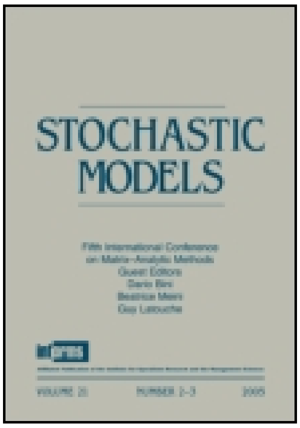


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# Distribution of busy period in stochastic fluid models

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# DISTRIBUTION OF BUSY PERIOD IN STOCHASTIC FLUID MODELS*

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## ABSTRACT

We consider the busy period in a stochastic fluid flow model with infinite buffer where the input and output rates are controlled by a finite homogeneous Markov process. We derive an explicit expression for the distribution of the busy period and we obtain an algorithm to compute it which exhibits nice numerical properties.

*Key Words*: Stochastic fluid model; Busy period; Markov process; Numerical analysis

## 1. INTRODUCTION

Stochastic fluid models (SFM) are widely applied to capture the queueing behaviour of packet switched networks with large buffers (1). An SFM is composed by a buffer and a background process that modulates the rate of the fluid accumulation in the buffer. The modulating process is commonly assumed to be a continuous time Markov chain (CTMC).

The transient analysis of SFMs, i.e., the analysis of fluid distribution in the buffer at time $t$, is a complex and computationally intensive task. The cardinality

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of the problem is characterized by the number of states of the modulating process. The majority of the published analysis approaches requires the spectral decomposition of a matrix of size of the state space (2,3). The applicability of this approach is limited by the computational complexity and potential numerical instability due to close eigenvalues. In some special cases it is possible to obtain an analytical solution exploiting the special behaviour of the modulating process (4). For general modulating processes a numerically stable recursive method was proposed in (5). There are other important transient measures of SFMs that are considered in the literature. The importance of the distribution of the busy period[1] was introduced in (6), where a SFM with two priorities is studied. In the considered model, the higher priority stream occupies the server capacity as long as there is "high priority fluid" in the buffer and the low priority stream gets service only when there is no high priority fluid in the buffer. The low priority stream is served with server vacation, where the server vacation is the busy period of the high priority stream.

In this paper we provide a stable numerical method to evaluate the distribution of the busy period in SFMs with infinite capacity. The remainder of the paper is organized as follows. Section 2 provides the proposed numerical procedure, while Section 3 introduces a numerical example.

## 2. STOCHASTIC FLUID MODELS

Let $\{Z(t),\ t \geq 0\}$ be an irreducible CTMC on a finite state space $S$ with generator $A = [a_{ij}]$ and let $a_i = -a_{ii}$. We denote by $\pi = (\pi_i)$ the stationary distribution of $\{Z(t)\}$. Whenever the CTMC stays in state $i$, the fluid level of the buffer is increasing at rate $d_i$. $d_i$ is often referred to as the drift or the effective rate of state $i$. When $d_i < 0$ it means that the fluid level is decreasing in the buffer. Of course, the fluid level cannot decrease below 0. $Q(t)$ denotes the level of fluid in the buffer at time $t$. The dynamics of the fluid level process $\{Q(t),\ t \geq 0\}$ can be described as follows:

$$\frac{dQ(t)}{dt} = d_{Z(t)} \quad \text{when} \quad Q(t) > 0,$$

$$\frac{dQ(t)}{dt} = max(d_{Z(t)}, 0) \quad \text{when} \quad Q(t) = 0.$$

[1]This measure is referred to as first passage time in (6), but we follow a different naming convention, because the analysis of the first passage time to a general fluid level is a more complicated problem than the analysis of the first passage time to empty buffer.

## 2.1 Analysis of Busy Period

A busy period is the period of time while the buffer contains a positive amount of fluid (Fig. 1). The very first busy period differs from the consecutive ones because the buffer is empty and the drift is positive at the beginning of all busy periods except the first one. The first busy period, starting from time 0, can have positive fluid level and/or negative drift at its beginning. Without loss of generality we devote attention only to the first busy period in this paper. The length of further busy periods can be obtained as the special case when the initial fluid level is 0 and the drift is positive. The random time $T$ is defined by

$$T = \inf\{t > 0 | Q(t) = 0\}.$$

The distribution of the random time $T$ conditioned on the initial fluid level and on the initial state $Z(0)$ is defined as:

$$F_i(t, x) = \Pr(T \leq t | Z(0) = i, Q(0) = x).$$

By the given model definition $T = 0$ if $Q(0) = 0$ and $d_i \leq 0$, otherwise $\Pr(T - 0) = 0$. We assume that the stability condition, $\sum_{j \in S} d_j \pi_j < 0$, is satisfied, so that the random time $T$ is finite a.s.

**Theorem 1.** *$F_i(t,x)$ satisfies the backward differential equation*

$$\frac{\partial F_i(t,x)}{\partial t} - d_i \frac{\partial F_i(t,x)}{\partial x} = \sum_{k \in S} a_{ik} F_k(t,x) \quad \textit{if} \quad x > 0 \tag{1}$$

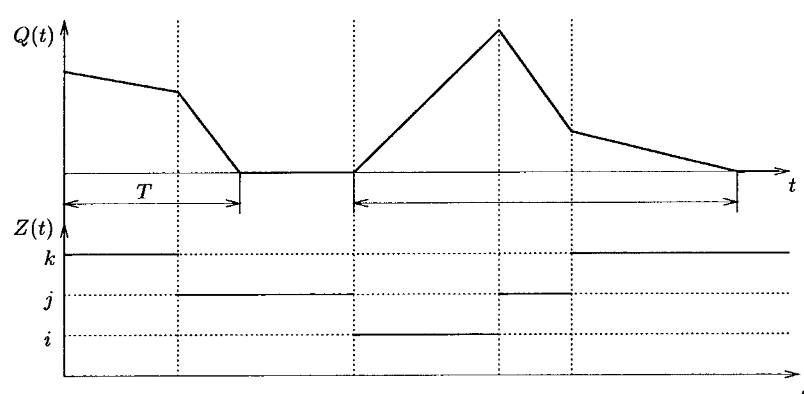


***Figure 1.*** Busy periods of a stochastic fluid model.

*with initial conditions*

$$F_i(t,0) = 1 \ \ if \ \ t \geq 0 \ \ and \ \ d_i \leq 0,$$

$$F_i(0,x) = 0 \ \ if \ \ x > 0,$$

$$F_i(0,0) = 0 \ \ if \ \ d_i > 0.$$

**Proof.** As it is provided in (6), the backward argument that describes the evolution of the process is:

$$F_i(t,x) = (1 - a_i\Delta)F_i(t-\Delta, x + d_i\Delta) + \sum_{k\in S, k\neq i} a_{ik}\Delta F_k(t-\Delta, x+d_i\Delta) + o(\Delta),$$

which gives the theorem by algebraic manipulations and letting $\Delta \rightarrow 0$. □

Let $|S|$ be the number of states in $S$ and let $m+1, m < |S|$ be the number of distinct values among all the effective rates $d_i$. These $m+1$ distinct effective rates are denoted by $r_0, r_1, \ldots, r_m$ and ordered as follows

$$r_m > r_{m-1} > \ldots > r_v \geq 0 > r_{v-1} > \ldots > r_1 > r_0,$$

where $v$ is the number of negative effective rates. The state space $S$ of the process $\{Z(t)\}$ can then be divided into $m+1$ disjoint subsets $B_m, B_{m-1}, \ldots, B_0$ where $B_i$ is composed by the states $i$ of $S$ having the same effective rate $r_i$, that is $B_i = \{j \in S | d_j = r_i\}$. $|B_i|$ denotes the cardinality of subset $B_i$.

If $v = 0$ the buffer never becomes empty after time 0, so we have $T = \infty$. Thus, we suppose without loss of generality that $v \geq 1$.

With this notation, we have, with probability 1,

$$T \in \begin{cases} \bigcup_{j=0}^{v-1} \left[ -\frac{x}{r_j}, -\frac{x}{r^+_{j+1}} \right) & \text{if } x > 0, \\ [0,\infty) & \text{if } x = 0, \end{cases}$$

where $r^+_{j+1} = r_{j+1}$ for $j = 0, \ldots, v-2$ and $r^+_{j+1} = 0$ for $j = v-1$, so that $-x/r^+_v = +\infty$.

For $x > 0$, the distribution of $T$ has $v$ jumps at points $-x/r_j$ for $j = 0, \ldots, v-1$. If $x = 0$, the distribution of $T$ has only one jump at point 0. For $x > 0$, the jump at point $-x/r_j$ corresponds to a sojourn of the Markov process $\{Z(t)\}$ in the subset $B_j$ that starts at time 0 and ends after time $-x/r_j$ (Fig. 2). These jumps are

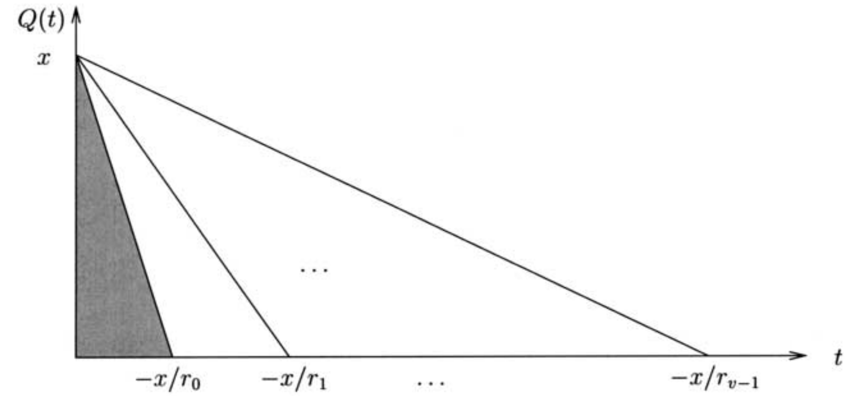


***Figure 2.*** Paths corresponding to the jumps of $F_i(t,x)$.

given for $x > 0$ and $j = 0, \ldots, v-1$ by

$$\Pr\left(T = -\frac{x}{r_j} | Z(0) = i, Q(0) = x\right) = \begin{cases} h_{B_j}(i)\, e^{-A_{B_jB_j}\frac{x}{r_j}} 1_{B_j} & \text{if } i \in B_j, \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

where $A_{B_jB_j}$ is the sub-infinitesimial generator of dimension $|B_j|$ obtained from $A$ by considering only the internal transitions of the subset $B_j$, $1_{B_j}$ is the column vector of dimension $|B_j|$ with all its entries equal to 1 and $h_{B_j}(i)$ is the row vector of dimension $|B_j|$ with entry $i$ equal to 1 and all other entries equal to 0.

Let $P$ be the transition probability matrix of the uniformized Markov chain associated to $\{Z(t)\}$ and by $\lambda$ the uniformization rate which verifies $\lambda \geq \max(a_i, i \in S)$. The matrix $P$ is then related to $A$ by $P = I + A/\lambda$, where $I$ denotes the identity matrix. In the following, to simplify notation, we will consider $\{Z(t)\}$ as the uniformized process. For every $i, j = 0, \ldots, m$, we denote by $P_{B_iB_j}$ the submatrix of $P$ containing the transition probabilities from states of $B_i$ to states of $B_j$.

The distribution of the first time the buffer becomes empty, $T$, is given in the following theorem that applies the same approach as in (5). The notation $0_{B_l}$ stands for the null column vector of dimension $|B_l|$.

**Theorem 2.** *For every $i \in S$ and $x > 0$, we have*

$$F_i(t,x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} p_j^k (1-p_j)^{n-k} b_i^{(j)}(n,k), \tag{3}$$

*where* $j = 0, 1, \ldots, v-1$ *is such that*

$$t \in \left[-\frac{x}{r_j}, -\frac{x}{r_{j+1}^+}\right).$$

*Then*

$$p_j = \frac{x + r_j t}{(r_j - r_{j+1}^+)l}.$$

*The coefficients* $b_i^{(j)}(n,k)$ *are given by the following recursive expressions on the column vectors* $b_{B_l}^{(j)}(n,k) = (b_i^{(j)}(n,k))_{i \in B_l}$ *for* $0 \le l \le m$ *and* $0 \le j \le v-1$,

for $j+1 \le l \le m$ :

for $n \ge 0 : b_{B_l}^{(0)}(n,0) = 0_{B_l}$ and $b_{B_l}^{(j)}(n,0) = b_{B_l}^{(j-1)}(n,n)$ for $j > 0$,
for $1 \le k \le n$ :

$$b_{B_l}^{(j)}(n,k) = \frac{r_l - r_{j+1}^+}{r_l - r_j} b_{B_l}^{(j)}(n,k-1) + \frac{r_{j+1}^+ - r_j}{r_l - r_j} \sum_{i=0}^{m} P_{B_l B_i} b_{B_i}^{(j)}(n-1,k-1),$$

for $0 \le l \le j$ :

for $n \ge 0 : b_{B_l}^{(v-1)}(n,n) = 1_{B_l}$ and $b_{B_l}^{(j)}(n,n) = b_{B_l}^{(j+1)}(n,0)$ *for* $j < v-1$,
for $0 \le k \le n-1$ :

$$b_{B_l}^{(j)}(n,k) = \frac{r_j - r_l}{r_{j+1}^+ - r_l} b_{B_l}^{(j)}(n,k+1) + \frac{r_{j+1}^+ - r_j}{r_{j+1}^+ - r_l} \sum_{i=0}^{m} P_{B_l B_i} b_{B_i}^{(j)}(n-1,k).$$

**Proof.** See Appendix A. □

The special case when the initial fluid level is 0 (i.e. $Q(0) = 0$) is considered in the following corollary.

**Corollary 3.** *For* $v = 1$ *and for every* $i \in S$, *we have*

$$F_i(t,0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} b_i(n,n), \tag{4}$$

*where the coefficients* $b_i(n,k)$ *are given by the following recursive expressions on the column vectors* $b_{B_l}(n,k)$ *for* $0 \le l \le m$, for $v \leq l \leq m$ :

for $n \geq 0 : b_{B_l}(n,0) = 0_{B_l}$,
for $1 \leq k \leq n$ :

$$b_{B_l}(n,k) = \frac{r_l}{r_l - r_{v-1}} b_{B_l}(n,k-1) + \frac{-r_{v-1}}{r_l - r_{v-1}} \sum_{i=0}^{m} P_{B_l B_i} b_{B_i}(n-1,k-1), \quad (5)$$

for $0 \leq l \leq v-1$ :

for $n \geq 0 : b_{B_l}(n,n) = 1_{B_l}$,
for $0 \leq k \leq n-1$ :

$$b_{B_l}(n,k) = \frac{r_l - r_{v-1}}{r_l} b_{B_l}(n,k+1) + \frac{r_{v-1}}{r_l} \sum_{i=0}^{m} P_{B_l B_i} b_{B_i}(n-1,k). \quad (6)$$

**Proof.** When $x = 0$ we have $T \in [0, +\infty)$. This corresponds to the case $j = v-1$ in Theorem 2. By taking $x = 0$ and $j = v-1$ in equation (3) we get relation (4) since in this case $p_j = 1$. The recurrence relations satisfied by the $b_{B_l}(n,k)$ are then easily obtained by taking $j = v-1$ in the recurrence relations of Theorem 2. □

Note that the relations (5) and (6) are convex combinations of vectors since we have

$$0 \leq \frac{r_l}{r_l - r_{v-1}} = 1 - \frac{-r_{v-1}}{r_l - r_{v-1}} \leq 1, \quad \text{for } v \leq l \leq m,$$

and

$$0 \leq \frac{r_l - r_{v-1}}{r_l} = 1 - \frac{r_{v-1}}{r_l} \leq 1, \quad \text{for } 0 \leq l \leq v-1.$$

## 2.2 Properties of the Numerical Procedure

In practical applications the analysis of busy period with initially empty buffer and only one negative drift ($v = 1$) is much more common. Fortunately, both the computational complexity and the memory requirement of the numerical method based on Corollary 3 are significantly less in this case. The computational complexity of the analysis procedure of this particular case can be further reduced using the results provided in the following theorem.

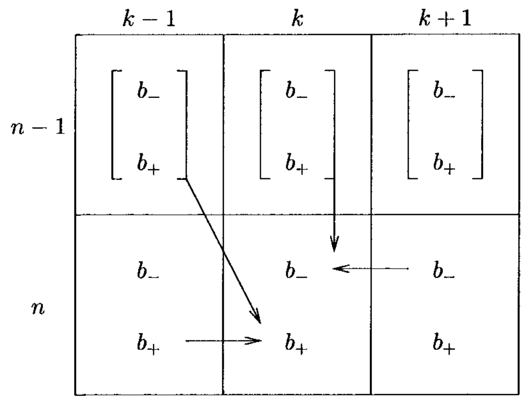


***Figure 3.*** Computation of the $b_i(n,k)$.

**Theorem 4.** *The $b_{B_l}(n,k)$ vectors piecewise satisfy the following inequalities:*

(a) $0_{B_l} \le b_{B_l}(n,k) \le 1_{B_l}$ *for* $0 \le l \le m, n \ge 0, 0 \le k \le n$,
(b) $b_{B_l}(n,k) \le b_{B_l}(n+1,k+1)$ *for* $0 \le l \le m, n \ge 0, 0 \le k \le n$,
(c) $b_{B_l}(n,k) \ge b_{B_l}(n+1,k)$ *for* $0 \le l \le m, n \ge 0, 0 \le k \le n$,
(d) $b_{B_l}(n,k) \le b_{B_l}(n,k+1)$ *for* $0 \le l \le m, n \ge 0, 0 \le k < n$,
(e) $b_{B_l}(n,k) \ge b_{B_l}(n+1,k-1)$ *for* $0 \le l \le m, n \ge 0, 1 \le k \le n$,
(f) $\lim_{n\to\infty} b_{B_l}(n,n) = 1_{B_l}$ *for* $0 \le l \le m$.

**Proof.** See Appendix B. □

The computation of the $b_i(n,k)$ can be illustrated using Fig. 3. In this figure, we represent the $b_i(n,k)$ using column vectors $b_-(n,k) = (b_{B_l}(n,k))_{0\le l\le v-1}$ and $b_+(n,k) = (b_{B_l}(n,k))_{v\le l\le m}$ and we show graphically the relations (5) and (6) used for the computation of $b_+(n,k)$ and $b_-(n,k)$.

Suppose, without any loss of generality, that the initial state $i$ is fixed. For a given error tolerance $\varepsilon$, we define integer $N'$ as

$$N' = \min\left\{ n \in \mathrm{N} \left| (1 - b_i(n,n)) \left(1 - \sum_{r=0}^{n} e^{-\lambda t} \frac{(\lambda t)^r}{r!}\right) \le \varepsilon \right. \right\}.$$

Note that the value of $N'$ will be known only a posteriori since it depends on the $b_i(n,k)$. An upper bound of $N'$, available a priori, that is before the computation of the $b_i(n,k)$, is the classical truncation step of the Poisson series given by

$$N = \min\left\{ n \in \mathrm{N} \left| \left( 1 - \sum_{r=0}^{n} e^{-\lambda t} \frac{(\lambda t)^r}{r!} \right) \leq \varepsilon \right. \right\}. \tag{7}$$

From Theorem 4, inequality (a), we obtain $N' \leq N$. Using the truncation step $N'$, we get

$$F_i(t,0) = 1 - \sum_{n=0}^{N'} e^{-\lambda t} \frac{(\lambda t)^n}{n!} + \sum_{n=0}^{N'} e^{-\lambda t} \frac{(\lambda t)^n}{n!} b_i(n,n) - e(N'),$$

where the rest of the series $e(N')$ satisfies

$$e(N') = \sum_{n=N'+1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} (1 - b_i(n,n))$$
$$\leq (1 - b_i(N',N')) \left( 1 - \sum_{n=0}^{N'} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \right) \leq \varepsilon.$$

Another way to reduce the computational complexity is to avoid the calculation of the vectors $b_{B_l}(n,k)$ when all of their components are less than or equal to a given value $\varepsilon'$. It is easy to check based on expression (5) that if the vectors $b_{B_i}(n,k-1)$ and $b_{B_l}(n-1,k-1)$ have all their entries less than or equal to $\varepsilon'$ then the vector $b_{B_l}(n,k)$ has also all its entries less than or equal to $\varepsilon'$. The same result holds for the appropriate terms in expression (6). This property is due to the fact that both expressions (5) and (6) are convex combinations of vectors. This property together with Theorem 4 suggest us to further reduce the computation of $b_{B_l}(n,k)$ vectors. More precisely, let us define, for a given value of $\varepsilon'$ the integers $N_0$, $N_1$,… and $N''$ as

$$N_0 = \min\{1 \leq n \leq N'' - 1 | b_{B_l}^{[0]}(n,0) \leq \varepsilon' 1_{B_l} \text{ for } l = 0, \ldots, v-1\}, \tag{8}$$

for $h \geq 1$,

$$N_h = \min\{N_{h-1} + 1 \leq n \leq N'' - 1 | b_{B_l}^{[h]}(n,h) \leq \varepsilon' 1_{B_l} \text{ for } l = 0, \ldots, v-1\}, \tag{9}$$

and, for the fixed initial state $i$,

$$N'' = \min\left\{ n \in N \left| \left(1 - b_i^{[H+1]}(n,n)\right) \left( 1 - \sum_{r=0}^{n} e^{-\lambda t} \frac{(\lambda t)^r}{r!} \right) \right| \leq \varepsilon \right\}, \tag{10}$$

where,

(1) for $0 \leq l \leq m$, $0 \leq n \leq N_0$, and $0 \leq k \leq n$ :

$$b_{B_l}^{[0]}(n,k) = b_{B_l}(n,k),$$

(2) for $h \geq 1$, $0 \leq l \leq m$ and $h \leq k \leq N_{h-1}$ :

$$b_{B_l}^{[h]}(N_{h-1},k) = b_{B_l}^{[h-1]}(N_{h-1},k),$$

(3) for $h \geq 1$ and $0 \leq l \leq m$ :
for $v \leq l \leq m, N_{h-1}+1 \leq n$ and $h+1 \leq k \leq n$ :

$$b_{B_i}^{[h]}(n,h) = 0_{B_l},$$

$$b_{B_l}^{[h]}(n,k) = \frac{r_l}{r_l - r_{v-1}} b_{B_l}^{[h]}(n,k-1) + \frac{-r_{v-1}}{r_l - r_{v-1}} \sum_{i=0}^{m} P_{B_l B_i} b_{B_i}^{[h]}(n-1,k-1),$$

for $0 \leq l \leq v-1, N_{h-1}+1 \leq n$ and $h \leq k \leq n-1$ :

$$b_{B_l}^{[h]}(n,n) = 1_{B_l}$$

$$b_{B_l}^{[h]}(n,k) = \frac{r_l - r_{v-1}}{r_l} b_{B_l}^{[h]}(n,k+1) + \frac{r_{v-1}}{r_l} \sum_{i=0}^{m} P_{B_l B_i} b_{B_i}^{[h]}(n-1,k),$$

(4) the index of the greatest considered $N_h$ is

$$H = \max\{h | N_h \leq N'' - 1\}.$$

In the above list, Item (1) represents the initialization step for $h = 0$, and Item (2) for $h \geq 1$. Item (3) provides the application of (5) and (6) and the approximation of the negligible vectors. The proposed numerical procedure calculates $b_{B_l}^{[h]}(n,k)$ instead of $b_{B_l}(n,k)$ for $N_{h-1} < n \leq N_h$ with an initial value provided for $b_{B_l}^{[h]}(N_{h-1},k)$ by Item 2). Finally, Item (4) defines the greatest level of reduction used in the numerical method. Note that $N'$ as well as $N_h (0 \leq 1 \leq H)$ and $H$ are obtained during the execution of the numerical procedure (a posteriori). This is how the mutually dependence in the definition of $N''$ and $H$ is resolved. By

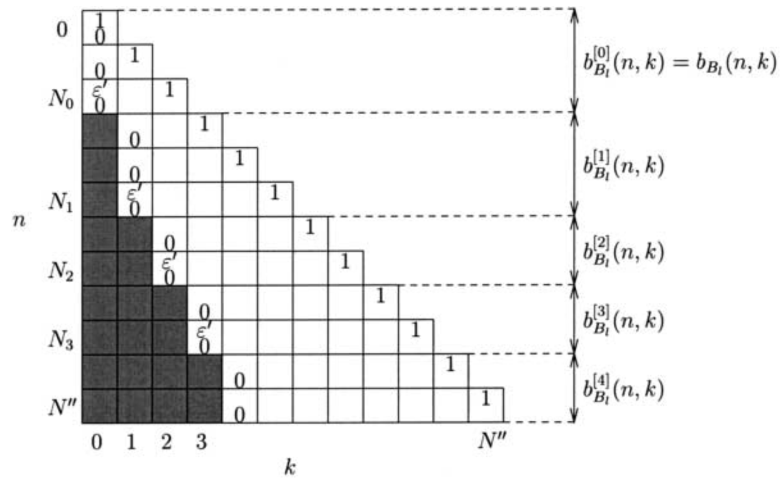


***Figure 4.*** In gray, the cells $(n,k)$ that are not computed.

the definition of $N_h$, we have $h + 1 \leq N_h \leq N'' - 1$, so

$$H \leq N'' - 2. \tag{11}$$

All these mechanisms are illustrated in Fig. 4, where $H = 3$. In this figure, we represent the initial conditions for vectors $b_-(n,k)$ and $b_+(n,k)$ described in Fig. 3. The vector 1 means that we have $b_-^{[h]}(n,n) = 1_-$ and $\varepsilon'$ means that $b_-^{[h]}(N_h,h) \leq \varepsilon' 1_-$ by definition of $N_h$. We obtain, in particular, from relation (5), that

$$b_+^{[h+1]}(N_h + 1, h + 1) \leq \varepsilon' 1_+$$

and so, to avoid its computation we set $b_+^{[h+1]}(N_h + 1, h + 1) = 0_+$ and we also set $b_+^{[h+1]}(n, h + 1) = 0_+$ for $n \geq N_h + 2$. The cells in gray, in Fig. 4, are not calculated. Let us now evaluate the error introduced by the use of the $b_{B_l}^{[h]}(n,k)$ instead of the $b_{B_l}(n,k)$. It is easy to check that Theorem 4 is still valid for all the $b_{B_l}^{[h]}(n,k), h = 0, \ldots, H + 1$. It follows in particular that the integer $N''$ exists. Moreover, for $h \geq 1$, $n \geq N_{h-1} + 1$, $h \leq k \leq n$, and $l = 0, \ldots, m$, we have

$$b_{B_l}^{[h]}(n,k) \leq b_{B_l}^{[h-1]}(n,k)$$

since we start the computation of the $b_{B_l}^{[h]}(n,k)$ when $b_{B_l}^{[h-1]}(N_{h-1}, h - 1) \leq \varepsilon'$ for $l = 0, \ldots, v - 1$, and we set $b_{B_l}^{[h]}(N_{h-1} + 1, h) = 0$ for $l = v, \ldots, m$. Based on these remarks, it can be easily checked by induction that for every $i \in S, n \geq 0$, $0 \leq$

$k \leq n$, and $h = 1, \ldots, H+1$, we have

$$0 \leq b_i^{[h-1]}(n,k) - b_i^{[h]}(n,k) \leq \varepsilon'. \tag{12}$$

In order to simplify writing, we define $N_{-1} = -1$ and $N_{H+1} = N''$. The quantity that is really computed is $F_i(t)$ which is given by

$$\tilde{F}_i(t) = 1 - \sum_{n=0}^{N''} e^{-\lambda t}\frac{(\lambda t)^n}{n!} + \sum_{h=0}^{H+1}\sum_{n=N_{h-1}+1}^{N_h} e^{-\lambda t}\frac{(\lambda t)^n}{n!} b_i^{[h]}(n,n).$$

Let us denote by $E$ the error so obtained. We have

$$\begin{aligned} E &= F_i(t,0) - \tilde{F}_i(t) \\ &= \sum_{n=N''+1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}(1 - b_i(n,n)) - \sum_{h=0}^{H+1}\sum_{n=N_{h-1}+1}^{N_h} e^{-\lambda t}\frac{(\lambda t)^n}{n!}(b_i(n,n) \\ &\quad - b_i^{[h]}(n,n)). \end{aligned}$$

We denote respectively by $e_1\,(N'')$ and $e_2\,(N'')$ the first and second terms of the right hand side. From Theorem 4, (12) and (10), we have that

$$\begin{aligned} 0 \leq e_1(N'') &= \sum_{n=N''+1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}(1 - b_i(n,n)) \\ &\leq (1 - b_i(N'',N''))\left(1 - \sum_{n=0}^{N''} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\right) \\ &\leq (1 - b_i^{[H+1]}(N'',N''))\left(1 - \sum_{n=0}^{N''} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\right) \leq \varepsilon. \end{aligned}$$

For $h = 0$, we have by definition $=b_i(n,n) = b_i^{[0]}(n,n)$, and for $h \geq 1$, we have

$$0 \leq b_i(n,n) - b_i^{[h]}(n,n) = \sum_{u=1}^{h}(b_i^{[u-1]}(n,n) - b_i^{[u]}(n,n)) \leq h\varepsilon'.$$

***Table 1.*** Algorithm for the Computation of the Busy Period Distribution

```
input: ε, i, t_1 < · · · < t_M.
output: F̃_i(t_j), for j = 1, ..., M.
Compute N from relation (7) with t = t_M;
N'' = N;
ε' = ε/(N − 1);
for l = 0 to v − 1 do b_{B_l}(0,0) = 1_{B_l}; endfor
for l = v to m do b_{B_l}(0,0) = 0_{B_l}; endfor
h = 0;
for n = 1 to N do
   for l = 0 to v − 1 do b_{B_l}(n,n) = 1_{B_l}; endfor
   for k = n − 1 downto h do
      for l = 0 to v − 1 do compute b_{B_l}(n,k) from relation (6); endfor
   endfor
   for l = v to m do b_{B_l}(n,h) = 0_{B_l}; endfor
   for k = h + 1 to n do
      for l = v to m do compute b_{B_l}(n,k) from relation (5); endfor
   endfor
   if (b_{B_l}(n,h) ≤ ε' 1_{B_l}, ∀l = 0, ..., v − 1) then N_h = n; h = h + 1; endif
   if ((1 − b_i(n,n))(1 − Σ_{r=0}^{n} e^{−λt_M}((λt_M)^r/r!)) ≤ ε) then N'' = n; break; endif
endfor
for j = 1 to M do F̃_i(t_j) = 1 − Σ_{n=0}^{N''} e^{−λt_j}((λt_j)^n/n!) + Σ_{n=0}^{N''} e^{−λt_j}((λt_j)^n/n!) b_i(n,n);
```



Thus, we get from inequalities (12) and (11),

$$0 < e_2(N'') = \sum_{h=1}^{H+1} \sum_{n=N_{h-1}+1}^{N_h} e^{-\lambda t} \frac{(\lambda t)^n}{n!} (b_i(n,n) - b_i^{[h]}(n,n))$$

$$\leq \varepsilon' \sum_{h=1}^{H+1} h \sum_{n=N_{h-1}+1}^{N_h} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \leq \varepsilon' \sum_{h=1}^{H+1} \sum_{n=N_{h-1}+1}^{N_{H+1}} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \leq (H+1)\varepsilon'$$

$$\leq (N'' - 1)\varepsilon' \leq (N-1)\varepsilon'.$$

By choosing $\varepsilon' = \varepsilon/(N-1)$, where $N$ is known a priori, we get $0 \leq e_1(N'') \leq \varepsilon$ and $0 \leq e_2(N'') \leq \varepsilon$ so

$$|E| = |e_1(N'') - e_2(N'')| \leq \varepsilon.$$

The pseudocode of the algorithm is given in Table 1. In this algorithm, the $b_{B_l}^{[h]}(n,k)$ are computed successively for the different values of $h$ and are all stored in the $b_{B_l}(n,k)$ according to Fig. 4 and thanks to Item 2).

*Remark.* The truncation levels $N$, $N'$ and $N''$ are in fact functions of $t$. In order to compute $F_i(t,0)$ for several values of $t$, say $t_1 < \cdots < t_M$, we only

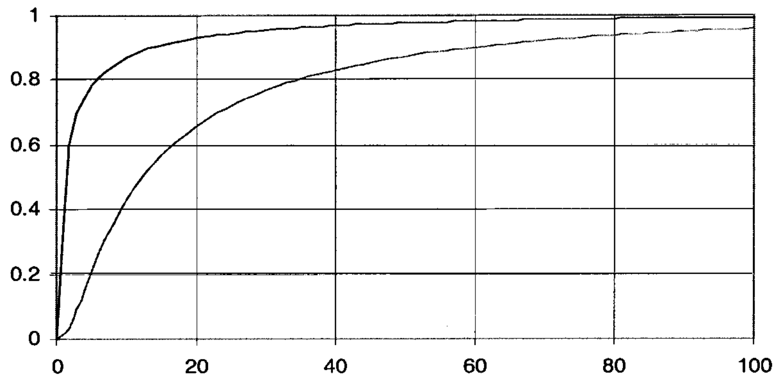


***Figure 5.*** Distribution of the busy period with different initial states.

need to determine these truncation levels for the highest value $t_M$ since the rest of the Poisson series, which is used to bound the errors, is an increasing function of $t$.

## 3. NUMERICAL EXAMPLE

The distribution of the busy period and its dependence on the initial state of the busy period are analyzed in this section. The considered fluid process is generated by $m$ identical on-off sources whose on and off periods are exponentially distributed with parameters $\beta$ and $\gamma$, respectively. The sources generate fluid at rate $\theta$ during their on period, and do not generate any fluid during their off period. The fluid generated by the sources is driven to an infinite buffer whose exit rate is $c$. In this case the Markov chain that determines the fluid accumulation has $m+1$ states. Assuming the states are numbered from 0 to $m$ according to the number of on sources ($Z(t) =$ on-sources) the drift of state $i$ is $i\theta - c$. Since the busy ratio of a source is $\gamma/(\gamma+\beta)$ the utilization of the fluid system is

$$\rho = \frac{\theta m \gamma}{c(\gamma+\beta)}.$$

Figure 5 depicts the distribution of the busy period of the fluid system with the following set of parameters: $m = 4$, $\beta = 1$, $\gamma = 0.2$, $\theta = 1$, $c = 0.8(\rightarrow \rho = 0.833333)$, $Q(0) = 0$, and $\varepsilon = 10^{-5}$. The upper line represents the case when the initial state at the beginning of the busy period is the one with minimal on sources (i.e., $Z(0) = \min\{i | i\theta - c > 0\}$), which is $Z(0) = 1$, while the lower line represents the case when all the sources are in the on state at the beginning of the busy period, i.e., $Z(0) = m$.

To evaluate the benefit of the numerical procedure based on Theorem 4 the same fluid model with identical on-off sources is evaluated with $Z(0) = m$. The obtained uniformization rate is $\lambda = 4$ and the maximal time at which the distribution is evaluated is $t = 100$ (i.e., $\lambda t = 400$).

With these parameters the truncation of the randomization method with respect to $\varepsilon$ is at $N = 488$ and the value of $N'$ truncation is obtained at 470.

The computational cost of an iteration cycle reduces significantly when the procedure using the truncation steps $N_h$ is used. With this procedure, we get $N'' = 470$, $H = 160$. Some of the values of the $N_h$ are: $N_0 = 80$, $N_1 = 88$, $N_{50} = 243$, $N_{100} = 352$, $N_{150} = 450$ and the last one is $N_{160} = 469$. The number of cells whose calculation is avoided is equal to $\sum_{h=0}^{H}(N - N_h) = 27512$. This number represents approximately 25% of the total number of cells, which is $(N'' + 1)\times(N'' + 2)/2 = 111156$.

## APPENDIX A: PROOF OF THEOREM 2

For $x > 0$ and $t \in \left(-\frac{x}{r_j}, \frac{x}{r_{j+1}^+}\right)$ for $j = 0, 1, \ldots, v - 1$, we write the solution of equation (1) for every $i \in S$, as

$$F_i(t,x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} p_j^k (1 - p_j)^{n-k} b_i^{(j)}(n,k),$$

and we determine the relations that must be satisfied by the $b_i^{(j)}(n,k)$. We then have

$$\begin{aligned}
\frac{\partial F_i(t,x)}{\partial t} =& -\lambda F_i(t,x) \\
&+ e^{-\lambda t} \frac{\partial}{\partial t}\left[\sum_{n=1}^{\infty} \frac{\left(\frac{\lambda}{r_j - r_{j+1}^+}\right)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (x + r_j t)^k (-x - r_{j+1}^+ t)^{n-k} b_i^{(j)}(n,k)\right] \\
=& -\lambda F_i(t,x) + e^{-\lambda t}\left[\sum_{n=1}^{\infty} \frac{\left(\frac{\lambda}{r_j - r_{j+1}^+}\right)^n}{n!} \sum_{k=1}^{n} k \binom{n}{k}\right. \\
&\times (x + r_j t)^{k-1} (-x - r_{j+1}^+ t)^{n-k} r_j b_i^{(j)}(n,k) \\
&\left. - \sum_{n=1}^{\infty} \frac{\left(\frac{\lambda}{r_j - r_{j+1}^+}\right)^n}{n!} \sum_{k=0}^{n-1} (n-k) \binom{n}{k} (x + r_j t)^k (-x - r_{j+1}^+ t)^{n-k-1} r_{j+1}^+ b_i^{(j)}(n,k)\right]
\end{aligned}$$





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which leads to

$$\frac{\partial F_i(t,x)}{\partial t} = -\lambda F_i(t,x) + \frac{\lambda}{r_j - r_{j+1}^+}\sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{k=0}^{n}\binom{n}{k} p_j^k (1-p_j)^{n-k}$$

$$\times (r_j b_i^{(j)}(n+1,k+1) - r_{j+1}^+ b_i^{(j)}(n+1,k)).$$

In the same way, we have

$$\frac{\partial F_i(t,x)}{\partial x} = \frac{\lambda}{r_j - r_{j+1}^+}\sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{k=0}^{n}\binom{n}{k} p_j^k (1-p_j)^{n-k}$$

$$\times (b_i^{(j)}(n+1,k+1) - b_i^{(j)}(n+1,k)).$$

Using the uniformization technique, we have

$$\sum_{r\in S} a_{ir} F_r(t,x) = -\lambda F_i(t,x) + \lambda \sum_{r\in S} p_{ir} F_r(t,x).$$

It follows that if the $b_i^{(j)}(n,k)$ are such that

$$(r_j - d_i) b_i^{(j)}(n+1,k+1) + (d_i - r_{j+1}^+) b_i^{(j)}(n+1,k)$$

$$= (r_j - r_{j+1}^+)\sum_{r\in S} p_{ir} b_r^{(j)}(n,k) \qquad (13)$$

then equation (1) is satisfied.

The recurrence relation (13) can also be written as follows, for $j = 0, \ldots, v-1$.

For $i \in B_{j+1} \cup \cdots \cup B_m$,

$$b_i^{(j)}(n,k) = \frac{d_i - r_{j+1}^+}{d_i - r_j} b_i^{(j)}(n,k-1) + \frac{r_{j+1}^+ - r_j}{d_i - r_j}\sum_{r\in S} p_{ir} b_r^{(j)}(n-1,k-1),$$

and for $i \in B_0 \cup \cdots \cup B_j$,

$$b_i^{(j)}(n,k) = \frac{r_j - d_i}{r_{j+1}^+ - d_i} b_i^{(j)}(n,k+1) + \frac{r_{j+1}^+ - r_j}{r_{j+1}^+ - d_i}\sum_{r\in S} p_{ir} b_r^{(j)}(n-1,k).$$

Using matrix and vector notations, we get for $j = 0, \ldots, v-1$ and for $j+1 \le l \le m$ :

$$b_{B_l}^{(j)}(n,k) = \frac{r_l - r_{j+1}^+}{r_l - r_j} b_{B_l}^{(j)}(n,k-1) + \frac{r_{j+1}^+ - r_j}{r_l - r_j} \sum_{i=0}^{m} P_{B_l B_i} b_{B_i}^{(j)}(n-1,k-1),$$

for $0 \le l \le j$ :

$$b_{B_l}^{(j)}(n,k) = \frac{r_j - r_l}{r_{j+1}^+ - r_l} b_{B_l}^{(j)}(n,k+1) + \frac{r_{j+1}^+ - r_j}{r_{j+1}^+ - r_l} \sum_{i=0}^{m} P_{B_l B_i} b_{B_i}^{(j)}(n-1,k).$$

To get the initial conditions for the $b_i^{(j)}(n,k)$, we consider the jumps of $F_i(t,x)$ given by relation (2) in which we write

$$e^{-A_{B_j B_j}\frac{x}{r_j}} = \sum_{n=0}^{\infty} e^{\lambda \frac{x}{r_j}} \frac{\left(-\lambda \frac{x}{r_j}\right)^n}{n!} P_{B_j B_j}^n.$$

For every $j = 0, 1, \ldots, v-1$ we have $p_j = 0$ when $t = -x/r_j$ and $p_j \to 1$ when $t \to -x/r_{j+1}^+, t < -x/r_{j+1}^+$, and so

$$F_i\left(-\frac{x}{r_j}, x\right) = \sum_{n=0}^{\infty} e^{\lambda \frac{x}{r_j}} \frac{\left(-\lambda \frac{x}{r_j}\right)^n}{n!} b_i^{(j)}(n,0)$$

and, for $j < v-1$,

$$\lim_{t \to \frac{-x}{r_{j+1}}, t < \frac{x}{r_{j+1}}} F_i(t,x) = \sum_{n=0}^{\infty} e^{\lambda \frac{x}{r_{j+1}}} \frac{\left(-\lambda \frac{x}{r_{j+1}}\right)^n}{n!} b_i^{(j)}(n,n).$$

For $j = 0$, we get from relation (2),

$$b_i^{(0)}(n,0) = \begin{cases} h_{B_0}(i) P_{B_0 B_0}^n 1_{B_0} & \text{if } i \in B_0, \\ 0 & \text{otherwise.} \end{cases}$$

For $j = 1, \ldots, v-1$, we get

$$F_i\left(-\frac{x}{r_j}, x\right) = \begin{cases} \lim_{t \to \frac{-x}{r_j}, t < \frac{-x}{r_j}} F_i(t,x) + h_{B_j}(i) e^{-A_{B_j B_j}\frac{x}{r_j}} & \text{if } i \in B_j, \\ \lim_{t \to \frac{-x}{r_j}, t < \frac{-x}{r_j}} F_i(t,x) & \text{otherwise.} \end{cases}$$

From relation (2) it follows that

$$b_i^{(j)}(n,0) = \begin{cases} b_i^{(j-1)}(n,n) + h_{B_j}(i) P^n_{B_j B_j} 1_{B_j} & \text{if } i \in B_j, \\ b_i^{(j-1)}(n,n) & \text{otherwise,} \end{cases}$$

that is,

$$b_{B_j}^{(j)}(n,0) = b_{B_j}^{(j-1)}(n,n) + P^n_{B_j B_j} 1_{B_j},$$

$$b_{B_l}^{(j)}(n,0) = b_{B_l}^{(j-1)}(n,n) \text{ for } l \neq j.$$

Last we consider the case where $j = v-1$, that is when

$$t \in \left[\frac{-x}{r_{v-1}}, +\infty\right).$$

In this case, since $r_{j+1}^{+} = 0$, we get when $x \to 0$, with $x > 0$, $p_j \to 1$ and so

$$\lim_{x \to 0, x > 0} F_i(t,x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} b_i^{(v-1)}(n,n).$$

It follows from the initial condition expressed in Theorem 1 that

$$b_i^{(v-1)}(n,n) = 1 \text{ if } d_i \leq 0,$$

that is

$$b_{B_l}^{(v-1)}(n,n) = 1_{B_l} \text{ for } l = 0, \ldots, v-1.$$

## APPENDIX B: PROOF OF THEOREM 4

(a) These inequalities are immediate since the relations (5) and (6) are convex combinations of the vectors $b_{B_l}(n,k)$ that initially have their entries equal to 0 or 1.

(b) The relation is immediate for $n = 0$ since we have $b_{B_l}(1,1) \geq 0 = b_{B_l}(0,0)$ for $l = v, \ldots, m$ and $b_{B_l}(1,1) = b_{B_l}(0,0) = 1_{B_l}$ for $l = 0, \ldots, v-1$.

Suppose the relation is satisfied at level $n-1$, $n \geq 1$, that is suppose that for all $l = 0, \ldots, m$ and $0 \leq k \leq n$, we have $b_{B_l}(n,k) \geq b_{B_l}(n-1,k-1)$.

For $l = 0, \dots, v-1$, we have $b_{B_l}(n+1, n+1) = b_{B_l}(n, n) = 1_{B_l}$ which means that the relation is satisfied at level $n$ for $k = n$. Suppose the relation is satisfied at level $n$ for the integer $k+1$, that is suppose that we have $b_{B_l}(n+1, k+2) \geq b_{B_l}(n, k+1)$. Let us define

$$p = \frac{r_l - r_{v-1}}{r_l}.$$

We have $p \in [0, 1]$. Using the relation (6), we get

$$\begin{aligned} b_{B_l}(n+1, k+1) \; &- b_{B_l}(n, k) = p(b_{B_l}(n+1, k+2) - b_{B_l}(n, k+1)) \\ &+ (1-p)\sum_{i=0}^{m} P_{B_l B_i}(b_{B_i}(n, k+1) - b_{B_i}(n-1, k)) \geq 0, \end{aligned}$$

from the recurrence hypothesis.

For $l = v, \dots, m$, we have $b_{B_l}(n+1, 1) \geq 0 = b_{B_l}(n, 0)$ which means that the relation is satisfied at level $n$ for $k - 0$. Suppose the relation is satisfied at level $n$ for the integer $k-1$, that is suppose that we have $b_{B_l}(n+1, k) \geq b_{B_l}(n, k-1)$. Let us define

$$q = \frac{r_l}{r_l - r_{v-1}}.$$

We have $q \in [0, 1]$. Using the relation (6), we get

$$\begin{aligned} b_{B_l}(n+1, k+1) \; &- b_{B_l}(n, k) = q(b_{B_l}(n+1, k) - b_{B_l}(n, k-1)) \\ &+ (1-q)\sum_{i=0}^{m} P_{B_l B_i}(b_{B_i}(n, k) - b_{B_i}(n-1, k)) \leq 0, \end{aligned}$$

from the recurrence hypothesis.

(c) The relation is immediate for $n = 0$ since we have $b_{B_l}(1, 0) = b_{B_l}(0, 0) = 0$ for $l = v, \dots, m$, and $b_{B_l}(1, 0) \leq b_{B_l}(0, 0) = 1B_l$, for $1 = 0, \dots, v-1$.

Suppose the relation is satisfied at level $n-1$, $n \geq 1$, that is suppose that for all $l = 0, \dots, m$ and $0 \leq k \leq n-1$, we have $b_{B_l}(n, k) \leq b_{B_l}(n-1, k)$.

For $l = 0, \dots, v-1$, we have $b_{B_l}(n+1, n) \leq b_{B_l}(n, n) = 1_{B_l}$ which means that the relation is satisfied at level $n$ for $k = n$. Suppose the relation is satisfied at level $n$ for the integer $k+1$, that is suppose that we have $b_{B_l}(n+1, k+1) \leq b_{B_l}(n, k+1)$. Let us define

$$p = \frac{r_l - r_{v-1}}{r_l}.$$

We have $p \in [0,1]$. Using the relation (6), we get

$$b_{B_l}(n+1,k) - b_{B_l}(n,k) = p(b_{B_l}(n+1,k+1) - b_{B_l}(n,k+1)) + (1-p)\sum_{i=0}^{m} P_{B_l B_i}(b_{B_i}(n,k) - b_{B_i}(n-1,k)) \leq 0,$$

from the recurrence hypothesis.

For $l = v, \ldots, m$, we have $b_{B_l}(n+1,0) = b_{B_l}(n,0) = 0$ which means that the relation is satisfied at level $n$ for the $k = 0$. Suppose the relation is satisfied at level $n$ for the integer $k-1$, that is suppose that we have $b_{B_l}(n+1,k-1) \leq b_{B_l}(n,k-1)$. Let us define

$$q = \frac{r_l}{r_l - r_{v-1}}.$$

We have $q \in [0,1]$. Using the relation (5), we get

$$b_{B_l}(n+1,k) - b_{B_l}(n,k) = q(b_{B_l}(n+1,k-1) - b_{B_l}(n,k-1)) + (1-q)\sum_{i-0}^{m} P_{B_l B_i}(b_{B_i}(n,k-1) - b_{B_i}(n-1,k-1)) \leq 0,$$

from the recurrence hypothesis.

(d) directly follows from (b) and (c).

(e) directly follows from (c) and (d).

(f) From inequality (b), we deduce that for every $i \in S$, the sequence $b_i(n,n)$ is increasing. Moreover, from inequality (a), we have $b_i(n,n) \leq 1$ so, the sequence $b_i(n,n)$ converges when $n$ goes to infinity. For every $i \in S$, we denote by $l_i$ the limit of the sequence $b_i(n,n)$. We then have

$$F_i(t,0) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} b_i(n,n) \to l_i \text{ when } t \to \infty.$$

In another hand, since we have assumed that the stability condition $\sum_{i\in S} d_i \pi_i < 0$ is satisfied, we have $F_i(t,0) \to 1$ when $t \to \infty$. Thus, we conclude that for every $i \in S, l_i = 1$.

## REFERENCES


1. Roberts, J.W.; Mocci, U.; Virtamo, J. (Eds). *Broadband Network Teletraffic. Performance Evaluation and Design of Broadband Multiservice Networks*; Lecture Notes in Computer Science. Springer: Heidelberg, 1995; Vol. 1155. Final report of the COST 242 action.
2. Ren, Q.; Kobayashi, H. Transient Solutions for the Buffer Behavior in Statistical Multiplexing. Perf. Eval. **1995**, *23*, 65–87.

3. Kobayashi, H.; Ren, Q. A Mathematical Theory for Transient Analysis of Communication Networks. IEICE Trans. Commun. **1992**, *E75-B* (12), 1266–1276.
4. Anick, D.; Mitra, D.; Sondhi, M.M. Stochastic Theory of a Data-Handling System with Multiple Sources. Bell System Tech. J. **1982**, *61* (8), 1871–1894.
5. Sericola, B. Transient Analysis of Stochastic Fluid Models. Perf. Eval. **1998**, *23*, 245–263.
6. Narayanan, A.; Kulkarni, V.G. First Passages Times in Fluid Models with an Application to Two Priority Fluid Systems. *Proceedings of IPDS'96, Urbana-Champaign, Illinois, USA, Sept. 4–6, 1996*; IEEE Computer Society Press: Los Alamitos, CA, 1996; 166–175.

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