

## OCCUPATION TIMES IN MARKOV PROCESSES

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### ABSTRACT

In a homogeneous finite-state Markov process, we consider the occupation times, that is, the times spent by the process in given subsets of the state space during a finite interval of time. We first derive the distribution of the occupation time of one subset and then we generalize that result to the joint distribution of occupation times of different subsets of the state space by the use of order statistics from the uniform distribution. Next, we consider the distribution of weighted sums of occupation times. We obtain forward and backward equations describing the behavior of these weighted sums and we show how these lead to simple expressions for that distribution.

### 1 INTRODUCTION

Let  $X = \{X_u, u \geq 0\}$  be a homogeneous Markov process with finite state space  $S$ . The occupation time of a subset  $U \subset S$  over  $[0, t)$  is defined as the random variable

$$W_t = \int_0^t 1_{\{X_u \in U\}} du,$$

where  $1_{\{c\}} = 1$ , if condition  $c$  holds and 0 otherwise. That random variable has drawn much attention as it is also known as the interval availability in

reliability and dependability theory. In [2] an expression for the distribution of  $W_t$  was obtained using uniform order statistics on  $[0, t)$ . From a computational point of view that expression is very interesting; various methods were developed to compute it even in the case of denumerable state spaces (see [2], [10], [8], [9] and the references therein).

We first recall how the joint distribution of the pair  $(W_t, X_t)$  was obtained in [2] by using the forward and backward equations associated with the uniformized Markov chain of the process  $X$ . We then generalize that result to the joint distribution of  $W_t^1, \dots, W_t^m, X_t$ , where  $W_t^i$  is the occupation time of a subset  $B_i$  over the interval  $[0, t)$ . Finally, we consider a weighted sum of occupation times, that is the random variable  $Y_t$  defined by

$$Y_t = \int_0^t \rho(X_u) du,$$

where for each  $i \in S$ ,  $\rho(i)$  is a nonnegative constant. The quantity  $Y_t$  arises in the performability analysis in reliability and dependability theory (see [3], [5] and the references therein). Here we derive backward and forward equations describing the behavior of the joint distribution of  $(Y_t, X_t)$ . These partial differential equations are then solved. We show that they lead to simple expressions for the joint distribution of  $(Y_t, X_t)$ .

The remainder of the paper is organized as follows. In the next section, we consider the joint distribution of uniform order statistics and the joint conditional distribution of the jumps in a Poisson process and we recall how they are related. In Section 3, we consider the case  $m = 1$ , to obtain the distribution of occupation time for a discrete time Markov chain. That distribution, combined with the results of Section 2, leads to a simple expression for the joint distribution of the pair  $(W_t, X_t)$ . In Section 4, the results of Section 3 are generalized to the case  $m > 1$ . Finally, Section 5 deals with the distribution of the pair  $(Y_t, X_t)$ .

## 2 ORDER STATISTICS

### 2.1 The Uniform Distribution

We consider the order statistics of uniform random variables on  $[0, t)$ . Formally, let  $X_1, X_2, \dots, X_n$  be  $n$  i.i.d. uniform random variables on  $[0, t)$ . If the random variables  $X_1, X_2, \dots, X_n$  are rearranged in ascending order of magnitude and written as

$$X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)},$$

we call  $X_{(i)}$  the  $i$ th order statistic,  $i = 1, 2, \dots, n$ .

Let  $F_r(x)$  be the distribution of  $X_{(r)}$ . We know that (see for instance [1]), for  $x \in (0, t)$ ,

$$F_r(x) = \mathbb{P}\{X_{(r)} \leq x\} = \sum_{i=r}^n \binom{n}{i} \left(\frac{x}{t}\right)^i \left(1 - \frac{x}{t}\right)^{n-i} \quad (1)$$

In [1] it is shown that the joint density of  $X_{(l_1)}, X_{(l_1+l_2)}, \dots, X_{(l_1+l_2+\dots+l_k)}$  is given, for  $1 \leq k \leq n$ ,  $1 \leq l_1 + l_2 + \dots + l_k \leq n$  ( $l_i \geq 1$ ), and  $x_1 \leq x_2 \leq \dots \leq x_k$  ( $x_i \in (0, t)$ ), by

$$g_{l_1, l_2, \dots, l_k}(x_1, x_2, \dots, x_k) = \frac{n! \left(\frac{1}{t}\right)^k \left(\frac{x_1}{t}\right)^{l_1-1} \left(\frac{x_2 - x_1}{t}\right)^{l_2-1} \dots \left(\frac{x_k - x_{k-1}}{t}\right)^{l_k-1} \left(1 - \frac{x_k}{t}\right)^{n-(l_1+l_2+\dots+l_k)}}{(l_1-1)!(l_2-1)!\dots(l_k-1)!(n-(l_1+l_2+\dots+l_k))!}.$$

Furthermore, if for fixed  $1 \leq k \leq n$  and  $1 \leq l_1 + l_2 + \dots + l_k \leq n$  ( $l_i \geq 1$ ), we define  $Y_1 = X_{(l_1)}$  and  $Y_i = X_{(l_1+l_2+\dots+l_i)} - X_{(l_1+l_2+\dots+l_{i-1})}$  for  $i = 2, \dots, k$ , then the joint density function of  $Y_1, Y_2, \dots, Y_k$  is given, for  $0 < s_1 + s_2 + \dots + s_k < t$  ( $s_i \in (0, t)$ ), by

$$h_{l_1, l_2, \dots, l_k}(s_1, s_2, \dots, s_k) = g_{l_1, l_2, \dots, l_k}(s_1, s_1 + s_2, \dots, s_1 + \dots + s_k),$$

that is,

$$h_{l_1, l_2, \dots, l_k}(s_1, s_2, \dots, s_k) = \frac{n! \left(\frac{1}{t}\right)^k \left(\frac{s_1}{t}\right)^{l_1-1} \left(\frac{s_2}{t}\right)^{l_2-1} \dots \left(\frac{s_k}{t}\right)^{l_k-1} \left(1 - \frac{s_1 + s_2 + \dots + s_k}{t}\right)^{n-(l_1+l_2+\dots+l_k)}}{(l_1-1)!(l_2-1)! \dots (l_k-1)!(n-(l_1+l_2+\dots+l_k))!}.$$

In particular, for  $k = n$ , we get the joint density function of the spacings  $Y_1 = X_{(1)}, Y_2 = X_{(2)} - X_{(1)}, \dots, Y_n = X_{(n)} - X_{(n-1)}$ , denoted by  $h(x_1, x_2, \dots, x_n)$  by setting  $l_1 = l_2 = \dots = l_n = 1$  in the preceding expression, that is

$$h(x_1, x_2, \dots, x_n) = \begin{cases} \frac{n!}{t^n} & \text{if } \sum_{i=1}^n x_i \leq t \\ 0 & \text{otherwise.} \end{cases}$$

By writing  $Y_{n+1} = t - X_{(n)}$ , this also determines the (degenerate) joint density of  $Y_1, Y_2, \dots, Y_n, Y_{n+1}$  on the set

$$x_i \geq 0 \quad (i = 1, \dots, n, n+1) \quad \sum_{i=1}^{n+1} x_i = t.$$

The joint distribution  $H_{l_1, l_2, \dots, l_k}(s_1, s_2, \dots, s_k)$  of  $Y_{l_1}, Y_{l_2}, \dots, Y_{l_k}$  is given in the following lemma.

**Lemma 2.1** *For  $1 \leq k \leq n$ ,  $1 \leq l_1 + l_2 + \dots + l_k \leq n$  ( $l_i \geq 1$ ), and  $0 < s_1 + s_2 + \dots + s_k < t$  ( $s_i \in (0, t)$ ), we have that*

$$H_{l_1, l_2, \dots, l_k}(s_1, s_2, \dots, s_k) = \sum_{\substack{i_1 \geq l_1, i_2 \geq l_2, \dots, i_k \geq l_k, \\ i_1 + i_2 + \dots + i_k \leq n}} \frac{n! \left(\frac{s_1}{t}\right)^{i_1} \left(\frac{s_2}{t}\right)^{i_2} \dots \left(\frac{s_k}{t}\right)^{i_k} \left(1 - \frac{s_1 + s_2 + \dots + s_k}{t}\right)^{n-(i_1+i_2+\dots+i_k)}}{i_1! i_2! \dots i_k! (n - (i_1 + i_2 + \dots + i_k))!}.$$

**Proof.** It suffices to show that

$$\frac{\partial^k H_{l_1, l_2, \dots, l_k}(s_1, s_2, \dots, s_k)}{\partial s_1 \partial s_2 \dots \partial s_k} = h_{l_1, l_2, \dots, l_k}(s_1, s_2, \dots, s_k).$$

To simplify notation, we define

$$\theta_{n; i_1, i_2, \dots, i_k}^{t; s_1, s_2, \dots, s_k} = \frac{\left(\frac{s_1}{t}\right)^{i_1} \left(\frac{s_2}{t}\right)^{i_2} \dots \left(\frac{s_k}{t}\right)^{i_k} \left(1 - \frac{s_1 + s_2 + \dots + s_k}{t}\right)^{n-(i_1+i_2+\dots+i_k)}}{i_1! i_2! \dots i_k! (n - (i_1 + i_2 + \dots + i_k))!}.$$

We obtain

$$\begin{aligned}
 \frac{\partial H_{l_1, l_2, \dots, l_k}(s_1, s_2, \dots, s_k)}{\partial s_k} &= \frac{n!}{t} \sum_{\substack{i_1 \geq l_1, i_2 \geq l_2, \dots, i_k \geq l_k, \\ i_1 + i_2 + \dots + i_k \leq n}} \theta_{n; i_1, i_2, \dots, i_{k-1}, i_k-1}^{t; s_1, s_2, \dots, s_k} \\
 &\quad - \frac{n!}{t} \sum_{\substack{i_1 \geq l_1, i_2 \geq l_2, \dots, i_k \geq l_k, \\ i_1 + i_2 + \dots + i_k \leq n-1}} \theta_{n-1; i_1, i_2, \dots, i_k}^{t; s_1, s_2, \dots, s_k} \\
 &= \frac{n!}{t} \sum_{\substack{i_1 \geq l_1, i_2 \geq l_2, \dots, i_{k-1} \geq l_{k-1}, i_k \geq l_k-1, \\ i_1 + i_2 + \dots + i_k \leq n-1}} \theta_{n-1; i_1, i_2, \dots, i_k}^{t; s_1, s_2, \dots, s_k} \\
 &\quad - \frac{n!}{t} \sum_{\substack{i_1 \geq l_1, i_2 \geq l_2, \dots, i_k \geq l_k, \\ i_1 + i_2 + \dots + i_k \leq n-1}} \theta_{n-1; i_1, i_2, \dots, i_k}^{t; s_1, s_2, \dots, s_k} \\
 &= \frac{n!}{t} \sum_{\substack{i_1 \geq l_1, i_2 \geq l_2, \dots, i_{k-1} \geq l_{k-1}, \\ i_1 + i_2 + \dots + i_{k-1} \leq n-1}} \theta_{n-1; i_1, i_2, \dots, i_{k-1}, l_k-1}^{t; s_1, s_2, \dots, s_k}
 \end{aligned}$$

where the second equality is obtained by the change of variable  $i_k \rightarrow i_k + 1$ . By successively iterating the same argument with respect to variables  $s_{k-1}, \dots, s_2$ , we obtain

$$\frac{\partial^{k-1} H_{l_1, l_2, \dots, l_k}(s_1, s_2, \dots, s_k)}{\partial s_k \partial s_{k-1} \dots \partial s_2} = \frac{n!}{t^{k-1}} \sum_{i_1=l_1}^{n-k+1} \theta_{n-k+1; i_1, l_2-1, \dots, l_{k-1}-1, l_k-1}^{t; s_1, s_2, \dots, s_k}$$

and finally,

$$\begin{aligned}
 &\frac{\partial^k H_{l_1, l_2, \dots, l_k}(s_1, s_2, \dots, s_k)}{\partial s_k \dots \partial s_2 \partial s_1} \\
 &= \frac{n!}{t^k} \sum_{i_1=l_1}^{n-k+1} \theta_{n-k+1; i_1-1, l_2-1, \dots, l_k-1}^{t; s_1, s_2, \dots, s_k} - \frac{n!}{t^k} \sum_{i_1=l_1}^{n-k} \theta_{n-k; i_1, l_2-1, \dots, l_k-1}^{t; s_1, s_2, \dots, s_k} \\
 &= \frac{n!}{t^k} \sum_{i_1=l_1-1}^{n-k} \theta_{n-k; i_1, l_2-1, \dots, l_k-1}^{t; s_1, s_2, \dots, s_k} - \frac{n!}{t^k} \sum_{i_1=l_1}^{n-k} \theta_{n-k; i_1, l_2-1, \dots, l_k-1}^{t; s_1, s_2, \dots, s_k} \\
 &= \frac{n!}{t^k} \theta_{n-k; l_1-1, l_2-1, \dots, l_k-1}^{t; s_1, s_2, \dots, s_k} \\
 &= h_{l_1, l_2, \dots, l_k}(s_1, s_2, \dots, s_k),
 \end{aligned}$$

where the second equality is obtained by the change of variable  $i_1 \rightarrow i_1 + 1$ . ■

### 2.2 The Poisson Process

Let  $\{N_t, t \in \mathbb{R}\}$  be a Poisson process of rate  $\lambda$  and  $T_0, T_0 + T_1, \dots, T_0 + T_1 + \dots + T_{n-1}$ , be the first  $n$  instants of jumps of  $\{N_t\}$  in  $[0, t)$ . It is well-known, see [4], that the density of the conditional distribution of  $T_0, T_1, \dots, T_{n-1}$ , given  $\{N_t = n\}$  is

$$f(x_0, x_1, \dots, x_{n-1}) = \begin{cases} \frac{n!}{t^n} & \text{if } \sum_{i=0}^{n-1} x_i \leq t \\ 0 & \text{otherwise.} \end{cases}$$

That is also, as seen in the previous subsection, the joint density of the order statistics from the uniform distribution on  $(0, t)$ .

If we write  $T_n = t - (T_0 + T_1 + \dots + T_{n-1})$ , this also determines the (degenerate) joint density function of  $T_0, T_1, \dots, T_{n-1}, T_n$  on the set

$$x_i \geq 0 \quad (i = 0, \dots, n - 1, n) \quad \sum_{i=0}^n x_i = t.$$

The symmetric role of the variables  $x_0, x_1, \dots, x_{n-1}, x_n$  shows that the random variables  $T_0, T_1, \dots, T_{n-1}$  are exchangeable. It follows from relation (1) that, for  $1 \leq l \leq n$ ,  $\{i_1, \dots, i_l\} \subset \{0, 1, \dots, n\}$ , and  $s \in (0, t)$ , we have

$$\begin{aligned} \mathbb{P}\{T_{i_1} + \dots + T_{i_l} \leq s \mid N_t = n\} &= \mathbb{P}\{T_0 + \dots + T_{l-1} \leq s \mid N_t = n\} \\ &= \mathbb{P}\{X_{(l)} \leq s\} \\ &= \sum_{k=l}^n \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}. \end{aligned} \tag{2}$$

More generally, let  $k$  be an integer such that  $1 \leq k \leq n$  and let  $l_1, l_2, \dots, l_k$  be integers such that  $1 \leq l_1 + l_2 + \dots + l_k \leq n$  ( $l_i \geq 1$ ). For any subset  $\{i_1, i_2, \dots, i_{l_1+l_2+\dots+l_k}\}$  of  $\{0, 1, \dots, n - 1, n\}$ , the vectors

$$\left( \sum_{j=1}^{l_1} T_{i_j}, \sum_{j=l_1+1}^{l_1+l_2} T_{i_j}, \dots, \sum_{j=l_1+l_2+\dots+l_{k-1}+1}^{l_1+l_2+\dots+l_k} T_{i_j} \right)$$

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and

$$\left( \sum_{j=0}^{l_1-1} T_j, \sum_{j=l_1}^{l_1+l_2-1} T_j, \dots, \sum_{j=l_1+l_2+\dots+l_{k-1}}^{l_1+l_2+\dots+l_{k-1}} T_j \right)$$

have the same conditional distribution given that  $N_t = n$ . By lemma 2.1, for  $0 < s_1 + s_2 + \dots + s_k < t$  ( $s_i \in (0, t)$ ), we thus get

$$\begin{aligned} \mathbb{P}\left\{ \sum_{j=1}^{l_1} T_j \leq s_1, \sum_{j=l_1+1}^{l_1+l_2} T_j \leq s_2, \dots, \sum_{j=l_1+l_2+\dots+l_{k-1}+1}^{l_1+l_2+\dots+l_k} T_j \leq s_k \mid N_t = n \right\} = \\ \sum_{\substack{i_1 \geq l_1, i_2 \geq l_2, \dots, i_k \geq l_k, \\ i_1 + i_2 + \dots + i_k \leq n}} \frac{n! \left(\frac{s_1}{t}\right)^{i_1} \left(\frac{s_2}{t}\right)^{i_2} \dots \left(\frac{s_k}{t}\right)^{i_k} \left(1 - \frac{s_1 + s_2 + \dots + s_k}{t}\right)^{n-(i_1+i_2+\dots+i_k)}}{i_1! i_2! \dots i_k! (n - (i_1 + i_2 + \dots + i_k))!}. \end{aligned} \tag{3}$$

### 3 DISTRIBUTION OF OCCUPATION TIMES

Let  $X = \{X_u, u \geq 0\}$  be a homogeneous Markov process with finite state space  $S$ . The process  $X$  is characterized by its infinitesimal generator  $A$  and its initial probability distribution  $\alpha$ . We denote by  $Z = \{Z_n, n \geq 0\}$  the uniformized Markov chain [7] associated to the Markov process  $X$ , with the same initial distribution  $\alpha$ . Its transition probability matrix  $P$  is related to the matrix  $A$  by the relation  $P = I + A/\lambda$ , where  $I$  is the identity matrix and  $\lambda$  satisfies  $\lambda \geq \max\{-A_{i,i}, i \in S\}$ . The rate  $\lambda$  is the rate of the Poisson process  $\{N_u, u \geq 0\}$ , independent of  $Z$ , that counts the number of transitions of process  $\{Z_{N_u}, u \geq 0\}$  over  $[0, t)$ . It is well-known that the processes  $\{Z_{N_u}\}$  and  $X$  are stochastically equivalent. We consider a partition  $S = U \cup D$ ,  $U \cap D = \emptyset$ , of the state space  $S$  and we study the occupation time in the subset  $U$ .

#### 3.1 The Discrete Time Case

For the Markov chain  $Z = \{Z_n, n \geq 0\}$ , the random variable  $V_n$  is the total

number of states of  $U$  visited during the  $n$  first transitions of  $Z$ , that is

$$V_n = \sum_{k=0}^n 1_{\{Z_k \in U\}}.$$

The following theorem gives the backward equations for the behavior of the pair  $(V_n, Z_n)$ . For every  $i \in S$ , the notation  $\mathbb{P}_i$  denotes the conditional probability given that  $Z_0 = i$ , that is  $\mathbb{P}_i\{\cdot\} = \mathbb{P}\{\cdot \mid Z_0 = i\}$ .

**Theorem 3.1** For  $n \geq 1$ , and  $1 \leq k \leq n$ , we have that

$$\begin{aligned} \text{for } i \in U, \mathbb{P}_i\{V_n \leq k, Z_n = j\} &= \sum_{l \in S} P_{i,l} \mathbb{P}_l\{V_{n-1} \leq k-1, Z_{n-1} = j\}, \\ \text{for } i \in D, \mathbb{P}_i\{V_n \leq k, Z_n = j\} &= \sum_{l \in S} P_{i,l} \mathbb{P}_l\{V_{n-1} \leq k, Z_{n-1} = j\}. \end{aligned}$$

**Proof.** By using the Markov property and the homogeneity of  $Z$ , we get that

$$\begin{aligned} \mathbb{P}_i\{V_n \leq k, Z_n = j\} &= \sum_{l \in S} P_{i,l} \mathbb{P}_i\{V_n \leq k, Z_n = j \mid Z_1 = l\} \\ &= \sum_{l \in S} P_{i,l} \mathbb{P}_l\{V_{n-1} \leq k - 1_{\{i \in U\}}, Z_{n-1} = j\}. \end{aligned}$$

■

The following theorem gives the forward equations for the probabilities associated with the pair  $(V_n, Z_n)$ .

**Theorem 3.2** For  $n \geq 1$ , and  $1 \leq k \leq n$ , we have that

$$\begin{aligned} \text{for } j \in U, \mathbb{P}_i\{V_n \leq k, Z_n = j\} &= \sum_{l \in S} \mathbb{P}_i\{V_{n-1} \leq k-1, Z_{n-1} = l\} P_{l,j}, \\ \text{for } j \in D, \mathbb{P}_i\{V_n \leq k, Z_n = j\} &= \sum_{l \in S} \mathbb{P}_i\{V_{n-1} \leq k, Z_{n-1} = l\} P_{l,j}. \end{aligned}$$

**Proof.** By the same arguments, we get that

$$\begin{aligned} \mathbb{P}\{V_n \leq k, Z_n = j, Z_0 = i\} &= \mathbb{P}\{V_{n-1} \leq k - 1_{\{j \in U\}}, Z_n = j, Z_0 = i\} \\ &= \sum_{l \in S} \mathbb{P}\{V_{n-1} \leq k - 1_{\{j \in U\}}, Z_n = j, Z_{n-1} = l, Z_0 = i\} \\ &= \sum_{l \in S} \mathbb{P}\{V_{n-1} \leq k - 1_{\{j \in U\}}, Z_0 = i \mid Z_{n-1} = l\} \mathbb{P}\{Z_n = j, Z_{n-1} = l\} \\ &= \sum_{l \in S} \mathbb{P}\{V_{n-1} \leq k - 1_{\{j \in U\}}, Z_{n-1} = l, Z_0 = i\} P_{l,j}. \end{aligned}$$

We thus obtain the desired relation by conditioning on  $Z_0$ .

■



For  $n \geq 0$ , and  $k \geq 0$ , we introduce the matrix  $F(n, k) = \{F_{i,j}(n, k)\}$  defined by

$$F_{i,j}(n, k) = \mathbb{P}\{V_n \leq k, Z_n = j \mid Z_0 = i\}.$$

The results of Theorems 3.1 and 3.2 can be easily expressed in matrix notation. We decompose the matrices  $P$  and  $F(n, k)$  with respect to the partition  $\{U, D\}$  as

$$P = \begin{pmatrix} P_U & P_{UD} \\ P_{DU} & P_D \end{pmatrix} \quad \text{and} \quad F(n, k) = \begin{pmatrix} F_U(n, k) & F_{UD}(n, k) \\ F_{DU}(n, k) & F_D(n, k) \end{pmatrix}.$$

The result of Theorem 3.1 can now be written as

$$\begin{pmatrix} F_U(n, k) & F_{UD}(n, k) \\ F_{DU}(n, k) & F_D(n, k) \end{pmatrix} = \begin{pmatrix} P_U & P_{UD} \\ P_{DU} & P_D \end{pmatrix} F(n-1, k-1)$$

$$\begin{pmatrix} F_U(n, k) & F_{UD}(n, k) \\ F_{DU}(n, k) & F_D(n, k) \end{pmatrix} = \begin{pmatrix} P_U & P_{UD} \\ P_{DU} & P_D \end{pmatrix} F(n-1, k)$$

or

$$F(n, k) = \begin{pmatrix} P_U & P_{UD} \\ 0 & 0 \end{pmatrix} F(n-1, k-1) + \begin{pmatrix} 0 & 0 \\ P_{DU} & P_D \end{pmatrix} F(n-1, k).$$

In the same way, Theorem 3.2 can be written as

$$\begin{pmatrix} F_U(n, k) \\ F_{DU}(n, k) \end{pmatrix} = F(n-1, k-1) \begin{pmatrix} P_U \\ P_{DU} \end{pmatrix}$$

$$\begin{pmatrix} F_{UD}(n, k) \\ F_D(n, k) \end{pmatrix} = F(n-1, k) \begin{pmatrix} P_{UD} \\ P_D \end{pmatrix}$$

or

$$F(n, k) = F(n-1, k-1) \begin{pmatrix} P_U & 0 \\ P_{DU} & 0 \end{pmatrix} + F(n-1, k) \begin{pmatrix} 0 & P_{UD} \\ 0 & P_D \end{pmatrix}.$$

The initial conditions are

$$F(n, 0) = \begin{pmatrix} 0 & 0 \\ 0 & (P_D)^n \end{pmatrix}, \quad \text{for } n \geq 0.$$

Note that for all  $k \geq n+1$ ,  $F(n, k) = P^n$ .

### 3.2 The Continuous Time Case

We consider the Markov process  $X = \{X_t, t \geq 0\}$  and the occupation time  $W_t$  of the subset  $U$  in  $[0, t)$ , that is

$$W_t = \int_0^t 1_{\{X_u \in U\}} du.$$

That random variable represents the time spent by the process  $X$  in the subset  $U$  during the interval  $[0, t)$ . The joint distribution of the pair  $(W_t, X_t)$  is given by the following theorem.

**Theorem 3.3** *For every  $i, j \in S$ , for  $t > 0$ , and  $s \in [0, t)$ , we have that*

$$\begin{aligned} & \mathbb{P}\{W_t \leq s, X_t = j \mid X_0 = i\} \\ &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^n \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k} \mathbb{P}\{V_n \leq k, Z_n = j \mid Z_0 = i\}. \end{aligned} \quad (4)$$

**Proof.** For  $s < t$ , we have that

$$\begin{aligned} & \mathbb{P}\{W_t \leq s, X_t = j \mid X_0 = i\} \\ &= \sum_{n=0}^{\infty} \mathbb{P}_i\{W_t \leq s, N_t = n, X_t = j\} \\ &= \sum_{n=0}^{\infty} \mathbb{P}_i\{W_t \leq s, N_t = n, Z_n = j\} \quad \text{since } \{X_t\} \text{ and } \{Z_{N_t}\} \text{ are equivalent} \\ &= \sum_{n=0}^{\infty} \mathbb{P}_i\{N_t = n\} \mathbb{P}_i\{W_t \leq s, Z_n = j \mid N_t = n\} \\ &= \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\} \mathbb{P}_i\{W_t \leq s, Z_n = j \mid N_t = n\} \\ &= \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\} \sum_{l=0}^{n+1} \mathbb{P}_i\{W_t \leq s, V_n = l, Z_n = j \mid N_t = n\} \\ &= \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\} \sum_{l=0}^{n+1} \mathbb{P}_i\{V_n = l, Z_n = j\} \mathbb{P}_i\{W_t \leq s \mid V_n = l, Z_n = j, N_t = n\} \\ &= \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\} \sum_{l=0}^n \mathbb{P}_i\{V_n = l, Z_n = j\} \mathbb{P}_i\{W_t \leq s \mid V_n = l, Z_n = j, N_t = n\}. \end{aligned}$$

The fourth and sixth equalities follow from the independence of the processes  $\{Z_n\}$  and  $\{N_t\}$  and the fact that  $X_0 = Z_0$ . The last equality follows from the

fact that if  $l = n + 1$ , we trivially have that  $V_n = n + 1$  and  $N_t = n$  imply that  $W_t = t$ . We so get  $\mathbb{P}\{W_t \leq s \mid V_n = n + 1, Z_n = j, N_t = n\} = 0$ , since  $s < t$ . Let us consider now the expression  $\mathbb{P}_i\{W_t \leq s \mid V_n = l, Z_n = j, N_t = n\}$ . For fixed  $i, j \in S$  and  $0 \leq l \leq n$ , we define the set

$$G_{l,n}^{i,j} = \left\{ \widehat{z} = (i, z_1, \dots, z_{n-1}, j) \in S^{n+1} \left| \begin{array}{l} l \text{ entries of } \widehat{z} \text{ are in } U \text{ and} \\ n + 1 - l \text{ entries of } \widehat{z} \text{ are in } D \end{array} \right. \right\}$$

and we denote by  $\widehat{Z}$  the random vector  $(Z_0, \dots, Z_n)$ . We then have  $\mathbb{P}_i\{W_t \leq s \mid V_n = l, Z_n = j, N_t = n\}$

$$\begin{aligned} &= \sum_{\widehat{z} \in G_{l,n}^{i,j}} \mathbb{P}\{W_t \leq s \mid \widehat{Z} = \widehat{z}, V_n = l, N_t = n\} \mathbb{P}_i\{\widehat{Z} = \widehat{z} \mid V_n = l, Z_n = j, N_t = n\} \\ &= \sum_{\widehat{z} \in G_{l,n}^{i,j}} \mathbb{P}\{W_t \leq s \mid \widehat{Z} = \widehat{z}, V_n = l, N_t = n\} \mathbb{P}_i\{\widehat{Z} = \widehat{z} \mid V_n = l, Z_n = j\}, \end{aligned}$$

where the last equality follows from the independence of  $\{Z_n\}$  and  $\{N_t\}$ . We denote by  $T_0, T_0 + T_1, \dots, T_0 + T_1 + \dots + T_{n-1}$ , the first  $n$  instants of jumps of the Poisson process  $\{N_t\}$  over  $[0, t)$  and we set  $T_n = t - (T_0 + T_1 + \dots + T_{n-1})$ . Then,

$$\begin{aligned} \mathbb{P}\{W_t \leq s \mid \widehat{Z} = \widehat{z}, V_n = l, N_t = n\} &= \mathbb{P}\left\{\sum_{j=1}^l T_{i_j} \leq s \mid \widehat{Z} = \widehat{z}, V_n = l, N_t = n\right\} \\ &= \mathbb{P}\left\{\sum_{j=1}^l T_{i_j} \leq s \mid N_t = n\right\}, \end{aligned}$$

where the distinct indices  $i_1, \dots, i_l \in \{0, 1, \dots, n\}$  correspond to the  $l$  entries of  $\widehat{z}$  that are in  $U$  and the last equality is due to the independence of the processes  $\{Z_n\}$  and  $\{N_t\}$ . For  $l = 0$ , we obtain the correct result, which is equal to 1 by using the convention  $\sum_a^b(\dots) = 0$  if  $a > b$ . From relation (2) we get, for  $l = 0, \dots, n$ ,

$$\begin{aligned} \mathbb{P}\{T_{i_1} + \dots + T_{i_l} \leq s \mid N_t = n\} &= \mathbb{P}\{T_0 + \dots + T_{l-1} \leq s \mid N_t = n\} \\ &= \sum_{k=l}^n \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}. \end{aligned}$$

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Again, the convention  $\sum_a^b(\dots) = 0$  for  $a > b$  allows us to cover the cases  $l = 0$  and  $l = n + 1$ . Finally, we obtain that

$$\begin{aligned} & \mathbb{P}_i\{W_t \leq s \mid V_n = l, Z_n = j, N_t = n\} \\ &= \sum_{\hat{z} \in G_{i,n}^{l,j}} \sum_{k=l}^n \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k} \mathbb{P}\{\hat{Z} = \hat{z} \mid V_n = l, Z_n = j, Z_0 = i\} \\ &= \sum_{k=l}^n \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k} \end{aligned}$$

That is, since  $\mathbb{P}\{N_t = n\} = e^{-\lambda t}(\lambda t)^n/n!$ ,

$$\begin{aligned} & \mathbb{P}\{W_t \leq s, X_t = j \mid X_0 = i\} \\ &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{l=0}^n \sum_{k=l}^n \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k} \mathbb{P}\{V_n = l, Z_n = j \mid Z_0 = i\} \\ &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^n \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k} \sum_{l=0}^k \mathbb{P}\{V_n = l, Z_n = j \mid Z_0 = i\} \\ &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^n \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k} \mathbb{P}\{V_n \leq k, Z_n = j \mid Z_0 = i\}. \end{aligned}$$

■

## 4 JOINT DISTRIBUTION OF OCCUPATION TIMES

Next, we partition the state space  $S$  into  $m + 1$  subsets  $B_0, B_1, \dots, B_m$ .

### 4.1 The Discrete Time Case

We consider the random variables  $V_n^i$  defined by

$$V_n^i = \sum_{k=0}^n 1_{\{Z_k \in B_i\}}.$$

The next theorem gives the backward equation for the joint distribution of the  $V_n^i$  and  $Z_n$ .

**Theorem 4.1** For  $r = 1, \dots, m, n \geq 1$ , and  $0 \leq k_1, \dots, k_m \leq n$  ( $k_r \geq 1$ ), we have

$$\begin{aligned} & \text{for } i \in B_r, \mathbb{P}_i\{V_n^1 \leq k_1, \dots, V_n^r \leq k_r, \dots, V_n^m \leq k_m, Z_n = j\} = \\ & \quad \sum_{l \in S} P_{i,l} \mathbb{P}_l\{V_{n-1}^1 \leq k_1, \dots, V_{n-1}^r \leq k_r - 1, \dots, V_{n-1}^m \leq k_m, Z_{n-1} = j\}, \\ & \text{for } i \in B_0, \mathbb{P}_i\{V_n^1 \leq k_1, \dots, V_n^m \leq k_m, Z_n = j\} = \\ & \quad \sum_{l \in S} P_{i,l} \mathbb{P}_l\{V_{n-1}^1 \leq k_1, \dots, V_{n-1}^m \leq k_m, Z_{n-1} = j\}. \end{aligned}$$

**Proof.** By  $\widehat{V}_n$  and  $\widehat{k}$  we denote the vectors  $(V_n^1, \dots, V_n^m)$  and  $(k_1, \dots, k_m)$  respectively and by  $e_i, i = 1, \dots, m$ , the unit row vector of dimension  $m$  whose  $i$ th entry is 1. The proof follows the same steps as that of Theorem 3.1. We have

$$\begin{aligned} \mathbb{P}\{\widehat{V}_n \leq \widehat{k}, Z_n = j \mid Z_0 = i\} &= \sum_{l \in S} P_{i,l} \mathbb{P}_i\{\widehat{V}_n \leq \widehat{k}, Z_n = j \mid Z_1 = l\} \\ &= \sum_{l \in S} P_{i,l} \mathbb{P}_l\{\widehat{V}_{n-1} \leq \widehat{k} - e_r 1_{\{i \in B_r\}}, Z_{n-1} = j\}. \end{aligned}$$

■

The theorem that follows gives the forward equation for the joint distribution of the  $V_n^i$  and  $Z_n$ .

**Theorem 4.2** For  $r = 1, \dots, m, n \geq 1$ , and  $0 \leq k_1, \dots, k_m \leq n$  ( $k_r \geq 1$ ), we have

$$\begin{aligned} & \text{for } j \in B_r, \mathbb{P}_i\{V_n^1 \leq k_1, \dots, V_n^r \leq k_r, \dots, V_n^m \leq k_m, Z_n = j\} = \\ & \quad \sum_{l \in S} \mathbb{P}_i\{V_{n-1}^1 \leq k_1, \dots, V_{n-1}^r \leq k_r - 1, \dots, V_{n-1}^m \leq k_m, Z_{n-1} = l\} P_{l,j}, \\ & \text{for } j \in B_0, \mathbb{P}_i\{V_n^1 \leq k_1, \dots, V_n^m \leq k_m, Z_n = j\} = \\ & \quad \sum_{l \in S} \mathbb{P}_i\{V_{n-1}^1 \leq k_1, \dots, V_{n-1}^m \leq k_m, Z_{n-1} = l\} P_{l,j}. \end{aligned}$$

**Proof.** Using the notation of the proof of Theorem 4.1, we follow the same steps as in Theorem 3.2. We have that

$$\begin{aligned} \mathbb{P}\{\widehat{V}_n \leq \widehat{k}, Z_n = j, Z_0 = i\} &= \mathbb{P}\{\widehat{V}_{n-1} \leq \widehat{k} - e_r 1_{\{j \in B_r\}}, Z_n = j, Z_0 = i\} \\ &= \sum_{l \in S} \mathbb{P}\{\widehat{V}_{n-1} \leq \widehat{k} - e_r 1_{\{j \in B_r\}}, Z_n = j, Z_{n-1} = l, Z_0 = i\} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{l \in S} \mathbb{P}\{\widehat{V}_{n-1} \leq \widehat{k} - e_r 1_{\{j \in B_r\}}, Z_0 = i \mid Z_{n-1} = l\} \mathbb{P}\{Z_n = j, Z_{n-1} = l\} \\
 &= \sum_{l \in S} \mathbb{P}\{\widehat{V}_{n-1} \leq \widehat{k} - e_r 1_{\{j \in B_r\}}, Z_{n-1} = l, Z_0 = i\} P_{l,j}.
 \end{aligned}$$

By conditioning on  $Z_0$ , we obtain the desired relation. ■

By  $F(n, k_1, \dots, k_m)$ , for  $n \geq 0$  and  $k_r \geq 0$ , we denote the matrix with  $(i, j)$  entry

$$F_{i,j}(n, k_1, \dots, k_m) = \mathbb{P}\{V_n^1 \leq k_1, \dots, V_n^m \leq k_m, Z_n = j \mid Z_0 = i\}.$$

The results of Theorems 4.1 and 4.2 can be conveniently expressed in matrix notation. We first decompose the matrices  $P$  and  $F(n, k_1, \dots, k_m)$  with respect to the partition  $\{B_0, B_1, \dots, B_m\}$  of the state space  $S$  as

$$P = \{P_{B_r B_h}\}_{0 \leq r, h \leq m} \quad \text{and} \quad F(n, k_1, \dots, k_m) = \{F_{B_r B_h}(n, k_1, \dots, k_m)\}_{0 \leq r, h \leq m}.$$

The result of Theorem 4.1 can then be written as

$$F_{B_r B_h}(n, k_1, \dots, k_m) = \sum_{l=0}^m P_{B_r B_l} F_{B_l B_h}(n-1, k_1, \dots, k_r - 1_{\{r \neq 0\}}, \dots, k_m),$$

and that of Theorem 3.2 as

$$F_{B_r B_h}(n, k_1, \dots, k_m) = \sum_{l=0}^m F_{B_r B_l}(n-1, k_1, \dots, k_h - 1_{\{h \neq 0\}}, \dots, k_m) P_{B_l B_h}.$$

The initial conditions are given by

$$F(n, 0, \dots, 0) = \begin{pmatrix} 0 & 0 \\ 0 & (P_{B_0 B_0})^n \end{pmatrix}, \text{ for } n \geq 0.$$

Note that in the case  $k_1 + \dots + k_m \geq n+1$ , with  $k_i \leq n$ , for  $i = 1, \dots, m$ , the  $m$ -dimensional joint distribution of  $V_n^1, \dots, V_n^m$  can be expressed as a combination of the  $h$ -dimensional joint distributions of the  $V_n^i$  for  $h = 1, \dots, m-1$ . That observation is based on the following general result.

For any random variables  $U_1, \dots, U_m$  and any event  $A$ , we have

$$\begin{aligned}
 \mathbb{P}\{U_1 \leq x_1, \dots, U_m \leq x_m, A\} &= \sum_{E \subset \{1, \dots, m\}} (-1)^{m-|E|+1} \mathbb{P}\{U_l \leq x_l; l \in E, A\} \\
 &+ (-1)^m \mathbb{P}\{U_1 > x_1, \dots, U_m > x_m, A\}, \quad (5)
 \end{aligned}$$

where the inclusion is strict, that is  $E \neq \{1, \dots, m\}$ , and where, for convenience, we set  $\mathbb{P}\{U_l \leq x_l; l \in \emptyset, A\} = \mathbb{P}\{A\}$ . For the random variables  $V_n^i$ ,

$$\mathbb{P}\{V_n^1 > k_1, \dots, V_n^m > k_m, Z_n = j \mid Z_0 = i\} = 0, \text{ if } k_1 + \dots + k_m \geq n + 1,$$

so, in that case, we get the desired result,

$$\mathbb{P}_i\{\widehat{V}_n \leq \widehat{k}, Z_n = j\} = \sum_{E \subset \{1, \dots, m\}} (-1)^{m-|E|+1} \mathbb{P}_i\{V_n^l \leq k_l; l \in E, Z_n = j\}. \quad (6)$$

### 4.2 The Continuous Time Case

We consider the random variables  $W_t^i, i = 1, \dots, m$ , defined by

$$W_t^i = \int_0^t 1_{\{X_u \in B_i\}} du,$$

whose joint distribution with  $X_t$  is given in the next theorem.

**Theorem 4.3** *For every  $i, j \in S$ , for every  $t > 0$ , and  $s_1, \dots, s_m \in [0, t]$  such that  $s_1 + s_2 + \dots + s_m < t$ , we have*

$$\mathbb{P}\{W_t^1 \leq s_1, \dots, W_t^m \leq s_m, X_t = j \mid X_0 = i\} = \sum_{\substack{k_1 \geq 0, \dots, k_m \geq 0, \\ k_1 + \dots + k_m \leq n}} \frac{\sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k_1 \geq 0, \dots, k_m \geq 0, \\ k_1 + \dots + k_m \leq n} n! \theta_{n; k_1, k_2, \dots, k_m}^{t; s_1, s_2, \dots, s_m} \mathbb{P}_i\{V_n^1 \leq k_1, \dots, V_n^m \leq k_m, Z_n = j\}}{k_1! k_2! \dots k_m! (n - (k_1 + k_2 + \dots + k_m))!} \quad (7)$$

where

$$\theta_{n; k_1, k_2, \dots, k_m}^{t; s_1, s_2, \dots, s_m} = \frac{\left(\frac{s_1}{t}\right)^{k_1} \left(\frac{s_2}{t}\right)^{k_2} \dots \left(\frac{s_m}{t}\right)^{k_m} \left(1 - \frac{s_1 + \dots + s_m}{t}\right)^{n - (k_1 + \dots + k_m)}}{k_1! k_2! \dots k_m! (n - (k_1 + k_2 + \dots + k_m))!}.$$

**Proof.** We define the vectors  $\widehat{W}_t = (W_t^1, \dots, W_t^m)$ ,  $\widehat{V}_n = (V_n^1, \dots, V_n^m)$ , and  $\widehat{s} = (s_1, \dots, s_m)$ . Inequality between vectors means component-wise inequality.

For  $n \geq 0$ , we define the set  $E_n$  by

$$E_n = \{\widehat{l} = (l_1, l_2, \dots, l_m) \in \mathbb{N}^m \mid l_1 + l_2 + \dots + l_m \leq n\}.$$

We have that

$$\begin{aligned}
 & \mathbb{P}\{\widehat{W}_t \leq \widehat{s}, X_t = j \mid X_0 = i\} \\
 = & \sum_{n=0}^{\infty} \mathbb{P}_i\{\widehat{W}_t \leq \widehat{s}, N_t = n, X_t = j\} \\
 = & \sum_{n=0}^{\infty} \mathbb{P}_i\{\widehat{W}_t \leq \widehat{s}, N_t = n, Z_n = j\} \quad \text{since } \{X_t\} \text{ and } \{Z_{N_t}\} \text{ are equivalent} \\
 = & \sum_{n=0}^{\infty} \mathbb{P}_i\{N_t = n\} \mathbb{P}_i\{\widehat{W}_t \leq \widehat{s}, Z_n = j \mid N_t = n\} \\
 = & \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\} \mathbb{P}_i\{\widehat{W}_t \leq \widehat{s}, Z_n = j \mid N_t = n\} \\
 = & \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\} \sum_{\widehat{l} \in E_n} \mathbb{P}_i\{\widehat{W}_t \leq \widehat{s}, \widehat{V}_n = \widehat{l}, Z_n = j \mid N_t = n\} \\
 = & \sum_{n=0}^{\infty} \mathbb{P}\{N_t = n\} \sum_{\widehat{l} \in E_n} \mathbb{P}_i\{\widehat{V}_n = \widehat{l}, Z_n = j\} \mathbb{P}_i\{\widehat{W}_t \leq \widehat{s} \mid \widehat{V}_n = \widehat{l}, Z_n = j, N_t = n\}.
 \end{aligned}$$

The fourth and last equalities follow from the independence of the processes  $\{Z_n\}$  and  $\{N_t\}$  and the fact that  $X_0 = Z_0$ . In the fifth equality, the summation in  $\widehat{l}$  should be over  $E_{n+1}$  but it can be restricted to  $E_n$ . If  $l_1 + l_2 + \dots + l_m = n + 1$ , then  $\widehat{V}_n = \widehat{l}$  and  $N_t = n$  imply that  $V_n^1 + \dots + V_n^m = n + 1$  and so that  $W_t^1 + \dots + W_t^m = t$ . As  $s_1 + \dots + s_m < t$  that implies  $\mathbb{P}\{\widehat{W}_t \leq \widehat{s} \mid \widehat{V}_n = \widehat{l}, N_t = n\} = 0$ . Now consider the expression  $\mathbb{P}_i\{\widehat{W}_t \leq \widehat{s} \mid \widehat{V}_n = \widehat{l}, Z_n = j, N_t = n\}$ . For  $\widehat{l} = (l_1, l_2, \dots, l_m) \in E_n$ , and  $i, j \in S$ , we define the set

$$G_{\widehat{l}, n}^{i, j} = \left\{ \widehat{z} = (i, z_1, \dots, z_{n-1}, j) \in S^{n+1} \left| \begin{array}{l} l_1 \text{ entries of } \widehat{z} \text{ are in } B_1, \dots, \\ l_m \text{ entries of } \widehat{z} \text{ are in } B_m \text{ and} \\ n + 1 - (l_1 + \dots + l_m) \text{ are in } B_0 \end{array} \right. \right\},$$

and we denote the random vector  $(Z_0, \dots, Z_n)$  by  $\widehat{Z}$ . We have that

$$\begin{aligned}
 & \mathbb{P}_i\{\widehat{W}_t \leq \widehat{s} \mid \widehat{V}_n = \widehat{l}, Z_n = j, N_t = n\} \\
 = & \sum_{\widehat{z} \in G_{\widehat{l}, n}^{i, j}} \mathbb{P}\{\widehat{W}_t \leq \widehat{s} \mid \widehat{Z} = \widehat{z}, \widehat{V}_n = \widehat{l}, N_t = n\} \mathbb{P}_i\{\widehat{Z} = \widehat{z} \mid \widehat{V}_n = \widehat{l}, Z_n = j, N_t = n\} \\
 = & \sum_{\widehat{z} \in G_{\widehat{l}, n}^{i, j}} \mathbb{P}\{\widehat{W}_t \leq \widehat{s} \mid \widehat{Z} = \widehat{z}, \widehat{V}_n = \widehat{l}, N_t = n\} \mathbb{P}_i\{\widehat{Z} = \widehat{z} \mid \widehat{V}_n = \widehat{l}, Z_n = j\}
 \end{aligned}$$



where the last equality follows from the independence of the processes  $\{Z_n\}$  and  $\{N_t\}$ . It follows that

$$\mathbb{P}\{\widehat{W}_t \leq \widehat{s} \mid \widehat{Z} = \widehat{z}, \widehat{V}_n = \widehat{l}, N_t = n\} = \mathbb{P}\{\widehat{T}(\widehat{l}) \leq \widehat{s} \mid \widehat{Z} = \widehat{z}, \widehat{V}_n = \widehat{l}, N_t = n\},$$

where

$$\widehat{T}(\widehat{l}) = \left( \sum_{j=1}^{l_1} T_{i_j}, \sum_{j=l_1+1}^{l_1+l_2} T_{i_j}, \dots, \sum_{j=l_1+l_2+\dots+l_{m-1}+1}^{l_1+l_2+\dots+l_m} T_{i_j} \right).$$

Again using the independence of  $\{Z_n\}$  and  $\{N_t\}$  and relation (3), we obtain

$$\begin{aligned} \mathbb{P}\{\widehat{T}(\widehat{l}) \leq \widehat{s} \mid \widehat{Z} = \widehat{z}, \widehat{V}_n = \widehat{l}, N_t = n\} &= \mathbb{P}\{\widehat{T}(\widehat{l}) \leq \widehat{s} \mid N_t = n\} \\ &= \sum_{\substack{k_1 \geq l_1, k_2 \geq l_2, \dots, k_m \geq l_m, \\ k_1 + k_2 + \dots + k_m \leq n}} n! \theta_{n; k_1, k_2, \dots, k_m}^{t; s_1, s_2, \dots, s_m}. \end{aligned}$$

Note that if one of the  $l_i$ 's is zero, the corresponding entry of the vector  $\widehat{T}(\widehat{l})$  becomes zero and the preceding formula still holds. Indeed, suppose for simplicity that  $l_m = 0$ , then

$$\begin{aligned} \sum_{\substack{k_1 \geq l_1, \dots, k_m \geq 0, \\ k_1 + \dots + k_m \leq n}} n! \theta_{n; k_1, k_2, \dots, k_m}^{t; s_1, s_2, \dots, s_m} &= \sum_{\substack{k_1 \geq l_1, \dots, k_{m-1} \geq l_{m-1}, \\ k_1 + \dots + k_{m-1} \leq n}} n! \sum_{k_m=0}^{n-(k_1+\dots+k_{m-1})} \theta_{n; k_1, \dots, k_m}^{t; s_1, \dots, s_m} \\ &= \sum_{\substack{k_1 \geq l_1, \dots, k_{m-1} \geq l_{m-1}, \\ k_1 + \dots + k_{m-1} \leq n}} n! \theta_{n; k_1, \dots, k_{m-1}}^{t; s_1, \dots, s_{m-1}}. \end{aligned}$$

Note also that if all the  $l_i$ 's are zero, all the entries of  $\widehat{T}(\widehat{l})$  are zero and the formula still holds since

$$\sum_{\substack{k_1 \geq 0, k_2 \geq 0, \dots, k_m \geq 0, \\ k_1 + k_2 + \dots + k_m \leq n}} n! \theta_{n; k_1, k_2, \dots, k_m}^{t; s_1, s_2, \dots, s_m} = 1.$$

Putting these results together, we obtain

$$\begin{aligned} \mathbb{P}_i\{\widehat{W}_t \leq \widehat{s} \mid \widehat{V}_n = \widehat{l}, Z_n = j, N_t = n\} &= \sum_{\widehat{z} \in G_{l, n}^{i, j}} \sum_{\substack{k_1 \geq l_1, k_2 \geq l_2, \dots, k_m \geq l_m, \\ k_1 + k_2 + \dots + k_m \leq n}} n! \theta_{n; k_1, k_2, \dots, k_m}^{t; s_1, s_2, \dots, s_m} \mathbb{P}_i\{\widehat{Z} = \widehat{z} \mid \widehat{V}_n = \widehat{l}, Z_n = j\} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{k_1 \geq l_1, k_2 \geq l_2, \dots, k_m \geq l_m, \\ k_1 + k_2 + \dots + k_m \leq n}} n! \theta_{n; k_1, k_2, \dots, k_m}^{t; s_1, s_2, \dots, s_m} \sum_{\widehat{z} \in G_{l, n}^{i, j}} \mathbb{P}_i \{ \widehat{Z} = \widehat{z} \mid \widehat{V}_n = \widehat{l}, Z_n = j \} \\
 &= \sum_{\substack{k_1 \geq l_1, k_2 \geq l_2, \dots, k_m \geq l_m, \\ k_1 + k_2 + \dots + k_m \leq n}} n! \theta_{n; k_1, k_2, \dots, k_m}^{t; s_1, s_2, \dots, s_m},
 \end{aligned}$$

and so,

$$\begin{aligned}
 &\mathbb{P} \{ \widehat{W}_t \leq \widehat{s}, X_t = j \mid X_0 = i \} \\
 &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{\widehat{l} \in E_n} \sum_{\substack{k_1 \geq l_1, k_2 \geq l_2, \dots, k_m \geq l_m, \\ k_1 + k_2 + \dots + k_m \leq n}} n! \theta_{n; k_1, k_2, \dots, k_m}^{t; s_1, s_2, \dots, s_m} \mathbb{P}_i \{ \widehat{V}_n = \widehat{l}, Z_n = j \} \\
 &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{\widehat{k} \in E_n} n! \theta_{n; k_1, k_2, \dots, k_m}^{t; s_1, s_2, \dots, s_m} \sum_{\substack{l_1 \geq k_1, l_2 \geq k_2, \dots, l_m \geq k_m, \\ l_1 + l_2 + \dots + l_m \leq n}} \mathbb{P}_i \{ \widehat{V}_n = \widehat{l}, Z_n = j \} \\
 &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{\widehat{k} \in E_n} n! \theta_{n; k_1, k_2, \dots, k_m}^{t; s_1, s_2, \dots, s_m} \mathbb{P} \{ \widehat{V}_n \leq \widehat{l} \} \\
 &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{\substack{k_1 \geq 0, k_2 \geq 0, \dots, k_m \geq 0, \\ k_1 + k_2 + \dots + k_m \leq n}} n! \theta_{n; k_1, k_2, \dots, k_m}^{t; s_1, s_2, \dots, s_m} \mathbb{P}_i \{ \widehat{V}_n \leq \widehat{l}, Z_n = j \}.
 \end{aligned}$$

■

From relation (7) the distribution  $\mathbb{P}_i \{ W_t^1 \leq s_1, \dots, W_t^m \leq s_m, X_t = j \}$  is differentiable with respect to  $t$  and also with respect to  $s_1, s_2, \dots, s_m$  for  $t > 0$ ,  $s_1, \dots, s_m \in (0, t)$ , and  $s_1 + \dots + s_m \in (0, t)$ . Moreover, if  $s_1 + s_2 + \dots + s_m \geq t$ , then trivially

$$\mathbb{P} \{ W_t^1 > s_1, \dots, W_t^m > s_m, X_t = j \mid X_0 = i \} = 0,$$

so that relation (6) applies by replacing the  $V_n^l$  and the  $k_l$  by the  $W_t^l$  and the  $s_l$  respectively.

## 5 WEIGHTED SUMS OF OCCUPATION TIMES

A constant performance level or reward rate  $\rho(i)$  is associated with each state

$i$  of  $S$ . We consider the random variable  $Y_t$  defined by

$$Y_t = \int_0^t \rho(X_u) du.$$

We denote by  $m + 1$  the number of distinct rewards and their values by

$$r_0 < r_1 < \dots < r_{m-1} < r_m.$$

We then have  $Y_t \in [r_0 t, r_m t]$  with probability one. Without loss of generality, we may set  $r_0 = 0$ . That can be easily done by considering the random variable  $Y_t - r_0 t$  instead of  $Y_t$  and the reward rates  $r_i - r_0$  instead of  $r_i$ . As in Section 4, the state space  $S$  is partitioned into subsets  $B_0, \dots, B_m$ . The subset  $B_l$  contains the states with reward rate  $r_l$ , that is  $B_l = \{i \in S | \rho(i) = r_l\}$ . With this notation,

$$Y_t = \sum_{l=1}^m r_l \int_0^t \mathbb{1}_{\{X_u \in B_l\}} du = \sum_{l=1}^m r_l W_t^l. \tag{8}$$

As the distribution of each  $W_t^l$  has at most two jumps at 0 and  $t$ , the distribution of  $Y_t$  has at most  $m + 1$  jumps at the points  $r_0 t = 0, r_1 t, \dots, r_m t$ . For  $t > 0$ , the jump at point  $x = r_l t$  is equal to the probability that the process  $X$ , starting in subset  $B_l$ , stays in the subset  $B_l$  during all of  $[0, t)$ , that is

$$\mathbb{P}\{Y_t = r_l t\} = \alpha_{B_l} e^{A_{B_l B_l} t} \mathbb{1}_{B_l} \text{ for } t > 0,$$

where  $\mathbb{1}_{B_l}$  is the column vector of dimension  $|B_l|$  with all components equal to 1. For every  $i, j \in S$ , and  $t > 0$ , we define the functions  $F_{i,j}(t, x)$  by

$$F_{i,j}(t, x) = \mathbb{P}\{Y_t > x, X_t = j \mid X_0 = i\},$$

and we introduce the matrix  $F(t, x) = \{F_{i,j}(t, x)\}$ . Using the partition  $B_m, B_{m-1}, \dots, B_0$ , the matrices  $A, P$ , and  $F(t, x)$  can be written as

$$A = \{A_{B_u B_v}\}_{0 \leq u, v \leq m} ; P = \{P_{B_u B_v}\}_{0 \leq u, v \leq m} ; F(t, x) = \{F_{B_u B_v}(t, x)\}_{0 \leq u, v \leq m}.$$

Note that for  $t > 0$ , and  $0 \leq l \leq m$ ,

$$\mathbb{P}\{Y_t = r_l t, X_t = j \mid X_0 = i\} = \begin{cases} (e^{A_{B_l B_l} t})_{i,j} & \text{if } i, j \in B_l, \\ 0 & \text{otherwise,} \end{cases}$$

that is

$$\mathbb{P}\{Y_t = r_t t, X_t = j \mid X_0 = i\} = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} (P_{B_t B_t})_{i,j}^n \mathbf{1}_{\{i,j \in B_t\}}. \tag{9}$$

The distribution  $F_{i,j}(t, x)$  can be obtained from relation (8), using the joint distribution of the  $W_t^l$  obtained in Section 4.2. From relation (7),  $F_{i,j}(t, x)$  is differentiable with respect to  $x$  and  $t$  in the domain

$$E = \{(t, x) ; t > 0 \text{ and } x \in \bigcup_{l=1}^m (r_{l-1}t, r_l t)\}.$$

The initial conditions are given, for  $t > 0$ , by

$$F_{i,j}(t, 0) = \mathbb{P}\{X_t = j \mid X_0 = i\} - \mathbb{P}\{Y_t = 0, X_t = j \mid X_0 = i\},$$

that is, in matrix notation,

$$F_{B_u B_v}(t, 0) = (e^{At})_{B_u B_v} - e^{A_{B_0 B_0} t} \mathbf{1}_{\{u=v=0\}},$$

which can also be written as

$$F_{B_u B_v}(t, 0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left[ (P^n)_{B_u B_v} - (P_{B_0 B_0})^n \mathbf{1}_{\{u=v=0\}} \right]. \tag{10}$$

### 5.1 Backward and Forward Equations

In what follows, we derive backward and forward equations satisfied by the distribution of the pair  $(Y_t, X_t)$ . First, we recall some well-known and useful results in the following lemma. Remember that  $\{N(t)\}$  is a Poisson process of rate  $\lambda$ , independent of the Markov chain  $Z$ . We denote by  $N(t, t + s)$  the number of transitions during the interval  $[t, t + s)$ .

**Lemma 5.1**

$$\mathbb{P}\{N(t, t + s) = 0 \mid X_t = j\} = e^{-\lambda s} \tag{11}$$

$$\mathbb{P}\{X_{t+s} = j, N(t, t + s) = 1 \mid X_t = i\} = P_{i,j} \lambda s e^{-\lambda s} \tag{12}$$

$$\mathbb{P}\{N(t, t + s) \geq 2 \mid X_0 = i\} = o(s). \tag{13}$$

The following theorem establishes the forward equation for the pair  $(Y_t, X_t)$ .

**Theorem 5.2** For  $t > 0$ ,  $i, j \in S$ ,  $1 \leq h \leq m$ , and  $x \in (r_{h-1}t, r_h t)$ , we have

$$\frac{\partial F_{i,j}(t, x)}{\partial t} = -\rho(j) \frac{\partial F_{i,j}(t, x)}{\partial x} + \sum_{k \in S} F_{i,k}(t, x) A_{k,j}. \quad (14)$$

**Proof.** By conditioning on the number of transitions in  $[t, t+s)$ , we have

$$\begin{aligned} \mathbb{P}_i\{Y_{t+s} > x, X_{t+s} = j\} &= \mathbb{P}_i\{Y_{t+s} > x, X_{t+s} = j, N(t, t+s) = 0\} \\ &\quad + \mathbb{P}_i\{Y_{t+s} > x, X_{t+s} = j, N(t, t+s) = 1\} \\ &\quad + \mathbb{P}_i\{Y_{t+s} > x, X_{t+s} = j, N(t, t+s) \geq 2\}. \end{aligned}$$

We separately consider these three terms. For the first term, since  $X_{t+s} = j$  and  $N(t, t+s) = 0$  is equivalent to  $X_t = j$  and  $N(t, t+s) = 0$ , we have

$$\begin{aligned} \mathbb{P}_i\{Y_{t+s} > x, X_{t+s} = j, N(t, t+s) = 0\} &= \mathbb{P}_i\{Y_{t+s} > x, X_t = j, N(t, t+s) = 0\} \\ &= \mathbb{P}_i\{Y_{t+s} > x \mid X_t = j, N(t, t+s) = 0\} \mathbb{P}_i\{X_t = j, N(t, t+s) = 0\} \\ &= \mathbb{P}_i\{Y_t > x - \rho(j)s \mid X_t = j, N(t, t+s) = 0\} \mathbb{P}_i\{X_t = j, N(t, t+s) = 0\} \\ &= \mathbb{P}_i\{Y_t > x - \rho(j)s \mid X_t = j\} \mathbb{P}_i\{X_t = j, N(t, t+s) = 0\} \\ &= \mathbb{P}_i\{Y_t > x - \rho(j)s, X_t = j\} \mathbb{P}_i\{N(t, t+s) = 0 \mid X_t = j\} \\ &= \mathbb{P}\{N(t, t+s) = 0 \mid X_t = j\} F_{i,j}(t, x - \rho(j)s) \\ &= e^{-\lambda s} F_{i,j}(t, x - \rho(j)s) \\ &= (1 - \lambda s) F_{i,j}(t, x - \rho(j)s) + o(s). \end{aligned}$$

The second equality follows from the fact that, if  $X_t = j$  and  $N(t, t+s) = 0$ , we have  $Y_{t+s} = Y_t + \rho(j)s$ . The third and fifth follow from the Markov property, and the sixth from relation (11). For the second term denoted by  $G(s)$ , we define

$$G_k(s) = \mathbb{P}_i\{Y_{t+s} > x \mid X_t = k, X_{t+s} = j, N(t, t+s) = 1\}.$$

We then have

$$\begin{aligned} G(s) &= \mathbb{P}_i\{Y_{t+s} > x, X_{t+s} = j, N(t, t+s) = 1\} \\ &= \sum_{k \in S} G_k(s) \mathbb{P}_i\{X_t = k, X_{t+s} = j, N(t, t+s) = 1\} \end{aligned}$$

Let us define  $\rho_{\min} = \min\{\rho(i)\}$  and  $\rho_{\max} = \max\{\rho(i)\}$ .

As  $Y_t + \rho_{\min}s \leq Y_{t+s} \leq Y_t + \rho_{\max}s$ , we get

$$\mathbb{P}_i\{Y_t > x - \rho_{\min}s \mid X_t = k, X_{t+s} = j, N(t, t+s) = 1\} \leq G_k(s),$$

and

$$G_k(s) \leq \mathbb{P}_i\{Y_t > x - \rho_{\max}s \mid X_t = k, X_{t+s} = j, N(t, t+s) = 1\}.$$

Using the Markov property,

$$\mathbb{P}_i\{Y_t > x - \rho_{\min}s \mid X_t = k\} \leq G_k(s) \leq \mathbb{P}_i\{Y_t > x - \rho_{\max}s \mid X_t = k\}.$$

We thus obtain

$$\sum_{k \in S} F_{i,k}(t, x - \rho_{\min}s) U_{k,j}(s) \leq G(s) \leq \sum_{k \in S} F_{i,k}(t, x - \rho_{\max}s) U_{k,j}(s),$$

where  $U_{k,j}(s) = \mathbb{P}\{X_{t+s} = j, N(t, t+s) = 1 \mid X_t = k\}$ . From relation (12),

$$\lim_{s \rightarrow 0} \frac{U_{k,j}(s)}{s} = \lambda P_{k,j},$$

so we obtain

$$\lim_{s \rightarrow 0} \frac{G(s)}{s} = \lambda \sum_{k \in S} F_{i,k}(t, x) P_{k,j}.$$

For the third term, we have by relation (13),

$$\mathbb{P}_i\{Y_{t+s} > x, X_{t+s} = j, N(t, t+s) \geq 2\} \leq \mathbb{P}_i\{N(t, t+s) \geq 2\} = o(s).$$

Combining the three terms, we obtain

$$\begin{aligned} \frac{F_{i,j}(t+s, x) - F_{i,j}(t, x)}{s} &= \frac{(1 - \lambda s)F_{i,j}(t, x - \rho(j)s) - F_{i,j}(t, x)}{s} + \frac{G(s)}{s} + \frac{o(s)}{s} \\ &= \frac{F_{i,j}(t, x - \rho(j)s) - F_{i,j}(t, x)}{s} - \lambda F_{i,j}(t, x - \rho(j)s) + \frac{G(s)}{s} + \frac{o(s)}{s} \end{aligned}$$

If now  $s$  tends to 0, we get

$$\frac{\partial F_{i,j}(t, x)}{\partial t} = -\rho(j) \frac{\partial F_{i,j}(t, x)}{\partial x} - \lambda F_{i,j}(t, x) + \lambda \sum_{k \in S} F_{i,k}(t, x) P_{k,j}.$$

Since  $P = I + A/\lambda$ , we obtain that

$$\frac{\partial F_{i,j}(t, x)}{\partial t} = -\rho(j) \frac{\partial F_{i,j}(t, x)}{\partial x} + \sum_{k \in S} F_{i,k}(t, x) A_{k,j}.$$

■

**Corollary 5.3** For  $t > 0$ ,  $0 \leq p \leq m$ ,  $i \in B_p$ ,  $j \in S$ ,  $1 \leq h \leq m$ , and  $x \in (r_{h-1}t, r_h t)$ , we have

$$F_{i,j}(t, x) = \sum_{k \in S} \int_0^t F_{i,k}(t-u, x - \rho(j)u) \lambda e^{-\lambda u} du P_{k,j} + e^{-\lambda t} \mathbf{1}_{\{h \leq p\}} \mathbf{1}_{\{i=j\}}. \quad (15)$$

**Proof.** Consider equation (14) and the functions  $\varphi_{i,j}$  defined by

$$\varphi_{i,j}(u) = F_{i,j}(t-u, x - \rho(j)u) e^{-\lambda u}.$$

Differentiating with respect to  $u$  yields

$$\varphi'_{i,j}(u) = e^{-\lambda u} \left[ -\frac{\partial F_{i,j}}{\partial t} - \rho(j) \frac{\partial F_{i,j}}{\partial x} \right] (t-u, x - \rho(j)u) - F_{i,j}(t-u, x - \rho(j)u) \lambda e^{-\lambda u}$$

which, by (14) and the relation  $A = -\lambda(I - P)$ , gives

$$\begin{aligned} \varphi'_{i,j}(u) &= -\sum_{k \in S} F_{i,k}(t-u, x - \rho(j)u) A_{k,j} e^{-\lambda u} - F_{i,j}(t-u, x - \rho(j)u) \lambda e^{-\lambda u} \\ &= -\sum_{k \in S} F_{i,k}(t-u, x - \rho(j)u) \lambda e^{-\lambda u} P_{k,j} \end{aligned}$$

Integrating that expression between 0 and  $t$  gives

$$\varphi_{i,j}(t) - \varphi_{i,j}(0) = -\sum_{k \in S} \int_0^t F_{i,k}(t-u, x - \rho(j)u) \lambda e^{-\lambda u} du P_{k,j}.$$

Finally, we have  $\varphi_{i,j}(0) = F_{i,j}(t, x)$  and

$$\varphi_{i,j}(t) = F_{i,j}(0, x - \rho(j)t) e^{-\lambda t} = e^{-\lambda t} \mathbf{1}_{\{x - \rho(j)t < 0\}} \mathbf{1}_{\{i=j\}} = e^{-\lambda t} \mathbf{1}_{\{h \leq p\}} \mathbf{1}_{\{i=j\}}.$$

■

We next derive the backward equation for the evolution of the pair  $(Y_t, X_t)$ .

**Theorem 5.4** For  $t > 0$ ,  $0 \leq p \leq m$ ,  $i \in B_p$ ,  $j \in S$ ,  $1 \leq h \leq m$ , and  $x \in (r_{h-1}t, r_h t)$ ,

$$F_{i,j}(t, x) = \sum_{k \in S} P_{i,k} \int_0^t F_{k,j}(t-u, x - \rho(i)u) \lambda e^{-\lambda u} du + e^{-\lambda t} \mathbf{1}_{\{h \leq p\}} \mathbf{1}_{\{i=j\}}. \quad (16)$$

**Proof.** Let  $T_1$  be the sojourn time in the initial state. We have

$$F_{i,j}(t, x) = \int_0^\infty \mathbb{P}\{Y_t > x, X_t = j \mid T_1 = u, X_0 = i\} \lambda e^{-\lambda u} du.$$

If  $u \geq t$  and  $X_0 = i$ , we have  $Y_t = \rho(i)t = r_p t$  and  $\mathbb{P}\{X_t = j \mid T_1 = u, X_0 = i\} = 1$ , if  $i = j$  and 0 otherwise. Moreover, as  $r_p t > x$  is equivalent to  $r_p t \geq r_h t$ , that is  $h \leq p$ , we obtain

$$F_{i,j}(t, x) = \int_0^t \mathbb{P}\{Y_t > x, X_t = j \mid T_1 = u, X_0 = i\} \lambda e^{-\lambda u} du + e^{-\lambda t} \mathbf{1}_{\{h \leq p\}} \mathbf{1}_{\{i=j\}}.$$

Now,

$$\begin{aligned} & \mathbb{P}\{Y_t > x, X_t = j \mid T_1 = u, X_0 = i\} \\ &= \sum_{k \in S} \mathbb{P}\{Y_t > x, X_t = j \mid X_u = k, T_1 = u, X_0 = i\} \mathbb{P}\{X_u = k \mid T_1 = u, X_0 = i\}. \end{aligned}$$

For the second factor in the summand, we have that

$$\begin{aligned} \mathbb{P}\{X_u = k \mid T_1 = u, X_0 = i\} &= \mathbb{P}\{X_{T_1} = k \mid T_1 = u, X_0 = i\} \\ &= \mathbb{P}\{Z_1 = k \mid T_1 = u, Z_0 = i\} \\ &= P_{i,k}. \end{aligned}$$

For the first factor,  $T_1 = u$  and  $X_0 = i$  imply  $Y_u = \rho(i)u$ , so that

$$\begin{aligned} & \mathbb{P}\{Y_t > x, X_t = j \mid X_u = k, T_1 = u, X_0 = i\} \\ &= \mathbb{P}\left\{\int_u^t \rho(X_v) dv > x - \rho(i)u, X_t = j \mid X_u = k, T_1 = u, X_0 = i\right\} \\ &= \mathbb{P}\left\{\int_u^t \rho(X_v) dv > x - \rho(i)u, X_t = j \mid X_u = k\right\} \\ &= \mathbb{P}\{Y_{t-u} > x - \rho(i)u, X_{t-u} = j \mid X_0 = k\} \\ &= F_{k,j}(t-u, x - \rho(i)u), \end{aligned}$$



where the second equality follows from the Markov property and the third by homogeneity. Combining these results, we obtain relation (16). ■

**Corollary 5.5** *For  $t > 0$ ,  $i, j \in S$ ,  $1 \leq h \leq m$  and  $x \in (r_{h-1}t, r_h t)$  we have*

$$\frac{\partial F_{i,j}(t, x)}{\partial t} = -\rho(i) \frac{\partial F_{i,j}(t, x)}{\partial x} + \sum_{k \in S} A_{i,k} F_{k,j}(t, x). \tag{17}$$

**Proof.** Consider equation (16) with  $i \in B_p$ ,  $0 \leq p \leq m$  and  $j \in S$ . Differentiating  $F_{i,j}(t, x)$  with respect to  $t$ , we obtain

$$\begin{aligned} \frac{\partial F_{i,j}(t, x)}{\partial t} &= \sum_{k \in S} P_{i,k} \int_0^t \frac{\partial F_{k,j}(t-u, x-\rho(i)u)}{\partial t} \lambda e^{-\lambda u} du \\ &\quad + \sum_{k \in S} P_{i,k} F_{k,j}(0, x-\rho(i)t) \lambda e^{-\lambda t} - \lambda e^{-\lambda t} \mathbf{1}_{\{h \leq p\}} \mathbf{1}_{\{i=j\}} \end{aligned}$$

Next, differentiating  $F_{i,j}(t, x)$  with respect to  $x$ , we get

$$\frac{\partial F_{i,j}(t, x)}{\partial x} = \sum_{k \in S} P_{i,k} \int_0^t \frac{\partial F_{k,j}(t-u, x-\rho(i)u)}{\partial x} \lambda e^{-\lambda u} du.$$

Consider the functions  $\psi_{k,j}$  and  $\varphi_{k,j}$  defined by

$$\psi_{k,j}(u) = F_{k,j}(t-u, x-\rho(i)u), \quad \text{and} \quad \varphi_{k,j}(u) = \psi_{k,j}(u) e^{-\lambda u}.$$

Note that  $\psi_{i,j}(t) = \mathbf{1}_{\{h \leq p\}} \mathbf{1}_{\{i=j\}}$ , so (16) can be written as

$$\varphi_{i,j}(0) = \lambda \sum_{k \in S} P_{i,k} \int_0^t \varphi_{k,j}(u) du + \varphi_{i,j}(t) \tag{18}$$

Differentiating  $\psi_{k,j}$  with respect to  $u$ , we get

$$\psi'_{k,j}(u) = -\frac{\partial F_{k,j}(t-u, x-\rho(i)u)}{\partial t} - \rho(i) \frac{\partial F_{k,j}(t-u, x-\rho(i)u)}{\partial x}.$$

We thus obtain

$$\begin{aligned} \frac{\partial F_{i,j}(t, x)}{\partial t} + \rho(i) \frac{\partial F_{i,j}(t, x)}{\partial x} &= \lambda \sum_{k \in S} P_{i,k} \left[ \varphi_{k,j}(t) - \int_0^t \psi'_{k,j}(u) e^{-\lambda u} du \right] - \lambda \varphi_{i,j}(t) \\ &= \lambda \sum_{k \in S} P_{i,k} \varphi_{k,j}(0) - \lambda^2 \int_0^t \varphi_{k,j}(u) du - \lambda \varphi_{i,j}(t) \\ &= \lambda \sum_{k \in S} P_{i,k} \varphi_{k,j}(0) - \lambda \varphi_{i,j}(0) \\ &= \lambda \sum_{k \in S} P_{i,k} F_{k,j}(t, x) - \lambda F_{i,j}(t, x) \end{aligned}$$

$$= \sum_{k \in S} A_{i,k} F_{k,j}(t, x),$$

where the second equality is obtained by integration by parts and the third follows from relation (18).  $\blacksquare$

Let  $D$  be the diagonal matrix with the reward rates  $\rho(i)$  on the diagonal and  $F(t, x)$  the matrix  $\{F_{i,j}(t, x)\}$ . In matrix notation, the forward and backward equations (14) and (17) become

$$\frac{\partial F(t, x)}{\partial t} = -\frac{\partial F(t, x)}{\partial x} D + F(t, x) A, \quad (19)$$

and

$$\frac{\partial F(t, x)}{\partial t} = -D \frac{\partial F(t, x)}{\partial x} + A F(t, x). \quad (20)$$

These are hyperbolic partial differential equations having a unique solution on the domain  $E$  with the initial condition given by relation (10), see for instance [6].

## 5.2 Solutions

The solution to equation (19) is given by the following theorem:

**Theorem 5.6** *For every  $t > 0$ , and  $x \in [r_{h-1}t, r_h t)$ , for  $1 \leq h \leq m$ ,*

$$F(t, x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^n \binom{n}{k} x_h^k (1 - x_h)^{n-k} C^{(h)}(n, k), \quad (21)$$

where  $x_h = \frac{x - r_{h-1}t}{(r_h - r_{h-1})t}$  and the matrices  $C^{(h)}(n, k) = (C_{B_u B_v}^{(h)}(n, k))_{0 \leq u, v \leq m}$  are given by the recurrence relations

for  $0 \leq u \leq m$ , and  $h \leq v \leq m$  :

$$\text{for } n \geq 0 : C_{B_u B_v}^{(1)}(n, 0) = (P^n)_{B_u B_v}, \quad C_{B_u B_v}^{(h)}(n, 0) = C_{B_u B_v}^{(h-1)}(n, n), \quad \text{for } h > 1,$$

for  $1 \leq k \leq n$  :

$$C_{B_u B_v}^{(h)}(n, k) = \frac{r_v - r_h}{r_v - r_{h-1}} C_{B_u B_v}^{(h)}(n, k-1) + \frac{r_h - r_{h-1}}{r_v - r_{h-1}} \sum_{w=0}^m C_{B_u B_w}^{(h)}(n-1, k-1) P_{B_w B_v}, \quad (22)$$

for  $0 \leq u \leq m$ , and  $0 \leq v \leq h - 1$  :

$$\text{for } n \geq 0 : C_{B_u B_v}^{(m)}(n, n) = 0_{B_u B_v}, \quad C_{B_u B_v}^{(h)}(n, n) = C_{B_u B_v}^{(h+1)}(n, 0), \text{ for } h < m,$$

for  $0 \leq k \leq n - 1$  :

$$C_{B_u B_v}^{(h)}(n, k) = \frac{r_{h-1} - r_v}{r_h - r_v} C_{B_u B_v}^{(h)}(n, k+1) + \frac{r_h - r_{h-1}}{r_h - r_v} \sum_{w=0}^m C_{B_u B_w}^{(h)}(n-1, k) P_{B_w B_v}. \tag{23}$$

**Proof.** For  $t > 0$ , and  $x \in (r_{h-1}t, r_h t)$ ,  $1 \leq h \leq m$ , we write the solution of equation (19) as

$$F(t, x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^n \binom{n}{k} x_h^k (1 - x_k)^{n-k} C^{(h)}(n, k),$$

and we establish the relations that the matrices  $C^{(h)}(n, k)$  must satisfy. So,

$$\begin{aligned} \frac{\partial F(t, x)}{\partial t} &= -\lambda F(t, x) + \frac{\lambda}{r_h - r_{h-1}} \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^n \binom{n}{k} x_h^k (1 - x_h)^{n-k} \\ &\quad \times \left[ r_h C^{(h)}(n+1, k) - r_{h-1} C^{(h)}(n+1, k+1) \right], \end{aligned}$$

and

$$\begin{aligned} \frac{\partial F(t, x)}{\partial x} &= \frac{\lambda}{r_h - r_{h-1}} \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^n \binom{n}{k} x_h^k (1 - x_h)^{n-k} \\ &\quad \times \left[ C^{(h)}(n+1, k+1) - C^{(h)}(n+1, k) \right]. \end{aligned}$$

Since  $A = -\lambda(I - P)$ , we obtain  $F(t, x)A = -\lambda F(t, x) + \lambda F(t, x)P$ , that is,

$$F(t, x)A = -\lambda F(t, x) + \lambda \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^n \binom{n}{k} x_h^k (1 - x_h)^{n-k} C^{(h)}(n, k)P.$$

It follows that if the matrices  $C^{(h)}(n, k)$  satisfy

$$C^{(h)}(n+1, k+1)[D - r_{h-1}I] = C^{(h)}(n+1, k)[D - r_h I] + (r_h - r_{h-1})C^{(h)}(n, k)P, \tag{24}$$

then equation (19) is satisfied. For every  $1 \leq h \leq m$ , and  $0 \leq u \leq m$ , the recurrence relation (24) can also be written as follows:

If  $h \leq v \leq m$ , then

$$C_{B_u B_v}^{(h)}(n, k) = \frac{r_v - r_h}{r_v - r_{h-1}} C_{B_u B_v}^{(h)}(n, k-1) + \frac{r_h - r_{h-1}}{r_v - r_{h-1}} \sum_{w=0}^m C_{B_u B_w}^{(h)}(n-1, k-1) P_{B_w B_v},$$

and if  $0 \leq v \leq h - 1$ , then

$$C_{B_u B_v}^{(h)}(n, k) = \frac{r_{h-1} - r_v}{r_h - r_v} C_{B_u B_v}^{(h)}(n, k+1) + \frac{r_h - r_{h-1}}{r_h - r_v} \sum_{w=0}^m C_{B_u B_w}^{(h)}(n-1, k) P_{B_w B_v}.$$

To get the initial conditions for the  $C^{(h)}(n, k)$ , we consider the jumps of  $F(t, x)$ . We first consider the jump at  $x = r_0 t = 0$ . For  $t > 0$ , at  $x = 0$ , that is, for  $h = 1$ , relation (21) yields that

$$F(t, 0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} C^{(1)}(n, 0).$$

It follows from (10) that for  $0 \leq u, v \leq m$ ,

$$C_{B_u B_v}^{(1)}(n, 0) = (P^n)_{B_u B_v} - (P_{B_0 B_0})^n 1_{\{u=v=0\}}. \tag{25}$$

In particular, that implies that for every  $0 \leq u \leq m$ ,

$$C_{B_u B_v}^{(1)}(n, 0) = (P^n)_{B_u B_v}, \text{ for } 1 \leq v \leq m.$$

Next, we consider the jumps at  $x = r_h t$ ,  $1 \leq h \leq m - 1$ . For  $t > 0$ ,  $1 \leq h \leq m - 1$ , and  $i, j \in S$ , we have that

$$F_{i,j}(t, r_h t) = \lim_{x \nearrow r_h t} F_{i,j}(t, x) - \mathbb{P}\{Y_t = r_h t, X_t = j \mid X_0 = i\}.$$

From (21) and (9), we obtain

$$C_{B_u B_v}^{(h+1)}(n, 0) = C_{B_u B_v}^{(h)}(n, n) - (P_{B_h B_h})^n 1_{\{u=v=h\}}. \tag{26}$$

In particular, that implies that for every  $0 \leq u \leq m$ ,

$$C_{B_u B_v}^{(h)}(n, 0) = C_{B_u B_v}^{(h-1)}(n, n), \text{ for } 1 < h \leq v \leq m,$$

and

$$C_{B_u B_v}^{(h)}(n, n) = C_{B_u B_v}^{(h+1)}(n, 0), \text{ for } 0 \leq v \leq h - 1 < m - 1.$$

Finally, we consider the jump at  $x = r_m t$ , that is, for  $h = m$ . For  $t > 0$ ,

$$0 = F_{i,j}(t, r_m t) = \lim_{x \rightarrow r_m t} F_{i,j}(t, x) - \mathbb{P}\{Y_t = r_m t, X_t = j \mid X_0 = i\},$$

which, as in the preceding case, leads to

$$C_{B_u B_v}^{(m)}(n, n) = (P_{B_m B_m})^n 1_{\{u=v=m\}}. \tag{27}$$

That implies that for every  $0 \leq u \leq m$ ,

$$C_{B_u B_v}^{(m)}(n, n) = 0, \text{ for } 0 \leq v \leq m - 1.$$

■

**Corollary 5.7** *For  $1 \leq h \leq m$ ,  $n \geq 0$ , and  $0 \leq k \leq n$ , the matrices  $C^{(h)}(n, k) = \left(C_{B_u B_v}^{(h)}(n, k)\right)_{0 \leq u, v \leq m}$  satisfy the following recurrence relations for  $h \leq u \leq m$ , and  $0 \leq v \leq m$ :*

$$\text{for } n \geq 0 : C_{B_u B_v}^{(1)}(n, 0) = (P^n)_{B_u B_v}, \quad C_{B_u B_v}^{(h)}(n, 0) = C_{B_u B_v}^{(h-1)}(n, n), \text{ for } h > 1,$$

for  $1 \leq k \leq n$ :

$$C_{B_u B_v}^{(h)}(n, k) = \frac{r_u - r_h}{r_u - r_{h-1}} C_{B_u B_v}^{(h)}(n, k-1) + \frac{r_h - r_{h-1}}{r_u - r_{h-1}} \sum_{w=0}^m P_{B_u B_w} C_{B_w B_v}^{(h)}(n-1, k-1),$$

for  $0 \leq u \leq h - 1$ , and  $0 \leq v \leq m$ :

$$\text{for } n \geq 0 : C_{B_u B_v}^{(m)}(n, n) = 0_{B_u B_v}, \quad C_{B_u B_v}^{(h)}(n, n) = C_{B_u B_v}^{(h+1)}(n, 0), \text{ for } h < m,$$

for  $0 \leq k \leq n - 1$ :

$$C_{B_u B_v}^{(h)}(n, k) = \frac{r_{h-1} - r_u}{r_h - r_u} C_{B_u B_v}^{(h)}(n, k+1) + \frac{r_h - r_{h-1}}{r_h - r_u} \sum_{w=0}^m P_{B_u B_w} C_{B_w B_v}^{(h)}(n-1, k).$$

**Proof.** The proof is the same as that of Theorem 5.6 using equations (20) and (21). We thus obtain that the matrices  $C^{(h)}(n, k)$  satisfy the relation

$$[D - r_{h-1}I]C^{(h)}(n+1, k+1) = [D - r_h I]C^{(h)}(n+1, k) + (r_h - r_{h-1})PC^{(h)}(n, k). \tag{28}$$

For every  $1 \leq h \leq m$ , and  $0 \leq v \leq m$ , the relation (28) may also be written as follows:

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If  $h \leq u \leq m$ , then

$$C_{B_u B_v}^{(h)}(n, k) = \frac{r_u - r_h}{r_u - r_{h-1}} C_{B_u B_v}^{(h)}(n, k-1) + \frac{r_h - r_{h-1}}{r_u - r_{h-1}} \sum_{w=0}^m P_{B_u B_w} C_{B_w B_v}^{(h)}(n-1, k-1),$$

and if  $0 \leq u \leq h-1$ , then

$$C_{B_u B_v}^{(h)}(n, k) = \frac{r_{h-1} - r_u}{r_h - r_u} C_{B_u B_v}^{(h)}(n, k+1) + \frac{r_h - r_{h-1}}{r_h - r_u} \sum_{w=0}^m P_{B_u B_w} C_{B_w B_v}^{(h)}(n-1, k).$$

As in the proof of Theorem 5.6, we consider the jumps of  $F(t, x)$ . Relation (25) implies that, for every  $0 \leq v \leq m$ ,

$$C_{B_u B_v}^{(1)}(n, 0) = (P^n)_{B_u B_v}, \text{ for } 1 \leq u \leq m.$$

Relation (26) implies that, for every  $0 \leq v \leq m$ ,

$$C_{B_u B_v}^{(h)}(n, 0) = C_{B_u B_v}^{(h-1)}(n, n), \text{ for } 1 < h \leq u \leq m,$$

and

$$C_{B_u B_v}^{(h)}(n, n) = C_{B_u B_v}^{(h+1)}(n, 0), \text{ for } 0 \leq u \leq h-1 < m-1.$$

Finally, (27) implies that, for every  $0 \leq v \leq m$ ,

$$C_{B_u B_v}^{(m)}(n, n) = 0, \text{ for } 0 \leq u \leq m-1.$$

■

The following corollary gives an upper bound for the matrices  $C^{(h)}(n, k)$ . If  $M$  and  $K$  are square matrices of the same dimension, the notation  $M \leq K$  means element-wise inequality.

**Corollary 5.8** *For every  $n \geq 0$ ,  $0 \leq k \leq n$ , and  $1 \leq h \leq m$ ,*

$$0 \leq C^{(h)}(n, k) \leq P^n.$$

**Proof.** The proof is by a two-stage induction; first over  $n$ , then, for fixed  $n$ , over  $k$ , by using the recurrence relation in Theorem 5.6, or equivalently in Corollary 5.7. The result clearly holds for  $n = 0$ . Note that in (22), that is,

for  $h \leq v$ , we have

$$0 \leq \frac{r_v - r_h}{r_v - r_{h-1}} = 1 - \frac{r_h - r_{h-1}}{r_v - r_{h-1}} \leq 1,$$

and in (23), that is, for  $v \leq h - 1$ , we have

$$0 \leq \frac{r_{h-1} - r_v}{r_h - r_v} = 1 - \frac{r_h - r_{h-1}}{r_h - r_v} \leq 1.$$

Consider first the case  $v \leq h - 1$ . The result holds for the pair  $(n, n)$ , since  $C_{B_u B_v}^{(m)}(n, n) = 0$ . Suppose the result holds for  $n - 1$  and for the pair  $(n, k + 1)$ , then from (23), we get  $C_{B_u B_v}^{(h)}(n, k) \geq 0$ , and

$$\begin{aligned} C_{B_u B_v}^{(h)}(n, k) &= \frac{r_{h-1} - r_v}{r_h - r_v} C_{B_u B_v}^{(h)}(n, k + 1) + \frac{r_h - r_{h-1}}{r_h - r_v} \sum_{w=0}^m C_{B_u B_w}^{(h)}(n - 1, k) P_{B_w B_v} \\ &\leq \frac{r_{h-1} - r_v}{r_h - r_v} (P^n)_{B_u B_v} + \frac{r_h - r_{h-1}}{r_h - r_v} \sum_{w=0}^m (P^{n-1})_{B_u B_w} P_{B_w B_v} \\ &= \frac{r_{h-1} - r_v}{r_h - r_v} (P^n)_{B_u B_v} + \frac{r_h - r_{h-1}}{r_h - r_v} (P^n)_{B_u B_v} \\ &= (P^n)_{B_u B_v}. \end{aligned}$$

The same argument is used in the case  $h \leq v$  from relation (22). Moreover, the relations

$$C_{B_u B_v}^{(h)}(n, 0) = C_{B_u B_v}^{(h-1)}(n, n), \text{ for } 1 < h \leq v \leq m,$$

and

$$C_{B_u B_v}^{(h)}(n, n) = C_{B_u B_v}^{(h+1)}(n, 0), \text{ for } 0 \leq v \leq h - 1 < m - 1,$$

are used to account for both cases  $v \leq h - 1$  and  $h \leq v$ . ■

In numerical procedures, that result is particularly useful in avoiding overflow problems.

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