W^*(s)M^*(s)[(\mathcal{Q}^* - D_{\mu-4\sigma^2})\underline{f}](X^*(s)) \, ds\right) \\ &\quad - E\left(\int_0^t \lambda(X^*(s))M^*(s)f(X^*(s)) \, ds\right) \\ &= -E(W^*(0)M^*(0)f(X^*(0))) + E(W^*(t)M^*(t)f(X^*(t))). \end{aligned} \tag{5.15}$$

We are now going to study each term of this equality.

The second term on the left-hand side of (5.15) is equal to

$$\begin{aligned} &E\left(\int_0^t \lambda(X^*(s))M^*(s)f(X^*(s)) \, ds\right) \\ &= E\left(\int_0^t \exp\left(-\int_0^s [2\mu(X^*(v)) + \sigma(X^*(v))^2] \, dv + \int_0^s 2\sigma(X^*(v)) \, dB_v^*\right) \right. \\ &\quad \left. \times \lambda(X^*(s))f(X^*(s)) \, ds\right) \\ &= \int_0^t E\left(\exp\left(-\int_0^s [2\mu(X^*(v)) + \sigma(X^*(v))^2] \, dv + \int_0^s 2\sigma(X^*(v)) \, dB_v^*\right) \right. \\ &\quad \left. \times \lambda(X^*(s))f(X^*(s))\right) \, ds. \end{aligned} \tag{5.16}$$

But, again using a conditioning argument,

$$\begin{aligned}
 & \mathbb{E} \left(\exp \left(- \int_0^S [2\mu(X^*(v)) + \sigma(X^*(v))]^2 dv + \int_0^S 2\sigma(X^*(v)) dB_v^* \right) \lambda(X^*(s)) f(X^*(s)) \right) \\
 &= \mathbb{E} \left(\mathbb{E} \left(\exp \left(- \int_0^S [2\mu(X^*(v)) + \sigma(X^*(v))]^2 dv + \int_0^S 2\sigma(X^*(v)) dB_v^* \right) \right. \right. \\
 &\quad \left. \left. \times \lambda(X^*(s)) f(X^*(s)) \mid X^* \right) \right) \\
 &= \mathbb{E} \left(\mathbb{E} \left(\exp \left(- \int_0^S [2\mu(X^*(v)) + \sigma(X^*(v))]^2 dv + \int_0^S 2\sigma(X^*(v)) dB_v^* \right) \mid X^* \right) \right. \\
 &\quad \left. \times \lambda(X^*(s)) f(X^*(s)) \right) \\
 &= \mathbb{E} \left(\mathbb{E} \left(\exp \left(- \int_0^S [2\mu(X^*(v)) - \sigma(X^*(v))]^2 dv \right) \mid X^* \right) \lambda(X^*(s)) f(X^*(s)) \right) \\
 &= \mathbb{E} \left(\mathbb{E} \left(\exp \left(- \int_0^S [2\mu(X^*(v)) - \sigma(X^*(v))]^2 dv \right) \lambda(X^*(s)) f(X^*(s)) \mid X^* \right) \right) \\
 &= \mathbb{E} \left(\exp \left(- \int_0^S [2\mu(X^*(v)) - \sigma(X^*(v))]^2 dv \right) \lambda(X^*(s)) f(X^*(s)) \right). \tag{5.17}
 \end{aligned}$$

Thus, from (5.16) and (5.17),

$$\begin{aligned}
 & \mathbb{E} \left(\int_0^t \lambda(X^*(s)) M^*(s) f(X^*(s)) ds \right) \\
 &= \int_0^t \mathbb{E} \left(\exp \left(- \int_0^S [2\mu(X^*(v)) - \sigma(X^*(v))]^2 dv \right) \lambda(X^*(s)) f(X^*(s)) \right) ds \\
 &= \mathbb{E} \left(\int_0^t \exp \left(- \int_0^S [2\mu(X^*(v)) - \sigma(X^*(v))]^2 dv \right) \lambda(X^*(s)) f(X^*(s)) ds \right).
 \end{aligned}$$

The limit of this expression as t tends to $+\infty$ is the first moment of the steady regime of the model with release rate $2\mu(X(v)) - \sigma(X(v))^2$, variance function $2\sigma(X(v))$, and input rate $\lambda(X(s))f(X(s))$. Thus (3.2) yields that

$$\lim_{t \rightarrow \infty} \mathbb{E} \left(\int_0^t \lambda(X^*(s)) M^*(s) f(X^*(s)) ds \right) = \pi (D_{2\mu - \sigma^2} - Q^*)^{-1} F \Lambda \mathbf{1},$$

where

$$F = \text{diag}(f(1), \dots, f(N)).$$

Note that $D_{2\mu - \sigma^2} - Q^*$ is invertible because Condition 5.1 implies that $2\mu(i) - \sigma(i)^2 \geq 0$ for all $i \in S$ and $2\mu(i) - \sigma(i)^2 > 0$ for some $i \in S$.

We now consider the term $E(W^*(t)M^*(t)f(X^*(t)))$ in (5.15) and show that it tends to 0 as t tends to $+\infty$. Since f is bounded, we just need to show that $E(W^*(t)M^*(t))$ tends to 0. But

$$\begin{aligned} &M^*(t)W^*(t) \\ &= \exp\left(-\int_0^t [2\mu(X^*(v)) + \sigma(X^*(v))^2] dv + \int_0^t 2\sigma(X^*(v)) dB_v^*\right) \\ &\quad \times \int_t^\infty \exp\left(-\int_t^s \left[\mu(X^*(v)) + \frac{\sigma(X^*(v))^2}{2}\right] dv + \int_t^s \sigma(X^*(v)) dB_v^*\right) \lambda(X^*(s)) ds \\ &= \exp\left(-\int_0^t \left[\mu(X^*(v)) + \frac{\sigma(X^*(v))^2}{2}\right] dv + \int_0^t \sigma(X^*(v)) dB_v^*\right) \\ &\quad \times \int_t^\infty \exp\left(-\int_0^s \left[\mu(X^*(v)) + \frac{\sigma(X^*(v))^2}{2}\right] dv + \int_0^s \sigma(X^*(v)) dB_v^*\right) \lambda(X^*(s)) ds \\ &\leq \exp\left(-\int_0^t \left[\mu(X^*(v)) + \frac{\sigma(X^*(v))^2}{2}\right] dv + \int_0^t \sigma(X^*(v)) dB_v^*\right) \\ &\quad \times \int_0^\infty \exp\left(-\int_0^s \left[\mu(X^*(v)) + \frac{\sigma(X^*(v))^2}{2}\right] dv + \int_0^s \sigma(X^*(v)) dB_v^*\right) \lambda(X^*(s)) ds \\ &= \exp\left(-\int_0^t \left[\mu(X^*(v)) + \frac{\sigma(X^*(v))^2}{2}\right] dv + \int_0^t \sigma(X^*(v)) dB_v^*\right) W^*(0). \end{aligned}$$

Then, by the Cauchy–Schwartz inequality,

$$\begin{aligned} &E(M^*(t)W^*(t)) \\ &\leq E\left(\exp\left(-\int_0^t [2\mu(X^*(v)) + \sigma(X^*(v))^2] dv + \int_0^t 2\sigma(X^*(v)) dB_v^*\right)\right)^{1/2} E(W^*(0)^2)^{1/2} \\ &= E(M^*(t))^{1/2} E(W^2)^{1/2}. \end{aligned}$$

Now, by Lemma 5.3, $E(M^*(t))$ tends to 0 as t tends to infinity. Thus, $E(W^*(t)M^*(t)f(X^*(t)))$ tends to 0 as t tends to infinity.

Let us now focus on $E(W^*(0)M^*(0)f(X^*(0))) = E(Wf(X(0)))$ in (5.15). Applying (3.1) with $h = f$ yields that

$$\begin{aligned} E(W^*(0)M^*(0)f(X^*(0))) &= E(Wf(X(0))) \\ &= \pi F(D_\mu - Q^*)^{-1} \Lambda \mathbf{1}. \end{aligned}$$

In view of this analysis, we may then let $t \rightarrow +\infty$ in (5.15) to get

$$\begin{aligned} &E\left(\int_0^\infty W^*(s)M^*(s)[(Q^* - D_{\mu-4\sigma^2})\underline{f}](X^*(s)) ds\right) - \pi(D_{2\mu-\sigma^2} - Q^*)^{-1} F \Lambda \mathbf{1} \\ &= -\pi F(D_\mu - Q^*)^{-1} \Lambda \mathbf{1}. \end{aligned} \tag{5.18}$$

We now take $F = 2(Q^* - D_{\mu-4\sigma^2})^{-1} \Lambda$. Thus $\underline{f} = 2(Q^* - D_{\mu-4\sigma^2})^{-1} \Lambda \mathbf{1}$. In that case, with this expression for f and since W^2 is equal to (5.7) in distribution, we have

$$E\left(\int_0^\infty W^*(s)M^*(s)[(Q^* - D_{\mu-4\sigma^2})\underline{f}](X^*(s)) ds\right) = E(W^2).$$

Combining this equality with (5.18) yields (5.13).

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