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# A Finite Buffer Fluid Queue Driven by a Markovian Queue

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**Abstract.** We consider a finite buffer fluid queue receiving its input from the output of a Markovian queue with finite or infinite waiting room. The input flow into the fluid queue is thus characterized by a Markov modulated input rate process and we derive, for a wide class of such input processes, a procedure for the computation of the stationary buffer content of the fluid queue and the stationary overflow probability. This approach leads to a numerically stable algorithm for which the precision of the result can be specified in advance.

**Keywords:** fluid queue, Markovian queue, Markov process, overflow probability

## 1. Introduction

In performance evaluation of telecommunication and computer systems, fluid queues models with Markov modulated input rates have been widely used in many papers, see among others [1,3,7,9,10]. The traffic arriving to a network queue has already traversed parts of the network and has been modified along its traversal. In such cases, it is the output from a queue which forms the input to the next network element.

In the most important part of the literature on this subject, see, for instance, [3,7] and the references therein, the state space of the Markov process that modulates the input rate in the fluid queue is supposed to be finite. The case where this state space is infinite has been analyzed in [1,10] for the $M/M/1$ queue, in [9] for a birth and death process, and in [6] for a general Markovian queue with the condition that only one state must have its input rate smaller than its output rate. In all these papers dealing with an infinite state space, the capacity of fluid queue has always been supposed to be infinite.

In this paper, we consider the problem of a finite buffer fluid queue driven by a Markovian queue. As in [6], the only requirement needed on the Markov process that modulates the input and output rates is that it has a single state such that the input rate is smaller than the output rate of the fluid queue and that it has a uniform infinitesimal generator, that is, the supremum of the output rates of the states is bounded. These Markov processes include not only the well-known $M/M/1/L$, $M/M/K/L$, $M/PH/1/L$ and $M/PH/K/L$ queues with finite ($L < \infty$) or infinite ($L = \infty$) waiting room but also the superposition of on–off sources with exponential off periods and phase-type on periods.

Nevertheless, our method can not be used if the Markov process that modulates the input and output rates in the fluid queue has more than one state with a negative effective input rate. This is the case if the Markovian queue is for instance the $PH/PH/K$ queue with non-Poisson arrivals. In this queue the number of states corresponding to 0 customer in the queue is equal to the number of phases of the arrival process and this number is at least equal to 2 if we suppose that the arrival process is not a Poisson process.

The method used here to obtain the distribution of the stationary buffer content of the fluid queue and so the stationary overflow probability is neither based on spectral analysis nor on the use of Bessel functions as done in [1,9,10], but, as in [6], a direct approach is used which leads to simple recursions. This method is particularly interesting by the fact that it uses only additions and multiplications of positive numbers bounded by one. Thus we obtain a stable algorithm which moreover gives the result with a precision that can be specified in advance.

The rest of the paper is organized as follows. In the next section, we describe the model and the notation used. Then, in a first subsection, we briefly recall the results obtained in [6] which correspond to the infinite buffer case. The finite buffer case is considered in a second subsection where we obtain the distribution of the stationary buffer content as a function of the same distribution in the infinite buffer case. In section 3, we present numerical illustrations of these results and we compare the finite and infinite buffer cases.

## 2. Model description and notation

We consider a fluid queueing model with a buffer of capacity $B$, with $B \leqslant \infty$, for which the input and output rates are controlled by a homogeneous Markov process $\{X_t,\ t \geqslant 0\}$ on the state space $S$ with infinitesimal generator denoted by $A$ and stationary probability distribution denoted by $\pi = (\pi_i)$.

Let $r_i$ be the input rate and $c_i$ be the output rate when the Markov process $\{X_t\}$ is in state $i$. We denote by $\theta_i$ the effective input rate of state $i$, that is, $\theta_i = r_i - c_i$. We suppose that for every $i \in S$ we have $\theta_i \neq 0$. It is well known that the case where $\theta_i = 0$ for some $i$ can be reduced to this one (see, for instance, [6]).

We assume in this paper that the state space $S$ contains only one state with negative effective input rate. This state is denoted by 0 and thus we have $S = \{0\} \cup S^+$ with $\theta_0 < 0$ and $\theta_i > 0$ for $i \in S^+$. It is, moreover, assumed that $\inf\{\theta_i \mid \theta_i > 0\} > 0$.

We denote by $\rho$ the traffic intensity which is given by

$$\rho = \frac{\sum_{i \in S} r_i \pi_i}{\sum_{i \in S} c_i \pi_i}.$$

We will suppose that the stability condition $\rho < 1$ is satisfied, so that the limiting behavior of the buffer content exists when $B = \infty$. We denote by $X$ the stationary state of the Markov process $\{X_t\}$ and by $Q_B$ (respectively $Q$) the stationary amount of fluid in the buffer when $B < \infty$ (respectively $B = \infty$).

Let $F_j(x) = \Pr\{X = j,\ Q \leqslant x\}$ and $F_{j,B}(x) = \Pr\{X = j,\ Q_B \leqslant x\}$. We then have the following differential equations, see, for instance, [3], for all $j \in S$

$$\theta_j \frac{\mathrm{d}F_j(x)}{\mathrm{d}x} = \sum_{i \in S} F_i(x)A(i, j) \qquad \text{for } x > 0, \tag{1}$$

$$\theta_j \frac{\mathrm{d}F_{j,B}(x)}{\mathrm{d}x} = \sum_{i \in S} F_{i,B}(x)A(i, j) \quad \text{for } 0 < x < B. \tag{2}$$

The initial conditions are given by $F_j(0) = F_{j,B}(0) = 0$ for every $j \in S^+$ and we also have $F_{0,B}(B^-) = \pi_0$.

We assume that $\sup\{-A(i,i),\ i \in S\}$ is finite and we denote by $P$ the transition probability matrix of the uniformized Markov chain [5] with respect to the uniformization rate $\lambda$ which verifies $\lambda \geqslant \sup\{-A(i,i),\ i \in S\}$. The matrix $P$ is then related to $A$ by $P = I + A/\lambda$, where $I$ denotes the identity matrix.

### *2.1. The infinite buffer case*

We consider in this subsection the case where $B = \infty$. We recall in the following propositions the main results that have been obtained in [6]. We suppose that the stability condition $\rho < 1$ is satisfied.

**Proposition 2.1.** For every $j \in S$,

$$F_j(x) = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda x/\theta} \frac{(\lambda x/\theta)^n}{n!} b_j(n), \tag{3}$$

where $\theta = \inf\{\theta_i \mid \theta_i > 0\}$ and the coefficients $b_j(n)$ are given by the following recursive expression:

$$b_0(0) = \frac{\sum_{j \in S} \theta_j \pi_j}{\theta_0} = (1-\rho)\frac{\sum_{j \in S} c_j \pi_j}{-\theta_0} \quad \text{and} \quad b_j(0) = 0 \quad \text{for } j \in S^+,$$

and for $n \geqslant 1$ and $j \in S$,

$$b_j(n) = \left(1 - \frac{\theta}{\theta_j}\right) b_j(n-1) + \frac{\theta}{\theta_j} \sum_{i \in S} b_i(n-1)P(i, j). \tag{4}$$

**Proposition 2.2.** For every $n \geqslant 0$,

$$b_0(n) = b_0(0) + \frac{\sum_{j \in S^+} \theta_j b_j(n)}{-\theta_0}. \tag{5}$$

**Proposition 2.3.** For every $j \in S$, the sequence $b_j(n)$ is nonnegative, increasing and converges to $\pi_j$.

The algorithm for the computation of the stationary buffer content uses these three propositions and is presented in [6].

### 2.2. *The finite buffer case*

We consider in this subsection the case where the capacity $B$ of the fluid queue is finite. In the case where the stability condition $\rho < 1$ is satisfied, the finite buffer case is related to the infinite buffer case by the following theorem.

**Theorem 2.4.** If $\rho < 1$, then

$$F_{j,B}(x) = \pi_0 \frac{F_j(x)}{F_0(B)}, \quad j \in S, \ x \in [0, B). \tag{6}$$

*Proof.* Since $F_j(x)$ satisfies equation (1), the function $F_{j,B}(x) = \pi_0 F_j(x)/F_0(B)$ satisfies equation (2). The initial conditions for $j = 0$, $x = B^-$ and $j \in S^+$, $x = 0$ being also satisfied, this completes the proof. □

Let us consider the functions $F_B(x) = \Pr\{Q_B \leqslant x)$ and $F(x) = \Pr\{Q \leqslant x\}$ for $x \in [0, B)$. We then have

$$F(x) = \sum_{j \in S} F_j(x) = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda x/\theta} \frac{(\lambda x/\theta)^n}{n!} b(n), \tag{7}$$

where $b(n) = \sum_{j \in S} b_j(n)$. The sequence $b(n)$ increases and converges to 1 when $n$ goes to infinity. By summing over $j$ in relation (6), we get

$$F_B(x) = \pi_0 \frac{F(x)}{F_0(B)}.$$

The computation of $F_B(x)$ can then be done as follows. We denote respectively by $F(N, x)$ and $F_0(N, B)$ the sums of the $N + 1$ first terms of relations (7) and (3), that is,

$$F(N, x) = \sum_{n=0}^{N} \mathrm{e}^{-\lambda x/\theta} \frac{(\lambda x/\theta)^n}{n!} b(n) \quad \text{and} \quad F_0(N, B) = \sum_{n=0}^{N} \mathrm{e}^{-\lambda B/\theta} \frac{(\lambda B/\theta)^n}{n!} b_0(n).$$

We then have

$$F(x) = F(N, x) + e(N) \quad \text{and} \quad F_0(B) = F_0(N, B) + e_0(N),$$

where $e(N)$ and $e_0(N)$ are the rests of the series $F(x)$ and $F_0(B)$, respectively. Let $\varepsilon'$ be a given error tolerance, we define integer $N$ as

$$N = \min\left\{ n \in \mathbb{N} \;\middle|\; \sum_{i=0}^{n} \mathrm{e}^{-\lambda B/\theta} \frac{(\lambda B/\theta)^i}{i!} \geqslant 1 - \varepsilon' \right\}. \tag{8}$$

We thus have

$$
\begin{aligned}
e(N) &= \sum_{n=N+1}^{\infty} \mathrm{e}^{-\lambda x/\theta}\,\frac{(\lambda x/\theta)^n}{n!}\,b(n) \leqslant \sum_{n=N+1}^{\infty} \mathrm{e}^{-\lambda x/\theta}\,\frac{(\lambda x/\theta)^n}{n!} \\
&\leqslant \sum_{n=N+1}^{\infty} \mathrm{e}^{-\lambda B/\theta}\,\frac{(\lambda B/\theta)^n}{n!} \leqslant \varepsilon',
\end{aligned}
$$

and

$$
e_0(N) = \sum_{n=N+1}^{\infty} \mathrm{e}^{-\lambda B/\theta}\,\frac{(\lambda B/\theta)^n}{n!}\,b_0(n) \leqslant \pi_0 \sum_{n=N+1}^{\infty} \mathrm{e}^{-\lambda B/\theta}\,\frac{(\lambda B/\theta)^n}{n!} \leqslant \pi_0\varepsilon'.
$$

**Proposition 2.5.** For each $x \in [0, B)$,

$$
\left| F_B(x) - \pi_0\,\frac{F(N,x)}{F_0(N,B)} \right| \leqslant \frac{\pi_0^2\varepsilon'}{b_0(0)^2(1-\varepsilon')}.
$$

*Proof.*

$$
\begin{aligned}
\left| F_B(x) - \pi_0\,\frac{F(N,x)}{F_0(N,B)} \right| &= \pi_0 \left| \frac{F(x)}{F_0(B)} - \frac{F(N,x)}{F_0(N,B)} \right| \\
&= \pi_0\,\frac{|F(x)F_0(N,B) - F(N,x)F_0(B)|}{F_0(B)F_0(N,B)} \\
&= \pi_0\frac{|F_0(N,B)e(N) - F(N,x)e_0(N)|}{F_0(B)F_0(N,B)}.
\end{aligned}
$$

The numerator of the fraction is less or equal than $\pi_0\varepsilon'$, since $F_0(N,B) \leqslant \pi_0$, $e(N) \leqslant \varepsilon'$ and $F(N,x) \leqslant 1$, $e_0(N) \leqslant \pi_0\varepsilon'$. For the denominator, the sequence $b_0(n)$ being increasing, we have

$$
F_0(B) = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda B/\theta}\,\frac{(\lambda B/\theta)^n}{n!}\,b_0(n) \geqslant b_0(0),
$$

and

$$
F_0(N,B) = \sum_{n=0}^{N} \mathrm{e}^{-\lambda B/\theta}\,\frac{(\lambda B/\theta)^n}{n!}\,b_0(n) \geqslant b_0(0) \sum_{n=0}^{N} \mathrm{e}^{-\lambda B/\theta}\,\frac{(\lambda B/\theta)^n}{n!} \geqslant b_0(0)\big(1-\varepsilon'\big),
$$

which completes the proof. □

This proposition allows us to compute the distribution $F_B(x)$ with an error $\varepsilon$ than can be specified in advance. To do that, it suffices to take

$$
\varepsilon' = \frac{b_0(0)^2\varepsilon}{b_0(0)^2\varepsilon + \pi_0^2}.
$$

Moreover, the computation can be easily done by using the algorithm described in [6] to compute the quantities $F(N, x)$ and $F_0(N, B)$.

## 3. Numerical results

As shown in [6], our method applies for a wide class of Markov modulated input rate processes with a high precision on the numerical results. We consider here a fluid queue driven by an $M/M/1$ queue and we compare the distribution of the stationary buffer content for both finite and infinite buffer fluid queues. The arrival rate of the $M/M/1$ queue is $\beta = 0.45$, its service rate is $\gamma = 1$. The random variable $X$ represents the

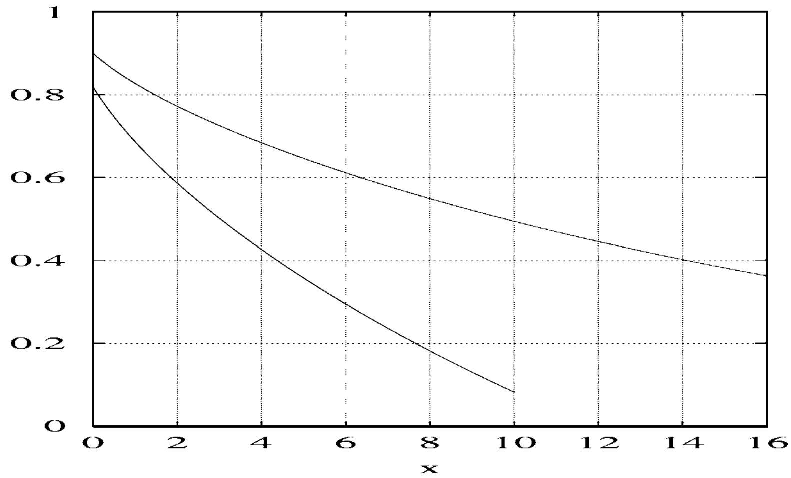


Figure 1. From top to the bottom: $\Pr\{Q > x\}$ and $\Pr\{Q_{10} > x\}$ versus $x$.

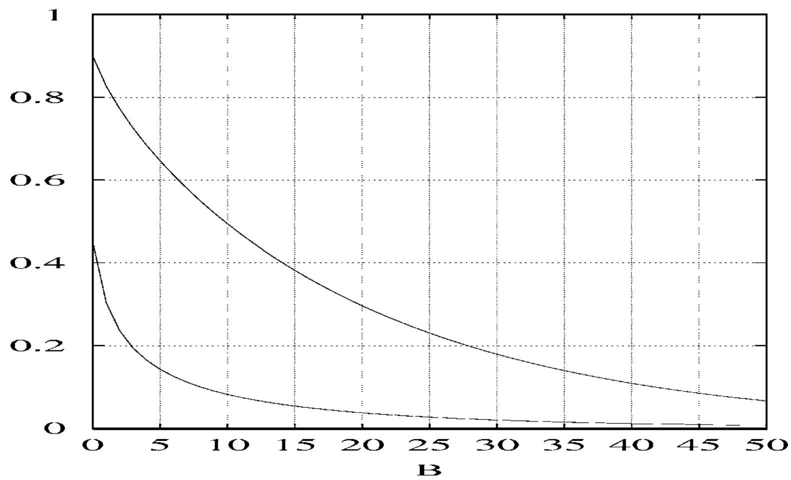


Figure 2. From top to the bottom: $\Pr\{Q > B\}$ and $\Pr\{Q_B = B\}$ versus $B$ for small values of $B$.

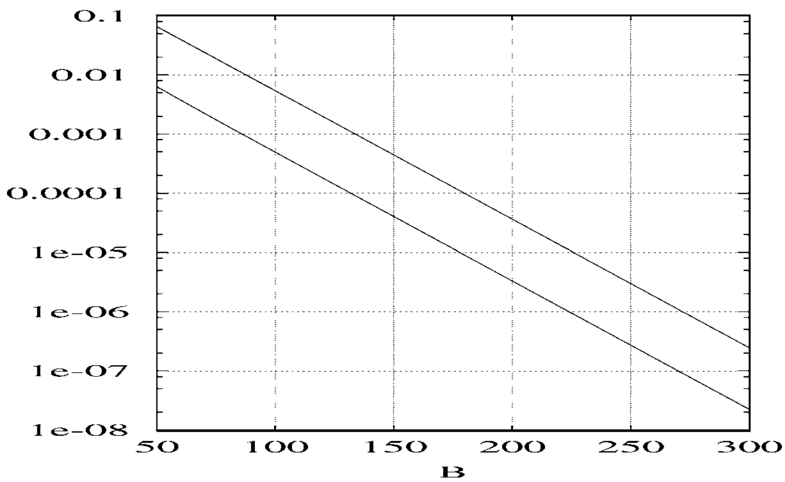


Figure 3. From top to the bottom: $\Pr\{Q > B\}$ and $\Pr\{Q_B = B\}$ versus $B$ for large values of $B$.

stationary number of customers in the $M/M/1$ queue, and so we have $\Pr\{X = i\} = \pi_i = (1 - \beta/\gamma)(\beta/\gamma)^i$, for every $i \geqslant 0$. The input rate in the fluid queue is given by $r_0 = 0$ and $r_i = 2$ for $i \geqslant 1$, and the output is supposed to be constant given by $c_i = 1$ for every $i \geqslant 0$. The effective input rate is thus $\theta_0 = -1$ and $\theta_i = 1$ for every $i \geqslant 1$. These values lead to the traffic intensity in the fluid queue $\rho = 0.9$.

Figure 1 shows the stationary buffer content of the fluid queue in both cases where the buffer of the fluid queue is of infinite capacity and when it is of capacity $B = 10$. Note that for $x \geqslant 10$, we have $\Pr\{Q_{10} > x\} = 0$. This jump corresponds to the overflow probability $\Pr\{Q_{10} = 10\}$. In this figure, the precision of the results is $\varepsilon = 10^{-5}$.

Figures 2 and 3 show the stationary buffer content of the infinite buffer fluid queue and the corresponding overflow probability in the finite buffer fluid queue. The precision of the results is $\varepsilon = 10^{-5}$ in figure 2 and $\varepsilon = 10^{-10}$ in figure 3.

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