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STATIONARY SOLUTION TO THE FLUID QUEUE FED BY AN M/M/1 QUEUE

N. BARBOT,* *IRISA, Rennes*
B. SERICOLA,* ** *IRISA and INRIA, Rennes*

Abstract

We consider an infinite-capacity buffer receiving fluid at a rate depending on the state of an M/M/1 queue. We obtain a new analytic expression for the joint stationary distribution of the buffer level and the state of the M/M/1 queue. This expression is obtained by the use of generating functions which are explicitly inverted. The case of a finite capacity fluid queue is also considered.

Keywords: Fluid queue; M/M/1 queue; generating functions

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1. Introduction

Markov-modulated fluid flow models have turned out to be very useful in analysing performance issues in telecommunication systems. These models are composed of a buffer and a continuous-time Markov chain that controls the input and service rates of the fluid in the buffer. In most studies dealing with the analysis of such fluid queues, the state space of the background Markov chain is supposed to be finite; see, for instance, [4], [8] and the references therein. We consider here an infinite-capacity fluid queue where the input rate is a function of the state of the server in an M/M/1 queue and where the service rate is constant. As suggested in [10], this model might represent a Poisson stream of packet arrivals, where the packet length is exponentially distributed. This stream is buffered in a queue and served with a constant rate. The output process of this M/M/1 queue forms the input process of the fluid queue.

The stationary behaviour of that fluid queue has been analysed in several papers. Although approaches are different, the fluid level distribution is generally obtained as an integral expression. First, in [10], Virtamo and Norros solved the well-known infinite differential system by studying the continuous spectrum of a key matrix. Secondly, Adan and Resing [1] considered the background process as an alternating renewal process, corresponding to the successive idle and busy periods of the M/M/1 queue. By renewal theory arguments, the fluid level distribution is given in terms of an integral of Bessel functions. They also obtained the expression of Virtamo and Norros via an integral representation of Bessel functions.

More general input processes are applied to that fluid model. In [9], van Doorn and Scheinhart studied a fluid queue fed by an infinite-state birth–death process. They solve the infinite differential system by the use of orthogonal polynomials with respect to a signed measure which is given explicitly in the case of the M/M/1 queue and leads to the same integral expression obtained in [10] and [1]. In [7], the authors considered a fluid queue driven by a general

Received 17 October 2001; revision received 21 March 2002.
* Postal address: IRISA, Campus universitaire de Beaulieu, 35042 Rennes cedex, France.
** Email address: bruno.sericola@irisa.fr

Markovian queue with the hypothesis that only one state has a negative drift. By using the differential system, the fluid level distribution is obtained in terms of a series and coefficients computed by means of recurrence relations. Moreover, this study was extended to the finite buffer case in [6]. Finally, the transient distribution of that fluid queue and the convergence to the stationary distribution has been analysed in [2].

In this paper, we obtain a new analytic expression for the joint stationary distribution of the buffer level and the state of the M/M/1 queue. This expression is obtained by writing the solution in terms of a matrix exponential and then by using generating functions that are explicitly inverted. In the following section, we present the model and a matrix analytic expression of the solution in terms of a series. In Section 3, we derive the generating function associated with the successive powers of a key matrix used in the solution. Next, in Section 4, by using the method developed in [3], this generating function is explicitly inverted and we obtain a very simple expression of the joint distribution. In Section 5, we use these results and those of [6] to obtain a similar expression for the case of a finite-capacity fluid queue.

2. Formulation of the model

We consider a fluid queue with an infinite buffer for which the service rate is a constant $c > 0$, and the input rate is governed by an M/M/1 queue with arrival rate λ and service rate μ. During the busy periods of the M/M/1 queue, the input rate in the fluid queue is positive, denoted by r, while during the idle periods no fluid enters the queue. We suppose that $r > c$ in order to avoid the trivial case where the queue remains always empty.

We denote by $\{X_t,\ t \geq 0\}$ the continuous-time Markov chain counting the number of customers in the M/M/1 queue. Its infinitesimal generator is denoted by A. The nonzero entries of the matrix A are $A_{0,0} = -\lambda$, $A_{0,1} = \lambda$ and, for $j \geq 1$,

$$A_{j,j-1} = \mu, \qquad A_{j,j} = -(\lambda + \mu), \qquad A_{j,j+1} = \lambda.$$

The drifts of that fluid queue represent the difference between the input and service rates. Let d_j be the drift when the M/M/1 queue is in state j. We thus have $d_0 = -c$ and $d_j = r - c$ for every $j \geq 1$. We denote by D the diagonal matrix containing these drifts.

Since we are concerned with the stationary behaviour of that fluid queue, we suppose that the stability condition is satisfied, that is,

$$\rho = \frac{\lambda r}{\mu c} < 1,$$

where ρ is the traffic intensity. We denote, respectively, by X and Q the stationary state of the Markov chain $\{X_t,\ t \geq 0\}$ and the stationary amount of fluid in the buffer.

Let $F_j(x) = \Pr\{X = j,\ Q \leq x\}$. It is easy to see that, for $j \geq 1$, we have $F_j(0) = 0$ and it has been shown in [7] that $F_0(0) = 1 - \rho$. It is well known (see for instance [10]) that the functions F_j satisfy, for $x > 0$, the following system of differential equations:

$$\begin{aligned} -cF_0'(x) &= -\lambda F_0(x) + \mu F_1(x), \\ (r-c)F_j'(x) &= \lambda F_{j-1}(x) - (\lambda + \mu)F_j(x) + \mu F_{j+1}(x) \quad \text{for } j \geq 1, \end{aligned} \tag{1}$$

where $F_j'(x)$ denotes the derivative of $F_j(x)$ with respect to x. Let $F(x)$ be the infinite row vector containing the $F_j(x)$. This system can also be written as $F'(x) = F(x)AD^{-1}$ and its solution is given by

$$F(x) = F(0)\exp(AD^{-1}x). \tag{2}$$

Let I be the identity matrix. Using a method similar to the uniformization technique, we introduce the matrix T defined by $T = I + (1/\theta)AD^{-1}$, where $\theta = (\lambda + \mu)/(r - c)$. We then have, from (2), for every $j \geq 0$,

$$\begin{aligned} F_j(x) &= (1-\rho)(\exp(AD^{-1}x))_{0,j} \\ &= (1-\rho)\exp(-\theta x)(\exp(\theta T x))_{0,j} \\ &= (1-\rho)\sum_{n=0}^{\infty}\exp(-\theta x)\frac{(\theta x)^n}{n!}T^n_{0,j}, \end{aligned} \tag{3}$$

where $T^n_{0,j}$ denotes the $(0, j)$ entry of the matrix T^n.

In what follows, we focus on the calculation of $T^n_{0,j}$ using generating functions.

3. Generating functions

In this section, we first recall the definition and properties of the generating functions which are then used in a second subsection to obtain the generating function associated with the matrix T.

3.1. Definition and properties

Let us consider the complex matrices M indexed on $\mathbb{N} \times \mathbb{N}$. We define

$$\nu(M) = \sup_{i \in \mathbb{N}} \sum_{j=0}^{\infty} |M_{ij}|$$

and denote by $\mathcal{M}$ the set of infinite complex matrices M such that $\nu(M)$ is finite. Then ν is a norm on $\mathcal{M}$ and $(\mathcal{M}, \nu)$ is a Banach algebra. With each $M \in \mathcal{M}$, we associate the complex function Φ_M, called the potential kernel of M or generating function, defined by

$$\Phi_M(z) = \sum_{k=0}^{\infty} M^k z^k$$

for all z such that $|z| < 1/\nu(M)$. Note that, for $M \in \mathcal{M}$ and z such that $|z| < 1/\nu(M)$, we have $\Phi_M(z) \in \mathcal{M}$ since $\nu(\Phi_M(z)) \leq 1/(1 - |z|\nu(M)) < +\infty$.

The following lemma is a classical straightforward result, so we give it without proof.

Lemma 1. *For every matrix H, $H\Phi_M$ is the only solution to the matrix equation*

$$X(z) = H + zX(z)M$$

for all z such that $|z| < 1/\nu(M)$.

We shall also need the following result which will be used along with Lemma 1.

Lemma 2. *For every M and N in $\mathcal{M}$, we have*

$$\Phi_{M+N}(z) = \Phi_M(z) + z\Phi_{M+N}(z)N\Phi_M(z)$$

for all z such that $|z| < \min\{1/\nu(M), 1/\nu(M+N)\}$.

Proof. See [3].

Let us now introduce some notation. We define the infinite matrices V, W, R, S and P by

$$V_{i,j} = I_{i+1,j}, \qquad W_{i,j} = I_{i,j+1}, \qquad R_{i,j} = I_{i,0} I_{0,j}, \qquad S_{i,j} = I_{i,1} I_{0,j}$$

for i and $j \in \mathbb{N}$ and

$$P = I + \frac{A}{\lambda + \mu}.$$

The matrix P is referred to as the transition probability matrix of the uniformized Markov chain associated with the M/M/1 queue. If p and q are defined by

$$p = \frac{\lambda}{\lambda + \mu} \quad \text{and} \quad q = \frac{\mu}{\lambda + \mu},$$

then the nonzero entries of P are

$$P_{0,0} = q, \qquad P_{0,1} = p, \qquad P_{i,i-1} = q, \qquad P_{i,i+1} = p \quad \text{for } i \geq 1.$$

The stability condition $\rho < 1$ and the fact that $r > c$ implies that $\lambda < \mu$ and so $p < q$.

Lemma 3. *We have $T = P + U$ where $U = (pR - qS)r/c$.*

Proof. From the definition of T, it suffices to write A and D^{-1} as

$$A = (\lambda + \mu)(P - I),$$
$$D^{-1} = \frac{1}{r - c}\left(I - \frac{r}{c} R\right)$$

and to use the fact that $PR = qR + qS$.

It is easy to check that

$$\nu(T) = \max\{1 + pr/c, p + q(r - c)/c\} > 1.$$

Using Lemma 2, we obtain

$$\Phi_T(z) = \Phi_P(z) + z\Phi_T(z)U\Phi_P(z) \tag{4}$$

for all z such that $|z| < 1/\nu(T)$, since $\nu(P) = 1$. We define the matrix $L(z)$ as

$$L(z) = U\Phi_P(z).$$

For $|z| < 1$, we have

$$\nu(L(z)) = \nu(U\Phi_P(z)) \leq \frac{\nu(U)}{1 - |z|} = \frac{rq}{c(1 - |z|)},$$

so for $|z| < c/(qr + c)$ we have $|z| < 1/\nu(L(z))$ and $L(z) \in \mathcal{M}$. Thus, we may apply Lemma 1 to (4) with $X(z) = \Phi_T(z)$, $H = \Phi_P(z)$, $M = L(z)$ and we get

$$\Phi_T(z) = \Phi_P(z)\Phi_{L(z)}(z) \tag{5}$$

for $|z| < c/(qr + c)$.

3.2. Calculation of Φ_T

In this section, we derive a simple expression for the potential kernel Φ_T given in (5). We first recall the expression for Φ_P obtained in [3].

The Catalan numbers c_n are defined for every $n \in \mathbb{N}$ by

$$c_n = \binom{2n}{n}\frac{1}{n+1}.$$

The generating function associated with the sequence of these numbers,

$$C(z) = \sum_{n=0}^{\infty} c_n z^n,$$

converges for all z such that $|z| \leq \frac{1}{4}$ and can be written as

$$C(z) = \frac{1-\sqrt{1-4z}}{2z}. \tag{6}$$

Lemma 4. *Let $|z| < 1$ and $\eta(z) = C(pqz^2)$. Let $X(z)$ and $Y(z)$ be the infinite matrices defined by*

$$X_{i,j}(z) = (qz\eta(z))^i(pz\eta(z))^j$$

and

$$Y(z) = \sum_{k=0}^{\infty} W^k X(z) V^k.$$

For all z such that $|z| < \frac{1}{2}$, we have

$$\Phi_P(z) = \eta(z)\left(Y(z) + \frac{qz\eta(z)}{1-qz\eta(z)}X(z)\right). \tag{7}$$

Proof. See [3].

Theorem 1. *For all z such that $|z| < \frac{1}{2}$, we have*

$$L(z) = \frac{\eta(z)}{1-qz\eta(z)}UX(z) \tag{8}$$

and

$$\Phi_{L(z)}(z) = I + \frac{z}{1-\rho qz\eta(z)}L(z). \tag{9}$$

Proof. Let us fix z such that $|z| < \frac{1}{2}$. Since $RW = SW = 0$, we have, by the definitions of $X(z)$ and $Y(z)$,

$$RY(z) = RX(z),$$
$$SY(z) = SX(z),$$

which gives, by the definition of the matrix U,

$$UY(z) = UX(z).$$

The definition of $L(z)$ and Lemma 4 lead to

$$\begin{aligned} L(z) &= U\Phi_P(z) \\ &= \eta(z)\left(UY(z) + \frac{qz\eta(z)}{1-qz\eta(z)}UX(z)\right) \\ &= \eta(z)\left(1 + \frac{qz\eta(z)}{1-qz\eta(z)}\right)UX(z) \\ &= \frac{\eta(z)}{1-qz\eta(z)}UX(z), \end{aligned}$$

which proves (8).

Consider now the successive powers $L^k(z)$ of the matrix $L(z)$. Observing that

$$X(z)RX(z) = X(z), \tag{10}$$

$$X(z)SX(z) = pz\eta(z)X(z), \tag{11}$$

we easily obtain

$$\begin{aligned} L^2(z) &= \left(\frac{\eta(z)}{1-qz\eta(z)}\right)^2 (UX(z))^2 \\ &= \left(\frac{r\eta(z)}{c(1-qz\eta(z))}\right)^2 (pRX(z) - qSX(z))^2 \\ &= \left(\frac{r\eta(z)}{c(1-qz\eta(z))}\right)^2 p(1-qz\eta(z))(pRX(z) - qSX(z)) \\ &= \frac{rp\eta(z)}{c}L(z) \\ &= \rho q\eta(z)L(z). \end{aligned}$$

It follows by induction that, for every $k \geq 0$,

$$L^{k+1}(z) = (\rho q\eta(z))^k L(z).$$

Since $|z| < \frac{1}{2}$, it is easy to check, from (6), that $|\eta(z)| \leq 2$ and so, that $|qz\eta(z)| < 1$. Thus, we obtain

$$\begin{aligned} \Phi_{L(z)}(z) &= \sum_{k=0}^{\infty} L^k(z)z^k \\ &= I + z\sum_{k=0}^{\infty}(\rho qz\eta(z))^k L(z) \\ &= I + \frac{z}{1-\rho qz\eta(z)}L(z), \end{aligned}$$

which completes the proof.

Theorem 2. *For $|z| < \min\{\frac{1}{2}, c/(qr+c)\}$, we have*

$$\Phi_T(z) = \eta(z)\left(Y(z) + \frac{qz\eta(z)((1+\rho-\rho qz\eta(z))X(z) - (r/c)WX(z))}{(1-qz\eta(z))(1-\rho qz\eta(z))}\right). \tag{12}$$

Proof. Let z be such that $|z| < \min\{\frac{1}{2}, c/(qr+c)\}$. Substituting (7) and (9) into (5), we obtain

$$\Phi_T(z) = \eta(z)\left(Y(z) + \frac{qz\eta(z)}{1-qz\eta(z)}X(z)\right)\left(I + \frac{z}{1-\rho qz\eta(z)}L(z)\right). \tag{13}$$

Now, since $VR = 0$ and $VS = R$, we obtain from (10) and (11) that

$$\begin{aligned} Y(z)RX(z) &= X(z), \\ Y(z)SX(z) &= pz\eta(z)X(z) + WX(z). \end{aligned}$$

From (8),

$$\begin{aligned} Y(z)L(z) &= \frac{r\eta(z)}{c(1-qz\eta(z))}(pY(z)RX(z) - qY(z)SX(z)) \\ &= \rho q\eta(z)X(z) - \frac{rq\eta(z)}{c(1-qz\eta(z))}WX(z), \end{aligned} \tag{14}$$

and using (10) and (11),

$$\begin{aligned} X(z)L(z) &= \frac{r\eta(z)}{c(1-qz\eta(z))}(pX(z)RX(z) - qX(z)SX(z)) \\ &= \rho q\eta(z)X(z). \end{aligned} \tag{15}$$

Putting (14) and (15) in (13), we obtain

$$\begin{aligned} \Phi_T(z) &= \eta(z)Y(z) + \frac{z\eta(z)}{1-\rho qz\eta(z)}Y(z)L(z) \\ &\quad + \frac{qz\eta^2(z)}{1-qz\eta(z)}X(z) + \frac{qz^2\eta^2(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))}X(z)L(z) \\ &= \eta(z)Y(z) + \frac{\rho qz\eta^2(z)}{1-\rho qz\eta(z)}X(z) - \frac{rqz\eta^2(z)}{c(1-qz\eta(z))(1-\rho qz\eta(z))}WX(z) \\ &\quad + \frac{qz\eta^2(z)}{1-qz\eta(z)}X(z) + \frac{\rho q^2z^2\eta^3(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))}X(z) \\ &= \eta(z)Y(z) - \frac{rqz\eta^2(z)}{c(1-qz\eta(z))(1-\rho qz\eta(z))}WX(z) \\ &\quad + \frac{qz\eta^2(z)(1+\rho-\rho qz\eta(z))}{(1-qz\eta(z))(1-\rho qz\eta(z))}X(z), \end{aligned}$$

which is the desired result.

4. Explicit solution

We obtain in this section a closed-form expression for $T^n_{0,j}$ and so also for $F_j(x)$. For that purpose, we need the following well-known lemma which gives an analytical expression of the powers of $\eta(z)$.

Lemma 5. *For every $k \geq 1$ and $|z| \leq \frac{1}{4}$, we have*

$$C^k(z) = \sum_{n=0}^{\infty} s(k,n) z^n,$$

where

$$s(k,n) = k\frac{(2n+k-1)!}{n!(n+k)!}.$$

Proof. See [5, p. 154].

The integers $s(k,n)$ are referred to as the ballot numbers.

Theorem 3. *For every $j \geq 0$,*

$$T^n_{0,j} = \begin{cases} 0 & \text{if } n < j, \\ \dfrac{p^j}{(1-\rho)q^j} \displaystyle\sum_{k=0}^{\lfloor (n-j)/2 \rfloor} s(n-2k+1,k) p^k q^{n-k} (1-\rho^{n-j-2k+1}) & \text{if } n \geq j, \end{cases}$$

where $\lfloor u \rfloor$ denotes the largest integer less than or equal to the real number u.

Proof. Let z be such that $|z| < \min\{\frac{1}{2}, c/(qr+c)\}$. Since the first row of the matrix $WX(z)$ has all its entries equal to zero, we have from (12), for every $j \in \mathbb{N}$,

$$(\Phi_T(z))_{0,j} = \eta(z) Y_{0,j}(z) + \frac{qz\eta^2(z)(1+\rho-\rho qz\eta(z))}{(1-qz\eta(z))(1-\rho qz\eta(z))} X_{0,j}(z).$$

By the definitions of $X(z)$ and $Y(z)$ we easily check that

$$X_{0,j}(z) = Y_{0,j}(z) = (pz\eta(z))^j.$$

So, we obtain

$$(\Phi_T(z))_{0,j} = \frac{p^j z^j \eta^{j+1}(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))}.$$

For $|z| < \frac{1}{2}$, we have $|qz\eta(z)| < 1$ and therefore, using the Cauchy product of two series, we obtain

$$\begin{aligned} (\Phi_T(z))_{0,j} &= \frac{p^j z^j \eta^{j+1}(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))} \\ &= p^j z^j \eta^{j+1}(z) \sum_{n=0}^{\infty} \sum_{k=0}^{n} (\rho qz\eta(z))^k (qz\eta(z))^{n-k} \\ &= \frac{p^j z^j \eta^{j+1}(z)}{1-\rho} \sum_{n=0}^{\infty} (qz\eta(z))^n (1-\rho^{n+1}) \\ &= \frac{p^j}{(1-\rho)q^j} \sum_{n=j}^{\infty} q^n z^n \eta^{n+1}(z)(1-\rho^{n-j+1}). \end{aligned}$$

From Lemma 5, we have

$$\eta^{n+1}(z) = C^{n+1}(pqz^2) = \sum_{k=0}^{\infty} s(n+1,k) p^k q^k z^{2k},$$

which leads, by changing the order of summations, to

$$\begin{aligned}(\Phi_T(z))_{0,j} &= \frac{p^j}{(1-\rho)q^j} \sum_{k=0}^{\infty}\sum_{n=j}^{\infty} s(n+1,k) p^k q^{n+k} z^{n+2k} (1-\rho^{n-j+1}) \\ &= \frac{p^j}{(1-\rho)q^j} \sum_{k=0}^{\infty}\sum_{n=2k+j}^{\infty} s(n-2k+1,k) p^k q^{n-k} z^n (1-\rho^{n-2k-j+1}).\end{aligned}$$

Exchanging the order of summations again, we get

$$(\Phi_T(z))_{0,j} = \frac{p^j}{(1-\rho)q^j} \sum_{n=j}^{\infty} z^n \sum_{k=0}^{\lfloor (n-j)/2 \rfloor} s(n-2k+1,k) p^k q^{n-k} (1-\rho^{n-2k-j+1}),$$

which completes the proof.

Using (3) and Theorem 3, we obtain, for every $j \in \mathbb{N}$ and $x \geq 0$,

$$F_j(x) = \frac{p^j}{q^j} \sum_{n=j}^{\infty} \exp(-\theta x) \frac{(\theta x)^n}{n!} \sum_{k=0}^{\lfloor (n-j)/2 \rfloor} s(n-2k+1,k) p^k q^{n-k} (1-\rho^{n-2k-j+1}), \quad (16)$$

that is,

$$F_j(x) = \frac{p^j}{q^j} \sum_{n=j}^{\infty} \exp(-\theta x) \frac{(\theta x)^n}{n!} \sum_{k=0}^{\lfloor (n-j)/2 \rfloor} \frac{n-2k+1}{n-k+1} \binom{n}{k} p^k q^{n-k} (1-\rho^{n-2k-j+1}).$$

For $i, j \in \mathbb{N}$ and $t \geq 0$, we denote by $P_{i,j}(t, \lambda, \mu)$ the transition probability at time t of the M/M/1 queue with arrival rate λ and service rate μ, that is,

$$P_{i,j}(t,\lambda,\mu) = \Pr\{X_t = j \mid X_0 = i\}.$$

It was shown in [3] that

$$P_{0,j}(t,\lambda,\mu) = \frac{p^j}{q^j} \sum_{n=j}^{\infty} \exp(-(\lambda+\mu)t) \frac{(\lambda+\mu)^n t^n}{n!} \sum_{k=0}^{\lfloor (n-j)/2 \rfloor} s(n-2k+1,k) p^k q^{n-k}, \quad (17)$$

and, in particular, for $j = 0$,

$$P_{0,0}(t,\lambda,\mu) = 1 - \frac{\lambda}{\mu} \sum_{n=1}^{\infty} \exp(-(\lambda+\mu)t) \frac{(\lambda+\mu)^n t^n}{n!} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{2k}{k} \frac{p^{k+1} q^k}{k+1}.$$

In the following corollaries, we show that the distribution of the stationary buffer content of the fluid queue can be expressed as a function of the transient behaviour of the M/M/1 queue.

Corollary 1. *For every $j \in \mathbb{N}$ and $x \geq 0$, we have*

$$F_j(x) = P_{0,j}\left(\frac{x}{r-c}, \lambda, \mu\right) - \rho^{j+1} \exp\left(-\left(\frac{\mu}{r} - \frac{\lambda}{c}\right)x\right) P_{0,j}\left(\frac{x}{r-c}, \frac{\mu c}{r}, \frac{\lambda r}{c}\right).$$

Proof. The proof is immediate from (16) and (17).

Corollary 2. *For every $x \geq 0$, we have*

$$\Pr\{Q \leq x\} = \frac{rF_0(x)}{r-c} - \frac{c(1-\rho)}{r-c}.$$

Proof. Consider the differential system (1). By summing over index j, we get

$$-cF_0'(x) + (r-c)\sum_{j=1}^{\infty} F_j'(x) = 0.$$

Integrating from 0 to x, we obtain

$$(r-c)\sum_{j=1}^{\infty}(F_j(x) - F_j(0)) = c(F_0(x) - F_0(0)).$$

Since $F_j(0) = 0$ for $j \geq 1$ and $F_0(0) = 1 - \rho$, we have

$$\Pr\{Q \leq x\} = \sum_{j=0}^{\infty} F_j(x) = \frac{1}{r-c}(rF_0(x) - c(1-\rho)).$$

The result follows by using (16) for $j = 0$.

5. Finite buffer case

We consider in this section the case where the capacity B of the buffer is finite. We denote by Q_B the stationary fluid buffer level and we define $F_{j,B}(x) = \Pr\{X = j, Q_B \leq x\}$. The following result gives an interesting relation between $F_{j,B}(x)$ and $F_j(x)$.

Theorem 4. *If $\rho < 1$, we have for every $j \in \mathbb{N}$ and $0 \leq x < B$,*

$$F_{j,B}(x) = (1 - \lambda/\mu)\frac{F_j(x)}{F_0(B)}.$$

Proof. See [6].

Using this result and Corollary 2, we easily obtain that, if $\rho < 1$ and $0 \leq x < B$,

$$\Pr\{Q_B \leq x\} = \frac{(1-\lambda/\mu)(rF_0(x) - c(1-\rho))}{(r-c)F_0(B)}.$$

The overflow probability is thus given by

$$\Pr\{Q_B = B\} = 1 - \Pr\{Q_B < B\} = \frac{c(1-\rho)}{r-c}\left(\frac{1-\lambda/\mu}{F_0(B)} - 1\right).$$

References

[1] ADAN, I. AND RESING, J. (1996). Simple analysis of a fluid queue driven by an M/M/1 queue. *Queueing Systems* **22,** 171–174.

[2] BARBOT, N. AND SERICOLA, B. (2001). Transient analysis of a fluid queue driven by an M/M/1 queue. In *Proc. 9th Internat. Conf. Telecommun. Systems: Modeling and Analysis* (ICTS'9, Dallas, TX), pp. 399–411.

[3] LEGUESDRON, P., PELLAUMAIL, J., RUBINO, G. AND SERICOLA, B. (1993). Transient analysis of the M/M/1 queue. *Adv. Appl. Prob.* **25,** 702–713.

[4] MITRA, D. (1988). Stochastic theory of a fluid model of producers and consumers coupled by a buffer. *Adv. Appl. Prob.* **20,** 646–676.

[5] RIORDAN, J. (1968). *Combinatorial Identities.* John Wiley, New York.

[6] SERICOLA, B. (2001). A finite buffer fluid queue driven by a Markovian queue. *Queueing Systems* **38,** 213–220.

[7] SERICOLA, B. AND TUFFIN, B. (1999). A fluid queue driven by a Markovian queue. *Queueing Systems* **31,** 253–264.

[8] STERN, T. E. AND ELWALID, A. I. (1991). Analysis of separable Markov-modulated rate models for information-handling systems. *Adv. Appl. Prob.* **23,** 105–139.

[9] VAN DOORN, E. A. AND SCHEINHARDT, W. R. W. (1997). A fluid queue driven by an infinite-state birth–death process. In *Proc. 15th Internat. Teletraffic Cong.* (ITC'15, Washington, DC), eds V. Ramaswami and P. E. Wirth, Elsevier, Amsterdam, pp. 465–475.

[10] VIRTAMO, J. AND NORROS, I. (1994). Fluid queue driven by an M/M/1 queue. *Queueing Systems* **16,** 373–386.