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# Exact transient solution of an $M/M/1$ driven fluid queue

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The transient solution of a fluid queue driven by an $M/M/1$ queue is obtained explicitly using continued fractions. The probability that the buffer is empty at a specified time is also determined and illustrated graphically.

*Keywords*: Catalan numbers; Continued fractions; Recurrence relation

*C.R. Categories*: D.4.8; G.3.2

## 1. Introduction

Stochastic fluid flow models are increasingly used in the performance analysis of telecommunication and manufacturing models. The bursty nature of the traffic carried in high-speed networks requires an understanding of steady-state as well as transient behaviour of the system. Fluid models prove to be efficient for these goals.

The steady state behaviour of Markov-driven fluid queues has been extensively studied in the literature. The case where the state space is infinite has been analysed by, among others, Adan and Resing [1], Virtamo and Norros [2], Parthasarathy *et al.* [3] and Barbot and Sericola [4] for an $M/M/1$ queue, and by van Doorn and Schienhardt [5] and Sericola [6] for more general processes.

Steady state analysis provides some important information on the congestion of the statistical multiplexer, but it is not sufficient, for example, for controlling the congestion. Transient analysis will be of critical value in understanding the dynamic behaviour of the statistical multiplexer in controlling the congestion and in studies relating to the rate of convergence to steady state. The problem is motivated by the need to understand better the performance of fast packet switching in the asynchronous transfer mode (ATM), which will be adopted in the Broadband Integrated Service Digital Network (B-ISDN).

The transient analysis of stochastic fluid flow models presents a host of new challenges and opportunities to network designers and performance analysts. In most of the earlier work, the solutions were obtained in the Laplace domain and numerical methods were used to perform

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the inverse Laplace transform [7–9]. Sericola [10] derived the transient solution for fluid flow models controlled by a finite homogenous Markov process based on recurrence relations which is particularly useful for its numerical properties. However, exact solutions are useful in gaining insights and also for comparing the relative merits of different numerical techniques.

In this paper, we obtain the transient solution of a fluid queue driven by an $M/M/1$ queue in terms of a modified Bessel function of the first kind by employing continued fraction methodology. The probability that the buffer is empty at any specified time is determined by modifying the recurrence relation suggested by Barbot and Sericola [11], thereby reducing the computational complexity involved in its evaluation. Numerical investigations of this probability have revealed interesting features of the variations of this function with respect to time.

## 2. Model description

Consider a fluid queue driven by an $M/M/1$ queueing model $\{X(t), t \geq 0\}$, taking values in $S = \{0, 1, 2, \ldots\}$, where $X(t)$ denotes the number of customers in the system at time $t$. Let $\lambda$ and $\mu$ denote the mean arrival and service rates, respectively. The arrivals are in Poisson fashion and the service times are exponentially distributed.

During the busy periods of the server, a fluid commodity which we refer as 'credit' accumulates in an infinite capacity buffer at a constant rate $r > 0$. The credit buffer depletes the fluid during the idle periods of the server at a constant rate $r_0 < 0$ as long as the buffer is non-empty.

If $C(t)$ denotes the content of buffer at time $t$, the two-dimensional process $[\{X(t), C(t)\}, t \geq 0]$ constitutes a Markov process. Fluid models of this type are applied telecommunications for modelling network traffic and in the approximation of discrete stochastic queueing networks. For practical design and performance evaluation, it is essential to obtain information about the buffer occupancy distribution.

If $F_j(t, x) \equiv P[X(t) = j, \ \ C(t) \leq x], \quad j \in \mathcal{S}, \quad t, x \geq 0$, the Kolmogorov forward equations for the Markov process $[X(t), C(t)]$ are given by

$$\begin{aligned}
\frac{\partial F_0(t,x)}{\partial t} &= -r_0 \frac{\partial F_0(t,x)}{\partial x} - \lambda F_0(t,x) + \mu F_1(t,x) \\
\frac{\partial F_j(t,x)}{\partial t} &= -r \frac{\partial F_j(t,x)}{\partial x} + \lambda F_{j-1}(t,x) - (\lambda+\mu) F_j(t,x) + \mu F_{j+1}(t,x), \\
& \quad j \in \mathcal{S}\backslash\{0\}, t, x \geq 0 \qquad (1)
\end{aligned}$$

subject to the initial conditions

$$F_0(0, x) = 1, \quad F_j(0, x) = 0 \quad \text{for } j = 1, 2, 3, \ldots$$

and the boundary condition

$$F_j(t, 0) = q_j(t) \quad \text{for } \ j = 0, 1, 2, \ldots.$$

Here $q_j(t)$ represents the probability that at time $t$ the buffer is empty and the state of the background Markov process is $j$. The content of the buffer decreases and thus it becomes empty only when the net input rate of the fluid into the buffer is negative. Therefore when the buffer becomes empty at any time $t$, the background process should necessarily be in state zero corresponding to which the effective input rate is $r_0 < 0$. Hence we have $q_j(t) = 0$ for $j = 1, 2, 3 \ldots$ as $r > 0$ for $j = 1, 2, 3, \ldots$.

The transient distribution of the buffer content is given by

$$\Pr[C(t) > x] = 1 - \sum_{j=0}^{\infty} F_j(t, x).$$

In this sequel let $F_j^*(s, x)$ and $F_j^{**}(s, w)$ denote the single Laplace transform (with respect to $t$) and double Laplace transform (with respect to $t$ and $x$), respectively, of $F_j(t, x)$.

## 3. Transient solution

Let $P$ denote the transition probability matrix of the uniformized Markov chain associated with the process $[X(t), t \geq 0]$. If $p$ and $q$ are defined by

$$p = \frac{\lambda}{\lambda + \mu} \quad \text{and} \quad q = \frac{\mu}{\lambda + \mu}$$

the non-zero entries of the matrix $P$ are

$$P_{0,0} = q, \quad P_{0,1} = p \quad \text{and} \quad \text{for } i \geq 1, \quad P_{i,i-1} = q, \;\; P_{i,i+1} = p.$$

For every $i \geq 0$, let $(\Omega P^n)_i$ denote the $i$th entry of the row vector $\Omega P^n$ with $\Omega = (1, 0, 0, \ldots)$ denoting the initial probability vector.

The expression for the joint distribution of the buffer content of the fluid queue and the state of the $M/M/1$ queue is given by Barbot and Sericola [11] as

$$F_i(t, x) = \sum_{n=0}^{\infty} \mathrm{e}^{-(\lambda+\mu)t} \frac{[(\lambda + \mu)t]^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{rt}\right)^k \left(1 - \frac{x}{rt}\right)^{n-k} b_i(n, k) \tag{2}$$

where the coefficients $b_i(n, k)$ are given by the following recursive expressions. For $i = 0$

$$\begin{aligned} b_0(n, n) &= (\Omega P^n)_0 \quad \text{for } n \geq 0 \\ b_0(n, k) &= \frac{rq}{r - r_0}[b_0(n-1, k) + b_1(n-1, k)] - \frac{r_0}{r - r_0} b_0(n, k+1) \\ &\qquad \text{for} \quad n \geq 1, \;\; 0 \leq k \leq n-1. \end{aligned} \tag{3}$$

For $i \geq 1$

$$\begin{aligned} b_i(n, 0) &= 0 \quad \text{for } n \geq 0 \\ b_i(n, k) &= p b_{i-1}(n-1, k-1) + q b_{i+1}(n-1, k-1) \quad \text{for } n \geq 1, \;\; 1 \leq k \leq n. \end{aligned} \tag{4}$$

The following propositions and theorem present a simplified formula for evaluating the above recurrence relations and thereby reducing the computational complexity.

PROPOSITION 1 *For all $n \geq 1$, $0 \leq k \leq n-1$, we have*

$$b_0(n, k) - \beta^{n-k} b_0(n, n) = \alpha \sum_{i=k}^{n-1} [b_0(n-1, i) + b_1(n-1, i)] \beta^{i-k} \tag{5}$$

*where $\alpha = rq/(r - r_0)$ and $\beta = r_0/(r_0 - r)$.*

*Proof* Consider equation (3):

$$b_0(n,k) - \beta b_0(n,k+1) = \alpha[b_0(n-1,k) + b_1(n-1,k)] \quad \text{for } n \geq 1,\ 0 \leq k \leq n-1.$$

Multiplying the above equation by $\beta^{i-k}$ and summing over all $i$ from $k$ to $n-1$, we obtain

$$\sum_{i=k}^{n-1} \beta^{i-k} b_0(n,i) - \sum_{i=k}^{n-1} \beta^{i-k+1} b_0(n,i+1) = \alpha \sum_{i=k}^{n-1} [b_0(n-1,i) + b_1(n-1,i)]\beta^{i-k}$$

$$b_0(n,k) - \beta^{n-k} b_0(n,n) = \alpha \sum_{i=k}^{n-1} [b_0(n-1,i) + b_1(n-1,i)]\beta^{i-k}.$$

■

THEOREM 1 *For $i \geq 1$, $b_i(n,k) = 0$ for $0 \leq n < i$ and*

$$b_i(n,k) = \begin{cases} 0 & \text{if } 0 \leq k < i \\ p^i \displaystyle\sum_{l=0}^{\lfloor k-i/2 \rfloor} s(i,l) p^l q^l b_0(n-2l-i, k-2l-i) & \text{if } i \leq k \leq n \\ 0 & \text{if } k > n \end{cases} \tag{6}$$

*where the numbers $s(i,l)$ are referred to as ballot numbers and are given by*

$$s(i,l) = i \frac{(2l+i-1)!}{l!(i+l)!}. \tag{7}$$

*Proof* Consider equation (4):

$$b_i(n,k) = p b_{i-1}(n-1,k-1) + q b_{i+1}(n-1,k-1) \quad \text{for } n \geq 1,\ 1 \leq k \leq n.$$

Define

$$H_i(n,v) = \sum_{k=0}^{n} v^k b_i(n,k).$$

Then equation(4) reduces to

$$H_i(n,v) = pvH_{i-1}(n-1,v) + qvH_{i+1}(n-1,v) \quad i \geq 1, \quad n \geq 1. \tag{8}$$

Again, define

$$\phi_i(u,v) = \sum_{n=0}^{\infty} \frac{u^n H_i(n,v)}{n!}.$$

Then equation (8) reduces to

$$\phi_i'(u,v) = pv\phi_{i-1}(u,v) + qv\phi_{i+1}(u,v).$$

Laplace transformation of the above equation with respect to $u$ yields

$$z\phi_i^*(z,v) = pv\phi_{i-1}^*(z,v) + qv\phi_{i+1}^*(z,v).$$

Writing in the form of continued fractions, we obtain

$$\frac{\phi_i^*(z,v)}{\phi_{i-1}^*(z,v)} = \frac{pv}{z - qv[\phi_{i+1}^*(z,v)/\phi_i^*(z,v)]} = \frac{pv}{z-}\;\frac{pqv^2}{z-}\;\frac{pqv^2}{z-}\cdots.$$

Solving the above continued fractions, we obtain

$$\begin{aligned}\frac{\phi_i^*(z,v)}{\phi_{i-1}^*(z,v)} &= \frac{z-\sqrt{z^2-4pqv^2}}{2qv}, \quad i=1,2,3,\ldots\\ &= \frac{pv}{z}\left(\frac{1-\sqrt{1-4(pqv^2/z^2)}}{2(pqv^2/z^2)}\right)\\ &= \frac{pv}{z}C\left(\frac{pqv^2}{z^2}\right). \qquad (9)\end{aligned}$$

Before we proceed further, we a briefly discuss the function $C(z)$.

Let $C(z)$ be the complex function defined by

$$C(z) = \frac{1-\sqrt{1-4z}}{2z}.$$

For $|z| \le 1/4$, we have

$$C(z) = \sum_{n=0}^{\infty} c_n z^n$$

where the numbers $c_n$ are referred to as Catalan numbers and are given by

$$c_n = \binom{2n}{n}\frac{1}{n+1}.$$

More generally, for $k \ge 1$ and $|z| \le 1/4$, we have

$$C^k(z) = \sum_{n=0}^{\infty} s(k,n) z^n$$

where $s(k,n)$ denotes the ballot numbers given by equation (7).

Continuing our discussion from equation (9), we easily obtain, for $i \ge 1$ and $|pqv^2/z^2| < 1/4$,

$$\begin{aligned}\phi_i^*(z,v) &= \frac{pv}{z}C\left(\frac{pqv^2}{z^2}\right)\phi_{i-1}^*(z,v)\\ &= \frac{p^i v^i}{z^i}C^i\left(\frac{pqv^2}{z^2}\right)\phi_0^*(z,v)\\ &= \frac{p^i v^i}{z^i}\sum_{l=0}^{\infty} s(i,l)\left(\frac{pqv^2}{z^2}\right)^l \phi_0^*(z,v).\end{aligned}$$

Thus we have, for $i \geq 1$ and $|pqv^2/z^2| < 1/4$,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{H_i(n,v)}{z^{n+1}} &= \frac{p^i v^i}{z^i} \sum_{l=0}^{\infty} s(i,l) \left(\frac{pqv^2}{z^2}\right)^l \sum_{n=0}^{\infty} \frac{H_0(n,v)}{z^{n+1}} \\ &= p^i \sum_{l=0}^{\infty} s(i,l) p^l q^l v^{2l+i} \sum_{n=0}^{\infty} \frac{H_0(n,v)}{z^{2l+i+n+1}} \\ &= p^i \sum_{l=0}^{\infty} s(i,l) p^l q^l v^{2l+i} \sum_{n=2l+i}^{\infty} \frac{H_0(n-2l-i,v)}{z^{n+1}} \\ \sum_{n=0}^{\infty} \frac{H_i(n,v)}{z^{n+1}} &= p^i \sum_{n=i}^{\infty} \frac{1}{z^{n+1}} \sum_{l=0}^{\lfloor n-i/2 \rfloor} s(i,l) p^l q^l v^{2l+i} H_0(n-2l-i,v)\end{aligned}$$

where the last equality is obtained by exchanging the order of summation. This leads, for $i \geq 1$, to the following expression for $H_i(n, v)$:

$$H_i(n,v) = \begin{cases} 0 & \text{if } 0 \leq n < i \\ p^i \displaystyle\sum_{l=0}^{\lfloor n-i/2 \rfloor} s(i,l) p^l q^l v^{2l+i} H_0(n-2l-i,v) & \text{if } n \geq i. \end{cases}$$

This means, in particular, that $b_i(n, k) = 0$ for $i \geq 1$ and $0 \leq n < i$.

We consider now the case where $i \geq 1$ and $n \geq i$. By the definition of $H_i(n, v)$, we have

$$\begin{aligned}\sum_{k=0}^{n} v^k b_i(n,k) &= p^i \sum_{l=0}^{\lfloor n-i/2 \rfloor} s(i,l) p^l q^l v^{2l+i} \sum_{m=0}^{n-2l-i} v^m b_0(n-2l-i,m) \\ &= p^i \sum_{l=0}^{\lfloor n-i/2 \rfloor} s(i,l) p^l q^l \sum_{m=0}^{n-2l-i} v^{m+2l+i} b_0(n-2l-i,m) \\ &= p^i \sum_{l=0}^{\lfloor n-i/2 \rfloor} s(i,l) p^l q^l \sum_{k=2l+i}^{n} v^k b_0(n-2l-i,k-2l-i) \\ &= p^i \sum_{k=i}^{n} v^k \sum_{l=0}^{\lfloor \frac{k-i}{2} \rfloor} s(i,l) p^l q^l b_0(n-2l-i,k-2l-i)\end{aligned}$$

where the last equality is obtained by exchanging the order of summation. This leads, for $i \geq 1$ and $n \geq i$, to the following expression for $b_i(n, k)$:

$$b_i(n,k) = \begin{cases} 0 & \text{if } 0 \leq k < i \\ p^i \displaystyle\sum_{l=0}^{\lfloor k-i/2 \rfloor} s(i,l) p^l q^l b_0(n-2l-i,k-2l-i) & \text{if } i \leq k \leq n \\ 0 & \text{if } k > n. \end{cases}$$

In particular, for $i = 1$, the ballot numbers $s(1, l)$ are the Catalan numbers $c_l$. Thus

$$b_1(n,k) = \begin{cases} 0 & \text{if } k = 0 \\ \displaystyle\sum_{l=0}^{\lfloor k-1/2 \rfloor} \binom{2l}{l} \frac{p^{l+1}q^l}{l+1} b_0(n-2l-1, k-2l-1) & \text{if } 1 \le k \le n \\ 0 & \text{if } k > n. \end{cases} \tag{10}$$

■

Note that in the modified recurrence relation given by equation (6), $b_i(n, k)$ is explicitly expressed in terms of $b_0(\cdot, \cdot)$ with lower orders of $n$ and $k$ thereby reducing the computational complexity involved in the evaluation of $b_0(n, 0)$.

PROPOSITION 2 *For all $n \ge 0$, we have*

$$b_0(n,n) = (\Omega P^n)_0 = \begin{cases} \displaystyle\binom{2k}{k} p^k q^k + (q-p) \sum_{i=1}^{k} \binom{2k}{k-i} p^{k-i} q^{k+i-1}, & n = 2k \\ \displaystyle\binom{2k+1}{k} p^k q^{k+1} + (q-p) \sum_{i=1}^{k} \binom{2k+1}{k-i} p^{k-i} q^{k+i}, & n = 2k+1 \end{cases} \tag{11}$$

$$= 1 - \sum_{k=0}^{\lfloor n-1/2 \rfloor} \binom{2k}{k} \frac{p^{k+1}q^k}{k+1}. \tag{12}$$

*Proof* An explicit analytical expression for $(\Omega P^n)_0$ is presented here from a perspective discussed by Böhm [12]. The transition probability matrix $P$ is analysed in relation to the queueing process $Q(n)$ which is regarded as a random walk restricted by a weakly reflecting barrier at the origin having a state space the set of all non-negative integers. It is convenient to consider an *unrestricted* random walk $Z(n)$ having a state space the set of all integers and jumps of magnitude 1 and $-1$ with probabilities $p_1$ and $p_2$, respectively, such that $p_1 + p_2 = 1$. By exploiting the special features of the transition functions of $Z(n)$, we determine its relationship to the queueing process $Q(n)$. It is astonishingly simple to derive the distribution of the process $Q(n)$ from the distribution of the process $Z(n)$.

Let $\omega_k(n) = P[Z(n) = k]$ denote the $n$-step transition probabilities of $Z(n)$. Then

$$\omega_k(n) = [s^k] \left( p_1 s + \frac{p_2}{s} \right)^n$$

where $[s^k]$ is the coefficient operator. Using purely combinatorial approaches, an explicit formula for $\omega_k(n)$ is given by

$$\omega_k(n) = \sum_i \binom{2i-k}{i} \binom{n}{2i-k} p_1^i \, p_2^{i-k}.$$

The function $\omega_k(n)$ is quasi-symmetric:

$$\omega_{-k}(n) = \varrho^{-k} \omega_k(n)$$

where $\varrho = p_1/p_2$.

Let us assume that at time zero there are $m > 0$ customers waiting and let $\sigma(n) = \min_{0\le i\le n} Z(n)$. Then it is well known that

$$Q(n) = \max[Z(n) - \sigma(n), m + Z(n)].$$

Hence we have

$$\Pr[Q(n) < k | Q(0) = m] = \sum_{j>m} \omega_{k-j}(n) - \varrho^k \omega_{-k-j}(n). \tag{13}$$

Comparing the process $Q(t)$ with the uniformized Markov chain associated with the process $X(t)$, we have

$$(\Omega P^n)_0 = \Pr[Q(n) < 1 | Q(0) = 0].$$

For $p_1 = p$ and $p_2 = q$, considerable simplification of equation (13) yields

$$b_0(n,n) = (\Omega P^n)_0 = \begin{cases} \binom{2k}{k} p^k q^k + (q-p) \sum_{i=1}^{k} \binom{2k}{k-i} p^{k-i} q^{k+i-1}, & n = 2k \\ \binom{2k+1}{k} p^k q^{k+1} + (q-p) \sum_{i=1}^{k} \binom{2k+1}{k-i} p^{k-i} q^{k+i}, & n = 2k+1. \end{cases}$$

This can be simplified to

$$(\Omega P^n)_0 = 1 - \sum_{k=0}^{\lfloor n-1/2 \rfloor} \binom{2k}{k} \frac{p^{k+1} q^k}{k+1}$$

which was obtained by Leguesdron *et al.* [13] using a different approach. ■

From equation (2) the probability that the buffer is empty at time $t$ is given by

$$F_0(t,0) = q_0(t) = \mathrm{e}^{-(\lambda+\mu)t} \sum_{n=0}^{\infty} \frac{[(\lambda+\mu)t]^n}{n!} b_0(n,0) \tag{14}$$

where $b_0(n,0)$ for all $n \geq 1$ is given by

$$b_0(n,0) = \left(\frac{rq}{r-r_0}\right) \sum_{i=0}^{n-1} [b_0(n-1,i) + b_1(n-1,i)] \left(\frac{r_0}{r_0-r}\right)^i + \left(\frac{r_0}{r_0-r}\right)^n b_0(n,n).$$

The term $b_0(n-1,i)$ for $0 \leq i \leq n-1$ is recursively determined from equation (5) of proposition 1, the term $b_1(n-1,i)$ for $0 \leq i \leq n-1$ is obtained from equation (6) of theorem 1, and the term $b_0(n,n)$ for all $n$ is calculated using equation (11) of proposition 2.

The convergence of the transient solution given by equation (2) towards the stationary distribution subject to the stability condition

$$\rho = \frac{\lambda(r_0 - r)}{\mu r_0} < 1$$

is given by theorem 7 of [4] as

$$\lim_{t\to\infty} F_i(t,x) = \sum_{k\geq 0} \mathrm{e}^{(\lambda+\mu)x/r} \frac{[(\lambda+\mu)x]^k}{r^k\, k!}\, l_i(k), \quad \text{for all } i \geq 0 \text{ and } x \geq 0$$

where $l_i(k) = \lim_{n\to\infty} b_i(n, k)$. Therefore we conclude that

$$\lim_{t\to\infty} q_0(t) = \lim_{n\to\infty} b_0(n, 0) = 1 - \rho$$

which is numerically verified and illustrated graphically.

## 4. Numerical illustrations

In this section, we discuss the numerical investigation carried out to study the behaviour of $q_0(t)$ given by equation (14) with respect to time. The relations in propositions 1 and 2 and theorem 1 may speed up the computation as illustrated below. **Q1**

Define the truncation step as

$$N(\varepsilon, t) = \min\left\{ n \geq 0 \mid \sum_{h=0}^{n} \mathrm{e}^{-(\lambda+\mu)t} \frac{[(\lambda+\mu)t]^h}{h!} \geq 1 - \varepsilon \right\}. \tag{15}$$

It is easy to check that $N(\varepsilon, t)$ is an increasing function of $t$. Therefore if $q_0(t)$ has to be evaluated at $M$ points, say $t_1 < \cdots < t_M$, we only need to evaluate $b_0(n, 0)$ for $n = 0, 1, \ldots, N(\varepsilon, t_M)$ and compute

$$q_0^{(N)}(t) = \sum_{n=0}^{N(\varepsilon,t_M)} \mathrm{e}^{-(\lambda+\mu)t} \frac{[(\lambda+\mu)t]^n}{n!} b_0(n, 0). \tag{16}$$

Indeed, for every $t \leq t_M$, we have

$$\begin{aligned}
q_0(t) - q_0^{(N)}(t) &= \sum_{n=N(\varepsilon,t_M)+1}^{\infty} \mathrm{e}^{-(\lambda+\mu)t} \frac{[(\lambda+\mu)t]^n}{n!} b_0(n, 0) \\
&\leq \sum_{n=N(\varepsilon,t_M)+1}^{\infty} \mathrm{e}^{-(\lambda+\mu)t} \frac{[(\lambda+\mu)t]^n}{n!} \\
&= 1 - \sum_{n=0}^{N(\varepsilon,t_M)} \mathrm{e}^{-(\lambda+\mu)t} \frac{[(\lambda+\mu)t]^n}{n!} \\
&= 1 - \sum_{n=0}^{N(\varepsilon,t)} \mathrm{e}^{-(\lambda+\mu)t} \frac{[(\lambda+\mu)t]^n}{n!} \\
&\leq \varepsilon
\end{aligned}$$

where the first inequality comes from the fact that the $b_i(n, k)$ are between 0 and 1 as shown in [10].

The algorithm developed to illustrate the variations in the form of graphs is as follows.

ALGORITHM
**input :** $t_1 < \cdots < t_M, \varepsilon$
**output :** $q_0^{(N)}(t_1) < \cdots < q_0^{(N)}(t_M)$
Compute $N = N(\varepsilon, t_M)$ from equation (15).
$b_0(0, 0) = 1; b_1(0, 0) = 0$
**for** = $n = 1$ **to** $N$ **do**

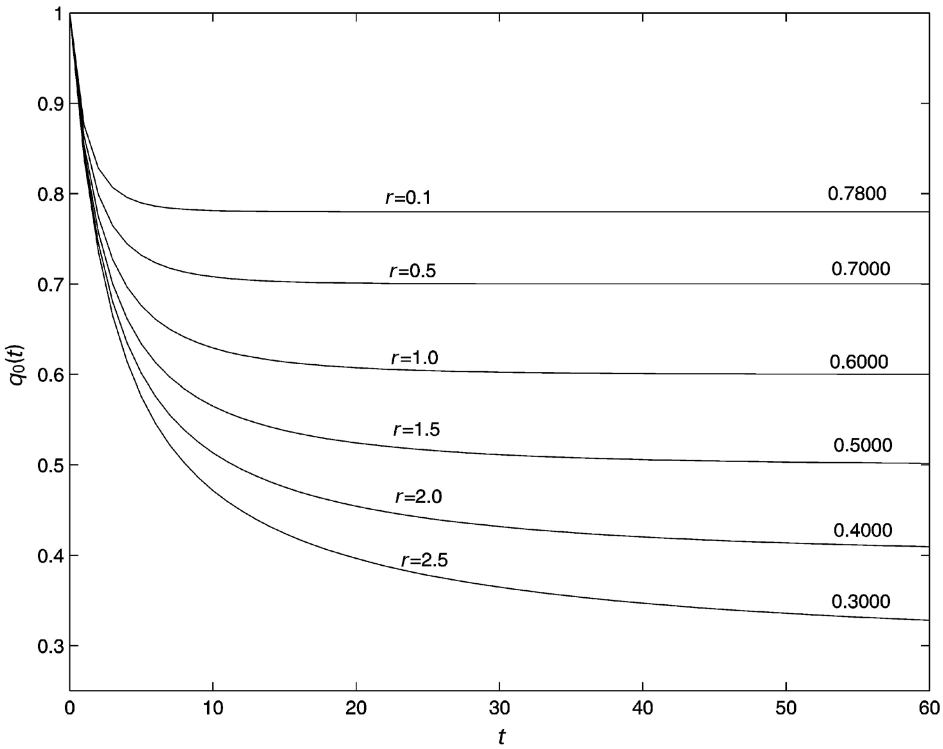


Figure 1. Variation of $q_0(t)$ with $t$ for varying values of $r$ with $\lambda = 0.2$, $\mu = 1$ and $r_0 = -1$.

Compute $b_0(n, n)$ from equation (12).
**for** $k = n - 1$ **step** -1 **to** 0 **do**
Compute $b_0(n, k)$ from equation (3).
**endfor**
$b_1(n, 0) = 0$
**for** $k = 1$ **to** $n$ **do**
Compute $b_1(n, k)$ from equation (10).
**endfor**
**for** = $i = 1$ **to** $M$ **do**
Compute $q_0^{(N)}(t)$ from equation (16).
**endfor**
**endfor**

Figure 1 shows the variation of the function $q_0(t)$ with $t$ for the parameter values $\lambda = 0.2$, $\mu = 1$, $r_0 = -1$ and $\varepsilon = 10^{-5}$, and varying values of $r$. As the net input rate of the fluid increases, the probability with which the buffer becomes empty over a period of time decreases faster and hence $q_0(t)$ approaches a limiting value $1 - \rho$, as seen in figure 1.

## 5. Analytical solution

In this section, we obtain an explicit transient solution for the fluid model under consideration by employing the method of continued fractions to solve the governing system of partial differential equations. Laplace transformation of equation (1) with respect to $t$ yields

$$sF_0^*(s,x) - F_0(0,x) + r_0\frac{\partial F_0^*}{\partial x}(s,x) = -\lambda F_0^*(s,x) + \mu F_1^*(s,x)$$

$$sF_j^*(s,x) - F_j(0,x) + r\frac{\partial F_j^*}{\partial x}(s,x) = \lambda F_{j-1}^*(s,x) - (\lambda+\mu)F_j^*(s,x) + \mu F_{j+1}^*(s,x)$$

$$\text{for } j = 1, 2, 3, \ldots. \tag{17}$$

Taking the Laplace transform of equations (17), again with respect to $x$, we obtain,

$$sF_0^{**}(s,w) - \frac{1}{w} + r_0 w F_0^{**}(s,w) - r_0 F_0^*(s,0) = -\lambda F_0^{**}(s,w) + \mu F_1^{**}(s,w)$$

$$sF_j^{**}(s,w) + rwF_j^{**}(s,w) - rF_j^*(s,0) = \lambda F_{j-1}^{**}(s,w) - (\lambda+\mu)F_j^{**}(s,w) + \mu F_{j+1}^{**}(s,w).$$

Rewriting the above system of equations, we obtain

$$(s + r_0 w + \lambda)F_0^{**}(s,w) - \mu F_1^{**}(s,w) = \frac{1}{w} + r_0 q_0^*(s)$$

$$-\lambda F_{j-1}^{**}(s,w) + (s+\lambda+\mu+rw)F_j^{**}(s,w) - \mu F_{j+1}^{**}(s,w) = 0.$$

These equations can be conveniently rewritten in the form of continued fractions as follows:

$$F_0^{**}(s,w) = \frac{1/w + r_0 q_0^*(s)}{s + r_0 w + \lambda - \mu(F_1^{**}(s,w)/F_0^{**}(s,w))}$$

$$\frac{F_j^{**}(s,w)}{F_{j-1}^{**}(s,w)} = \frac{\lambda}{s+\lambda+\mu+rw - \mu(F_{j+1}^{**}(s,w)/F_j^{**}(s,w))} \quad \text{for } j = 1, 2, 3, \ldots.$$

Define

$$f(s,w) = \frac{1}{s+\lambda+\mu+rw-}\quad \frac{\lambda\mu}{s+\lambda+\mu+rw-}\quad \frac{\lambda\mu}{s+\lambda+\mu+rw-}\cdots$$

$$= \frac{1}{s+\lambda+\mu+rw-\lambda\mu f(s,w)}.$$

That is,

$$\lambda\mu[f(s,w)]^2 - (s+\lambda+\mu+rw)f(s,w) + 1 = 0.$$

Solving the above quadratic equation, we obtain

$$f(s,w) = \frac{(s+\lambda+\mu+rw) - \sqrt{(s+\lambda+\mu+rw)^2 - 4\lambda\mu}}{2\lambda\mu}.$$

Using the above definition, we have

$$\frac{F_j^{**}(s,w)}{F_{j-1}^{**}(s,w)} = \lambda f(s,w) \quad \text{for } \; j = 1, 2, 3\ldots \tag{18}$$

and hence

$$F_0^{**}(s,w) = \frac{1/w + r_0 q_0^*(s)}{s + wr_0 + \lambda - \lambda\mu f(s,w)}$$

$$= \frac{1/w + r_0 q_0^*(s)}{s + wr_0 + \lambda - ((s+\lambda+\mu+rw) - \sqrt{(s+\lambda+\mu+rw)^2 - 4\lambda\mu}/2)}.$$

Let us denote $w + (s + \lambda + \mu)/r = \theta$ and $(2\sqrt{\lambda\mu})/r = \nu$; then

$$F_0^{**}(s, w) = \frac{1/w + r_0 q_0^*(s)}{r_0\,(w + (s+\lambda)/r_0) - r/2(\theta - \sqrt{\theta^2 - \nu^2})}$$
$$= \left[q_0^*(s) + \frac{1}{r_0 w}\right] \sum_{k=0}^{\infty} \left(\frac{r}{2r_0}\right)^k \frac{(\theta - \sqrt{\theta^2 - \nu^2})^k}{(w + (s+\lambda)/r_0)^{k+1}} \quad \text{for} \left|\frac{\lambda\mu f(s, w)}{r_0 w + s + \lambda}\right| < 1.$$

Also from equation (18), we have

$$F_j^{**}(s, w) = [\lambda f(s, w)]^j F_0^{**}(s, w) \quad \text{for } j = 1, 2, 3 \ldots$$
$$= \left(\frac{r}{2\mu}\right)^j \left[(q_0^*(s) + \frac{1}{r_0 w}\right] \sum_{k=0}^{\infty} \left(\frac{r}{2r_0}\right)^k \frac{(\theta - \sqrt{\theta^2 - \nu^2})^{j+k}}{(w + (s+\lambda)/r_0)^{k+1}}.$$

The inversion of the above equation is quite cumbersome, and after considerable simplification we obtain the following theorem.

THEOREM 2 *For every $t \geq 0$ and $x \in [0, rt)$, we have*

$$F_0(t, x) = \mathrm{e}^{(-\lambda x/r_0)} q_0\left(t - \frac{x}{r_0}\right) + \mathrm{e}^{-\lambda t} - \mathrm{e}^{(-\lambda x/r_0)} \mathrm{e}^{-\lambda(t - x/r_0)}$$
$$+ \sum_{k=1}^{x} \left(\frac{r}{2r_0}\right)^k \int_0^{\infty} \mathrm{e}^{[-(\lambda+\mu)/r](x-y)} \frac{\nu^k k I_k[\nu(x-y)]}{k!(x-y)}$$
$$\times \left[y^k \mathrm{e}^{(-\lambda y/r_0)} H\left(t - \frac{x-y}{r} - \frac{y}{r_0}\right) q_0\left(t - \frac{x-y}{r} - \frac{y}{r_0}\right)\right.$$
$$\left. + r_0^k e^{-\lambda[t-(x-y)/r]} \left(t - \frac{x-y}{r}\right)^k\right] \mathrm{d}y. \tag{19}$$

*and*

$$F_j(t, x) = \left(\frac{r}{2\mu}\right)^j \sum_{k=0}^{\infty} \left(\frac{r}{2r_0}\right)^k \int_0^x \mathrm{e}^{[-(\lambda+\mu)/r](x-y)} (j+k) \frac{\nu^{j+k} I_{j+k}[\nu(x-y)]}{k!(x-y)}$$
$$\times \left[y^k \mathrm{e}^{-\lambda y/r_0} H\left(t - \frac{x-y}{r} - \frac{y}{r_0}\right) q_0\left(t - \frac{x-y}{r} - \frac{y}{r_0}\right)\right.$$
$$\left. + r_0^k \mathrm{e}^{-\lambda[t-(x-y)/r]} \left(t - \frac{x-y}{r}\right)^k\right] \mathrm{d}y \quad \textit{for } j = 1, 2, \ldots \tag{20}$$

*where $H(\cdot)$ denotes the Heaviside function.*

It is verified that as $t \to \infty$, the transient probabilities given by equations (19) and (20) tend to the stationary solution given by equations (16) and (20), respectively, of [3] subject to the understanding $\lim_{t\to\infty} q_0(t) = F_0(0)$ in the stationary case.


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